Magnus and Neudecker ([1], Ch. 11, Theorem 4) show that for any real symmetric $n \times n$ matrix $\sum_1^T x_t x_t'$

$$\mu_1 \leq \frac{\alpha'\left(\sum_1^T x_t x_t'\right)\alpha}{\alpha'\alpha} \leq \mu_n \tag{5}$$

the expression in the middle of the inequality being known as the Rayleigh quotient. Obviously the lower bound of $\alpha'(\sum_1^T x_t x_t')\alpha/\alpha'\alpha$ can be achieved by choosing α to be the eigenvector associated with μ_1, the smallest eigenvalue of $\sum_1^T x_t x_t'$. By normalizing α the result follows.

REFERENCES

1. Magnus, J.R. & H. Neudecker. *Matrix Differential Calculus with Applications in Statistics and Econometrics.* New York: Wiley, 1988.
2. Sargan, D. *Lectures on Advanced Econometric Theory*, Meghnad Desai (ed.). Oxford: Basil Blackwell Ltd., 1988.

NOTE

Excellent solutions have been proposed independently by Juan J. Dolado, R.W. Farebrother, Bruce Hansen (the poser of the problem), Peter T. Kim, and Liqun Wang.

It may be worth noting that the solution to this problem is the first principal component of $\sum_1^T x_t x_t'$.

The following historical comments were made in R.W. Farebrother's solution: It is interesting to note that Wald [5] discussed the role of PLS in the errors in the variables model, mentioning the work of Adcock [1] and Pearson [3], and that Stigler [4] has reprinted the early papers on PLS by Adcock [1] and Kummell [2].

REFERENCES

1. Adcock, R.J. Note on the method of least squares. *The Analyst* 4 (1877): 183–184; 5(1878): 21–22 and 53–54.
2. Kummell, C.H. Reduction of observation equations which contain more than one observed quantity. *The Analyst* 6(1879): 97–105.
3. Pearson, K. On lines and planes of closest fit. *Philosophical Magazine* Ser 6, 2(1901): 559–572.
4. Stigler, S.M. *American Contributions to Mathematical Statistics in the Nineteenth Century.* New York: Arno Press, 1980.
5. Wald, A. The fitting of straight lines if both variables are subject to error. *Annals of Mathematical Statistics* 11 (1940): 284–300.

90.4.2. *Convergence to a Stochastic Integral*—Solution, proposed by Juan J. Dolado. Let u_{t_2} be a stochastic process with zero mean and constant variance σ_u^2, such that

$$T^{-3/2}\sum_1^T u_t^3 = o_p(1) \tag{1}$$

and

$$\sup_t \sum_{j=1}^{\infty} E|u_{t-j}(u_t^2 - \sigma_u^2)| < \infty. \tag{2}$$

Consider the partial sum $S_t = \sum_{j=1}^{t} u_j$ and assume the invariance principle holds:

$$S_T(r) = T^{-1/2}S_{[\mathrm{Tr}]} \Rightarrow B(r) \equiv BM(\sigma^2) \tag{3}$$

where B is a Brownian motion with long-run variance $\sigma^2 = \lim_{T\to\infty} E(T^{-1}S_T^2)$. Then, since $S_t = S_{t-1} + u_t$ ($S_0 = 0$), we have that

$$S_t^3 = S_{t-1}^3 + u_t^3 + 3S_{t-1}^2 u_t + 3S_{t-1}u_t^2. \tag{4}$$

Therefore

$$T^{-3/2}\sum_2^T S_{t-1}^2 u_t$$

$$= T^{-3/2}\left(\frac{1}{3}\right)\sum_2^T [S_t^3 - S_{t-1}^3 - 3S_{t-1}\sigma_u^2 - u_t^3 - 3S_{t-1}(u_t^2 - \sigma_u^2)] \tag{5}$$

where (see [1])

$$T^{-3/2}\sum_2^T (S_t^3 - S_{t-1}^3) = \sum_2^T [S_T(r)]_{(i-1)/T}^{i/T} \Rightarrow B(1)^3$$

$$\sigma_u^2 T^{-3/2}\sum_2^T S_{t-1} \Rightarrow \sigma_u^2 \int_0^1 B. \tag{6}$$

Finally, making use of the sufficient conditions (1) and (2), it is straightforward to show that the remaining two terms in (5) are $o_p(1)$, and thus negligible in the limiting distribution. Hence,

$$\sum_2^T S_{t-1}^2 u_t \Rightarrow \left(\frac{1}{3}\right)B(1)^3 + \sigma_u^2 \int_0^1 B. \tag{7}$$

By Ito's lemma (see, e.g., [2]) on stochastic differentiation, we know that if B_1 and B_2 are Brownian motions with instantaneous variance σ_1^2 and σ_2^2, we have that

$$d(B_1^2 B_2) = 2B_1 B_2\, dB_1 + B_1^2\, dB_2 + \tfrac{1}{2}[2B_2(dB_1)^2 + 4B_1\, dB_1\, dB_2]. \tag{8}$$

Then, making $B_1 \equiv B_2 \equiv B$ and taking the stochastic integral of (8) with limits (0,1), we get

$$B(1)^3 = 3\int B_2\, dB + 3\int B(dB)^2 = 3\int B_2\, dB + 3\sigma^2 \int B. \tag{9}$$

Thus, substituting (9) in (7) yields

$$\left(\frac{1}{3}\right)B(1)^3 - \sigma_u^2 \int_0^1 B = \int B^2\,dB + (\sigma^2 - \sigma_u^2)\int B \tag{10}$$

and the proof is completed.

NOTE

An excellent solution has been proposed independently by Bruce Hansen (the poser of the problem).

REFERENCES

1. Phillips, P.C.B. Time series regression with unit root. *Econometrica* 55 (1987): 277–302.
2. Cox, D.R. & H.D. Miller. *The Theory of Stochastic Processes.* New York: Wiley, 1965.

90.4.3. *The Limit Variance of g*—Solution, proposed by Eric Iksoon Im. Define k_1 and k_2 definite integrals, respectively, as

$$k_1 = \int_0^{\pi/2} \frac{1}{a^2\cos^2\phi + 1}\,d\phi; \qquad k_2 = \int_0^{\pi/2} \frac{1}{[a^2\cos^2\phi + 1]^2}\,d\phi. \tag{1}$$

Then, the analytic expressions of k_1 and k_2 in (1) are as follows (Dwight [2, equations 858.540 and 858.542, p. 219]):

$$k_1 = \frac{\pi}{2\sqrt{1 + a^2}}; \qquad k_2 = \frac{\pi(2 + a^2)}{4(1 + a^2)^{3/2}}. \tag{2}$$

Further define $g(\phi)$, $h(\phi)$, θ_1, and θ_2:

$$g(\phi) = \frac{d_i - 1}{(d_i - 1)\nu_0 + 1}; \qquad h(\phi) = \frac{(d_i - 1)^2}{[(d_i - 1)\nu_0 + 1]^2} \tag{3}$$

where d_i is defined as the ith largest eigenvalue of the matrix Q, defined in the problem, which can be expressed (Bellman, p. 66) as

$$d_i = d_i(\phi) = \frac{1}{2 + 2\cos(2\phi)} = \frac{1}{4\cos^2\phi}$$

in which

$$\phi = \phi(i) = \frac{\pi(T - i + 1)}{2T + 1},$$

and

$$\theta_1 = \lim_{T\to\infty} \sum_{i=1}^{T} g(\phi); \qquad \theta_2 = \lim_{T\to\infty} \sum_{i=1}^{T} h(\phi)\cdot \tag{4}$$