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Bending of Euler–Bernoulli beams using Eringen's integral formulation: A paradox resolved

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ABSTRACT

The Eringen nonlocal theory of elasticity formulated in differential form has been widely used to address problems in which size effect cannot be disregarded in micro- and nano-structured solids and nano-structures. However, this formulation shows some inconsistencies that are not completely understood. In this paper we formulate the problem of the static bending of Euler–Bernoulli beams using the Eringen integral constitutive equation. It is shown that, in general, the Eringen model in differential form is not equivalent to the Eringen model in integral form, and a general method to solve the problem rigorously in integral form is proposed. Beams with different boundary and load conditions are analyzed and the results are compared with those derived from the differential approach showing that they are different in general. With this integral formulation, the paradox that appears when solving the cantilever beam with the differential form of the Eringen model (increase in stiffness with the nonlocal parameter) is solved, which is one of the main contributions of the present work.

1. Introduction

The formalism based on the classical continuum mechanics has been widely used to develop powerful and reliable simulation tools to solve fundamental problems in several engineering fields such as civil, mechanical, aerospace, biomedical as well as in other applications of physical sciences. A basic feature of the local theory of continuum mechanics is that the stress at each point is related to the strain at the same point only. Therefore, a defining characteristic of this framework is that it is scale-free.

However, the matter is discrete and heterogeneous in nature. Materials used nowadays, like composites, functionally graded materials, polycrystalline solids, granular materials, and so on, all have inherent microstructures at different scales. Additionally, at high-frequency excitations, microstructural and size effects are observed in wave propagation in solids when the wavelength of a traveling wave becomes comparable with the scale of material heterogeneities (Gonella, Greene, & Liu, 2011). Moreover, modern technological applications involve the use of systems which can be devised as micro- or nano-structures, mainly in micro- or nano-electromechanical (MEMS or NEMS) devices (Martin, 1996), nano-machines (Bourlon, Glattli, Miko, Forro, & Bachtold, 2004; Drexler, 1992; Fennimore et al., 2003; Han, Globus, Jaffe, & Deardorff, 1997), as well as in biotechnology and biomedical fields (Saji, Choe, & Young, 2010). A main characteristic of these nanostructures is that their dimensions become comparable to the microstructural characteristics distances, thus the size effects are significant regarding their mechanical behavior.

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The above problems could be addressed using molecular dynamics (He, Liew, & Wei, 2007; Liew, Wei, & He, 2007; Tsai & Fang, 2007; Wei & Srivastava, 2004), but this approach requires a great computational effort, providing a motivation towards developing higher-order and nonlocal continuum mechanics theories able to capture the size effects by introducing intrinsic lengths in their formulations. Therefore, classical continuum mechanics, due to its inherent scale-free characteristic, cannot predict the size effect present in the above mentioned applications.

Apart from the first attempts to capture scale effects using the continuum theories through the works of Cauchy and Voigt in the 19th century, and the work of the Cosserat brothers in the first half of the 20th century, contributions by Mindlin and Tiersten (1962), Kröner (1963, 1967), Toupin (1963, 1964), Green and Rivlin (1964), Mindlin (1964, 1965), Krumhansl (1968), Mindlin and Eshel (1968), Kunin (1968), Eringen (1972a, 1972b), Eringen and Edelen (1972), constitute a major revival of the nonlocal and higher-order theories. The concept of nonlocal theory of linear elasticity was initially introduced by Kröner (1967), Krumhansl (1968), and Kunin (1968), and further developed by Eringen (1972a, 1972b), and Eringen and Edelen (1972). The basic feature of the nonlocal theories of elasticity is that the stress at each point is related to the strain at all points in the domain. This influence decreases as the distance between the point of interest and the neighboring points increases. The Eringen nonlocal integral constitutive equation describes the dependence of the stress at a point on the strain in the rest of the domain through a positive-decaying kernel function.

A differential constitutive theory, introduced by Eringen (1983), showed that for a specific class of kernel functions the nonlocal integral constitutive equation can be transformed into a differential form, much easier to manage than the integral model. From the pioneering work of Peddieson, Buchanan, and McNitt (Peddieson, Buchanan, & McNitt, 2003), and due to its simplicity, this differential Eringen nonlocal model has been widely used to analyze the static, buckling, and dynamic behavior of nanostructures. The list of papers is extremely long to be reported here. Nevertheless we cite some representative works.

As stated before, the Eringen nonlocal theory of elasticity formulated in differential form has been used to address the behavior of linear beams (Ke, Wang, & Wang, 2012; Loya, Lopez-Puente, Zaera, & Fernandez-Saez, 2009; Lu, 2007; Reddy, 2007; Wang, Zhang, & He, 2007; Wang, Zhang, Ramesh, & Kitipornchai, 2006; Xu, 2006), beams with von Kármán nonlinearity (Reddy, 2010; Reddy & El-Borgi, 2014), functionally graded beams (H. Salehipour, 2015; O. Rahmani, 2014; Reddy, El-Borgi, & Romanoff, 2014), beams under rotation (Aranda-Ruiz, Loya, & Fernández-Sáez, 2012; Murmu & Adhikari, 2010b; Narendar & Gopalakrishnan, 2011b; Pradhan & Murmu, 2010), rods (Kiani, 2010; Murmu & Adhikari, 2010a; Murmu & Pradhan, 2009b; Narendar, 2011; Narendar & Gopalakrishnan, 2010; Sun & Zhang, 2003), plates (Hosseini-Hashemi, Zare, & Nazemnezhad, 2013; Ke, Wang, & Wang, 2008; Murmu & Pradhan, 2009a), plates with von Kármán nonlinearity (Reddy, 2010), cylindrical shells (Hua, Liew, Wang, He, & Yakobson, 2008; Wang & Varadan, 2007; Wang & Wang, 2007), conical shells (Firouz-Abadi, Fotouhi, & Haddadpour, 2011; Liew et al., 2007; Tsai & Fang, 2007), rings (Moosavi, Mohammadi, Farajpour, & Shahidi, 2011; Wang & Duan, 2008), spherical shells (Ghavanloo & Fazelzadeh, 2013a; Vila, Zaera, & Fernandez-Saez, 2015; Zaera, Fernandez-Saez, & Loya, 2013), and particles (Ghavanloo & Fazelzadeh, 2013b), as well as carbon nanotubes (CNTS) (Ansari, Shahabodini, & Rouhi, 2013; Chen, Lee, & Eskandarian, 2004; Fleck & Hutchinson, 1997; Heireche, Tounsi, Benzair, Maachou, & Adda Berdia, 2008; Murmu & Pradhan, 2009c; Narendar & Gopalakrishnan, 2011a; Sudak, 2003; Zhou & Li, 2001).

Nevertheless, several authors have pointed out the inconsistent results obtained from the Eringen differential model regarding a cantilever beam when compared to other boundary conditions (Challamel & Wang, 2008; Challamel et al., 2014; Peddieson et al., 2003; Wang, Kitipornchai, Lim, & Eisenberger, 2008; Wang & Liew, 2007). For all boundary conditions except the cantilever, the model predicts softening effect (i.e. larger deflections and lower fundamental frequencies) as the nonlocal parameter is increased. Moreover, Lu, Lee, Lu, and Zhang (2006) showed that, depending on the nonlocal parameter, it is only possible to calculate a few natural frequencies of flexural vibrations of a cantilever beam. This last finding may be a consequence of the non self-adjoint characteristic of the Eringen differential operator (Challamel et al., 2014; Reddy, 2007).

Benvenuti and Simone (2013) found also inconsistent results regarding the behavior of a bar in tension. They observed that nonlocal solutions based on differential form of the Eringen theory are not consistent with the constitutive equation formulated in integral form. The reason is that in the transformation process of the constitutive equations from integral to differential forms, certain boundary conditions are not properly fulfilled (Polyanin & Manzhirov, 2008)

To overcome this paradoxical behavior, Challamel and Wang (2008) proposed a local/nonlocal moment-curvature relationship. In the same way, Challamel, Rakotomanana, and Le Marrec (2009) adopted a mixed local/nonlocal model to address the 1D wave propagation. In order to fit the dispersion relations obtained from a Von-Kármán lattice, Challamel et al. (2009) selected a negative value for the ”mixture” parameter leading to thermodynamic inconsistencies (Fafalis, Filopoulos, & Tsamasphyros, 2012). On the other hand, the two-phase constitutive model with both local and nonlocal phases was early proposed by Eringen (1987). This theory was further developed by Polizzotto (2001) who derived the variational principles governing the integral form. Using this two-phase constitutive model, Pisano and Fuschi (2003) and Benvenuti and Simone (2013) solved the problem of a bar in tension, and very recently Khodabakhshia and Reddy (2015) have presented the analysis of the static bending of Euler–Bernoulli beams subjected to different boundary and load conditions.

In this paper we formulate the problem of the static bending of Euler–Bernoulli beams using the Eringen integral constitutive equation. It is showed that, in general, the Eringen model in differential form is not equivalent to the Eringen model in integral form. Although this has been shown for a particular case (bending of beams), it also applies to other problems. A general method to solve rigorously the problem in integral form is proposed. Different boundary and load conditions are analyzed and the results have been compared whose derived from the widely used differential approach showing that they are different in general. With this integral formulation, the paradox that appears when solving the cantilever beam with the differential form of the Eringen model (increase in stiffness with the nonlocal parameter) is solved, being this one of the main outcomes of the work.

2. Governing equations

Here, an application of the Eringen integral model to the study of static bending of an Euler–Bernoulli beam is presented.

Hypotheses

We consider a beam of length L, uniform cross section A and constant Young modulus E. The variables x and w represent, respectively, the axial coordinate and the transverse displacement. The strain ε_x follows the kinematics of the Euler–Bernoulli beam

$$\varepsilon_x(x) = -z\frac{d^2w(x)}{dx^2} \tag{1}$$

The nonlocal constitutive equation σ_x is given by the Eringen integral equation

$$\sigma_x(x) = \int_0^L k(|x-\bar{x}|,\kappa)E\varepsilon_x(\bar{x})\,d\bar{x} \tag{2}$$

with the standard kernel

$$k(|x-\bar{x}|,\kappa) = \frac{1}{2\kappa}e^{-\frac{|x-\bar{x}|}{\kappa}} \tag{3}$$

κ being the non-local parameter, depending on an internal length scale a and on a constant e_0 appropriate to the material

$$\kappa = e_0 a \tag{4}$$

From the previous hypotheses, the following expression for the bending moment can be derived

$$M(x) = \int_A \sigma_x(x)z\,dA = -EI\int_0^L k(|x-\bar{x}|,\kappa)\frac{d^2w(\bar{x})}{d\bar{x}^2}d\bar{x} \tag{5}$$

where I is the moment of inertia of the cross section.

Derivation of the governing equation and BCs.

The governing equation and the boundary conditions are derived applying the Principle of Minimum Total Potential Energy, specializing the general formulation of Polizzotto (2001) for a 1D case.

Given the following expression for the strain energy

$$\mathcal{U}(w) = \int_0^L\int_A \frac{1}{2}\sigma_x(x)\varepsilon_x(x)dA\,dx = \frac{1}{2}EI\int_0^L\left[\int_0^L k(|x-\bar{x}|,\kappa)\frac{d^2w(\bar{x})}{d\bar{x}^2}d\bar{x}\right]\frac{d^2w(x)}{dx^2}dx \tag{6}$$

and for the work due to external loads $q(x)$

$$\mathcal{V}(w) = -\int_0^L q(x)w(x)dx \tag{7}$$

the total potential energy is

$$\Pi(w) = \mathcal{U}(w) + \mathcal{V}(w) \tag{8}$$

The Euler equation is determined by equating the first variation of $\Pi(w)$ to zero, i.e. $\delta\Pi(w) = 0$

$$\begin{aligned}&\frac{1}{2}EI\int_0^L\left[\int_0^L k(|x-\bar{x}|,\kappa)\frac{d^2\delta w(\bar{x})}{d\bar{x}^2}d\bar{x}\right]\frac{d^2w(x)}{dx^2}dx + \\ &\frac{1}{2}EI\int_0^L\left[\int_0^L k(|x-\bar{x}|,\kappa)\frac{d^2w(\bar{x})}{d\bar{x}^2}d\bar{x}\right]\frac{d^2\delta w(x)}{dx^2}dx - \int_0^L q(x)\delta w(x)dx = 0\end{aligned} \tag{9}$$

that can be written as

$$EI\int_0^L\left[\int_0^L k(|x-\bar{x}|,\kappa)\frac{d^2w(\bar{x})}{d\bar{x}^2}d\bar{x}\right]\frac{d^2\delta w(x)}{dx^2}dx - \int_0^L q(x)\delta w(x)dx = 0 \tag{10}$$

Integrating the first term twice by parts, we get

$$\begin{aligned}&\left[\left[EI\int_0^L k(|x-\bar{x}|,\kappa)\frac{d^2w(\bar{x})}{d\bar{x}^2}d\bar{x}\right]\delta\frac{dw(x)}{dx}\right]_0^L - \\ &\left[\frac{d}{dx}\left[EI\int_0^L k(|x-\bar{x}|,\kappa)\frac{d^2w(\bar{x})}{d\bar{x}^2}d\bar{x}\right]\delta w(x)\right]_0^L +\end{aligned}$$

$$\int_0^L \left[\frac{d^2}{dx^2} \left[EI \int_0^L k(|x-\bar{x}|,\kappa) \frac{d^2 w(\bar{x})}{d\bar{x}^2} d\bar{x} \right] - q(x) \right] \delta w(x) dx = 0 \tag{11}$$

By virtue of the Fundamental Lemma of Variational Calculus, we get the Euler equation

$$\frac{d^2}{dx^2} \left[EI \int_0^L k(|x-\bar{x}|,\kappa) \frac{d^2 w(\bar{x})}{d\bar{x}^2} d\bar{x} \right] - q(x) = 0 \tag{12}$$

or, in terms of the bending moment (Eq. (5))

$$\frac{d^2 M(x)}{dx^2} = -q(x) \tag{13}$$

with the following pairs of essential and natural boundary conditions:

$$w = 0; \quad or \quad \frac{d}{dx} \left[EI \int_0^L k(|x-\bar{x}|,\kappa) \frac{d^2 w(\bar{x})}{d\bar{x}^2} d\bar{x} \right] = 0 \tag{14}$$

and

$$\frac{dw}{dx} = 0; \quad or \quad EI \int_0^L k(|x-\bar{x}|,\kappa) \frac{d^2 w(\bar{x})}{d\bar{x}^2} d\bar{x} = 0 \tag{15}$$

The above equations can be written in nondimensional form using the following variables

$$\xi = \frac{x}{L}; \quad h = \frac{\kappa}{L}; \quad \bar{w} = \frac{w}{w_0}; \quad \bar{M} = \frac{M}{q_0 L^2}; \quad \bar{q} = \frac{q}{q_0} \tag{16}$$

with $w_0 = q_0 L^4/(EI)$ and q_0 a characteristic value of the transverse load. The nondimensional bending moment becomes

$$\bar{M}(\xi) = -\int_0^1 \frac{1}{2h} e^{-\frac{|\xi - s|}{h}} \frac{d^2 \bar{w}(s)}{ds^2} ds \tag{17}$$

Given the external load $\bar{q}(\xi)$, the bending moment can be obtained integrating the balance equation

$$\bar{M}''(\xi) = -\bar{q}(\xi) \tag{18}$$

thus

$$\bar{M}(\xi) = C_1 + C_2 \xi - \int_0^\xi (\xi - t) \bar{q}(t) dt \tag{19}$$

and the integral equation (17) permits to determine the function $\bar{w}''$, where $(\bullet)''$ indicates $d^2(\bullet)/d\xi^2$ for simplicity. It is worth to point out that Eq. (17), written as

$$\int_0^1 e^{-\frac{|\xi - s|}{h}} \bar{w}''(s) ds = -2h\bar{M}(\xi) \tag{20}$$

is of the general form

$$\int_a^b e^{\mu|\xi - s|} y(s) ds = f(\xi), \quad -\infty < a < b < \infty \tag{21}$$

with $a = 0$, $b = 1$, $\mu = -1/h$, $y = \bar{w}''$, $f = -2h\bar{M}$. As described in Polyanin and Manzhirov (2008), the integral equation is satisfied by the solution

$$y(\xi) = \frac{1}{2\mu} \left[f''(\xi) - \mu^2 f(\xi) \right] \tag{22}$$

Eq. (22) constitutes the well-known differential form of the nonlocal Eringen constitutive model, supposedly equivalent to its integral counterpart. However, as stated in Polyanin and Manzhirov (2008), for the solution in differential form to satisfy the integral equation, the following boundary conditions have to be fulfilled

$$f'(a) + \mu f(a) = 0, \quad f'(b) - \mu f(b) = 0 \tag{23}$$

It is evident that, in a general loading case, the previous conditions may not necessarily be satisfied by $f = -2h\bar{M}$, thus preventing to use the differential formulation of the Eringen model. This impediment has been highlighted in a bending problem, but it also applies to other Solid Mechanics problems when using the differential formulation of the Eringen model.

3. Proposed solution for the integral form

Once the discordance between the integral and the differential formulations of the nonlocal Eringen model – for a general loading case – has been underscored, a method to rigorously solve the original integral form is proposed. In a general case, $f(\xi)$ does not satisfy the conditions given by Eq. (23). As stated in Polyanin and Manzhirov (2008) a new function $F(\xi)$, based on $f(\xi)$ and written as

$$F(\xi) = f(\xi) + A\xi + B \tag{24}$$

where

$$A = \frac{1}{b\mu - a\mu - 2}\left[f'(a) + f'(b) + \mu f(a) - \mu f(b)\right] \tag{25}$$

$$B = -\frac{1}{\mu}\left[f'(a) + \mu f(a) + Aa\mu + A\right] \tag{26}$$

can be defined, which now satisfies the conditions

$$F'(a) + \mu F(a) = 0, \quad F'(b) - \mu F(b) = 0 \tag{27}$$

Taking advantage of this general solution, we may modify the governing Eq. (20) as follows

$$\int_0^1 e^{-\frac{|\xi - s|}{h}} \bar{w}''(s)ds = -2h\bar{M}(\xi) + A\xi + B - (A\xi + B) \tag{28}$$

Thus the solution of the integral equation may be derived as

$$\bar{w}''(\xi) = \bar{w}_1''(\xi) - A\,\bar{w}_A''(\xi) - B\,\bar{w}_B''(\xi) \tag{29}$$

$\bar{w}_1''(\xi)$, $\bar{w}_A''(\xi)$ and $\bar{w}_B''(\xi)$ being the solutions of the following integral equations

- For $\bar{w}_1''(\xi)$:

  $$\int_0^1 e^{-\frac{|\xi - s|}{h}} \bar{w}_1''(s)ds = -2h\bar{M}(\xi) + A\xi + B \tag{30}$$

  whose right-term follows the expression given by Eq. (24) and thus can be solved using Eq. (22).
- For $\bar{w}_A''(\xi)$:

  $$\int_0^1 e^{-\frac{|\xi - s|}{h}} \bar{w}_A''(s)ds = \xi \tag{31}$$

  that can be solved numerically.
- For $\bar{w}_B''(\xi)$:

  $$\int_0^1 e^{-\frac{|\xi - s|}{h}} \bar{w}_B''(s)ds = 1 \tag{32}$$

  that can be solved numerically.

The solution $\bar{w}_1''(\xi)$ depends on the particular loading case. Nevertheless, $\bar{w}_A''(\xi)$ and $\bar{w}_B''(\xi)$ are canonical functions (for a given value of the nonlocal parameter h), thus valid for any loading case.

Later on, the displacement $\bar{w}(\xi)$ is obtained as follows:

$$\bar{w}(\xi) = \bar{w}(0) + \bar{w}'(0)\xi + \int_0^{\xi} (\xi - s)\left(\bar{w}_1''(s) - A\bar{w}_A''(s) - B\bar{w}_B''(s)\right)ds \tag{33}$$

Likewise, the bending moment is derived as follows

$$\bar{M}(\xi) = -\int_0^1 \frac{1}{2h} e^{-\frac{|\xi - s|}{h}} \left(\bar{w}_1''(s) - A\,\bar{w}_A''(s) - B\,\bar{w}_B''(s)\right)ds \tag{34}$$

4. Selected results

To illustrate the differences in the results obtained by means of the integral form with those obtained with the differential form of the Eringen model, some cases are analyzed. Different boundary condition combinations (cantilever, fixed-pinned and simply supported) and loading states (point or distributed load) will be considered, some of them satisfying the conditions stated in Polyanin and Manzhirov (2008). The presented results show the vertical displacement of selected sections of the beam versus the nonlocal parameter h. Note that the case $h = 0$ corresponds to the classical (local) elasticity theory, and the solution for this value of h has not been obtained from the nonlocal integral approach because it leads to a singularity in the kernel.

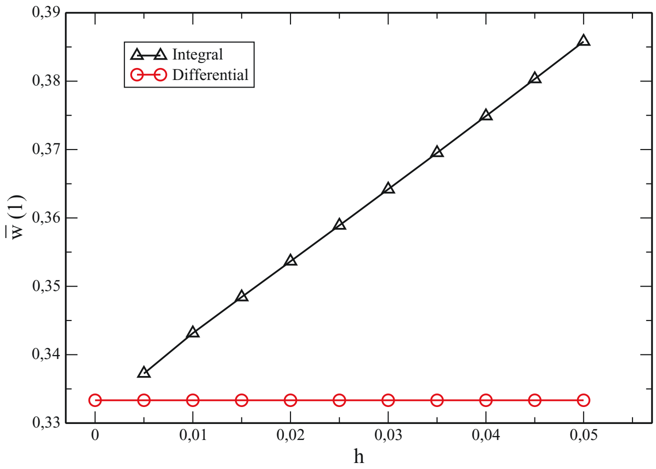

Fig. 1. Displacement of the section at $\xi = 1$ for a cantilever beam submitted to a point load at $\xi = 1$, versus nonlocal parameter *h*.

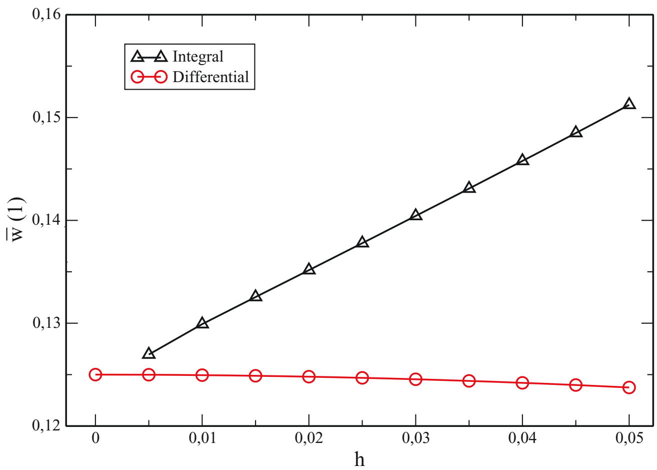

Fig. 2. Displacement of the section at $\xi = 1$ for a cantilever beam submitted to a uniform distributed load, versus nonlocal parameter *h*.

Cantilever beam: The first three conditions analyzed correspond to cantilever beams (clamped at $\xi = 0$) subjected to different loading cases. In all these configurations, conditions stated in Eq. (23) are not satisfied, showing discrepancies in the results obtained by the integral and differential models.

- *Cantilever beam with point load.* In case of a point load located at the free end ($\xi = 1$) of a cantilever beam, the maximum displacement obtained at this section using the integral form increases with the nonlocal parameter *h* (see Fig. 1).
  These results predict that flexibility of the beam increases with the small-scale parameter. This trend contrasts with the results obtained by means of the differential form, being the maximum displacement for this loading case (point load at the free end), independent of *h* (Wang & Liew (2007)).
- *Cantilever beam with distributed load.* In case of considering an uniform or a linear distributed load, $\bar{q}(\xi) = \xi$, (see Figs. 2 and 3, respectively), integral form shows that the stiffness of the beam decreases as well, whereas in the differential one the stiffness slightly increases.

The stiffening behaviour of cantilever beams with the nonlocal parameter when the differential formulation is used, opposite to that observed in other boundary conditions, has been defined as a paradox by other authors and could be a consequence of the improper transformation of the integral problem to the differential one. However, as we will see next, both approaches also provide different results for other boundary conditions if the requirements for $f(\xi)$ are not satisfied.

Simply-supported beam: Following, simply supported beam with two different loading states are analyzed.

- *Simply-supported beam with uniform distributed load.* For this case, displacement at $\xi = 1/4$ is shown in Fig. 4. As in previous cases, the particular conditions considered here lead to a wrong transformation of the integral problem into the differential one.
  Notice that in the particular case of displacement at $\xi = 1/2$, results obtained through both formulations are fully coincident (see Fig. 5).

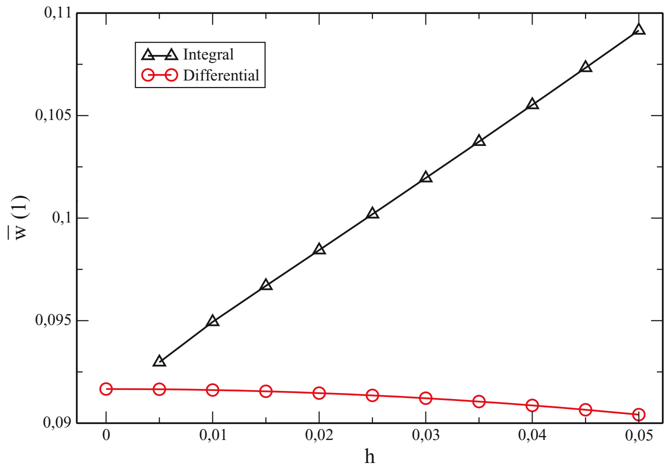

Fig. 3. Displacement of the section at $\xi = 1$ for a cantilever beam submitted to a triangular distributed load $\bar{q}(\xi) = \xi$, versus nonlocal parameter *h*.

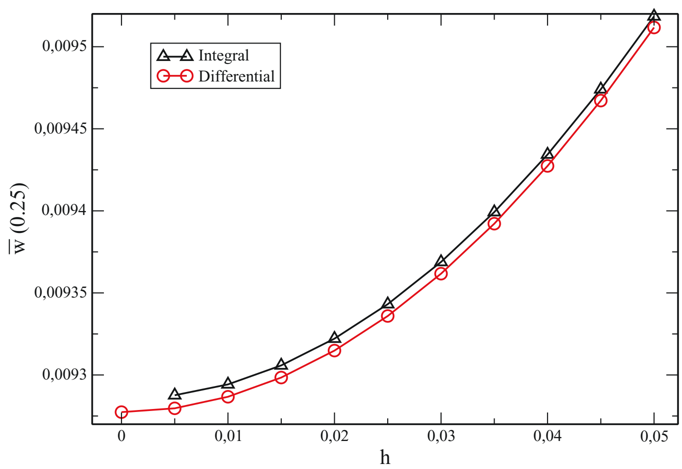

Fig. 4. Displacement of the section at $\xi = 1/4$ for a simply-supported beam submitted to a uniform distributed load, versus nonlocal parameter *h*.

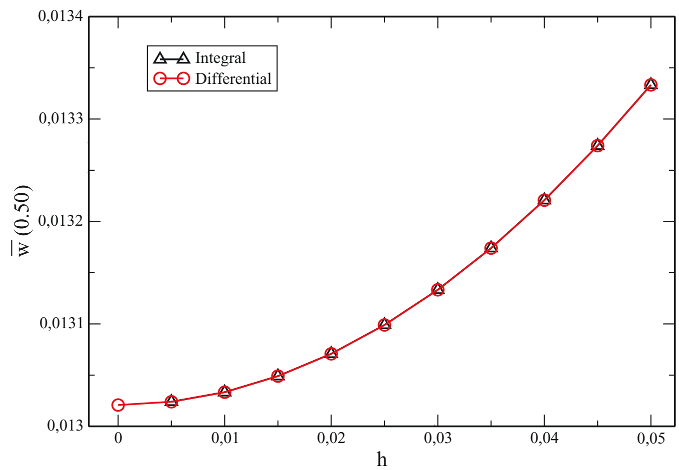

Fig. 5. Displacement of the section at $\xi = 1/2$ for a simply-supported beam submitted to a uniform distributed load, versus nonlocal parameter *h*.

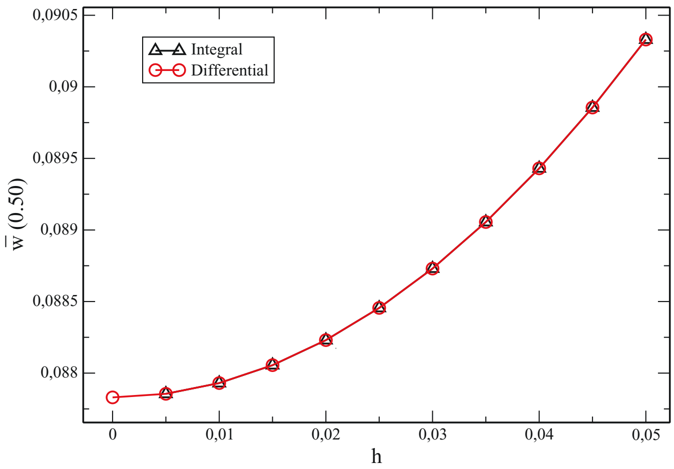

Fig. 6. Displacement of the section at $\xi = 1/2$ for a simply-supported beam submitted to an harmonic distributed load $\bar{q}(\xi) = 2\pi^2 \cos(2\pi\xi)$, versus nonlocal parameter *h*.

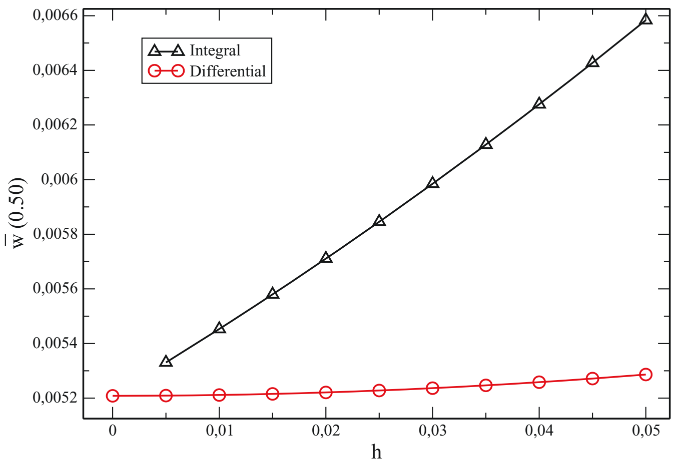

Fig. 7. Displacement of the section at $\xi = 1/2$ for a clamped-pinned beam submitted to a uniform distributed load, versus nonlocal parameter *h*.

- *Simply-supported beam with harmonic load.* Let us consider the following harmonic distributed load $\bar{q}(\xi) = 2\pi^2 \cos(2\pi\xi)$ and calculate the displacement at the mid-span of the beam. This load distribution satisfies the conditions stated in Polyanin and Manzhirov (2008) and therefore, the displacement obtained using both integral and differential nonlocal Eringen models are fully coincident (Fig. 6).

Clamped-pinned beam: The following case corresponds to a statically undetermined case with asymmetric boundary conditions.

- *Clamped-pinned beam with uniform distributed load.* In this case, the displacement at the mid-span of a clamped-pinned beam with an uniform distributed load has been calculated (Fig. 7). Here, both formulations show that the flexibility of the beam increases with the nonlocal parameter *h*. As in the cases previously commented, the load now considered does not satisfy the conditions stated in Polyanin and Manzhirov (2008) either.

As the analyzed cases shown, the comparison of results demonstrates that relevant discrepancies may appears when the conditions stated in Polyanin and Manzhirov (2008) are not satisfied, leading to an improper transformation from the integral to the differential formulation. In particular, the differential form leads to an apparent stiffness increasing in cantilever beams subjected to a distributed load, meanwhile opposite tendencies are observed in other boundary conditions. This paradox does not appears using the integral form.

5. Summary and concluding remarks

The paper shows the differences between the original nonlocal integral Eringen model and its differential counterpart, commonly supposed fully equivalent. The contrasts are illustrated through the static bending analysis of the Euler–Bernoulli beam, although it can be readily extended to other boundary value problems. An original methodology to rigorously solve the integral formulation is proposed, permitting to clear up some abnormal trends predicted by the differential formulation, which disappear when using the original integral form.

The following conclusions can be established:

- For a general loading case, the solution of the integral model is obtained by adding those of an associated differential problem and of two integral equations.
- For a given value of the non-local parameter, the solutions of the integral equations can be stated in terms of two canonical functions, valid for any loading case.
- The solutions of integral and differential forms of the Eringen model are coincident for specific loading cases (those satisfying the conditions stated in Polyanin & Manzhirov (2008)), for which the integral problem can be rigorously transformed into a differential problem.
- The paradox that appears when solving the cantilever beam with the differential form of the Eringen model (increase in stiffness with the nonlocal parameter) is resolved by using the integral form of the Eringen model.

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References

Ansari, R., Shahabodini, A., & Rouhi, H. (2013). A thickness-independent nonlocal shell model for describing the stability behavior of carbon nanotubes under compression. *Composite Structures, 100*, 323–331.
Aranda-Ruiz, J., Loya, J., & Fernández-Sáez, J. (2012). Bending vibrations of rotating nonuniform nanocantilevers using the eringen nonlocal elasticity theory. *Composite Structures, 95*, 2990–3001.
Benvenuti, E., & Simone, A. (2013). One-dimensional nonlocal and gradient elasticity: Closed-form solution and size effect. *Mechanics Research Communications, 48*, 46–51.
Bourlon, B., Glattli, D. C., Miko, C., Forro, L., & Bachtold, A. (2004). Carbon nanotube based bearing for rotational motions. *Nano Letters, 4*, 709–712.
Challamel, N., Rakotomanana, L., & Le Marrec, L. (2009). A dispersive wave equation using nonlocal elasticity. *Comptes Rendus - Mecanique, 337*, 591–595.
Challamel, N., & Wang, C. M. (2008). The small length scale effect for a non-local cantilever beam: a paradox solved. *Nanotechnology, 19*, 345703(7).
Challamel, N., Zhang, Z., Wang, C. M., Reddy, J. N., Wang, Q., Michelitsch, T., et al. (2014). On nonconservativeness of Eringen's nonlocal elasticity in beam mechanics: correction from a discrete-based approach. *Archive of Applied Mechanics, 84*, 1275–1292.
Chen, Y., Lee, J. D., & Eskandarian, A. (2004). Atomistic viewpoint of the applicability of microcontinuum theories. *International Journal of Solids and Structures, 41*(8), 2085–2097.
Drexler, K. E. (1992). *Nanosystems: Molecular machinery, manufacturing, and computation*. In Wiley (Ed.) New York.
Eringen, A. C. (1972a). Linear theory of nonlocal elasticity and dispersion of plane-waves. *International Journal of Engineering Science, 10*(5), 233–248.
Eringen, A. C. (1972b). Nonlocal polar elastic continua. *International Journal of Engineering Science, 10*, 1–16.
Eringen, A. C. (1983). On differential-equations of nonlocal elasticity and solutions of screw dislocation and surface-waves. *Journal of Applied Physics, 54*(9), 4703–4710.
Eringen, A. C. (1987). Theory of nonlocal elasticity and some applications. *Res Mechanica, 21*(4), 313–342.
Eringen, A. C., & Edelen, D. G. B. (1972). Nonlocal elasticity. *International Journal of Engineering Science, 10*(3), 233–248.
Fafalis, D., Filopoulos, S., & Tsamasphyros, G. (2012). On the capability of generalized continuum theories to capture dispersion characteristics at the atomic scale. *European Journal of Mechanics, A/Solids, 26*, 35–37.
Fennimore, A., Yuzvinsky, T. D., Han, W. Q., Fuhrer, M. S., Cumings, J., & Zettl, A. (2003). Rotational actuators based on carbon nanotubes. *Nature* 424–408.
Firouz-Abadi, R., Fotouhi, M., & Haddadpour, H. (2011). Free vibration analysis of nanocones using a nonlocal continuum model. *Physics Letters A, 375*, 3593–3598.
Fleck, N. A., & Hutchinson, J. W. (1997). Strain gradient plasticity. *Advances in Applied Mechanics, 33*, 295–361.
Ghavanloo, E., & Fazelzadeh, S. (2013a). Nonlocal elasticity theory for radial vibration of nanoscale spherical shells . *European Journal of Mechanics / A Solids, 41*, 37–42. doi:10.1016/j.euromechsol.2013.02.003.
Ghavanloo, E., & Fazelzadeh, S. (2013b). Radial vibration of free anisotropic nanoparticles based on nonlocal continuum mechanics. *Nanotechnology, 24*, 075702.
Gonella, S., Greene, M., & Liu, W. (2011). Characterization of heterogeneous solids via wave methods in computational microelasticity. *Journal of the Mechanics and Physics of Solids, 59*, 959–974.
Green, A. E., & Rivlin, R. S. (1964). Multipolar continuum mechanics. *Archive for rational mechanics and analysis, 17*(2), 113–147.
Salehipour, H., Shahidi, A., & Nahvi, H. (2015). Modified nonlocal elasticity theory for functionally graded materials. *International Journal of Engineering Science, 90*, 44–57.
Han, J., Globus, A., Jaffe, R., & Deardorff, G. (1997). Molecular dynamics simulation of carbon nanotubebased gear. *Nanotechnology, 8*, 95–102.
He, X. Q., Liew, K. M., & Wei, C. (2007). Mechanical properties of carbon nanocones. *Applied Physics Letters, 91*.261906–261906–3
Heireche, H., Tounsi, A., Benzair, A., Maachou, M., & Adda Berdia, E. (2008). Sound wave propagation in single-walled carbon nanotubes using nonlocal elasticity. *Physica E, 40*, 2791–2799.
Hosseini-Hashemi, S., Zare, M., & Nazemnezhad, R. (2013). An exact analytical approach for free vibration of mindlin rectangular nano-plates via nonlocal elasticity. *Composite Structures, 100*, 290–299. doi:10.1016/j.compstruct.2012.11.035.
Hua, Y. G., Liew, K. M., Wang, Q., He, X. Q., & Yakobson, B. I. (2008). Nonlocal shell model for elastic wave propagation in single- and double-walled carbon nanotubes. *Journal of the Mechanics and Physics of Solids, 56*, 3475–3485.
Ke, L.-L., Wang, Y.-S., & Wang, Z.-D. (2008). Non-local elastic plate theories. *Proceedings of the Royal Society A, 463*, 3225–3240.
Ke, L. L., Wang, Y. S., & Wang, Z. D. (2012). Nonlinear vibration of the piezoelectric nanobeams based on the nonlocal theory. *Composite Structures, 94*(6), 2038–2047. doi:10.1016/j.compstruct.2012.01.023.

Khodabakhshia, P., & Reddy, J. N. (2015). A unified integro-differential nonlocal model. *International Journal of Engineering Science, 95*, 60–75.
Kiani, K. (2010). Free longitudinal vibration of tapered nanowires in the context of nonlocal continuum theory via a perturbation technique. *Physica E, 43*(1), 387–397.
Kröner, E. (1963). On the physical reality of torque stresses in continuum mechanics. *International Journal of Engineering Science, 1*, 261–278.
Kröner, E. (1967). Elasticity theory of materials with long range cohesive forces. *International Journal of Solids and Structures, 3*, 731–742.
Krumhansl, J. (1968). Some considerations of the relation between solid state physics and generalized continuum mechanics. In E. Kröner (Ed.), *Mechanics of generalized continua. IUTAM symposia. Springer Berlin Heidelberg,* (pp. 298–311).
Kunin, I. A. (1968). The theory of elastic media with microstructure and the theory of dislocations. In E. Kröner (Ed.), *Mechanics of generalized continua. IUTAM symposia. Springer Berlin Heidelberg,* (pp. 321–329).
Liew, K. M., Wei, J. X., & He, X. Q. (2007). Carbon nanocones under compression: Buckling and post-buckling behaviors. *Physical Review B- Condensed Matter and Materials Physics, 75*, 195435.
Loya, J., Lopez-Puente, J., Zaera, R., & Fernandez-Saez, J. (2009). Free transverse vibrations of cracked nanobeams using a nonlocal elasticity model. *Journal of Applied Physics, 105*(4), 044309.
Lu, P. (2007). Dynamic analysis of axially prestressed micro/nanobeam structures based on nonlocal beam theory. *Journal of Applied Physics, 101*(7), 073504.
Lu, P., Lee, H. P., Lu, C., & Zhang, P. Q. (2006). Dynamic properties of flexural beams using a nonlocal elasticity model. *Journal of Applied Physics, 99*, 1–9.
Martin, C. R. (1996). Membrane-based synthesis of nanomaterials. *Chemistry of Materials, 8*, 1739–1746.
Mindlin, R. D. (1964). Micro-structure in linear elasticity. *Archive for Rational Mechanics and Analysis, 16*(1), 51–78.
Mindlin, R. D. (1965). Second gradient of strain and surface-tension in linear elasticity. *International Journal of Solids and Structures, 1*(4), 417–438.
Mindlin, R. D., & Eshel, N. (1968). On first strain-gradient theories in linear elasticity. *International Journal of Solids and Structures, 4*(1), 109–124.
Mindlin, R. D., & Tiersten, H. F. (1962). Effects of couple-stresses in linear elasticity. *Archive for Rational Mechanics and Analysis, 11*(5), 415–448.
Moosavi, H., Mohammadi, M., Farajpour, A., & Shahidi, S. H. (2011). Vibration analysis of nanorings using nonlocal continuum mechanics and shear deformable ring theory. *Physica E, 44*, 135–140.
Murmu, T., & Adhikari, S. (2010a). Non local effects in the longitudinal vibration of double-nanorod systems. *Physica E, 43*(1), 415–422.
Murmu, T., & Adhikari, S. (2010b). Scale-dependent vibration analysis of prestressed carbon nanotubes undergoing rotation. *Journal of Applied Physics, 108*, 123507.
Murmu, T., & Pradhan, S. C. (2009a). Small-scale effect on the free in-plane vibration of nanoplates by nonlocal continuum model. *Physica E, 41*(8), 1628–1633.
Murmu, T., & Pradhan, S. C. (2009b). Small-scale effect on the vibration of nonuniform nanocantilever based on nonlocal elasticity theory. *Physica E, 41*(8), 1451–1456.
Murmu, T., & Pradhan, S. C. (2009c). Thermo-mechanical vibration of a single-walled carbon nanotube embedded in an elastic medium based on nonlocal elasticity theory. *Computational Materials Science, 46*(4), 854–859.
Narendar, S. (2011). Terahertz wave propagation in uniform nanorods: a nonlocal continuum mechanics formulation including the effect of lateral inertia. *Physica E, 43*(4), 1015–1020.
Narendar, S., & Gopalakrishnan, S. (2010). Ultrasonic wave characteristics of nanorods via nonlocal strain gradient models. *Journal of Applied Physics, 107*(8), 084312.
Narendar, S., & Gopalakrishnan, S. (2011a). Critical buckling temperature of single-walled carbon nanotubes embedded in a one-parameter elastic medium based on nonlocal continuum mechanics. *Physica E, 43*(6), 1185–1191.
Narendar, S., & Gopalakrishnan, S. (2011b). Nonlocal wave propagation in rotating nanotube. *Physics, 1*, 17–25.
O. Rahmani, O. P. (2014). Analysis and modeling the size effect on vibration of functionally graded nanobeams based on nonlocal timoshenko beam theory. *International Journal of Engineering Science, 77*, 5570.
Peddieson, J., Buchanan, G. R., & McNitt, R. P. (2003). Application of nonlocal continuum models to nanotechnology. *International Journal of Engineering Science, 41*(3-5), 305–312.
Pisano, A. A., & Fuschi, P. (2003). Closed form solution for a nonlocal elastic bar in tension. *International Journal of Solids and Structures, 40*, 13–23.
Polizzotto, C. (2001). Nonlocal elasticity and related variational principles. *Mechanics Research Communications, 38*, 7359–7380.
Polyanin, A., & Manzhirov, A. (2008). *Handbook of integral equations*. New York: CRC Press.
Pradhan, S. C., & Murmu, T. (2010). Application of nonlocal elasticity and DQM in the flapwise bending vibration of a rotating nanocantilever. *Physica E, 42*, 1944–1949.
Reddy, J., El-Borgi, S., & Romanoff, J. (2014). Non-linear analysis of functionally graded microbeams using Eringens non-local differential model. *International Journal of Non-Linear Mechanics, 67*, 308–318.
Reddy, J. N. (2007). Nonlocal theories for bending, buckling and vibration of beams. *International Journal of Engineering Science, 45*(2-8), 288–307.
Reddy, J. N. (2010). Nonlocal nonlinear formulations for bending of classical and shear deformation theories of beams and plates. *International Journal of Engineering Science, 48*(11), 1507–1518.
Reddy, J. N., & El-Borgi, S. (2014). Eringen's nonlocal theories of beams accounting for moderate rotations. *International Journal of Engineering Science, 82*, 159–177.
Saji, V., Choe, H., & Young, K. (2010). Nanotechnology in biomedical applications-a review. *International Journal of Nano and Biomaterials, 3*, 119–139.
Sudak, L. J. (2003). Column buckling of multiwalled carbon nanotubes using nonlocal continuum mechanics. *Journal of Applied Physics, 94*(11), 7281–7287.
Sun, C. T., & Zhang, H. T. (2003). Size-dependent elastic moduli of platelike nanomaterials. *Journal of Applied Physics, 93*(2), 1212–1218.
Toupin, R. A. (1963). Elastic materials with couple-stresses. *Archive for Rational Mechanics and Analysis, 11*(5), 385–414.
Toupin, R. A. (1964). Theories of elasticity with couple-stress. *Archive for Rational Mechanics and Analysis, 17*(2), 85–112.
Tsai, P., & Fang, T. (2007). A molecular dynamics study of the nucleation, thermal stability and nanomechanics of carbon nanocones. *Nanotechnology, 18*, 105702.
Vila, J., Zaera, R., & Fernandez-Saez, J. (2015). Axisymmetric free vibration of closed thin spherical nanoshells with bending effects. *Journal of Vibration and Control*. doi:10.1177/1077546314565808.
Wang, C., Kitipornchai, S., Lim, C., & Eisenberger, M. (2008). Beam bending solutions based on nonlocal Timoshenko beam theory. *Journal of Engineering Mechanics, 134*(6), 475–481.
Wang, C. M., & Duan, W. H. (2008). Vibration analysis of nanorings using nonlocal continuum mechanics and shear deformable ring theory. *Journal of Applied Physics, 104*, 014303.
Wang, C. M., Zhang, Y. Y., & He, X. Q. (2007). Vibration of nonlocal Timoshenko beams. *Nanotechnology, 18*(10), 105401.
Wang, C. M., Zhang, Y. Y., Ramesh, S. S., & Kitipornchai, S. (2006). Buckling analysis of micro- and nano-rods/tubes based on nonlocal Timoshenko beam theory. *Journal of Physics D-Applied Physics, 39*(17), 3904–3909.
Wang, Q., & Liew, K. M. (2007). Application of nonlocal continuum mechanics to static analysis of micro- and nano-structures. *Physics Letters A, 363*(3), 236–242.
Wang, Q., & Varadan, V. (2007). Application of nonlocal elastic shell theory in wave propagation analysis of carbon nanotubes. *Smart Materials Structures, 16*, 178–190.
Wang, Q., & Wang, C. (2007). The constitutive relation and small scale parameter of nonlocal continuum mechanics for modelling carbon nanotubes. *Nanotechnology, 18*, 075702.
Wei, C., & Srivastava, D. (2004). Nanomechanics of carbon nanofibers: structural and elastic properties. *Applied Physics Letters, 85*, 2208—2210.
Xu, M. T. (2006). Free transverse vibrations of nano-to-micron scale beams. *Proceedings of the Royal Society A-Mathematical Physical and Engineering Sciences, 462*(2074), 2977–2995.
Zaera, R., Fernandez-Saez, J., & Loya, J. (2013). Axisymmetric free vibration of closed thin spherical nano-shell. *Composite Structures, 104*, 154–161.
Zhou, S. J., & Li, Z. Q. (2001). Length scales in the static and dynamic torsion of a circular cylindrical micro-bar. *Journal of Shandong University of Technology, 31*(5), 401–407.