A new numerical quadrature formula on the unit circle

E. Berriochoa · A. Cachafeiro · F. Marcellán

Abstract In this paper we study a quadrature formula for Bernstein–Szegő measures on the unit circle with a fixed number of nodes and unlimited exactness. Taking into account that the Bernstein–Szegő measures are very suitable for approximating an important class of measures we also present a quadrature formula for this type of measures such that the error can be controlled with a well-bounded formula.

Keywords Orthogonal polynomials · Szegő quadrature · Numerical integration

Mathematics Subject Classifications (2000) 33C47 · 42C05 · 65D30

This work was supported by Ministerio de Educación y Ciencia under grants number MTM2005-01320 (E. B. and A. C.) and MTM2006-13000-C03-02 (F. M.).

E. Berriochoa
Departamento de Matemática Aplicada I, Facultad de Ciencias,
Universidad de Vigo, 32004 Ourense, Spain
e-mail: esnaola@uvigo.es

A. Cachafeiro
Departamento de Matemática Aplicada I, E.T.S. Ingenieros Industriales,
Universidad de Vigo, 36310 Vigo, Spain
e-mail: acachafe@uvigo.es

F. Marcellán (✉)
Departamento de Matemáticas, Universidad Carlos III, 28911 Leganés, Spain
e-mail: pacomarc@ing.uc3m.es

1 Introduction

Gaussian quadrature formulas for measures supported on the real line constitute a classical subject that has been widely studied. It is well-known that Gaussian quadrature formulas allow to compute in an exact way the integrals of polynomials up to a certain degree depending on the nodes and the quadrature (Christoffel numbers) coefficients. Indeed, if $d\mu$ is a positive measure supported on the real line and $\{P_n(x)\}_{n\in\mathbb{N}}$ is the monic orthogonal polynomial sequence with respect to $d\mu$, then the quadrature formula $I_n(f) = \sum_{i=1}^{n} \lambda_j f(x_j)$, where $\{x_j\}_{j=1}^{n}$ are the zeros of $P_n(x)$, is such that $I_n(P) = \int_{\mathrm{supp}(\mu)} P(x)d\mu(x)$ for any polynomial P of degree less than or equal to $2n-1$. See for instance the monograph [4].

In the case of measures supported on the unit circle the analogue formulas are the so-called Szegő quadrature formulas. Now the nodes are the zeros of the para-orthogonal polynomials. These zeros have modulus 1, and the maximal domain of exactness in a space of Laurent polynomials depends on the nodes and the quadrature coefficients. If $d\mu$ is a positive measure on the unit circle $\mathbb{T}$, an n-point Szegő quadrature formula has the following form $I_n(f) = \sum_{j=1}^{n} A_j f(z_j)$, where the nodes belong to $\mathbb{T}$, that is, $|z_i| = 1,\ \forall i = 1, \cdots, n$, and $z_i \neq z_j$ for $i \neq j$. If $\{\Phi_n(z)\}_{n\in\mathbb{N}}$ is the monic orthogonal polynomial sequence with respect to $d\mu$ and the nodes are the zeros of the para-orthogonal polynomials $\Phi_n(z) + \tau\Phi_n^*(z)$, with $|\tau| = 1$, then $I_n(P) = \int_{\mathbb{T}} P(z)d\mu(z)$ for P belonging to the space of Laurent polynomials $\Lambda_{-(n-1),n-1}$. See for instance the following references: [1, 6, 7, 9], and [5]. Hence, in both situations, real line and unit circle, the construction of quadrature formulas needs the knowledge of the zeros of orthogonal or para-orthogonal polynomials and for increasing the dimension of the spaces of exactness new computations need to be done.

An alternative way of computing integrals with respect to measures supported on the unit circle is based on the zeros of the orthogonal polynomials, (see for example [2]). This way is particularly interesting when the measures of integration are of Bernstein–Szegő type. We will deal with such measures along this paper. The class of Bernstein–Szegő measures is an important class of measures on the unit circle such that the corresponding monic orthogonal polynomial sequences follow in a simple way. In this paper we construct a new quadrature formula for this type of measures which uses as nodes the zeros of an orthogonal polynomial of a fixed degree and which is exact for all polynomials. The quadrature formulas with more than very few nodes may be difficult to use because the weights are difficult to compute accurately, and the weights may be of large magnitude and therefore amplify errors in the function values. In the Gaussian numerical integration theory, a quadrature formula is said to be optimal if it is exact in a linear subspace of polynomials of degree as large as possible and it is asymptotically exact. In this sense the situation described for Bernstein–Szegő measures is better than the standard approach.

Moreover, since we can approach measures supported on $\mathbb{T}$ by Bernstein–Szegő measures, our method is a powerful tool for computing integrals for very general measures, for example those such that the absolutely continuous part of the measure is the inverse of an analytic function with a fast rate of convergence on $\mathbb{T}$.

The paper is organized as follows: In Section 2 we study quadrature formulas for Bernstein–Szegő measures. Indeed we prove that for each Bernstein–Szegő measure there exists a quadrature formula such that it is exact for all polynomials. The novelty is that we use the zeros of the orthogonal polynomials as nodes and we obtain exactness. Taking into account that the Bernstein–Szegő measures are very suitable for approximating an important class of measures we use their properties for studying new quadrature formulas. Thus, in Section 3, we present quadrature formulas for this new class of measures and we prove that the error can be controlled with a well-bounded formula. Finally, in Section 4 we show some numerical examples in order to point out the strength of these quadrature formulas.

2 Exact quadrature formulas for Bernstein–Szegő measures

The Bernstein–Szegő measures play a very important role in the theory of orthogonal polynomials on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. They are absolutely continuous measures with respect to the Lebesgue normalized measure:

$$d\mu(\theta) = \frac{d\theta}{2\pi|Q(e^{i\theta})|^2}, \tag{1}$$

where $Q(z)$ is an algebraic polynomial with zeros outside the closed unit disk $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$, i.e., they are rational modifications of the Lebesgue measure. For simplicity, in the sequel and throughout all of the paper, we will denote by $\frac{|dz|}{2\pi}$ the Lebesgue normalized measure on $\mathbb{T}$. In this sense the previous measure is written as $d\mu(z) = \frac{|dz|}{2\pi|Q(z)|^2}$ for $z \in \mathbb{T}$.

Several results concerning these measures and the corresponding sequences of orthogonal polynomials are well-known, (see [3] and [11]). Among them we point out:

(1) If the polynomial $Q(z)$ has degree m then the sequence of monic orthogonal polynomials with respect to the measure (1), $\{\Phi_n(z)\}_{n\in\mathbb{N}}$, is given by, (see [11]),

$$\Phi_n(z) = \frac{1}{\overline{Q(0)}} z^{n-m} Q^*(z), \quad \forall n \geq m.$$

We recall that the $*$ operator is defined by $P^*(z) = z^n \overline{P}(\frac{1}{z})$, assuming the degree of $P(z)$ is n, (see [3]).

Since $q_m(z) = \dfrac{1}{Q(0)} Q^*(z)$ is a monic polynomial, with zeros inside the unit disk $\mathbb{D}$, we can write

$$\Phi_n(z) = z^{n-m} q_m(z), \quad \forall n \geq m.$$

Notice that the zeros of $q_m(z)$ can be either simple or multiple.

(2) The Bernstein–Szegő measures are very suitable in order to approximate another type of measures. An important result concerning this fact is the following (see [10]).
Let $d\tau$ be a probability measure supported on $\partial\mathbb{D} = \mathbb{T}$ with orthonormal polynomial sequence $\{\chi_n\}_{n\in\mathbb{N}}$. Then for each n we get:

1.
$$\int_0^{2\pi} \frac{d\theta}{2\pi |\chi_n(e^{i\theta})|^2} = 1,$$

2.
$$d\tau_n(\theta) = \frac{d\theta}{2\pi |\chi_n(e^{i\theta})|^2} \to d\tau \;\text{ weakly when }\; n \to \infty.$$

In order to prove the main result concerning the existence of this new quadrature formula for Bernstein–Szegő measures, next we introduce some notation.

If $d\mu$ is a Bernstein–Szegő measure given by (1) we denote by $\{\Phi_n\}_{n\in\mathbb{N}}$ the monic orthogonal polynomial sequence and by $\{\varphi_n\}_{n\in\mathbb{N}}$ the sequence of orthonormal polynomials. If $\Phi_n(z) = z^{n-m} \prod_{i=1}^{s} (z - z_i)^{\nu_i}$, for $n \geq m$, with $z_i \neq z_j$ for $i \neq j$ and $\sum_{i=1}^{s} \nu_i = m$ we also denote by $\mathbf{V_m}$ the following Vandermonde matrix where the evaluations of the polynomials $\varphi_0(z), \cdots, \varphi_n(z)$ and its derivatives in the zeros $z_1, \cdots, z_s$ appear, i.e.

$$\mathbf{V_m} = \begin{pmatrix} \varphi_0(z_1) & \varphi_1(z_1) & \cdots & \varphi_{m-1}(z_1) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_0^{(\nu_1-1)}(z_1) & \varphi_1^{(\nu_1-1)}(z_1) & \cdots & \varphi_{m-1}^{(\nu_1-1)}(z_1) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_0(z_s) & \varphi_1(z_s) & \cdots & \varphi_{m-1}(z_s) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_0^{(\nu_s-1)}(z_s) & \varphi_1^{(\nu_s-1)}(z_s) & \cdots & \varphi_{m-1}^{(\nu_s-1)}(z_s) \end{pmatrix}. \tag{2}$$

Theorem 2.1 *Let $d\mu$ be a Bernstein–Szegő measure (1) such that its monic orthogonal polynomial sequence is $\Phi_n(z) = z^{n-m} \prod_{i=1}^{s} (z - z_i)^{\nu_i}$, for $n \geq m$, with $z_i \neq z_j$ for $i \neq j$ and $\sum_{i=1}^{s} \nu_i = m$. Then there exists a quadrature formula with nodes $\{z_1, \cdots, z_s\}$ which uses the values of the function and its derivatives* *in these nodes and such that it exactly integrates functions in the space of polynomials* $\mathbb{P}$, *that is,*

$$\int_{\mathbb{T}} P(z)d\mu(z) = \sum_{i=1}^{s}\sum_{j=0}^{\nu_i-1} \lambda_{i,j}P^{(j)}(z_i), \quad \textit{for every} \quad P(z) \in \mathbb{P}. \tag{3}$$

The coefficients $\{\lambda_{i,j}\}_{i=1,\cdots,s;\, j=0,\cdots,\nu_i-1}$ *are the entries of the first row of the inverse of the matrix* $\mathbf{V_m}$ *defined by* (2). *Moreover,* $\lambda_{i,\nu_i-1} \neq 0$ *for* $i = 1, \cdots, s$.

Proof Let P be a polynomial in $\mathbb{P}$. If the degree of P is M, we can write $P(z) = \sum_{k=0}^{M} a_k\varphi_k(z)$, where we denote by $\{\varphi_n\}_{n\in\mathbb{N}}$ the sequence of orthonormal polynomials with respect to the measure $d\mu$. Thus we get

$$\int_{\mathbb{T}} P(z)d\mu(z) = \int_{\mathbb{T}} \left(\sum_{k=0}^{M} a_k\varphi_k(z)\right) d\mu(z) = \sum_{k=0}^{M} a_k \int_{\mathbb{T}} \varphi_k(z)d\mu(z) = a_0.$$

On the other hand, if we compute the values of $P(z)$ and its derivatives in the nodes z_i, then we have for $1 \le i \le s$ and $0 \le j \le \nu_i - 1$

$$P^{(j)}(z_i) = \sum_{k=0}^{M} a_k\varphi_k^{(j)}(z_i) = \sum_{k=0}^{m-1} a_k\varphi_k^{(j)}(z_i). \tag{4}$$

Thus we have a linear system of m equations and m unknowns $a_0, \cdots, a_{m-1}$, which should be written in matrix form as follows

$$\mathbf{V_m} A_m = P_m,$$

where $\mathbf{V_m}$ was introduced in (2),

$$A_m = (a_0, \cdots, a_{\nu_1-1}, \cdots, a_{m-1})^T$$

and

$$P_m = \left(P(z_1), \cdots, P^{(\nu_1-1)}(z_1), \cdots, P^{(\nu_s-1)}(z_s)\right)^T.$$

Since the matrix of coefficients is non-singular, the system has a unique solution. If we denote by $(\lambda_{1,0}, \cdots, \lambda_{1,\nu_1-1}, \cdots, \lambda_{s,0}, \cdots, \lambda_{s,\nu_s-1})$ the first row of the inverse of $\mathbf{V_m}$ we get

$$a_0 = \sum_{i=1}^{s}\sum_{j=0}^{\nu_i-1} \lambda_{i,j}P^{(j)}(z_i).$$

Therefore our quadrature formula is exact in $\mathbb{P}$ because

$$\int_{\mathbb{T}} P(z)d\mu(z) = \sum_{i=1}^{s}\sum_{j=0}^{\nu_i-1} \lambda_{i,j}P^{(j)}(z_i).$$

In order to prove the property of the quadrature coefficients, let assume that $\lambda_{i,\nu_i-1} = 0$ for some i. For simplicity, take $i = 1$, that is, $\lambda_{1,\nu_1-1} = 0$. Then the quadrature formula becomes

$$\int_{\mathbb{T}} P(z)d\mu(z) = \sum_{j=0}^{\nu_1-2} \lambda_{1,j}P^{(j)}(z_1) + \sum_{i=2}^{s}\sum_{j=0}^{\nu_i-1} \lambda_{i,j}P^{(j)}(z_i), \quad \text{for every} \quad P(z) \in \mathbb{P}.$$

Now we consider the polynomial $q_{m-1}(z) = (z-z_1)^{\nu_1 - 1}(z-z_2)^{\nu_2} \cdots (z-z_s)^{\nu_s} \in \mathbb{P}_{m-1}$ and we compute $\int_{\mathbb{T}} z^m q_{m-1}(z) \overline{z}^k d\mu(z) = \int_{\mathbb{T}} z^{m-k} q_{m-1}(z) d\mu(z)$ for $k = 0, \cdots, m$. Since the quadrature formula is exact we deduce

$$\int_{\mathbb{T}} z^{m-k} q_{m-1}(z) d\mu(z) = \sum_{j=0}^{\nu_1 - 2} \lambda_{1,j} (z^{m-k} q_{m-1}(z))^{(j)}(z_1) + \sum_{i=2}^{s} \sum_{j=0}^{\nu_i - 1} \lambda_{i,j} (z^{m-k} q_{m-1}(z))^{(j)}(z_i) = 0.$$

Thus $z^m q_{m-1}(z)$ is orthogonal to z^k for $k = 0, \cdots, m$ and therefore it can be written like

$$z^m q_{m-1}(z) = \sum_{k=m}^{2m-1} a_k \Phi_k(z) = \Phi_m(z) R(z)$$

with $R(z) \in \mathbb{P}_{m-1}$.
Hence $z^m = (z - z_1) R(z)$ and since $z_1 \neq 0$, (notice that $\Phi_m(z) = \frac{1}{Q(0)} Q^*(z)$), we get a contradiction. Therefore $\lambda_{1,\nu_1 - 1} \neq 0$. □

Some remarks concerning the preceding theorem can be pointed out:

1. The particular case corresponding to one of the simplest Bernstein–Szegő measures, that is, when $m = 1$ is mentioned in [10], p. 129.
2. Although the preceding quadrature is not always interpolatory it is exact in the whole space of polynomials.
3. The quadrature coefficients and the nodes are determined by the coefficients of the polynomial $\varphi_m(z)$.
4. In the particular case when the nodes are simple, the quadrature formula is exact in the space $\mathbb{P}_{m-1}$ of polynomials of degree less than or equal to $m - 1$ and, therefore, it is an interpolatory quadrature which is exact in the linear space $\mathbb{P}$ of polynomials. Moreover, in this situation the coefficients $\lambda_i \neq 0$ for $i = 1, \cdots, m$.

3 Extension of the quadrature formulas to more general measures

Taking into account that the trigonometric polynomials are dense in the space of periodic continuous functions on $[0, 2\pi]$, we are going to use the Bernstein–Szegő measures to approximate an important class of measures.

Lemma 3.1 *Let $d\nu$ be a measure supported on $\mathbb{T}$, absolutely continuous with respect to the Lebesgue normalized measure, with weight function w, that is, $d\nu(z) = \frac{1}{2\pi} w(z)|dz|$.*

If we assume that w is continuous, then given $\epsilon > 0$ there exists a Bernstein–Szegő measure $d\mu(z) = \frac{|dz|}{2\pi|Q(z)|^2}$ such that

$$\left| w(z) - \frac{1}{|Q(z)|^2} \right| < \epsilon, \quad \textit{for every} \;\; z \in \mathbb{T}.$$

Proof It is a consequence of the density of the trigonometric polynomials in the space of periodic continuous functions on $[0, 2\pi]$ and the results by Fejér and Riesz about the representation of nonnegative trigonometric polynomials, (see [8]). □

Now we are in a position to prove the main result.

Theorem 3.2 *Let $d\nu$ be a measure supported on $\mathbb{T}$, absolutely continuous with respect to the Lebesgue normalized measure, with weight function w, which is continuous on $\mathbb{T}$. Given $\epsilon > 0$ there exists a quadrature formula, which uses $m = m(\epsilon, \nu)$ values of the function and its derivatives and m quadrature coefficients, of the following type:*

$$I_m(P) = \sum_{i=1}^{s} \sum_{j=0}^{\nu_i - 1} \lambda_{i,j} P^{(j)}(z_i).$$

Furthermore,

$$\left| \int_{\mathbb{T}} P(z) d\nu(z) - \sum_{i=1}^{s} \sum_{j=0}^{\nu_i - 1} \lambda_{i,j} P^{(j)}(z_i) \right| < \epsilon \parallel P \parallel_\infty, \quad \textit{for every} \;\; P \in \mathbb{P}. \tag{5}$$

Proof Applying the previous lemma, given $\epsilon > 0$ there exists a Bernstein–Szegő measure $d\mu(z) = \frac{|dz|}{2\pi|Q(z)|^2}$ such that $|w(z) - \frac{1}{|Q(z)|^2}| < \epsilon, \;\; \forall z \in \mathbb{T}$.

If the degree of $Q(z)$ is m, then we consider the zeros $z_1, \cdots, z_s$ of the corresponding orthogonal polynomial of degree m according to their multiplicities $\nu_1, \cdots, \nu_s$ and we construct the quadrature formula given in Theorem 2.1

$$I_m(P) = \sum_{i=1}^{s} \sum_{j=0}^{\nu_i - 1} \lambda_{i,j} P^{(j)}(z_i),$$

which is exact in $\mathbb{P}$. Then for $P \in \mathbb{P}$ we get

$$\left| \int_{\mathbb{T}} P(z) d\nu(z) - I_m(P) \right| = \left| \int_{\mathbb{T}} P(z) \left(w(z) - \frac{1}{|Q(z)|^2} \right) \frac{|dz|}{2\pi} \right| \leq \epsilon \parallel P \parallel_\infty . \qquad \square$$

Notice that we can use the preceding result for a large class of measures, that is, those that can be approximated by continuous weights on $\mathbb{T}$. For example, measures with associated weight function, which are continuous up to a finite number of points.

Corollary 1 *Let $d\nu_1$ be a measure on $\mathbb{T}$ absolutely continuous with respect to the Lebesgue normalized measure, with weight function w_1, such that, given $\epsilon_1 > 0$ and $\epsilon_2 > 0$ there exists a measure $d\nu$ on $\mathbb{T}$ absolutely continuous with respect to the Lebesgue normalized measure, with weight function w, which is continuous and such that $|w_1(z) - w(z)| < \epsilon_1$ up to a set of points with $d\nu_1$-measure and $d\nu$-measure less than ϵ_2.*

In such conditions given $\epsilon > 0$ there exists a quadrature formula which uses $m = m(\epsilon, \nu_1)$ values of the function and its derivatives and m quadrature coefficients such that

$$\left| \int_{\mathbb{T}} P(z) d\nu_1(z) - I_m(P) \right| \leq 2\epsilon \parallel P \parallel_\infty, \quad \textit{for every} \quad P \in \mathbb{P}.$$

Proof Given $\epsilon > 0$, let $\epsilon_1 > 0$ and $\epsilon_2 > 0$ be such that $\epsilon_1 + \epsilon_2 \leq \epsilon$. Now we choose the measure $d\nu$ satisfying the hypothesis. We apply to this measure Theorem 2 and we use the quadrature formula I_m in such a way that

$$\left| \int_{\mathbb{T}} P(z) d\nu(z) - I_m(P) \right| < \epsilon \parallel P \parallel_\infty, \quad \text{for every} \quad P \in \mathbb{P}.$$

Therefore

$$\begin{aligned} &\left| \int_{\mathbb{T}} P(z) d\nu_1(z) - I_m(P) \right| \\ &\quad = \left| \int_{\mathbb{T}} P(z) d\nu_1(z) - \int_{\mathbb{T}} P(z) d\nu(z) + \int_{\mathbb{T}} P(z) d\nu(z) - I_m(P) \right| \\ &\quad \leq \left| \int_{\mathbb{T}} P(z)(w_1(z) - w(z)) \frac{|dz|}{2\pi} \right| + \left| \int_{\mathbb{T}} P(z) d\nu(z) - I_m(P) \right| \\ &\quad < (\epsilon_1 + \epsilon_2) \parallel P \parallel_\infty + \epsilon \parallel P \parallel_\infty \leq 2\epsilon \parallel P \parallel_\infty . \end{aligned}$$

□

3.1 Applications

1. Let $P(\theta)$ be a trigonometric polynomial and let $w(\theta)$ be a weight function with a finite number of points of discontinuity on $[0, 2\pi]$. We are going to apply the preceding lemma and theorem to approximate the following integral:

   $$\int_0^{2\pi} P(\theta) w(\theta) d\theta.$$

   First we approximate the measure by a Bernstein–Szegő measure $d\mu$ and we determine the corresponding quadrature formula I_m.

If $P(\theta) = \sum_{k=0}^{M}(a_k \cos k\theta + b_k \sin k\theta)$, then for $z = e^{i\theta}$ we can write

$$P(\theta) = \sum_{k=0}^{M}\left(a_k \frac{z^k + z^{-k}}{2} + b_k \frac{z^k - z^{-k}}{2i}\right)$$
$$= \sum_{k=0}^{M}\left(\frac{a_k - ib_k}{2}\right) z^k + \sum_{k=0}^{M}\left(\frac{a_k + ib_k}{2}\right) z^{-k}.$$

Now, if we denote by $F(z) = \sum_{k=0}^{M}\left(\frac{a_k - ib_k}{2}\right) z^k$ and $G(\overline{z}) = \sum_{k=0}^{M}\left(\frac{a_k + ib_k}{2}\right) z^{-k}$ then using the quadrature formula I_m we compute

$$\int_{\mathbb{T}} (F(z) + G(\overline{z}))d\mu(z).$$

Hence we have approximated $\int_0^{2\pi} P(\theta)w(\theta)d\theta$.

2. Taking into account that a large class of functions can be approximated by polynomials on $\mathbb{T}$, and the fact that the quadrature formulas allow to integrate polynomials in $\overline{z}$, one can conclude that the domain of application of these quadratures is very wide.

4 Numerical examples

Let us consider a measure $d\nu$ which is a polynomial modification of the Lebesgue measure, that is, we consider the measure $d\nu(z) = \frac{1}{2\pi} w(z)|dz|$ with weight function $w(z) = |z - .2|^2$. First we are going to approximate the measure $d\nu$ by a Bernstein–Szegő measure $d\mu$ of the form $d\mu(z) = \frac{1}{2\pi |Q(z)|^2}|dz|$. According to Lemma 1, if we choose $\varepsilon = 10^{-7}$ then it is possible to obtain $Q(z) \in \mathbb{P}$ such that

$$\left| w(z) - \frac{1}{|Q(z)|^2} \right| < 10^{-7}, \quad \text{for every} \quad z \in \mathbb{T}.$$

Indeed, for $z \in \mathbb{T}$ we have $|z - .2|^2 = |1 - .2z|^2$ and $|1 - .2z|^2 = \frac{1}{|\sum_{n=0}^{\infty}(.2z)^n|^2}$. If we take $Q(z) = \sum_{n=0}^{10}(.2z)^n = \frac{1 - .2^{11}z^{11}}{1 - .2z}$ we deduce

$$\left| w(z) - \frac{1}{|Q(z)|^2} \right| = \left| |1 - .2z|^2 - \frac{|1 - .2z|^2}{|1 - .2^{11}z^{11}|^2} \right| = \frac{|1 - .2z|^2}{|1 - .2^{11}z^{11}|^2} |.2^{11}(z + \overline{z}) - .2^{22}|$$
$$\leq \frac{(1.2)^2[2(.2)^{11} + .2^{22}]}{.99} \leq .59578 \times 10^{-7} < 10^{-7}.$$

Therefore, in this case

$$d\mu(z) = \frac{1}{2\pi |\sum_{n=0}^{10} .2^n z^n|^2} |dz|$$

and the sequence $\{\Phi_n\}_{n\in\mathbb{N}}$ of monic orthogonal polynomials with respect to $d\mu$ is given by $\Phi_n(z) = z^{n-10}\Phi_{10}(z)$, for every $n \geq 10$, with

$$\Phi_{10}(z) = Q^*(z) = \sum_{n=0}^{10} .2^{10-n} z^n.$$

Next we determine the nodal system and the coefficients of our quadrature formula I_{10} in the following way:

1. The nodes $\{z_i\}_{i=1}^{10}$ are the zeros of $\Phi_{10}(z)$ and they are given in the vector nodes.
2. The quadrature coefficients $\{\lambda_i\}_{i=1}^{10}$ are the solutions of the following system

$$c_k = \sum_{i=1}^{10} \lambda_i z_i^k, \text{ for } k = 0, \cdots, 9,$$

where we denote by c_k the moments of the measure $d\mu$, that is, $c_k = \int_{\mathbb{T}} z^k d\mu(z)$. They are given in the vector coeff.

$$nodes = \begin{pmatrix} -0.191899 - 0.0563465\, I \\ -0.191899 + 0.0563465\, I \\ -0.130972 - 0.15115\, I \\ -0.130972 + 0.15115\, I \\ -0.028463 - 0.197964\, I \\ -0.028463 + 0.197964\, I \\ 0.083083 - 0.181926\, I \\ 0.083083 + 0.181926\, I \\ 0.168251 - 0.108128\, I \\ 0.168251 + 0.108128\, I \end{pmatrix} coeff = \frac{1}{2\pi} \begin{pmatrix} 1.16403 - 0.154488\, I \\ 1.16403 + 0.154488\, I \\ 0.983064 - 0.414416\, I \\ 0.983064 + 0.414416\, I \\ 0.678588 - 0.542769\, I \\ 0.678588 + 0.542769\, I \\ 0.347271 - 0.498797\, I \\ 0.347271 + 0.498797\, I \\ 0.0943028 - 0.296461\, I \\ 0.0943028 + 0.296461\, I \end{pmatrix}$$

Next we will compute integrals of the form $\int_{\mathbb{T}} f(z)|z - .2|^2|dz|$ for some analytic functions f by using our integration formula $I_{10}(f)$. We also compute the integrals by using Mathematica and we compare both results obtaining the error.

First we consider three elementary functions: $\exp(z)$, $\sin(z)$, and $\cos(z)$, for which it would be easy to obtain the exact value of the integrals. The results are displayed in the following table.

Function	Approx I_{10}	Approx NIntegrate	Bound error
$\exp(z)$	$\frac{1}{2\pi}(5.27788 + 2.25601 \times 10^{-14}\text{I})$	$\frac{1}{2\pi}(5.27788)$	10^{-9}
$\sin(z)$	$\frac{1}{2\pi}(-1.25664 - 5.37417 \times 10^{-15}\text{I})$	$\frac{1}{2\pi}(-1.25664)$	10^{-13}
$\cos(z)$	$\frac{1}{2\pi}(6.53451 + 2.72553 \times 10^{-14}\text{I})$	$\frac{1}{2\pi}(6.53451)$	10^{-13}

Next we consider the following three other examples: exp(sin(z)), sin(exp(z)), and exp(cos(z)) and the results are given in the next table.

Function	Approx I_{10}	Approx NIntegrate	Bound error
exp(sin(z))	$\frac{1}{2\pi}(5.27788 + 2.25722\times 10^{-14}\mathrm{I})$	$\frac{1}{2\pi}(5.27788+ 2.08167\times 10^{-16}\mathrm{I})$	10^{-9}
sin(exp(z))	$\frac{1}{2\pi}(4.81964 +2.0278\times 10^{-14}\mathrm{I})$	$\frac{1}{2\pi}(4.81964 + 7.63278\times 10^{-16}\mathrm{I})$	10^{-9}
exp(cos(z))	$\frac{1}{2\pi}(17.7626 + 7.3763\times 10^{-14}\mathrm{I})$	$\frac{1}{2\pi}(17.7626+1.66533\times 10^{-16}\mathrm{I})$	10^{-8}

We want to point out that for these last approximations the command NIntegrate of Mathematica is slower than our integration formula. Indeed for these calculations NIntegrate always needs more time of computation than the integration formula. This observation was confirmed several times on different computers.

We also want to point out that all the computations have been done with Mathematica 5.2, without using extra precision and we want to remark that the results obtained are like it could be expected, according to the theory developed in the previous section.

References

1. Cantero, M.J., Moral, L., Velázquez, L.: Measures and paraorthogonal polynomials on the unit circle. East J. Approx. **8**, 447–464 (2002)
2. Daruis, L., González-Vera, P., Marcellán, F.: Gaussian quadrature formulae on the unit circle. In: Proceedings of the 9th International Congress on Computational and Applied Mathematics (Leuven, 2000). J. Comput. Appl. Math. **140**, 159–183 (2002)
3. Freud, G.: Orthogonal Polynomials. Pergamon, New York (1971)
4. Gautschi, W.: Orthogonal polynomials. Computation and Approximation. Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford (2004)
5. Golinskii, L.: Quadrature formula and zeros of paraorthogonal polynomials on the unit circle. Acta Math. Hung. **96**, 169–186 (2002)
6. González Vera, P., Santos-León, J.C., Njåstad, O.: Some results about numerical quadratures on the unit circle. Adv. Comput. Math. **5**, 297–328 (1996)
7. Gragg, W.B.: Positive definite Toeplitz matrices, the Arnoldi process for isometric operators, and Gaussian quadrature on the unit circle. J. Comput. Appl. Math. **46**, 183–198 (1993)
8. Grenander, U., Szegő, G.: Toeplitz forms and their applications. Chelsea Publ. Company, 2nd (ed.), New York (1984)
9. Jones, W.B., Njåstad, O., Thron, W.J.: Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle. Bull. London Math. Soc. **21**, 113–152 (1989)
10. Simon, B.: Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory. Amer. Math. Soc. Coll. Publ. vol. 54, Amer. Math. Soc. Providence, RI (2005)
11. Szegő, G.: Orthogonal Polynomials. Amer. Math. Soc. Coll. Publ. vol. 23, 4th (ed.) Amer. Math. Soc. Providence, RI (1975)