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Before and after default: information and optimal portfolio via anticipating calculus

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Abstract

Default risk calculus emerges naturally in a portfolio optimization problem when the risky asset is threatened with a bankruptcy. The usual stochastic control techniques do not hold in this case and some additional assumptions are generally added to achieve the optimization in a before-and-after default context. We show how it is possible to avoid one of theses restrictive assumptions, the so-called *Jacod density hypothesis*, by using the framework of the forward integration. In particular, in the logarithmic utility case, in order to get the optimal portfolio the right condition it is proved to be the *intensity hypothesis*. We use the anticipating calculus to analyze the existence of the optimal portfolio for the logarithmic utility, and than under the assumption of existence of the optimal portfolio we prove the semimartingale decomposition of the risky asset in the filtration enlarged with the default process.

Keywords— Optimal portfolio; Default risk; Progressive enlargement; Forward integrals; Malliavin calculus.

1 Introduction

In this paper, we solve the problem of the optimal portfolio made of a riskless bond and a risky defaultable asset. The dynamics of the risky asset are driven by a Brownian motion W, a compensated Poisson random measure $\tilde{N}$ and a default process $H = (\mathbb{1}_{\{\tau \le t\}},\, 0 \le t \le T)$, with τ being some random time. We provide necessary and sufficient conditions for the existence of a local maximum portfolio in our market and we deduce the semimartingale decomposition of the processes W and $\tilde{N}$ in the progressive enlarged filtration $\mathbb{G} := \mathbb{F} \vee \sigma(\tau \wedge t)$ needed to take care of the default time. $\mathbb{F}$ denotes the minimum complete right-continuous filtration generated by W and $\tilde{N}$.

The first step in solving the optimal portfolio problem was taken by [37], who used standard stochastic control techniques, a geometric Brownian motion as the risky asset and the CRRA utility. [34] was able to give a general solution for both a complete and incomplete market, in

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which the risky asset is assumed to be a semimartingale process. [3] gave an explicit solution for different utilities under initial enlargement of filtrations satisfying the *Jacod density hypothesis*, which is generally known as the *density hypothesis* in a defaultable framework. In [5], the optimal portfolio problem is solved using a model in which the dynamics of the price follow a Lévy process with discontinuous trajectories. [13] added a default process which generates a bankruptcy and interrupts the trading. The knowledge of the default was not included in the agent's natural filtration, so they imposed that the random time associated to the default process satisfied the density hypothesis. We highlight [8] who changed the point of view of the optimal portfolio problem in a market where the investor had some privileged information. By assuming the existence of a local maximum solution, they proved the semimartingale decomposition in the enlarged filtration of the Brownian motion driving the risky asset. [12] was an extension of the latter reference in which a compensated Poisson random measure is also driving the risky asset.

Default risk analysis was strongly developed in the last twenty five years. In [38], it is studied the case in which the default time is a stopping time in the natural filtration of the price process, while in [10] they made a comparison with the case in which the default time is a stopping time only in the larger filtration $\mathbb{G}$. In [17], by assuming the default time as a specific hitting time, they computed explicitly the semimartingale decomposition of the process H in the filtration $\mathbb{G}$. Default risk analysis in the setting of optimal control problems is a current field of research, as in [16], where the immersed hypothesis is assumed within the progressive enlargement $\mathbb{G} \supset \mathbb{F}$. Then, in [4] this assumption is relaxed by introducing a weaker assumption of the density hypothesis. One of the aims of our work is to relax even more this assumption investigating also what happens when this hypothesis does not hold.

The objective of this paper is to give necessary and sufficient conditions for the existence of an optimal portfolio in a defaultable framework. Our main contribution is to carry out this development without assuming the density hypothesis. This extends existing results in the literature, and, in particular, we do it by incorporating the universal framework of the forward integration, where different information flows can be included at the same time. This allows to study for example the case where the default time is given by the maximum of the Brownian motion before the time horizon. In particular we prove that under the logarithmic utility assumption the existence of a local maximum for the before-default strategy is strictly related to the so-called *intensity hypothesis*, see Theorem 3.23 and Examples 3.24 and 3.26 for more details. We also propose a solution that uses the anticipating calculus in a non-standard way. Indeed, the Malliavin derivative D_t is usually introduced for random variables $F \in L^2(\boldsymbol{P}, \mathcal{F}_T)$ with the condition $D.F \in L^2(dt \times \boldsymbol{P})$. However, the default process does not always satisfy this condition, so we extend the domain of the derivative to a bigger Hida-Malliavin distribution space, $(\mathcal{S})^*$, in which $D.F \notin L^2(dt \times \boldsymbol{P})$ is allowed. We introduce the details of the anticipating calculus in the Subsection 3.1. In the second part of the paper, under a general utility function and by assuming the existence of an optimal portfolio, we prove that the Brownian motion and the compensated version of the Poisson random measure driving the risky asset are semimartingales in the filtration $\mathbb{G}$.

We work in a similar framework of [4] but without the assumption of the density hypothesis. With respect to [13], we expand the trading beyond the default time and manage to prove that there exists a measure $\boldsymbol{Q}$ such that the processes W and $\tilde{N}$ are semimartingales in $(\boldsymbol{Q}, \mathbb{G})$. We improve [12] using a general utility function and including the default process. We also extend the result in [8], which only consider purely continuous models. With respect to the literature on default risk, cfr. [18, 31, 32], this work is the first to introduce a progressive enlargement of filtrations without the density hypothesis associated to the default. With respect to [8], our model is also more general as it includes a Poisson compensated random measure. Finally, we extend the domain the relationship among Malliavin trace, Skorohod integral and forward integration to the space $(\mathcal{S})^*$.

The paper is organised as follows. In Section 2, we properly define the set-up and the main assumptions that hold in the rest of the paper. In Section 3, we state the optimization problem and the admissible set of strategies. We split the problem before and after the occurrence of a default and we rewrite the optimal portfolio problem as a different stochastic control functional. In Subsection 3.1, we introduce some notions about anticipating calculus in the context of the Hida-Malliavin distribution space. We also solve explicitly the problem for logarithmic utility in Proposition 3.20 and Theorem 3.23. In Section 4, we give sufficient conditions for the existence of the optimal strategy and the semimartingale decomposition for both the Brownian motion and the compensated Poisson random measure in the enlarged filtration for a general utility function. We end the paper with some remarkable conclusions. In the Appendix we give some details omitted in the anticipating calculus subsection, for both the Brownian motion and the compensated version of the Poisson random measure.

2 Model and Notation

We work in a complete filtered probability space $(\Omega, \mathcal{G}_T, \boldsymbol{P}, \mathbb{G})$, where the filtration $\mathbb{G}$ is assumed to be right-continuous. We assume that the agent is going to invest in a market composed by two assets in a finite horizon time $T > 0$. The first one is a risk-less bond and the second one is a risky-defaultable stock. The dynamics of both are given by the following SDEs,

$$\frac{dD_t}{D_t} = \rho_t dt \ , \quad D_0 = 1 \tag{2.1a}$$

$$\frac{d^- S_t}{S_{t-}} = \mu_t dt + \sigma_t d^- W_t + \int_{\mathbb{R}_0} \theta_t(z) \tilde{N}(d^- t, dz) + \kappa_t dH_t \ , \quad S_0 = s_0 > 0 \tag{2.1b}$$

where $W = (W_t,\, 0 \le t \le T)$ is a Brownian motion with $\mathbb{F}^W$ its natural filtration. The random field $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ is the compensated version of a Poisson random measure $N(dt, dz)$ with $\boldsymbol{E}[N(dt, dz)] = \nu(dz)dt$ and $\mathbb{F}^N$ its natural filtration. By $d^- W_t$ and $\tilde{N}(d^- t, dz)$ we denote the forward integration, we refer to the Subsection 3.1 for the details. We assume that W and $\tilde{N}$ are independent under the probability measure $\boldsymbol{P}$. Moreover, the Borel measure $\nu(dz)$ on $\mathbb{R}_0 := \mathbb{R} \backslash \{0\}$ is σ-finite and satisfies

$$\int_{\mathbb{R}_0} \left(1 \wedge z^2\right) \nu(dz) < +\infty \ ,$$

where $a \wedge b := \min\{a, b\}$. The filtration $\mathbb{F} = \{\mathcal{F}_t : 0 \le t \le T\}$ is generated by the Brownian motion and the compensated Poisson measure, and it is augmented by the zero $\boldsymbol{P}$-measure sets, $\mathcal{N}$:

$$\mathcal{F}_t := \sigma(W_s,\ N((s,t], B) :\ 0 \le s \le t, B \in \mathcal{B}(\mathbb{R}_0)) \vee \mathcal{N} \ .$$

By $\mathcal{B}(\mathbb{R}_0)$ we refer the Borel σ-algebra on $\mathbb{R}_0$.

The process $H = (H_t,\, 0 \le t \le T)$ represents the default of the risky asset and it is defined by $H_t = \mathbb{1}_{\{\tau \le t\}}$ where τ is assumed to be an $\mathcal{F}_T$-measurable random time. In this framework the market is incomplete as the risky asset has discontinuous paths provided by the jumps of the compensated Poisson random measure as well as the jump of the default process H. We denote by $\mathbb{G} = \{\mathcal{G}_t : 0 \le t \le T\}$ the minimum right-continuous and complete filtration such that τ is a $\mathbb{G}$-stopping time, i.e.,

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau \wedge t) \ .$$

Note that $\mathbb{G} \supset \mathbb{F}$ and it is a progressive enlargement of filtration only if τ is not an $\mathbb{F}$-stopping time. We define the (right-continuous and complete) filtration of an agent who only observes the price process S as $\mathbb{S} = \{\mathcal{S}_t : 0 \le t \le T\}$,

$$\mathcal{S}_t := \sigma(S_s :\ 0 \le s \le t) \vee \mathcal{N}\ .$$

Note that $\mathcal{S}_t \subset \mathcal{G}_t$ but, in general, it is not equal. Moreover, $\mathcal{S}_t \subset \mathcal{F}_t$ only if τ is an $\mathbb{F}$-stopping time, in which case, as we mentioned above, $\mathcal{F}_t = \mathcal{G}_t$. The different information flows require different assumptions on the nature of the driving noises, i.e., their being a semimartingale or not with respect to the various filtrations and consequently different assumptions of predictability on the integrands for an Itô type calculus. This justifies our use of the forward integration, which provides a universal framework where different filtrations can be considered at the same time and that reduces to the classical Itô integration when measurability and semimartingale conditions are matched. This approach was first introduced in [13]. Else, keeping to the framework of Itô integration one has to prove the desired property that the default process H is a semimartingale in $\mathbb{G}$ and the compensator satisfies some continuity conditions. In this case, the Doob-Meyer decomposition claims that there exists a unique $\mathbb{G}$-predictable increasing process A such that $J := (H_t - A_t,\ 0 \le t \le T)$ is a $\mathbb{G}$-martingale. To short the notation, we define the $\mathbb{F}$-*Azéma* supermartingale as

$$Z_t := \boldsymbol{P}(\tau > t|\mathcal{F}_t)\ . \tag{2.2}$$

As τ is $\mathcal{F}_T$-measurable, note that $Z_T = \mathbb{1}_{\{\tau > T\}} = 1 - H_T$. The following lemma gives an explicit expression for the compensator A, to see the details Proposition 2.15 of [2] can be consulted.

Lemma 2.1. *The process $J = H - A$, with*

$$J_t = H_t - \int_0^{t\wedge\tau} \frac{dA_s^\tau}{Z_{s-}}\ , \quad 0 \le t \le T\ , \tag{2.3}$$

is a $(\boldsymbol{P}, \mathbb{G})$-martingale, where A^τ is the $\mathbb{F}$-dual predictable projection of H.

Using [40], we know that the $\mathbb{F}$-dual predictable projection of H is a bounded variation process. As it is discussed in [19], (originally proved in [11]), if the compensator of H is continuous, then τ is not an $\mathbb{F}$-stopping time and it is a totally inaccessible $\mathbb{G}$-stopping time. In order to exploit some nice properties of the default time, in the literature there exist two main approaches: the first one is called the intensity approach and it is based on assuming that the process A is absolutely continuous with respect to the Lebesgue measure, i.e., there exists a $\mathbb{G}$-predictable process $\lambda = (\lambda_t,\, 0 \le t \le T)$ such that

$$\textit{(intensity hypothesis)} \quad H_t - \int_0^t \lambda_s ds\ , \quad 0 \le t \le T\ , \tag{2.4}$$

is a $\mathbb{G}$-martingale. Note that, by Lemma 2.1, the compensator always exists but under hypothesis (2.4) the singular and the jump part of the compensator are eliminated. The second one is called the density approach and it is based on assuming that the *Jacod density hypothesis* holds true, i.e., there exists a process $q(\eta) = (q_t(\eta),\, 0 \le t \le T)$ such that,

$$\textit{(density hypothesis)} \quad \boldsymbol{P}(\tau \in d\eta|\mathcal{F}_t) = q_t(\eta)\zeta(d\eta)\ , \tag{2.5}$$

being ζ the law of τ. In [18], it is proved that the density approach is stronger that the intensity one. In particular, assuming the density hypothesis (2.5) holds true, the process λ in (2.4) can be fully recovered. However, assuming (2.4), the process $q_t(\eta)$ can be recovered only in case $t \le \eta$. The density hypothesis is often assumed in default risk setting, we refer to the papers [4, 22, 23, 32] as main examples. The intensity hypothesis is also used in the literature, for example in [26, 36]. In some papers, the authors assume the so-called immersion property or $(\mathcal{H})$-hypothesis, in which the processes W and $\tilde{N}$ are $\mathbb{G}$-martingales. We refer to, e.g, [33] in the context of mean-variance hedging or Section 3.2 of [18] to see its relation with the density hypothesis.

In this work, a priori we assume neither the density nor the intensity approach, nor the $(\mathcal{H})$-hypothesis, as we work in the framework of forward integration. In such a way, we can consider cases such as the time in which the Brownian motion reaches its maximum, see Example 3.24 below for more details. In order to solve the problem for a general utility function, in Section 4 we assume the existence of a local maximum strategy for our problem and, as a consequence of the optimality, we prove that the intensity approach must hold.

Going back to the market model (2.1), we assume that the market coefficients are càglàd $\mathbb{G}$-adapted processes satisfying the following integrability condition

$$\boldsymbol{E}\left[\int_0^T |\rho_t| + |\mu_t| + |\sigma_t|^2 + \int_{\mathbb{R}_0} |\theta_t(z)|^2 \nu(dz)\, dt\right] < +\infty \ . \tag{2.6}$$

The default time τ satisfies the following condition

$$\boldsymbol{P}(\exists\, t \in [0,T] \text{ such that } \{\tau = t\} \text{ holds true and } N(\Delta t, B) > 0)) = 0 \tag{2.7}$$

for any $B \subset \mathbb{R}_0$ compact and $N(\Delta t, B) := N((0,t], B) - N((0,t), B)$. This means that the default time does not occur $\boldsymbol{P}$-a.s. at the same time of any jump of N. In particular, assumption (2.7) is used in Proposition 2.2 below to apply the Itô formula under forward integration.

By Lemma 4.4 of [30], any $\mathbb{G}$-adapted process $Y = (Y_t, 0 \le t \le T)$ can be expressed as $Y_t = Y_t^{\mathbb{F}} \mathbb{1}_{\{\tau > t\}} + Y_t(\tau)\mathbb{1}_{\{\tau \le t\}}$, where $Y^{\mathbb{F}}$ is an $\mathbb{F}$-adapted process and $Y_t(\tau)$ is $\mathcal{F}_t \otimes \sigma(\tau)$-measurable. Using this fact, we can express the following processes in a decomposition before and after the occurrence of the default time:

$$S_t = S_t^{\mathbb{F}} \mathbb{1}_{\{\tau > t\}} + S_t(\tau)\mathbb{1}_{\{\tau \le t\}} \tag{2.8a}$$
$$\rho_t = \rho_t^{\mathbb{F}} \mathbb{1}_{\{\tau > t\}} + \rho_t(\tau)\mathbb{1}_{\{\tau \le t\}} \tag{2.8b}$$
$$\mu_t = \mu_t^{\mathbb{F}} \mathbb{1}_{\{\tau > t\}} + \mu_t(\tau)\mathbb{1}_{\{\tau \le t\}} \tag{2.8c}$$
$$\sigma_t = \sigma_t^{\mathbb{F}} \mathbb{1}_{\{\tau > t\}} + \sigma_t(\tau)\mathbb{1}_{\{\tau \le t\}} \tag{2.8d}$$
$$\theta_t(z) = \theta_t^{\mathbb{F}}(z) \mathbb{1}_{\{\tau > t\}} + \theta_t(z,\tau)\mathbb{1}_{\{\tau \le t\}} \ . \tag{2.8e}$$

The integral with respect to the default process $H_t = \mathbb{1}_{\{\tau \le t\}}$ is defined as a Riemann-Stieltjes integral, i.e.,

$$\int_0^t \phi_s dH_s := \phi_\tau H_t = \phi_\tau \mathbb{1}_{\{\tau \le t\}} < \infty \quad \boldsymbol{P}\text{-a.s.} \ . \tag{2.9}$$

We also assume that $\kappa_t \neq 0$, to guarantee the presence of the default in the model. The following proposition is an adaptation of the version of Theorem 2.5 of [13] to the framework of defaultable assets presented in (2.1b).

Proposition 2.2. *Let Y be a process that satisfies the following equation,*

$$\frac{d^- Y_t}{Y_{t-}} = \mu_t dt + \sigma_t d^- W_t + \int_{\mathbb{R}_0} \theta_t(z) \tilde{N}(d^- t, dz) + \kappa_t dH_t \ , \tag{2.10}$$

where σ is forward integrable with respect to W, θ and $|\theta|$ are forward integrable with respect to $\tilde{N}$, (2.6) and (2.7) hold true and κ satisfies (2.9). Assume $f \in C^2(\mathbb{R})$ then,

$$\begin{aligned} df(Y_s) &= \left(f'(Y_{s-})Y_{s-}\mu_s + f''(Y_{s-})Y_{s-}^2\sigma_s^2\right) ds \\ &\quad + \int_{\mathbb{R}_0} f(Y_{s-} + Y_{s-}\theta_s(z)) - f(Y_{s-}) - f'(Y_{s-})Y_{s-}\theta_s(z)\, \nu(dz) ds \end{aligned}$$

$$+ f'(Y_{s-})Y_{s-}\sigma_s d^- W_s + \int_{\mathbb{R}_0} f(Y_{s-} + Y_{s-}\theta_s(z)) - f(Y_{s-})\, \tilde{N}(d^- s, dz)$$
$$+ \left(f(Y_{s-} + Y_{s-}\kappa_s) - f(Y_{s-})\right) dH_s \ . \tag{2.11}$$

Applying Proposition 2.2 to the dynamics of the risky asset, given by (2.1b), we get an explicit expression as follows,

$$\ln\frac{S_t}{s_0} = \int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2 + \int_{\mathbb{R}_0} \ln(1+\theta_s(z)) - \theta_s(z)\nu(dz)\right) ds$$
$$+ \int_0^t \sigma_s d^- W_s + \int_0^t \int_{\mathbb{R}_0} \ln(1+\theta_s(z))\tilde{N}(d^- s, dz) + \int_0^t \ln(1+\kappa_s) dH_s \ .$$

To have the process S well-defined, we assume that

$$-1 < \theta_t(z) \ , \quad dt \times \nu(dz) \times \boldsymbol{P} - \text{a.s.} \tag{2.12}$$
$$-1 < \kappa_t < \infty \ , \quad dt \times \boldsymbol{P} - \text{a.s.} \ . \tag{2.13}$$

In the next section we study the expected utility of the terminal wealth on the time horizon $[0,T]$. By combining the approaches developed in [13] and [32] we split the optimization problem (3.7) into two new problems: one before default in the stochastic interval $[[0, \tau\wedge T[[$, and the other one after default in the stochastic interval $[[\tau\wedge T, T]]$, where the latter interval can be empty. To do this, we need to rewrite the dynamics of the risky asset in terms of before and after default. The following proof differs from the one in [32] for using the forward integration.

Proposition 2.3. *The component $S^{\mathbb{F}}$ of S in* (2.8a) *satisfies the following SDE,*

$$\frac{dS_t^{\mathbb{F}}}{S_{t-}^{\mathbb{F}}} = \mu_t^{\mathbb{F}} dt + \sigma_t^{\mathbb{F}} dW_t + \int_{\mathbb{R}_0} \theta_t^{\mathbb{F}}(z)\tilde{N}(dt, dz), \ t \in [[0, \tau\wedge T[[\tag{2.14a}$$
$$S_0^{\mathbb{F}} = s_0 > 0 \ , \tag{2.14b}$$

and the component $S(\tau)$ satisfies

$$\frac{dS_t(\tau)}{S_{t-}(\tau)} = \mu_t(\tau)dt + \sigma_t(\tau)d^- W_t + \int_{\mathbb{R}_0} \theta_t(z,\tau)\tilde{N}(d^- t, dz), \quad t \in [[\tau\wedge T, T]] \tag{2.15a}$$
$$S_\tau(\tau) = S_{\tau-}^{\mathbb{F}}(1+\kappa_\tau) \ . \tag{2.15b}$$

Proof. As in (2.8a), we express the price process as $S_t = S_t^{\mathbb{F}}\mathbb{1}_{\{\tau>t\}} + S_t(\tau)\mathbb{1}_{\{\tau\le t\}}$. Under $\{\tau > t\}$ it is clear that $S_t^{\mathbb{F}}\mathbb{1}_{\{\tau>t\}} = S_t\mathbb{1}_{\{\tau>t\}}$, then by applying Proposition 2.2 we have the following

$$S_t^{\mathbb{F}}\mathbb{1}_{\{\tau>t\}} = S_0 \exp\Big\{ \int_0^t \mu_s - \frac{1}{2}\sigma_s^2 + \int_{\mathbb{R}_0} \ln(1+\theta_s(z)) - \theta_s(z)\,\nu(dz) ds$$
$$+ \int_0^t \sigma_s d^- W_s + \int_0^t \int_{\mathbb{R}_0} \ln(1+\theta_s(z))\tilde{N}(d^- s, dz)\Big\}\mathbb{1}_{\{\tau>t\}} \ ,$$
$$= S_0 \exp\Big\{ \int_0^t \mu_s^{\mathbb{F}} - \frac{1}{2}\left(\sigma_s^{\mathbb{F}}\right)^2 + \int_{\mathbb{R}_0} \ln(1+\theta_s^{\mathbb{F}}(z)) - \theta_s^{\mathbb{F}}(z)\,\nu(dz) ds$$
$$+ \int_0^t \sigma_s^{\mathbb{F}} dW_s + \int_0^t \int_{\mathbb{R}_0} \ln(1+\theta_s^{\mathbb{F}}(z))\tilde{N}(ds, dz)\Big\}\mathbb{1}_{\{\tau>t\}} \ ,$$

where, according to the decomposition of the market coeffients (2.8), we have used the following pathwise relation,

$$\exp\Big\{ \int_0^t \mu_s^{\mathbb{F}}\mathbb{1}_{\{\tau>s\}} + \mu_s(\tau)\mathbb{1}_{\{\tau\le s\}} ds\Big\}\mathbb{1}_{\{\tau>t\}} = \exp\Big\{ \int_0^t \mu_s^{\mathbb{F}} ds\Big\}\mathbb{1}_{\{\tau>t\}} \ .$$

Now, on the set $\{\tau \leq t \leq T\}$, we have

$$\begin{aligned} S_t(\tau)\mathbb{1}_{\{\tau\leq t\}} =& S_t\mathbb{1}_{\{\tau\leq t\}} = S_0 \exp\Big\{ \int_0^t \mu_s - \frac{1}{2}\sigma_s^2 ds \\ &+ \int_0^t \int_{\mathbb{R}_0} \ln(1+\theta_s(z)) - \theta_s(z)\,\nu(dz)ds + \int_0^t \sigma_s d^- W_s \\ &+ \int_0^t \int_{\mathbb{R}_0} \ln(1+\theta_s(z))\tilde{N}(d^-s, dz) + \ln(1+\kappa_\tau)\mathbb{1}_{\{\tau\leq t\}} \Big\} \mathbb{1}_{\{\tau\leq t\}} , \end{aligned}$$

where we used the following pathwise equality

$$(1+\kappa_\tau)^{\mathbb{1}_{\{\tau\leq t\}}}\mathbb{1}_{\{\tau\leq t\}} = (1+\kappa_\tau)\mathbb{1}_{\{\tau\leq t\}} .$$

As for the forward integrals, we use the following equality

$$\int_0^t \sigma_s d^- W_s = \int_0^t \mathbb{1}_{\{\tau > s\}}\sigma_s^{\mathbb{F}} d^- W_s + \int_0^t \mathbb{1}_{\{\tau\leq s\}}\sigma_s(\tau) d^- W_s$$

to get

$$\begin{aligned} S_t(\tau)\mathbb{1}_{\{\tau\leq t\}} =& S_{\tau-}^{\mathbb{F}}\,(1+\kappa_\tau) \exp\Big\{ \int_\tau^t \mu_s(\tau) - \frac{1}{2}\sigma_s^2(\tau) ds \\ &+ \int_\tau^t \int_{\mathbb{R}_0} \ln(1+\theta_s(z,\tau)) - \theta_s(z,\tau)\,\nu(dz)ds \\ &+ \int_\tau^t \sigma_s(\tau) d^- W_s + \int_\tau^t \int_{\mathbb{R}_0} \ln(1+\theta_s(z,\tau))\tilde{N}(d^-s, dz) \Big\} \mathbb{1}_{\{\tau\leq t\}} . \end{aligned}$$

The result then holds by expressing the equation above in differential form. □

Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ denote two probability measures. $\boldsymbol{E}$ indicates the expectation operator under measure $\boldsymbol{P}$ and $\boldsymbol{E_Q}$ the one under measure $\boldsymbol{Q}$. We denote by dt the Lebesgue measure and by $L^2(B, dt)$ the set of deterministic functions which satisfy $\|f\|_2^2 = \int_B f^2(t)dt < +\infty$, $B \subseteq \mathbb{R}$. Sometimes we omit B when it is clear from the context and we write $L^2(dt)$. We also use $L^2(\nu \times dt)$ in the product space with its associated norm. We define the space $L^2(\boldsymbol{P}, \mathcal{F}_T)$, or simply $L^2(\boldsymbol{P})$, as $L^2(\boldsymbol{P}) := \{F \in \mathcal{F}_T : \boldsymbol{E}[F^2] < +\infty\}$. In addition, we let $L^2(dt \times \boldsymbol{P}, \mathbb{F})$, or simply $L^2(dt \times \boldsymbol{P})$, to be the space of all $\mathbb{F}$-adapted processes X satisfying $\int_0^T \boldsymbol{E}\left[X_s^2\right] ds < +\infty$.

3 Optimization Problem

Using the previous set-up, it is assumed that an agent can control her portfolio by a *self-financing* process $\pi = (\pi_t,\, 0 \leq t \leq T)$, with the aim to maximize her expected utility gain at a finite terminal time $T > 0$. If we denote by $X^\pi = (X_t^\pi,\, 0 \leq t \leq T)$ the wealth of the portfolio of the investor under π, its dynamics are given by the following SDE, for $0 \leq t \leq T$,

$$\frac{d^- X_t^\pi}{X_{t-}^\pi} = (1-\pi_t)\frac{dD_t}{D_t} + \pi_t \frac{d^- S_t}{S_{t-}} , \quad X_0^\pi = x_0 . \tag{3.1}$$

By using the evolution of both assets given in (2.1) we get

$$\frac{d^- X_t^\pi}{X_{t-}^\pi} = (1-\pi_t)\rho_t dt + \pi_t \left(\mu_t dt + \sigma_t d^- W_t + \int_{\mathbb{R}_0} \theta_t(z)\tilde{N}(d^- t, dz) + \kappa_t dH_t \right)$$

where the SDE is well-defined on the probability space $(\Omega, \mathcal{G}_T, \boldsymbol{P}, \mathbb{G})$ within the progressive enlarged filtration. Before giving a proper definition of the set of processes π that we consider, we look for the natural conditions they should satisfy. If we apply the Itô formula for forward integration given in Proposition 2.2 to $\ln X_t^\pi$, we get,

$$\ln \frac{X_t^\pi}{x_0} = \int_0^t \left(\rho_s + \pi_s(\mu_s - \rho_s) - \frac{1}{2}\pi_s^2\sigma_s^2 + \int_{\mathbb{R}_0} \ln(1+\pi_s\theta_s(z)) - \pi_s\theta_s(z)\nu(dz) \right) ds$$
$$+ \int_0^t \pi_s\sigma_s d^- W_s + \int_0^t \int_{\mathbb{R}_0} \ln(1+\pi_s\theta_s(z))\tilde{N}(d^-s, dz) + \int_0^t \ln(1+\pi_s\kappa_s)dH_s \ , \quad (3.2)$$

provided that these integrals are well-defined. To assure this, we assume the following integrability conditions.

$$\boldsymbol{E}\left[\int_0^T |\rho_s| + |\pi_s||\mu_s - \rho_s| + \pi_s^2\sigma_s^2 \, ds\right] < +\infty \quad (3.3)$$
$$\boldsymbol{E}\left[\int_0^T \int_{\mathbb{R}_0} \pi_s^2\theta_s^2(z)\,\nu(dz)ds\right] < +\infty \ . \quad (3.4)$$

To guarantee that X^π is well-defined, we assume that for all π there exists $\epsilon^\pi > 0$ such that,

$$\min\{1 + \pi_t\theta_t(z), 1 + \pi_t\kappa_t\} > \epsilon^\pi \ , \quad 0 \le t \le T \ . \quad (3.5)$$

In order to carry out the optimization of the portfolio, we consider an *utility function* $U : \mathbb{R}^+ \to \mathbb{R}$ denoting the utility of the investor. We extend the domain of the function to $\mathbb{R}$ by assuming that $U(x) := -\infty$ if $x < 0$. In addition we assume that it is continuously differentiable, strictly increasing and strictly concave in its domain and satisfying the Inada conditions,

$$U'(0) := \lim_{x\to 0^+} U'(x) = +\infty \quad (3.6a)$$
$$U'(+\infty) := \lim_{x\to\infty} U'(x) = 0 \ . \quad (3.6b)$$

Now, we can define properly the optimization problem as the supremum of the expected utility gain of the agent's wealth at the finite horizon time T.

$$J(x_0, \pi) := \boldsymbol{E}\left[U(X_T^\pi)|X_0^\pi = x_0\right] \ , \quad \mathbb{V}_T^{\mathbb{E}} := \sup_{\pi\in\mathcal{A}(\mathbb{E})} J(x_0,\pi) \ , \quad \mathbb{E} \in \{\mathbb{F}, \mathbb{G}\} \ . \quad (3.7)$$

Finally we give the definition of the set $\mathcal{A}(\mathbb{E})$ of admissible strategies for the agent playing with information flow $\mathbb{E} \in \{\mathbb{F}, \mathbb{G}\}$.

Definition 3.1. *In a financial market whose coefficients verify* (2.6), (2.12) *and* (2.13), *we define the set* $\mathcal{A}(\mathbb{E})$ *of admissible strategies as the set which contains all the portfolios* π *satisfying* (3.3), (3.4) *and* (3.5) *such that,*

- π *is càglàd and adapted w.r.t. the filtration* $\mathbb{E}$.
- $\pi\sigma$ *is càglàd and forward integrable w.r.t.* W.
- $\pi\theta$, $\ln(1+\pi\theta)$ *and* $\frac{\pi\theta}{1+\pi\theta}$ *are càglàd and forward integrable w.r.t.* $\tilde{N}$.

Since in some cases we can characterize the optimal solution for the problem (3.7) only in a local sense, we introduce the definition of local maximum.

Definition 3.2. *We say that* $\pi \in \mathcal{A}(\mathbb{E})$ *is a local maximum for the problem* (3.7) *if*

$$\boldsymbol{E}[U(X_T^{\pi+y\beta})] \leq \boldsymbol{E}[U(X_T^{\pi})] \ , \tag{3.8}$$

for all bounded $\beta \in \mathcal{A}(\mathbb{E})$ *and* $|y| < \delta$, *for some* $\delta > 0$.

With the same procedure applied in Section 2, we rewrite the dynamics of the process X^π before and after default, we omit the proof because it is analogous to the Proposition 2.3. We recall the decomposition $X_t^\pi = X_t^{\pi,\mathbb{F}} \mathbb{1}_{\{\tau>t\}} + X_t^\pi(\tau)\mathbb{1}_{\{\tau\leq t\}}$.

Proposition 3.3. *The component* $X^{\pi,\mathbb{F}}$ *of* X *satisfies the following SDE,*

$$\frac{dX_t^{\pi,\mathbb{F}}}{X_{t-}^{\pi,\mathbb{F}}} = (1-\pi_t^{\mathbb{F}})\rho_t^{\mathbb{F}} dt + \pi_t^{\mathbb{F}} \frac{dS_t^{\mathbb{F}}}{S_{t-}^{\mathbb{F}}}, \quad t \in [[0, \tau \wedge T[[\tag{3.9a}$$

$$X_0^{\pi,\mathbb{F}} = x_0 \ , \tag{3.9b}$$

and the component $X^\pi(\tau)$

$$\frac{dX_t^\pi(\tau)}{X_{t-}^\pi(\tau)} = (1-\pi_t(\tau))\rho_t(\tau)dt + \pi_t(\tau)\frac{d^- S_t(\tau)}{S_{t-}(\tau)}, \quad t \in [[\tau \wedge T, T]] \tag{3.10a}$$

$$X_\tau^\pi(\tau) = X_{\tau-}^{\pi,\mathbb{F}}(1+\pi_\tau^{\mathbb{F}}\kappa_\tau) \ . \tag{3.10b}$$

Since $1 = \mathbb{1}_{\{\tau>T\}} + \mathbb{1}_{\{\tau\leq T\}}$, we can *split* the main problem in two new problems, one before and the other after the default time with respect to the horizon time T. In order to short the notation, we define

$$d\boldsymbol{P}^\tau(\eta) := \boldsymbol{P}(\tau \in d\eta \,|\, \tau \leq T) \ , \quad 0 \leq \eta \leq T \ . \tag{3.11}$$

Following [32] we get the next lemma, where $\boldsymbol{E}^{x_0}[\cdot] := \boldsymbol{E}[\cdot|X_0 = x_0]$.

Lemma 3.4. *The expected utility gain of the agent's wealth can be written as follows*

$$J(x_0,\pi) = \boldsymbol{E}^{x_0}\left[U\left(X_T^{\pi,\mathbb{F}}\right)(1-H_T) + \int_0^T \boldsymbol{E}\left[U\left(X_T^\pi(\eta)\right)|X_\eta^\pi(\eta)\right] d\boldsymbol{P}^\tau(\eta)\right] \ . \tag{3.12}$$

Proof.

$$\begin{aligned} J(x_0,\pi) &= \boldsymbol{E}^{x_0}[U(X_T^\pi)] = \boldsymbol{E}^{x_0}\left[U\left(X_T^\pi\right)\mathbb{1}_{\{\tau>T\}} + U\left(X_T^\pi\right)\mathbb{1}_{\{\tau\leq T\}}\right] \\ &= \boldsymbol{E}^{x_0}\left[U\left(X_T^{\pi,\mathbb{F}}\right)(1-H_T) + \boldsymbol{E}\left[U\left(X_T^\pi(\tau)\right)|\tau, X_\tau^\pi(\tau)\right]\mathbb{1}_{\{\tau\leq T\}}\right] \\ &= \boldsymbol{E}^{x_0}\left[U\left(X_T^{\pi,\mathbb{F}}\right)(1-H_T) + \int_0^T \boldsymbol{E}\left[U\left(X_T^\pi(\eta)\right)|\tau=\eta, X_\eta^\pi(\eta)\right] d\boldsymbol{P}^\tau(\eta)\right] \ . \end{aligned} \tag{3.13}$$

In the second equality we used the SDE (3.10). □

As every $\mathbb{G}$-admissible π can be written as $\pi = (\pi_t^{\mathbb{F}}\mathbb{1}_{\{\tau>t\}} + \pi_t(\tau)\mathbb{1}_{\{\tau\leq t\}}, 0 \leq t \leq T)$, we define the set

$$\mathcal{A}(\tau) = \{\pi(\tau) : \exists \pi^{\mathbb{F}} \in \mathcal{A}(\mathbb{F}) \text{ such that } \pi_\cdot^{\mathbb{F}}\mathbb{1}_{\{\tau>\cdot\}} + \pi_\cdot(\tau)\mathbb{1}_{\{\tau\leq\cdot\}} \in \mathcal{A}(\mathbb{G})\} \ .$$

We define the after default optimization problem as

$$\mathbb{V}(\eta,x) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}(\tau)} \boldsymbol{E}^{\eta,x}\left[U\left(X_T^\pi(\tau)\right)\right] \ , \ (\eta,x) \in [0,T]\times(0,\infty) \ , \tag{3.14}$$

where $\boldsymbol{E}^{\eta,x}[\cdot] := \boldsymbol{E}[\cdot|\tau = \eta, X_\tau^\pi(\tau) = x]$. Next proposition extends the results in [32] by not requiring the density hypothesis. Since we cannot assume absolutely continuity of the distribution of τ with respect to the Lebesgue measure we use the measure introduced in (3.11).

Proposition 3.5. *If $\mathbb{V}(\eta, x) < \infty$, $\boldsymbol{P}$-a.s. for all $(\eta, x) \in [0, T] \times (0, \infty)$, then*

$$\mathbb{V}_T^{\mathbb{G}} = \sup_{\pi^{\mathbb{F}} \in \mathcal{A}(\mathbb{F})} \boldsymbol{E}^{x_0} \left[U\left(X_T^{\pi,\mathbb{F}}\right)(1 - H_T) + \int_0^T \mathbb{V}\left(\eta, X_\eta^{\pi,\mathbb{F}}\left(1 + \pi_\eta^{\mathbb{F}} \kappa_\eta\right)\right) d\boldsymbol{P}^\tau(\eta) \right] \tag{3.15}$$

Proof. Let $\pi \in \mathcal{A}(\mathbb{G})$, we can compute the functional,

$$\begin{aligned} J(\pi, x_0) &= \boldsymbol{E}^{x_0} \left[U\left(X_T^{\pi,\mathbb{F}}\right)(1 - H_T) + \int_0^T \boldsymbol{E}[U\left(X_T^\pi(\eta)\right) | X_\eta^\pi(\eta)] d\boldsymbol{P}^\tau(\eta) \right] \\ &\leq \boldsymbol{E}^{x_0} \left[U\left(X_T^{\pi,\mathbb{F}}\right)(1 - H_T) + \int_0^T \mathbb{V}(\eta, X_\eta^\pi(\eta)) d\boldsymbol{P}^\tau(\eta) \right] \leq \widehat{\mathbb{V}} \end{aligned}$$

with

$$\widehat{\mathbb{V}} := \sup_{\pi^{\mathbb{F}} \in \mathcal{A}(\mathbb{F})} \boldsymbol{E}^{x_0} \left[U\left(X_T^{\pi,\mathbb{F}}\right)(1 - H_T) + \int_0^T \mathbb{V}(\eta, X_\eta^{\pi,\mathbb{F}}(1 + \pi_\eta^{\mathbb{F}} \kappa_\eta)) d\boldsymbol{P}^\tau(\eta) \right]$$

and where in the last inequality we used (3.10b). We have proved that $\mathbb{V}_T^{\mathbb{G}} \leq \widehat{\mathbb{V}}$, now we aim to prove the opposite inequality. We fix an arbitrary $\pi^{\mathbb{F}} \in \mathcal{A}(\mathbb{F})$. By the definition of the optimum $\mathbb{V}$, for any $\omega \in \Omega$, $\eta \in [0, T]$ and $\epsilon > 0$, there exists $\pi^{\epsilon,\omega}(\eta) \in \mathcal{A}(\eta)$ which is an ϵ-optimal strategy, i.e.,

$$\mathbb{V}(\eta, X_\eta^\pi(\eta)) - \epsilon \leq \boldsymbol{E}\left[U\left(X_T^{\pi^\epsilon}(\eta)\right) | \tau = \eta, X_\eta^{\pi^\epsilon}(\eta)\right]$$

By constructing the strategy $\pi_t^\epsilon = \pi_t^{\mathbb{F}} \mathbb{1}_{\{\tau > t\}} + \pi_t^\epsilon(\tau) \mathbb{1}_{\{\tau \leq t\}}$ and using the previous inequalities, we get

$$\begin{aligned} \mathbb{V}_T^{\mathbb{G}} &\geq \boldsymbol{E}\left[U\left(X^{\pi,\mathbb{F}}(T)\right)(1 - H_T) + \int_0^T \boldsymbol{E}\left[U\left(X_T^{\pi^\epsilon}(\eta)\right) | \tau = \eta, X_\eta^{\pi^\epsilon}(\eta)\right] d\boldsymbol{P}^\tau(\eta) \right] \\ &\geq \boldsymbol{E}\left[U\left(X^{\pi,\mathbb{F}}(T)\right)(1 - H_T) + \int_0^T \mathbb{V}\left(\eta, X_\eta^\pi(\eta)\right) d\boldsymbol{P}^\tau(\eta) \right] - \epsilon \end{aligned}$$

and we get the result, since ϵ is arbitrary. □

3.1 Anticipating calculus in the white noise framework

We introduce for the sake of completeness some notions of anticipating calculus in the context of the white noise analysis. We mainly follow the lines of Chapters 5 and 13 in [15]. The anticipating calculus is needed for the computations of Subsections 3.2 and 3.3.

3.1.1 Brownian motion

First, we introduce the *Hermite polynomials* as follows

$$h_n(s) := (-1)^n \exp\left(\frac{s^2}{2}\right) \frac{d^n}{ds^n} \exp\left(-\frac{s^2}{2}\right), \quad (s, n) \in [0, T] \times \mathbb{Z}^+,$$

and the *Hermite functions* defined as

$$e_{k+1}(s) := \frac{\pi^{-\frac{1}{4}}}{\sqrt{k!}} \exp\left(-\frac{s^2}{2}\right) h_k(\sqrt{2}s), \quad (s, k) \in [0, T] \times \mathbb{Z}^+ .$$

It can be proved that the family $\{e_k\}_{k \geq 1}$ is an orthonormal basis of $L^2(\mathbb{R}, dt)$. Let's consider also the following stochastic Itô integrals

$$\theta_k := \int_0^T e_k(s) dW_s .$$

Denoting by $\mathcal{J}$ we will refer the set of finite non-negative multi-index $\alpha = (\alpha_1, ..., \alpha_m)$ for $m \in \mathbb{N}$, we define

$$H_\alpha := \prod_{k=1}^{m} h_{\alpha_k}(\theta_k) \ , \quad \alpha = (\alpha_1, ..., \alpha_m) \in \mathcal{J} \ . \tag{3.16}$$

The family $\{H_\alpha\}_{\alpha\in\mathcal{J}}$ is an orthogonal basis for the space $L^2(\boldsymbol{P})$ in the sense of the following theorem. We refer to Theorem 2.2.3 and 2.2.4 of [24] for a detailed proof.

Theorem 3.6. *Let X be a $\mathcal{F}_T^W$-measurable random variable in $L^2(\boldsymbol{P})$, then there exists a unique sequence $\{a_\alpha\} \subset \mathbb{R}$ such that*

$$X = \sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha \ ,$$

and the norm can be computed as $\|X\|^2_{L^2(\boldsymbol{P})} = \sum \alpha! a_\alpha^2$, with $\alpha! = \alpha_1!\alpha_2!...\alpha_m!$.

In the next definition we introduce the Hida test function and the Hida-Malliavin distribution spaces, that will play an important role in our calculations.

Definition 3.7.

- *Let $f = \sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha \in L^2(\boldsymbol{P})$ be a random variable, we say that f belongs to the Hida test function Hilbert space $(\mathcal{S})_k$, for $k \in \mathbb{R}$, if*

$$\|f\|_k^2 := \sum_{\alpha\in\mathcal{J}} \alpha! \, a_\alpha^2 \prod_{j=1}^{m} (2j)^{k\alpha_j} < \infty \ .$$

We define the Hida test function space $(\mathcal{S}) = \bigcap_{k\in\mathbb{R}} (\mathcal{S})_k$ equipped with the projective topology.

- *Let $F = \sum_{\alpha\in\mathcal{J}} b_\alpha H_\alpha$ be a formal sum, we say that F belongs to the Hida-Malliavin distribution Hilbert space $(\mathcal{S})_{-q}$, for $q \in \mathbb{R}$, if*

$$\|F\|_{-q}^2 := \sum_{\alpha\in\mathcal{J}} \frac{\alpha! \, b_\alpha^2}{\prod_{j=1}^{m} (2j)^{q\alpha_j}} < \infty \ .$$

We define the Hida-Malliavin distribution space $(\mathcal{S})^* = \bigcup_{q\in\mathbb{R}} (\mathcal{S})_{-q}$ equipped with the inductive topology, i.e, convergence is studied with $\|\cdot\|_q$ for some $q \in \mathbb{R}$.

The space $(\mathcal{S})^*$ is the dual of $(\mathcal{S})$ and we consider the action of F on f as follows,

$$\langle F, f \rangle = \sum_{\alpha\in\mathcal{J}} a_\alpha b_\alpha \alpha! \ .$$

Note that the inclusions $(\mathcal{S}) \subset L^2(\boldsymbol{P}) \subset (\mathcal{S})^*$ holds true. We define the generalized expectation of $F = \sum b_\alpha H_\alpha \in (\mathcal{S})^*$ as $\boldsymbol{E}[F] := b_0$, and the generalized conditional expectation of F with respect to the σ-algebras of the natural filtration $\mathbb{F}^W$ as

$$\boldsymbol{E}\left[F|\mathcal{F}_t^W\right] := \sum_{\alpha\in\mathcal{J}} b_\alpha \boldsymbol{E}\left[H_\alpha|\mathcal{F}_t^W\right]$$

when it exists under the convergence in $(\mathcal{S})^*$. In addition, we define

$$\boldsymbol{E}[F|\mathcal{G}_t] := \sum_{\alpha\in\mathcal{J}} b_\alpha \boldsymbol{E}[H_\alpha|\mathcal{G}_t] \ , \quad \mathcal{F}_t \subset \mathcal{G}_t \ ,$$

with the convergence in $(\mathcal{S})^*$ and the object $Z := \boldsymbol{E}[H_\alpha|\mathcal{G}_t]$ satisfies $\boldsymbol{E}[\mathbb{1}_A Z] = \boldsymbol{E}[\mathbb{1}_A H_\alpha]$, $\forall A \in \mathcal{G}_t$. Next, we give the definition of Malliavin derivative in the space $(\mathcal{S})^*$.

Definition 3.8. *Let $F = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha \in (\mathcal{S})^*$ be a Hida-Malliavin distribution, then we define the Malliavin derivative as*

$$D_t F = \sum_{\alpha \in \mathcal{J}} \sum_{k=1}^{\infty} a_\alpha \alpha_k e_k(t) H_{\alpha - \varepsilon(k)} \ ,$$

whenever this sum converges in $(\mathcal{S})^*$ and where $\varepsilon(k) := (0, 0, ..., 1)$ is the canonical vector with 1 in the k-th position. We define the set $Dom(D_t)$ as its domain.

As it is pointed out in Chapter 6 of [15], the Definition 3.8 is a natural generalization to $(\mathcal{S})^*$ of the $L^2(\boldsymbol{P})$-Malliavin derivative as they coincide in the domain of the second one, usually denoted by $\mathbb{D}_{1,2}$, and we have $L^2(\boldsymbol{P}) \subset Dom(D_t) \subset (\mathcal{S})^*$. Let $Y = (Y_t, 0 \leq t \leq T) \in (\mathcal{S})^*$, if the following limit exists

$$D_{t^+} Y_t := \lim_{s \to t^+} D_s Y_t \tag{3.17}$$

in $(\mathcal{S})^*$, we say that $D_{t^+} Y_t$ is the *Malliavin trace* of Y.

Definition 3.9. *The singular (or pointwise) white noise $\mathbb{W} = \{\mathbb{W}_t : 0 \leq t \leq T\} \in (\mathcal{S})^*$ is defined as $\mathbb{W}_t = \sum_{k=1}^{\infty} e_k(t) H_{\varepsilon(k)}$.*

It can be proved that $\mathbb{W}_t$ is well-defined as an object in $(\mathcal{S})^*$ and it satisfies $\frac{d}{dt} W_t = \mathbb{W}_t$, see Section 5.2 of [15] to consult the details. Now that we have defined the notion of derivative we introduce different types of integration. The first one generalizes the Lebesgue integral for $L^2([0, T], dt)$.

Definition 3.10. *Suppose $Z : [0, T] \longrightarrow (\mathcal{S})^*$ has the property $\langle Z(t), \psi \rangle \in L^1([0, T], dt)$, for any $\psi \in (\mathcal{S})$. Then the integral $\int_0^T Z(t)dt$ is defined as the unique element of $(\mathcal{S})^*$ such that*

$$\left\langle \int_0^T Z(t)dt, \psi \right\rangle = \int_0^T \langle Z(t), \psi \rangle dt \ , \quad \psi \in (\mathcal{S}) \ .$$

The function Z is called dt-integrable in $(\mathcal{S})^*$.

In the next definition we introduce the notion of forward integral in the context of $(\mathcal{S})^*$.

Definition 3.11. *A càglàd stochastic process $Y = (Y_t, 0 \leq t \leq T)$ is forward integrable in $(\mathcal{S})^*$ with respect to the Brownian motion $W = (W_t, 0 \leq t \leq T)$ if the limit*

$$\int_0^T Y_t d^- W_t = \lim_{\epsilon \to 0} \int_0^T Y_t \frac{W_{t+\epsilon} - W_t}{\epsilon} dt$$

exists in $(S)^*$.

When the previous limit is taken in probability we say that the process is classically forward integrable. This object can be seen as a generalization of the Itô integral and it provides a universal framework that considers different filtrations at the same time, in the sense of Lemma A.1 in the Appendix A. We introduce an operation named Wick product within the space of distributions $(\mathcal{S})^*$.

Definition 3.12. *Let $F = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha$, $G = \sum_{\beta \in \mathcal{J}} b_\beta H_\beta \in (\mathcal{S})^*$, then we define the Wick product as*

$$F \diamond G := \sum_{\alpha, \beta \in \mathcal{J}} a_\alpha b_\beta H_{\alpha + \beta} \ .$$

With the previous definitions we state the Skorohod integral within $(\mathcal{S})^*$. It is motivated by Theorem 5.20 of [15], as a natural generalization of the definition of Skorohod integral for $L^2(dt \times \boldsymbol{P})$-processes.

Definition 3.13. *Let $Y = (Y_t,\, 0 \le t \le T)$ be $(\mathcal{S})^*$-valued, then we define the Skorohod integral as*

$$\int_0^T Y_t \diamond \mathbb{W}_t \, dt \ ,$$

provided the integrand is dt-integrable in $(\mathcal{S})^*$.

We state the following lemma because it will play a crucial role in the main result of the section, Theorem 3.15. It extends the domain of applicability of the Malliavin derivative by allowing that $D.F \in (\mathcal{S})^*$ and not only $D.F \in L^2(dt \times \boldsymbol{P})$.

Lemma 3.14. *Let $F \in L^2(\boldsymbol{P})$ be a $\mathcal{F}_T^W$-measurable such that $D_t F$ is dt-integrable in $(\mathcal{S})^*$, then*

$$F \diamond (W_t - W_s) = F \cdot (W_t - W_s) - \int_s^t D_u F du \ , \quad 0 \le s < t \le T \ . \tag{3.18}$$

Proof. We refer to the Appendix A for the details of the proof. □

We introduce the main result of this section. It provides an explicit relationship in $(\mathcal{S})^*$ between the forward integral, the Skorohod integral and the Malliavin trace. In the context of fractional white noise analysis it was already attempted in Theorem 3.7 of [9], however they require the strong assumption of $D.Y. \in L^2(dt \times \boldsymbol{P})$ that we managed to avoid.

Theorem 3.15. *Let $Y = (Y_t,\, 0 \le t \le T)$ be an $L^2(dt \times \boldsymbol{P}, \mathbb{G})$-adapted process, if $D_{t^+}Y_t$ is dt-integrable in $(\mathcal{S})^*$, then*

$$\int_0^T Y_t d^- W_t = \int_0^T Y_t \diamond \mathbb{W}_t dt + \int_0^T D_{t^+} Y_t dt \tag{3.19}$$

holds true in $(\mathcal{S})^*$.

Proof. We refer to the Appendix A for the details of the proof. □

3.1.2 Poisson random measure

We repeat the main ideas of the previous section but in the case of the compensated version of a Poisson random measure. In this case, we refer to the Appendix B for the details and we state only the main result and the most important definitions. The first one is the Malliavin derivative, it involves the polynomial basis $\{p_j\}$ defined in (B.1) and the $L^2(\boldsymbol{P})$-orthogonal basis $\{K_\alpha\}_{\alpha \in \mathcal{J}}$ defined in (B.3), analogously to the previous basis $\{H_\alpha\}_{\alpha \in \mathcal{J}}$.

Definition 3.16. *Let $F = \sum_{\alpha \in \mathcal{J}} a_\alpha K_\alpha \in (\mathcal{S})^*$, then we define the Malliavin derivative as*

$$D_{t,z} F := \sum_{\alpha \in \mathcal{J}} \sum_{i,j=1}^{\infty} a_\alpha \alpha_{z(i,j)} e_i(t) p_j(z)\, K_{\alpha - \varepsilon(i,j)} \ ,$$

whenever this sum converges in $(\mathcal{S})^*$ and the set $Dom(D_{t,z})$ as its domain.

If the limit $D_{t^+,z}\theta_t(z) := \lim_{s \to t^+} D_{s,z}\theta_t(z)$ exists in $(\mathcal{S})^*$, we say that $D_{t^+,z}\theta_t(z)$ is the Malliavin trace of $\theta_t(z)$.

Definition 3.17. *A càglàd stochastic process $\theta = \{\theta_t(z),\ 0 \le t \le T\}$ is forward integrable in $(\mathcal{S})^*$ with respect to the compensated Poisson random measure $\tilde{N}$ if the limit*

$$\int_0^T \int_{\mathbb{R}_0} \theta_s(z)\tilde{N}\left(d^- s, dz\right) := \lim_{m\to\infty} \int_0^T \int_{\mathbb{R}_0} \theta_s(z)\mathbb{1}_{\{z\in U_m\}}\tilde{N}(ds, dz)$$

exists in $(\mathcal{S})^*$, where $\{U_m : m \in \mathbb{N}\}$ is an increasing sequence of compact sets $U_m \subseteq \mathbb{R}_0$ with $\nu\left(U_m\right) < \infty$ such that $\lim_{m\to\infty} U_m = \mathbb{R}_0$.

We finally introduce the Skorohod integral in $(\mathcal{S})^*$. It involves the white noise $\mathbb{N}$, stated in Definition B.3, and the version of Wick product for the Poisson case, see Definition B.4.

Definition 3.18. *Let $Y = (Y_t,\ 0 \le t \le T) \in (\mathcal{S})^*$, then we define the Skorohod integral as*

$$\int_0^T \int_{\mathbb{R}_0} Y_t \diamond \tilde{\mathbb{N}}(t, z)\, \nu(dz)dt\ , \tag{3.20}$$

provided the integrand is $(\nu \times dt)$-integrable in $(\mathcal{S})^*$.

In the following, we state the main result of the section for the Poisson case. It gives an explicit relationship among the forward integral, the Skorohod integral and the Malliavin trace in $(\mathcal{S})^*$. We present the proof as a novel contribution, to the best of our knowledge, never attempted before. We need to apply a different argument compared with the Brownian case because we can not handle with the Hermite polynomials. In Sections 6.2 and 6.3 of [44] the Charlier polynomials are introduced playing a similar role for the Poisson process, however, only in case of $\nu(dz) = \lambda\delta_{z_0}(z)$ we can recover similar properties. We refer Lemma 15.5 of [15] as the statement that we are extending in the next theorem.

Theorem 3.19. *Let $\theta_t(z)$ be a forward integrable process such that $\theta_t(z) \in\ L^2(\boldsymbol{P})$ for every $(t, z) \in [0, T] \times \mathbb{R}_0$ and $D_{t^+,z}\theta_t(z)$, $D_{t^+,z}\theta_t(z) \diamond \tilde{\mathbb{N}}(t, z)$ are $(\nu \times dt)$-integrable in $(\mathcal{S})^*$, then*

$$\begin{aligned}\int_0^T \int_{\mathbb{R}_0} \theta_t(z)\tilde{N}(d^- t, dz) &= \int_0^T \int_{\mathbb{R}_0} \big(\theta_t(z) + D_{t^+,z}\theta_t(z)\big) \diamond \tilde{\mathbb{N}}(t, z)\, \nu(dz)dt \\ &\quad + \int_0^T \int_{\mathbb{R}_0} D_{t^+,z}\theta_t(z)\, \nu(dz)dt\end{aligned} \tag{3.21}$$

holds true in $(\mathcal{S})^*$.

Proof. Let U_m be a compact set of $\mathbb{R}_0$ such that $0 \notin U_m$. We consider a partition of $[0, T] \times U_m$ as

$$0 = t_0 < t_1 < \ldots < t_{l-1} < t_l = T\ , \quad z_0 < z_1 \le z_2 < z_3 \le \ldots \le z_{2k} < z_{2k+1}\ ,$$

and we have $U_m = \cup_{j=0}^k [z_{2j}, z_{2j+1}]$. We consider the following càglàd process

$$\theta_t^{l,k}(z) := \sum_{i,j=0}^{l,k} \theta_{t_i}(z_{2j})\mathbb{1}_{\{(t,z)\in[t_i,t_{i+1}]\times[z_{2j},z_{2j+1}]\}}\ ,$$

and we are going to show that the Equation (3.21) holds true:

$$\begin{aligned}\int_0^T \int_{\mathbb{R}_0} \theta_t^{l,k}(z)\tilde{N}(d^- t, dz) &= \lim_{m\to\infty} \int_0^T \int_{\mathbb{R}_0} \theta_t^{l,k}(z)\mathbb{1}_{\{z\in U_m\}}\tilde{N}(dt, dz) \\ &= \lim_{m\to\infty} \sum_{i,j=0}^{l,k} \int_{t_i}^{t_{i+1}} \int_{z_{2j}}^{z_{2j+1}} \theta_{t_i}^{l,k}(z_{2j})\tilde{N}(dt, dz)\end{aligned}$$

$$
\begin{aligned}
&= \lim_{m\to\infty} \sum_{i,j=0}^{l,k} \theta_{t_i}^{l,k}(z_{2j}) \left(\tilde{N}(t_{i+1}, [z_{2j}, z_{2j+1}]) - \tilde{N}(t_i, [z_{2j}, z_{2j+1}]) \right) \\
&= \lim_{m\to\infty} \sum_{i,j=0}^{l,k} \int_{t_i}^{t_{i+1}} \int_{z_{2j}}^{z_{2j+1}} \left(\theta_{t_i}^{l,k}(z_{2j}) + D_{u,z}\theta_{t_i}^{l,k}(z_{2j}) \right) \diamond \tilde{\mathbb{N}}(u,z)\, \nu(dz)du \\
&\quad + \lim_{m\to\infty} \int_{t_i}^{t_{i+1}} \int_{z_{2j}}^{z_{2j+1}} D_{u,z}\theta_{t_i}^{l,k}(z_{2j})\, \nu(dz)du \\
&= \lim_{m\to\infty} \int_0^T \int_{U_m} \left(\theta_t^{l,k}(z) + D_{u,z}\theta_t^{l,k}(z) \right) \diamond \tilde{\mathbb{N}}(u,z) \; + D_{u,z}\theta_t^{l,k}(z)\; \nu(dz)du \\
&= \int_0^T \int_{\mathbb{R}_0} \left(\theta_t^{l,k}(z) + D_{u,z}\theta_t^{l,k}(z) \right) \diamond \tilde{\mathbb{N}}(u,z) \; + D_{u,z}\theta_t^{l,k}(z)\; \nu(dz)du
\end{aligned}
$$

and the last limit is convergent because the functions are Bochner integrable in $[0,T]\times U_m$, see [39]. We finish the proof for the general case using an approximation argument. □

3.2 After Default: Logarithmic case

In this subsection we restrict our computations to the logarithmic utility, i.e. $U(x) = \ln x$ that allows to obtain more explicit results. In particular, we analyze the after default problem, which is stated as

$$
\mathbb{V}(\eta, x) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}(\tau)} \boldsymbol{E}^{\eta,x}\left[\ln X_T^{\pi}(\tau)\right] \;, \quad (\eta, x) \in [0,T]\times\mathbb{R}^+ \tag{3.22}
$$

where the random variable $\ln X_T^{\pi}(\tau)$ has explicit solution as follows,

$$
\begin{aligned}
\ln \frac{X_T^{\pi}(\tau)}{X_\tau^{\pi}(\tau)} &= \int_\tau^T \rho_s(\tau) + \pi_s(\tau)(\mu_s(\tau) - \rho_s(\tau)) - \frac{1}{2}\pi_s^2(\tau)\sigma_s^2(\tau)\, ds \\
&\quad + \int_\tau^T \int_{\mathbb{R}_0} \ln(1 + \pi_s(\tau)\theta_s(z,\tau)) - \pi_s(\tau)\theta_s(z,\tau)\, \nu(dz)ds \\
&\quad + \int_\tau^T \pi_s(\tau)\sigma_s(\tau) d^- W_s + \int_\tau^T \int_{\mathbb{R}_0} \ln(1 + \pi_s(\tau)\theta_s(z,\tau)) \tilde{N}(d^- s, dz) \;.
\end{aligned} \tag{3.23}
$$

The stochastic integrals with random intervals are treated as follows,

$$
\int_\tau^T \pi_s(\tau)\sigma_s(\tau) d^- W_s = \int_0^T \mathbb{1}_{\{\tau\le s\}} \pi_s(\tau)\sigma_s(\tau) d^- W_s \;.
$$

In the next proposition we solve the optimization problem (3.22) using the anticipating calculus described in Subsection 3.1.

Proposition 3.20. *There exists a local maximum for the problem* (3.22) *with initial condition* $(\eta, x) \in [0,T]\times(0,\infty)$ *if and only if it satisfies, for* $s \in [\eta, T]$,

$$
\begin{aligned}
\pi_s(\eta)\sigma_s^2(\eta) &= \mu_s(\eta) - \rho_s(\eta) + \boldsymbol{E}\left[D_{s^+}\sigma_s(\eta)\,|\,\mathcal{G}_s\right] + \sigma_s(\eta) \diamond \mathbb{W}_s \\
&\quad + \int_{\mathbb{R}_0} \frac{-\pi_s(\eta)\theta_s^2(z,\eta)}{1 + \pi_s(\eta)\theta_s(z,\eta)} + \boldsymbol{E}\left[D_{s^+,z} \frac{\theta_s(z,\eta)}{1 + \pi_s(\eta)\theta_s(z,\eta)}\,|\,\mathcal{G}_s\right] \nu(dz) \\
&\quad + \int_{\mathbb{R}_0} \left(\frac{\theta_s(z,\eta)}{1 + \pi_s(\eta)\theta_s(z,\eta)} + \boldsymbol{E}\left[D_{s^+,z} \frac{\theta_s(z,\eta)}{1 + \pi_s(\eta)\theta_s(z,\eta)}\,|\,\mathcal{G}_s\right] \right) \diamond \tilde{\mathbb{N}}(s,z)\nu(dz) \;.
\end{aligned} \tag{3.24}
$$

Proof. We use a perturbation argument as in Theorem 16.20 of [15]. We define

$$G_{\eta,x}(\pi) := \boldsymbol{E}^{\eta,x}\left[\ln \frac{X_T^\pi(\tau)}{X_\tau^\pi(\tau)}\right]$$

and aim to optimize the function $\pi \longrightarrow G_{\eta,x}(\pi)$ with (η, x) fixed. Let $\beta \in \mathcal{A}(\tau)$ bounded and consider $\delta > 0$ such that $\pi + \epsilon\beta \in \mathcal{A}(\tau)$ for $\epsilon \in (-\delta, \delta)$ and we consider the function $I(\epsilon) := G_{\eta,x}(\pi + \epsilon\beta)$. A local maximum strategy must satisfy the following first order condition,

$$\begin{aligned}
0 =& I'(0) = \lim_{\epsilon\to 0} \frac{I(\epsilon) - I(0)}{\epsilon} = \lim_{\epsilon\to 0} \frac{G_{\eta,x}(\pi + \epsilon\beta) - G_{\eta,x}(\pi)}{\epsilon} \\
=& \boldsymbol{E}^{\eta,x}\left[\int_\tau^T \beta_s(\mu_s(\tau) - \rho_s(\tau)) - \beta_s\pi_s(\tau)\sigma_s^2(\tau) - \int_{\mathbb{R}_0} \frac{\beta_s\pi_s(\tau)\theta_s^2(z,\tau)}{1 + \pi_s(\tau)\theta_s(z,\tau)}\nu(dz)\, ds\right] \\
&+ \boldsymbol{E}^{\eta,x}\left[\int_\tau^T \beta_s\sigma_s(\tau)d^-W_s + \int_\tau^T \int_{\mathbb{R}_0} \frac{\beta_s\theta_s(z,\tau)}{1 + \pi_s(\tau)\theta_s(z,\tau)}\tilde{N}(d^-s, dz)\right] .
\end{aligned}$$

We take $\beta_s = \xi_t \mathbb{1}_{\{t<s\le t+h\}}$ for $\eta \le t \le T - h$ and ξ_t any bounded $\mathcal{G}_t$-measurable random variable. Then the condition is written as,

$$\begin{aligned}
0 =& \boldsymbol{E}^{\eta,x}\left[\xi_t \int_t^{t+h} \mu_s(\tau) - \rho_s(\tau) - \pi_s(\tau)\sigma_s^2(\tau) - \int_{\mathbb{R}_0} \frac{\pi_s(\tau)\theta_s^2(z,\tau)}{1 + \pi_s(\tau)\theta_s(z,\tau)}\nu(dz)\, ds\right] \\
&+ \boldsymbol{E}^{\eta,x}\left[\xi_t \left(\int_t^{t+h} \sigma_s(\tau)d^-W_s + \int_t^{t+h} \int_{\mathbb{R}_0} \frac{\theta_s(z,\tau)}{1 + \pi_s(\tau)\theta_s(z,\tau)}\tilde{N}(d^-s, dz)\right)\right]
\end{aligned}$$

which, as ξ_t is general, is equivalent to

$$\begin{aligned}
0 =& \boldsymbol{E}^{\eta,x}\left[\int_t^{t+h} \mu_s(\tau) - \rho_s(\tau) - \pi_s(\tau)\sigma_s^2(\tau) - \int_{\mathbb{R}_0} \frac{\pi_s(\tau)\theta_s^2(z,\tau)}{1 + \pi_s(\tau)\theta_s(z,\tau)}\nu(dz)\, ds|\mathcal{G}_t\right] \\
&+ \boldsymbol{E}^{\eta,x}\left[\int_t^{t+h} \sigma_s(\tau)d^-W_s + \int_t^{t+h} \int_{\mathbb{R}_0} \frac{\theta_s(z,\tau)}{1 + \pi_s(\tau)\theta_s(z,\tau)}\tilde{N}(d^-s, dz)|\mathcal{G}_t\right] . \qquad (3.25)
\end{aligned}$$

By applying Theorems 3.15 and 3.19 we rewrite the stochastic integrals as dt-integral and $(\nu \times dt)$-integral in $(\mathcal{S})^*$ respectively, and by rearranging all the previous terms, we get the condition (3.24).

Conversely, it is enough to see that by assuming (3.24) we recover (3.25) and the perturbation argument holds true because of the concavity, i.e., $I''(\epsilon) < 0$ for $\epsilon \in (-\delta, \delta)$. □

Remark 3.21. *The domain $\mathbb{D}_{1,2}$ of the Malliavin derivative does not contain the indicator random variables, which will play a prominent role in Theorem 3.23 and Examples 3.24 and 3.26 below. This is why we need to extend the framework of our paper and consider the Malliavin derivative, the Skorohod integral and the forward integral in the space of Hida-Malliavin distributions, as we have done in Subsection 3.1. In this context, the white noises for the Brownian motion and the Poisson random measure arise naturally, as we have shown in the Proposition 3.20. Note that Lemma 5.1 of [7] guarantees that the conditional expectation of the Malliavin derivative of a $L^2(\boldsymbol{P})$-random variable is well defined as an element of $L^2(dt \times \boldsymbol{P})$. In addition, using the computations appearing in the proof of Lemma 2.5.7 of [24] we can assure the well definition of the Wick product of an $L^2(dt \times \boldsymbol{P})$ process with respect to the white noises.*

3.3 Before Default: Logarithmic Case

We remind that the random variable $X_{\tau-}^{\pi,\mathbb{F}}$ has explicit solution as follows,

$$\ln\left(\frac{X_{\tau-}^{\pi,\mathbb{F}}}{x_0}\right) = \int_0^T \mathbb{1}_{\{\tau>s\}}\left(\rho_s^{\mathbb{F}} + \pi_s^{\mathbb{F}}(\mu_s^{\mathbb{F}} - \rho_s^{\mathbb{F}}) - \frac{1}{2}\left(\pi_s^{\mathbb{F}}\sigma_s^{\mathbb{F}}\right)^2\right) ds$$

$$
\begin{aligned}
&+\int_0^T \int_{\mathbb{R}_0} \mathbb{1}_{\{\tau>s\}} \left(\ln(1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)) - \pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)\right) \nu(dz)ds \\
&+\int_0^T \mathbb{1}_{\{\tau>s\}}\pi_s^{\mathbb{F}}\sigma_s^{\mathbb{F}} d^- W_s + \int_0^T \int_{\mathbb{R}_0} \mathbb{1}_{\{\tau>s\}} \ln(1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z))\tilde{N}(d^- s, dz)\ . \qquad (3.26)
\end{aligned}
$$

Considering the logarithmic utility, Proposition 3.5 can be simplified as follows.

Corollary 3.22. *The optimization problem* (3.7)*, under* $U(x) = \ln x$ *and the information flow* $\mathbb{G}$*, can be written as follows*

$$
\mathbb{V}_T^{\mathbb{G}} = \boldsymbol{E}^{x_0}[h_\tau H_T] + \sup_{\pi^{\mathbb{F}}\in\mathcal{A}(\mathbb{F})} \boldsymbol{E}^{x_0}\left[\ln\left(X_{\tau-}^{\pi,\mathbb{F}}\right) + \ln\left(1+\pi_\tau^{\mathbb{F}}\kappa_\tau\right) H_T\right] \qquad (3.27)
$$

for some process h *with* $h_T = 0$ $\boldsymbol{P}$*-a.s.*

Proof. Let $\pi(\tau)$ the local maximum strategy defined in (3.24) and we consider the optimal value $\mathbb{V}_T^{\mathbb{G}}$ combined with (3.13), then,

$$
\begin{aligned}
\mathbb{V}_T^{\mathbb{G}} &= \sup_{\pi^{\mathbb{F}}\in\mathcal{A}(\mathbb{F})} \boldsymbol{E}^{x_0}\left[\ln\left(X_T^{\pi,\mathbb{F}}\right)(1-H_T) + \boldsymbol{E}\left[\ln\left(X_T^\pi(\tau)\right)|\tau, X_\tau^\pi(\tau)\right] H_T\right] \\
&= \sup_{\pi^{\mathbb{F}}\in\mathcal{A}(\mathbb{F})} \boldsymbol{E}^{x_0}\left[\ln\left(X_T^{\pi,\mathbb{F}}\right)(1-H_T) + \ln\left(X_\tau^\pi(\tau)\right) H_T + \boldsymbol{E}\left[\ln \frac{X_T^\pi(\tau)}{X_\tau^\pi(\tau)}\Big|\tau, X_\tau^\pi(\tau)\right] H_T\right] \\
&= \boldsymbol{E}^{x_0}\left[\boldsymbol{E}\left[\widetilde{h}_\tau(T)|\tau, X_\tau^\pi(\tau)\right] H_T\right] \\
&\quad + \sup_{\pi^{\mathbb{F}}\in\mathcal{A}(\mathbb{F})} \boldsymbol{E}^{x_0}\left[\ln\left(X_T^{\pi,\mathbb{F}}\right)(1-H_T) + \ln\left(X_{\tau-}^{\pi,\mathbb{F}}\left(1+\pi_\tau^{\mathbb{F}}\kappa_\tau\right)\right) H_T\right]
\end{aligned}
$$

where by $\widetilde{h}_\tau(T)$ we denote the right-hand side of (3.23) with the local maximum strategy $\pi(\tau)$. Note that by the integral expression $\widetilde{h}_T(T) = 0$ and we define $h_\tau := \boldsymbol{E}\left[\widetilde{h}_\tau(T)|\tau, X_\tau^\pi(\tau)\right]$. Finally, we point out that the first two terms can be expressed as follows,

$$
\ln\left(X_T^{\pi,\mathbb{F}}\right)(1-H_T) + \ln\left(X_{\tau-}^{\pi,\mathbb{F}}\right) H_T = \ln\left(X_{\tau-}^{\pi,\mathbb{F}}\right)\ ,
$$

where the right-hand side is defined in (3.26) and the result follows. □

In the next theorem we solve the before default optimization problem under the logarithmic utility. We characterize the existence of a local maximum by the equation (3.28) and the intensity hypothesis (2.4) on the default process.

Theorem 3.23. *There exists a local maximum portfolio for the problem* (3.27) *if and only if it satisfies the condition*

$$
\begin{aligned}
\pi_s^{\mathbb{F}}\left(\sigma_s^{\mathbb{F}}\right)^2 =& \mu_s^{\mathbb{F}} - \rho_s^{\mathbb{F}} - \int_{\mathbb{R}_0} \frac{\pi_s^{\mathbb{F}}(\theta_s^{\mathbb{F}}(z))^2}{1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)}\nu(dz) + \frac{\boldsymbol{E}\left[D_{s^+}\mathbb{1}_{\{\tau>s\}}\,|\mathcal{F}_s\right]}{Z_{s-}}\sigma_s^{\mathbb{F}} \\
&+ \frac{\boldsymbol{E}\left[D_{s^+,z}\mathbb{1}_{\{\tau>s\}}\,|\,\mathcal{F}_s\right]}{Z_{s-}} \int_{\mathbb{R}_0} \frac{\theta_s^{\mathbb{F}}(z)}{1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)}\nu(dz) + \frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}\frac{1}{Z_{s-}}\lambda_s \qquad (3.28)
\end{aligned}
$$

and the intensity hypothesis (2.4) *holds true on* τ *with intensity process* λ*.*

Proof. We proceed as in the proof of Proposition 3.20, applying a perturbation argument in order to achieve a condition for local optimality. By applying the Corollary 3.22, we redefine the functional

$$
G_{x_0}(\pi) := \boldsymbol{E}^{x_0}\left[\ln\left(X_\tau^{\pi,\mathbb{F}}\right) + \ln\left(1+\pi_\tau^{\mathbb{F}}\kappa_\tau\right) H_T\right]\ .
$$

Let $\beta \in \mathcal{A}(\mathbb{F})$ bounded and consider $\delta > 0$ such that $\pi^{\mathbb{F}} + \epsilon\beta \in \mathcal{A}(\mathbb{F})$ for $\epsilon \in (-\delta, \delta)$. We consider the function $I(\epsilon) := G_{x_0}(\pi^{\mathbb{F}}+\epsilon\beta)$. A local maximum strategy must satisfy the condition $0 = I'(0)$, which is equivalent to,

$$\begin{aligned}
0 =& I'(0) = \lim_{\epsilon\to 0} \frac{I(\epsilon) - I(0)}{\epsilon} = \lim_{\epsilon\to 0} \frac{G_{x_0}(\pi^{\mathbb{F}} + \epsilon\beta) - G_{x_0}(\pi^{\mathbb{F}})}{\epsilon} \\
=& \boldsymbol{E}^{x_0}\left[\int_0^T \mathbb{1}_{\{\tau>s\}}\beta_s\left(\mu_s^{\mathbb{F}} - \rho_s^{\mathbb{F}} - \pi_s^{\mathbb{F}}(\sigma_s^{\mathbb{F}})^2 - \int_{\mathbb{R}_0} \frac{\pi_s^{\mathbb{F}}(\theta_s^{\mathbb{F}}(z))^2}{1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)}\nu(dz)\right) ds\right] \\
&+ \boldsymbol{E}^{x_0}\left[\int_0^T \mathbb{1}_{\{\tau>s\}}\beta_s\sigma_s^{\mathbb{F}} d^- W_s + \int_0^T\int_{\mathbb{R}_0} \frac{\mathbb{1}_{\{\tau>s\}}\beta_s\theta_s^{\mathbb{F}}(z)}{1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)}\tilde{N}(d^-s, dz) + \frac{\beta_\tau\kappa_\tau}{1+\pi_\tau^{\mathbb{F}}\kappa_\tau}H_T\right] .
\end{aligned}$$

By applying Theorems 3.15 and 3.19, we compute the expectation of the forward integrals as follows

$$\begin{aligned}
\boldsymbol{E}^{x_0}\left[\int_t^{t+h} \mathbb{1}_{\{\tau>s\}}\beta_s\sigma_s^{\mathbb{F}} d^- W_s\right] &= \boldsymbol{E}^{x_0}\left[\int_t^{t+h} (D_{s^+}\mathbb{1}_{\{\tau>s\}})\beta_s\sigma_s^{\mathbb{F}} ds\right] \\
\boldsymbol{E}^{x_0}\left[\int_t^{t+h}\int_{\mathbb{R}_0} \frac{\mathbb{1}_{\{\tau>s\}}\beta_s\theta_s^{\mathbb{F}}(z)}{1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)}\tilde{N}(d^-s, dz)\right] &= \boldsymbol{E}^{x_0}\left[\int_t^{t+h}\int_{\mathbb{R}_0} D_{s^+,z}\mathbb{1}_{\{\tau>s\}}\frac{\beta_s\theta_s^{\mathbb{F}}(z)\nu(dz)ds}{1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)}\right]
\end{aligned}$$

because $\beta, \sigma^{\mathbb{F}}, \theta^{\mathbb{F}}, \pi^{\mathbb{F}}$ are $\mathbb{F}$-adapted processes and therefore their Malliavin derivative is null. Moreover, thanks to the $\mathbb{F}$-dual predictable projection we can rewrite

$$\boldsymbol{E}^{x_0}\left[\frac{\beta_\tau\kappa_\tau}{1+\pi_\tau^{\mathbb{F}}\kappa_\tau}H_T\right] = \boldsymbol{E}^{x_0}\left[\int_0^T \frac{\beta_s\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}dA_s^\tau\right] . \tag{3.29}$$

We take $\beta_s = \xi_t\mathbb{1}_{\{t<s\leq t+h\}}$ for ξ_t any bounded $\mathcal{F}_t$-measurable random variable. Then the condition is written as,

$$\begin{aligned}
0 =& \boldsymbol{E}^{x_0}\left[\int_t^{t+h} \mathbb{1}_{\{\tau>s\}}\left(\mu_s^{\mathbb{F}} - \rho_s^{\mathbb{F}} - \pi_s^{\mathbb{F}}(\sigma_s^{\mathbb{F}})^2 - \int_{\mathbb{R}_0} \frac{\pi_s^{\mathbb{F}}(\theta_s^{\mathbb{F}}(z))^2}{1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)}\nu(dz)\right) ds\right. \\
&+ \int_t^{t+h} (D_{s^+}\mathbb{1}_{\{\tau>s\}})\sigma_s^{\mathbb{F}} ds + \int_t^{t+h}\int_{\mathbb{R}_0} (D_{s^+,z}\mathbb{1}_{\{\tau>s\}})\frac{\theta_s^{\mathbb{F}}(z)}{1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)}\nu(dz)ds \\
&\left.+ \int_t^{t+h} \frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}dA_s^\tau|\mathcal{F}_t\right] .
\end{aligned} \tag{3.30}$$

The $\mathbb{F}$-dual predictable projection admits the following representation, see Lemma 2.1 of [21],

$$A_t^\tau = \int_0^t \lambda_s ds + A_t^{\tau,\perp} + \sum_{0\leq s\leq t} \Delta A_s^\tau , \tag{3.31}$$

where λ is an $\mathbb{G}$-predictable integrable process, $A^{\tau,\perp} = (A_t^{\tau,\perp}, 0 \leq t \leq T)$ is an increasing and continuous process such that $dA_s^{\tau,\perp} \perp ds$. Then,

$$\begin{aligned}
0 =& \boldsymbol{E}^{x_0}\left[\int_t^{t+h} \mathbb{1}_{\{\tau>s\}}\left(\mu_s^{\mathbb{F}} - \rho_s^{\mathbb{F}} - \pi_s^{\mathbb{F}}(\sigma_s^{\mathbb{F}})^2 - \int_{\mathbb{R}_0} \frac{\pi_s^{\mathbb{F}}(\theta_s^{\mathbb{F}}(z))^2}{1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)}\nu(dz)\right)\right. \\
&+ \int_{\mathbb{R}_0} (D_{s^+,z}\mathbb{1}_{\{\tau>s\}})\frac{\theta_s^{\mathbb{F}}(z)}{1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)}\nu(dz) + (D_{s^+}\mathbb{1}_{\{\tau>s\}})\sigma_s^{\mathbb{F}} + \frac{\kappa_s\lambda_s}{1+\pi_s^{\mathbb{F}}\kappa_s} ds \\
&\left.+ \int_t^{t+h} \frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}dA_s^{\tau,\perp} + \sum_{t\leq s\leq t+h} \frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}\Delta A_s^\tau|\mathcal{F}_t\right] .
\end{aligned} \tag{3.32}$$

We define the following $\mathbb{F}$-adapted process $Y = (Y_t,\, 0 \le t \le T)$,

$$Y_t := \int_0^t \boldsymbol{E}[\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_t]\left(\mu_s^{\mathbb{F}} - \rho_s^{\mathbb{F}} - \pi_s^{\mathbb{F}}(\sigma_s^{\mathbb{F}})^2 - \int_{\mathbb{R}_0} \frac{\pi_s^{\mathbb{F}}(\theta_s^{\mathbb{F}}(z))^2}{1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)}\nu(dz)\right)$$
$$+ \int_{\mathbb{R}_0} \boldsymbol{E}[D_{s^+,z}\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_t]\frac{\theta_s^{\mathbb{F}}(z)}{1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)}\,\nu(dz) + \boldsymbol{E}[D_{s^+}\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_t]\sigma_s^{\mathbb{F}}$$
$$+ \frac{\kappa_s\,\lambda_s}{1+\pi_s^{\mathbb{F}}\kappa_s}\,ds + \int_0^t \frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s} dA_s^{\tau,\perp} + \sum_{0\le s\le t} \frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}\Delta A_s^{\tau}\ , \tag{3.33}$$

which is a finite variation process and, by (3.32), it is an $\mathbb{F}$-martingale. So we conclude that it is a null process, for the absolutely continuous, the singular and the purely discontinuous parts. In particular, as $\kappa_s \neq 0$, we get that $A_s^{\tau,\perp} = \Delta A_s^{\tau} = 0$ and (2.4) holds true.

Conversely, we assume (3.28) and (2.4). Then we recover (3.30) and the perturbation arguments holds true by the concavity property of the functional I, i.e., $I''(0) < 0$. □

In the following example we illustrate some implications of the Theorem 3.23. In particular, we show a random time for which (2.4) is not satisfied as the compensator of the process H in the enlarged filtration $\mathbb{G}$ results singular to the Lebesgue measure. It follows that the optimal portfolio among the admissible strategies $\mathcal{A}(\mathbb{G})$ does not exist.

Example 3.24. *We consider the following example with*

$$\tau = \arg\max_{s\in[0,T]} W_s\ , \tag{3.34}$$

that is the time in which the Brownian motion reaches it maximum before the time horizon. This case was previously studied for example in [25], but it was never been considered in the optimal portfolio problem with progressive enlargement and defaultable framework. We denote by

$$M_{s,t} := \sup_{u\in[s,t]} W_u\ ,$$

and define $M_s := M_{0,s}$. It is clear that $\{\tau > s\} = \{M_s < M_{s,T}\}$ and so we can compute the Malliavin trace of the second term. Note that, as the random time depends only on the Brownian motion, we have $D_{s^+,z}\mathbb{1}_{\{\tau>s\}} = 0$ for all (s,z).

Lemma 3.25. *The Malliavin trace of $\mathbb{1}_{\{\tau>s\}}$ is as follows,*

$$D_{s^+}\mathbb{1}_{\{\tau>s\}} = \delta_{M_s}(M_{s,T})\mathbb{1}_{\{M_{s,T}>W_s\}}\ . \tag{3.35}$$

Moreover, its conditional expectation is computed as

$$\boldsymbol{E}\left[D_{s^+}\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s\right] = \frac{2e^{-\frac{(M_s-W_s)^2}{2(T-s)}}}{\sqrt{2\pi(T-s)}}\ . \tag{3.36}$$

Proof. By applying Corollary 5.3 of [7], as it is argued in the Example 5.3 of the same reference, we have

$$D_{s^+}\mathbb{1}_{\{\tau>s\}} = \delta_{M_s}(M_{s,T})D_{s^+}M_{s,T}\ .$$

So we need to compute the trace of $M_{s,T}$. In Proposition 2.1.10 of [41] it is proved that the Malliavin derivative of $M_{s,T}$ exists and it is computed as follows,

$$D_{s^+}M_{s,T} = \mathbb{1}_{\{M_{s,T}>W_s\}}\ ,$$

and we obtain (3.35). For the conditional expectation, we get

$$\begin{aligned}\boldsymbol{E}\left[D_{s^+}\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s\right] &= \boldsymbol{E}\left[\delta_{M_s}(M_{s,T})\mathbb{1}_{\{M_{s,T}>W_s\}}|\mathcal{F}_s\right] \\ &= \int_{W_s}^{\infty}\delta_{M_s}(m)f_s(m)dm = f_s(M_s)\ ,\end{aligned}$$

being f_s the density of the random variable $M_{s,T}$ given $\mathcal{F}_s$, which is equivalent to consider the variable M_{T-s} in the domain $(W_s, +\infty)$, and the result follows. □

In this example, we proceed as follows. We assume that there exists an admissible local maximum for the optimization problem with logarithmic utility and, as we are going to prove that the intensity hypothesis (2.4) *does not hold, we will get a contradiction. We carry out the perturbation argument to derive the condition for the local maximum strategy, as in the proof of Theorem 3.23. With respect to the term of the default process in the optimal strategy, we take the left-hand side of* (3.29) *and we rewrite it by using Lemma 3.2 of [35]. We can apply this result as, by [25], τ is a Honest time.*

$$\begin{aligned}\boldsymbol{E}\left[\frac{\beta_\tau\kappa_\tau}{1+\pi_\tau^{\mathbb{F}}\kappa_\tau}H_T\right] &= \boldsymbol{E}\left[\boldsymbol{E}\left[\frac{\beta_\tau\kappa_\tau}{1+\pi_\tau^{\mathbb{F}}\kappa_\tau}H_T|\mathcal{G}_t\right]\right] \\ &= \boldsymbol{E}\left[\frac{\beta_\tau\kappa_\tau}{1+\pi_\tau^{\mathbb{F}}\kappa_\tau}\mathbb{1}_{\{\tau\le t\}} - \mathbb{1}_{\{\tau>t\}}\frac{1}{Z_t}\boldsymbol{E}\left[\int_t^T\frac{\beta_s\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}dZ_s|\mathcal{F}_t\right]\right]\ ,\end{aligned}$$

and, as in the proof of Theorem 3.23, we consider $\beta_s = \xi_t\mathbb{1}_{\{t<s\le t+h\}}$ for ξ_t any bounded $\mathcal{F}_t$-measurable random variable. Note that, in the first term we have $\mathbb{1}_{\{\tau\le t\}}\mathbb{1}_{\{t<\tau\le t+h\}} = 0$. Then, by taking $\mathcal{F}_t$-conditional expectation we get,

$$\begin{aligned}\boldsymbol{E}\left[\frac{\beta_\tau\kappa_\tau}{1+\pi_\tau^{\mathbb{F}}\kappa_\tau}H_T\right] &= \boldsymbol{E}\left[\boldsymbol{E}\left[-\xi_t\frac{\mathbb{1}_{\{\tau>t\}}}{Z_t}\boldsymbol{E}\left[\int_t^{t+h}\frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}dZ_s|\mathcal{F}_t\right]|\mathcal{F}_t\right]\right] \\ &= -\boldsymbol{E}\left[\xi_t\frac{\boldsymbol{E}\left[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\right]}{Z_t}\boldsymbol{E}\left[\int_t^{t+h}\frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}dZ_s|\mathcal{F}_t\right]\right] \\ &= -\boldsymbol{E}\left[\xi_t\int_t^{t+h}\frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}dZ_s\right]\ ,\end{aligned}$$

where we have used that $\boldsymbol{E}\left[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\right] = \boldsymbol{P}(\tau>t|\mathcal{F}_t) = Z_t$. Then, looking at the right-hand side of (3.29) *we have proved that*

$$\boldsymbol{E}\left[\xi_t\int_t^{t+h}\frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}dA_s^\tau\right] = -\boldsymbol{E}\left[\xi_t\int_t^{t+h}\frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}dZ_s\right]$$

for any $\xi_t\in\mathcal{F}_t$ bounded, so we rewrite the default-term appearing in (3.30) *as*

$$\begin{aligned}\boldsymbol{E}\left[\int_t^{t+h}\frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}dA_s^\tau|\mathcal{F}_t\right] &= -\boldsymbol{E}\left[\int_t^{t+h}\frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}dZ_s|\mathcal{F}_t\right] &(3.37)\\ &= \sqrt{\frac{2}{\pi}}\boldsymbol{E}\left[\int_t^{t+h}\frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}\frac{dM_s}{\sqrt{T-s}}|\mathcal{F}_t\right]\ , &(3.38)\end{aligned}$$

where in the last equality we used the result in Section 5.6.6 of [29]. Since $dM_s\perp ds$, see for example page 522 in [42], (2.4) *is not satisfied. Then, the agent who is playing with the filtration $\mathbb{G}$ can take advantage of the increments of M in order to maximize her expected utility and we find no optimal strategy in $\mathcal{A}(\mathbb{G})$. See [20] for a detailed explanation of arbitrage and Honest times.*

However, we can proceed as in the after default with the $(\mathcal{S})^*$ *approach and try to look an optimal strategy by giving an explicit expression for the right-hand side of* (3.37)*. We define the process* $K_s := M_s - W_s$ *and we consider the expression appearing in the Example 4.1.7.5 of [29],*

$$g(s, K_s) := Z_s = \frac{2}{\sqrt{2\pi}} \int_{K_s/\sqrt{T-s}}^{+\infty} \exp(-y^2/2)dy \ .$$

We compute dZ_s *by applying the Itô Lemma with the following property of the partial derivatives:*

$$\frac{\partial^2 g}{\partial K^2}(s, K_s) = \frac{2K_s}{\sqrt{2\pi(T-s)^3}} \exp\left(-\frac{K_s^2}{2(T-s)}\right) = 2\frac{\partial g}{\partial s}(s, K_s) \ ,$$

so we conclude that

$$dZ_s = \frac{\partial g}{\partial K}(s, K_s) dK_s \ .$$

In (30.2.55) of page 462 in the monograph [43], the term dK_s *is computed as follows*

$$dK_s = sgn(W_s)\mathbb{1}_{\{W_s \neq 0\}} dW_s + dL_s^0 \ ,$$

being $L^0 = (L_t^0, 0 \leq t \leq T)$ *the local time at zero of the Brownian motion. According to Theorem 4.13 of [6], the previous functional takes the expression*

$$dL_t^0 = \left(\frac{1}{\sqrt{2\pi t}} - \int_0^t \frac{1}{\sqrt{2\pi(t-s)^3}} W_s e^{-\frac{W_s^2}{2(t-s)}} dW_s \right) dt \tag{3.39}$$

in $(\mathcal{S})^*$*. As* $d\langle K, K\rangle_t = dt$ $\boldsymbol{P}$*-a.s., we conclude that*

$$dZ_s = \frac{\partial g}{\partial K}(s, K_s) \left(\frac{dL_s^0}{ds} \right) ds + \frac{\partial g}{\partial K}(s, K_s) sgn(W_s) \mathbb{1}_{\{W_s \neq 0\}} dW_s$$

and, using the martingale property, we can reformulate (3.37) *as*

$$\boldsymbol{E}\left[\int_t^{t+h} \frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s} dZ_s | \mathcal{F}_t \right] = \boldsymbol{E}\left[\int_t^{t+h} \frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s} \frac{\partial g}{\partial K}(s, K_s) \frac{dL_s^0}{ds} ds | \mathcal{F}_t \right] \ .$$

Finally the before default strategy is

$$\begin{aligned} \pi_s^{\mathbb{F}} =& \frac{\mu_s^{\mathbb{F}} - \rho_s^{\mathbb{F}}}{(\sigma_s^{\mathbb{F}})^2} - \frac{1}{(\sigma_s^{\mathbb{F}})^2} \int_{\mathbb{R}_0} \frac{\pi_s^{\mathbb{F}} \left(\theta_s^{\mathbb{F}}(z)\right)^2}{1+\pi_s^{\mathbb{F}}\theta_s^{\mathbb{F}}(z)} \nu(dz) + \frac{1}{\sigma_s^{\mathbb{F}}} \frac{e^{-\frac{K_s^2}{2(T-s)}}}{Z_s\sqrt{T-s}} \\ &- \frac{1}{(\sigma_s^{\mathbb{F}})^2} \frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s} \frac{\partial g}{\partial K}(s, K_s) \frac{dL_s^0}{ds} \ , \quad 0 < s \leq T \ . \end{aligned} \tag{3.40}$$

We achieved an implicit formula for the local maximum portfolio under the progressive enlargement given by the time in which the Brownian motion reaches its maximum and we remind that this progressive enlargement does not satisfy neither the density nor the intensity hypothesis. However, note that $\frac{dL_s^0}{ds}$ *is not well-defined in* $L^2(dt \times \boldsymbol{P})$ *and we have gotten a strategy which is not in* $\mathcal{A}(\mathbb{G})$*, because it not satisfies the condition* (3.3)*.*

In the following example, we show a random time satisfying the intensity hypothesis (2.4) but not the density one (2.5). In this case, we get the condition characterizing a local maximum strategy.

Example 3.26. *We consider the following example with*

$$\tau := \sup_{t\in[0,T]} \left\{ W_t = \frac{W_T}{2} \right\} , \tag{3.41}$$

deeply studied in [1, 29]. The process Z is computed in Subsection 5.6.5 of [29] as

$$Z_t = 1 - h\left(\frac{|W_t|}{\sqrt{T-t}}\right) \quad , \quad h(x) := \sqrt{\frac{2}{\pi}} \int_0^x y^2 e^{-y^2/2} dy \ . \tag{3.42}$$

By the decomposition given by Proposition 1.46 of [2], we conclude that $A_t^\tau = h\left(\frac{|W_t|}{\sqrt{T-t}}\right)$ *and, by [29], the Itô Lemma is computed with the following result,*

$$dA_t^\tau = \frac{|W_t| e^{-\frac{W_t^2}{2(T-t)}}}{(T-t)^{3/2}} dt \ .$$

Then we conclude that (2.4) *holds true. Moreover, after Theorem 4.13 of [1], it is mentioned that the hypothesis* ($\mathcal{H}'$) *fails, so we have found an example satisfying* (2.4) *but not* (2.5). *In order to determine the optimality condition* (3.28) *before default, we need to compute the Malliavin derivative.*

Lemma 3.27. *The Malliavin trace of $\mathbb{1}_{\{\tau>s\}}$ is as follows,*

$$\begin{aligned} D_{s^+}\mathbb{1}_{\{\tau>s\}} = &- \delta_0(2m_{s,T} - W_T)\mathbb{1}_{\{W_T\geq 0\}} - \delta_0(W_T)\mathbb{1}_{\{2m_{s,T}\geq W_T\}} \\ &+ \delta_0(2M_{s,T} - W_T)\mathbb{1}_{\{W_T\leq 0\}} + \delta_0(W_T)\mathbb{1}_{\{2M_{s,T}\leq W_T\}} \ . \end{aligned} \tag{3.43}$$

Moreover, its conditional expectation is computed as

$$\begin{aligned} \boldsymbol{E}\left[D_{s^+}\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s\right] =&\sqrt{\frac{2}{\pi}}\left(-sgn(W_s)\frac{W_s^2}{\sqrt{(T-s)^3}}\exp\left(-\frac{W_s^2}{2(T-s)}\right)\right. \\ &\left. + \frac{|W_s|}{\sqrt{(T-s)^3}}\exp\left(-\frac{W_s^2}{2(T-s)}\right)\right) \end{aligned} \tag{3.44}$$

Proof. Note that, by using [29], we can rewrite $\mathbb{1}_{\{\tau>s\}}$ as

$$\mathbb{1}_{\{\tau>s\}} = 1 - \mathbb{1}_{\{\tau\leq s\}} = 1 - \mathbb{1}_{\{2m_{s,T}\geq W_T\}}\mathbb{1}_{\{W_T\geq 0\}} - \mathbb{1}_{\{2M_{s,T}\leq W_T\}}\mathbb{1}_{\{W_T\leq 0\}} \ .$$

Then we use the chain rule and we get,

$$\begin{aligned} D_{s^+}\mathbb{1}_{\{\tau>s\}} =& - D_{s^+}\left(\mathbb{1}_{\{2m_{s,T}\geq W_T\}}\mathbb{1}_{\{W_T\geq 0\}}\right) - D_{s^+}\left(\mathbb{1}_{\{2M_{s,T}\leq W_T\}}\mathbb{1}_{\{W_T\leq 0\}}\right) \\ =& - D_{s^+}\left(\mathbb{1}_{\{2m_{s,T}\geq W_T\}}\right)\mathbb{1}_{\{W_T\geq 0\}} - D_{s^+}\left(\mathbb{1}_{\{W_T\geq 0\}}\right)\mathbb{1}_{\{2m_{s,T}\geq W_T\}} \\ &- D_{s^+}\left(\mathbb{1}_{\{2M_{s,T}\leq W_T\}}\right)\mathbb{1}_{\{W_T\leq 0\}} - D_{s^+}\left(\mathbb{1}_{\{W_T\leq 0\}}\right)\mathbb{1}_{\{2M_{s,T}\leq W_T\}} \ , \end{aligned}$$

where the main computation is

$$\begin{aligned} D_{s^+}\mathbb{1}_{\{2M_{s,T}\leq W_T\}} = D_{s^+}\mathbb{1}_{\{2M_{s,T}-W_T\leq 0\}} &= -\delta_0(2M_{s,T} - W_T)D_{s^+}\left(2M_{s,T} - W_T\right) \\ &= -\delta_0(2M_{s,T} - W_T)\left(2\mathbb{1}_{\{M_{s,T}>W_s\}} - 1\right) \end{aligned}$$

and $D_{s^+}\mathbb{1}_{\{W_T\geq 0\}} = \delta_0(W_T)$ for any $s<T$. We get

$$\begin{aligned}
D_{s^+}\mathbb{1}_{\{\tau>s\}} =& -\delta_0(2m_{s,T}-W_T)\left(2\mathbb{1}_{\{m_{s,T}\leq W_s\}}-1\right)\mathbb{1}_{\{W_T\geq 0\}} - \delta_0(W_T)\mathbb{1}_{\{2m_{s,T}\geq W_T\}}\\
&+\delta_0(2M_{s,T}-W_T)\left(2\mathbb{1}_{\{M_{s,T}\geq W_s\}}-1\right)\mathbb{1}_{\{W_T\leq 0\}} + \delta_0(W_T)\mathbb{1}_{\{2M_{s,T}\leq W_T\}}\\
=& -\delta_0(2m_{s,T}-W_T)\mathbb{1}_{\{W_T\geq 0\}} - \delta_0(W_T)\mathbb{1}_{\{2m_{s,T}\geq W_T\}}\\
&+\delta_0(2M_{s,T}-W_T)\mathbb{1}_{\{W_T\leq 0\}} + \delta_0(W_T)\mathbb{1}_{\{2M_{s,T}\leq W_T\}}\ .
\end{aligned}$$

We finally compute the conditional expectation, We denote by $f^M_{s,T}(m,w)$ the density of $(M_{s,T},W_T|W_s)$ and by $f^m_{s,T}(m,w)$ the density of $(m_{s,T},W_T|W_s)$.

$$\begin{aligned}
\boldsymbol{E}[D_{s^+}\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s] =& -\int_{-\infty}^{W_s}\int_{m\wedge 0}^{+\infty}\delta_0(2m-w)f^m_{s,T}(m,w)dwdm\\
&-\int_{-\infty}^{W_s}\int_{m}^{2m}\delta_0(w)f^m_{s,T}(m,w)dwdm\\
&+\int_{W_s}^{\infty}\int_{-\infty}^{m\vee 0}\delta_0(2m-w)f^M_{s,T}(m,w)dwdm\\
&+\int_{W_s}^{\infty}\int_{2m}^{m}\delta_0(w)f^M_{s,T}(m,w)dwdm\\
=&\mathbb{1}_{\{W_s\geq 0\}}\left(\int_0^{W_s} f^m_{s,T}(m,2m)dm + f^m_{s,T}(0,0)\right)\\
&+\mathbb{1}_{\{W_s\leq 0\}}\left(\int_{W_s}^{0} f^M_{s,T}(m,2m)dm + f^M_{s,T}(0,0)\right)\ .
\end{aligned}$$

Finally, we aim to give a more explicit expression by computing the densities $f^M_{s,T}$ and $f^m_{s,T}$. We get,

$$\begin{aligned}
f^M_{s,T}(x,y) =&\boldsymbol{P}(M_{s,T}\in dx, W_T\in dy|\mathcal{F}_s)\\
=&\ \boldsymbol{P}(M_{s,T}-W_s\in dx-W_s, W_T-W_s\in dy-W_s|\mathcal{F}_s)\\
=&\ \boldsymbol{P}(M_{T-s}\in dx-W_s, W_{T-s}\in dy-W_s)\\
=&\sqrt{\frac{2}{\pi}}\frac{2x-y-W_s}{\sqrt{(T-s)^3}}\exp\left(-\frac{(2x-y-W_s)^2}{2(T-s)}\right)\ ,\quad x\geq W_s\ ,\ x\geq y\ ,
\end{aligned}$$

and we conclude

$$\begin{aligned}
\mathbb{1}_{\{W_s\leq 0\}}\boldsymbol{E}[D_{s^+}\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s] = \mathbb{1}_{\{W_s\leq 0\}}\sqrt{\frac{2}{\pi}}&\left(\frac{W_s^2}{\sqrt{(T-s)^3}}\exp\left(-\frac{W_s^2}{2(T-s)}\right)\right.\\
&\left.-\frac{W_s}{\sqrt{(T-s)^3}}\exp\left(-\frac{W_s^2}{2(T-s)}\right)\right)\ ,
\end{aligned}$$

and by reasoning analogously with $f^m_{s,T}$ the result follows. □

Then, the optimal strategy in $\mathcal{A}(\mathbb{F})$ exists if and only if the the following equation has a solution,

$$\pi^{\mathbb{F}}_s\left(\sigma^{\mathbb{F}}_s\right)^2 = \mu^{\mathbb{F}}_s - \rho^{\mathbb{F}}_s - \int_{\mathbb{R}_0}\frac{\pi^{\mathbb{F}}_s(\theta^{\mathbb{F}}_s(z))^2}{1+\pi^{\mathbb{F}}_s\theta^{\mathbb{F}}_s(z)}\nu(dz) + \frac{\boldsymbol{E}\left[D_{s^+}\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s\right]}{Z_s}\sigma^{\mathbb{F}}_s$$

$$+ \frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}\frac{1}{Z_s}\frac{|W_s|e^{-\frac{W_s^2}{2(T-s)}}}{(T-s)^{3/2}} \ , \tag{3.45}$$

where Z satisfies (3.42) *and* $\boldsymbol{E}[D_s\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s]$ *satisfies* (3.44)*.*

Remark 3.28. *In particular, if the market coefficient of the Poisson process is $\theta = 0$, we have the following equation for the local maximum $\pi^{\mathbb{F}}$,*

$$\pi_s^{\mathbb{F}} = \frac{\mu_s^{\mathbb{F}} - \rho_s^{\mathbb{F}}}{(\sigma_s^{\mathbb{F}})^2} + \frac{\boldsymbol{E}\left[D_{s^+}\mathbb{1}_{\{\tau>s\}}\,|\mathcal{F}_s\right]}{Z_s\sigma_s^{\mathbb{F}}} + \frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}\frac{1}{Z_s\,(\sigma_s^{\mathbb{F}})^2}\frac{|W_s|e^{-\frac{W_s^2}{2(T-s)}}}{(T-s)^{3/2}} \ . \tag{3.46}$$

To short the notation, we define

$$a_s := \frac{\mu_s^{\mathbb{F}} - \rho_s^{\mathbb{F}}}{(\sigma_s^{\mathbb{F}})^2} + \frac{\boldsymbol{E}\left[D_{s^+}\mathbb{1}_{\{\tau>s\}}\,|\mathcal{F}_s\right]}{Z_s\sigma_s^{\mathbb{F}}} \ , \quad b_s := \frac{1}{Z_s\,(\sigma_s^{\mathbb{F}})^2}\frac{|W_t|e^{-\frac{W_t^2}{2(T-t)}}}{(T-t)^{3/2}} \ ,$$

and the equation is reduced to

$$\pi_s^{\mathbb{F}} = a_s + \frac{\kappa_s}{1+\pi_s^{\mathbb{F}}\kappa_s}b_s \Longrightarrow \kappa_s\left(\pi_s^{\mathbb{F}}\right)^2 + (1-a_s\kappa_s)\pi_s^{\mathbb{F}} - (a_s + b_s\kappa_s) = 0 \ ,$$

and we solved it as

$$\begin{aligned}\pi_s^{\mathbb{F}} &= \frac{a_s\kappa_s - 1 + \sqrt{(1-a_s\kappa_s)^2 + 4\kappa_s(a_s+b_s\kappa_s)}}{2\kappa_s} \\ &= \frac{a_s\kappa_s - 1 + \sqrt{(1+a_s\kappa_s)^2 + 4b_s\kappa_s^2}}{2\kappa_s} \ ,\end{aligned}$$

where it can be checked that the positive solution is the only admissible that satisfies (3.5)*.*

4 Sufficient Approach

In Section 3 we showed that the existence of a local maximum for the logarithmic utility optimization problem is strictly related with (2.4). In this section, we are going to require as assumption the existence of such local maximum.

In order to conclude that the optimal problem is well-posed, we introduce some conditions in the following Assumption 4.1. The first one is obvious while the second one is needed to properly define the new measure $\boldsymbol{Q}$ in Theorem 4.4. The third and the fourth ones are needed to compute local maximum strategies by allowing the differentiation under the integral sign. We define the following process,

$$\begin{aligned}\Psi_t(y,\beta,\pi) := &\int_0^t \beta_s\left(\mu_s - \rho_s - (\pi_s + y\beta_s)\sigma_s^2 - \int_{\mathbb{R}_0}\frac{(\pi_s+y\beta_s)\theta_s^2(z)}{1+(\pi_s+y\beta_s)\theta_s(z)}\nu(dz)\right)ds \\ &+ \int_0^t \beta_s\sigma_s d^-W_s + \int_0^t\int_{\mathbb{R}_0}\frac{\beta_s\theta_s(z)}{1+(\pi_s+y\beta_s)\theta_s(z)}\tilde{N}(d^-z,ds) \\ &+ \int_0^t \frac{\beta_s\kappa_s\,dH_s}{1+(\pi_s+y\beta_s)\kappa_s}\end{aligned} \tag{4.1}$$

assuming that there exists some $\delta > 0$, which may depend on $\pi \in \mathcal{A}(\mathbb{E})$, such that $y \in (-\delta,\delta)$, $\pi + y\beta \in \mathcal{A}(\mathbb{E})$ for $\mathbb{E} \in \{\mathbb{F},\mathbb{G}\}$ and $\beta \in \mathcal{A}(\mathbb{E})$ is bounded.

Assumption 4.1. *For any $\pi \in \mathcal{A}(\mathbb{E})$,*

1. $\boldsymbol{E}[U(X_T^\pi)] < +\infty$.
2. $0 < U'(X_T^\pi)X_T^\pi < +\infty$, $\boldsymbol{P}-a.s.$
3. *The following family of processes is uniformly integrable,*
$$\{U'(X_T^{\pi+y\beta})X_T^{\pi+y\beta}|\Psi_T(y,\beta,\pi)|\}_{y\in(-\delta,\delta)} \ .$$
4. *The following family of processes is uniformly integrable,*
$$\Big\{U''\left(X_T^{\pi+y\beta}\right)(X_T^{\pi+y\beta})^2\Psi_T^2(y,\beta,\pi) \\ + U'\left(X_T^{\pi+y\beta}\right)X_T^{\pi+y\beta}\left(\Psi_T(y,\beta,\pi)+\frac{d}{dy}\Psi_T(y,\beta,\pi)\right)\Big\}_{y\in(-\delta,\delta)} \ .$$

Remark 4.2. *If we consider the particular case with $0 \le \pi_t \le 1$, $\forall t \in [0,T]$, logarithmic utility and bounded market coefficients, it can be shown that the previous assumption holds true.*

The following lemma is quite technical but we need it in order to simplify the following computations.

Lemma 4.3. *Let $\pi, \beta \in \mathcal{A}(\mathbb{E})$ be some admissible strategies with β bounded, and define the function*
$$g(y) := \boldsymbol{E}[U(X_T^{\pi+y\beta})] \ .$$
Then the first and second derivatives of the function g can be computed as follows
$$g'(y) = \frac{d}{dy}\boldsymbol{E}\left[U(X_T^{\pi+y\beta})\right] = \boldsymbol{E}[U'(X_T^{\pi+y\beta})X_T^{\pi+y\beta}\,\Psi_T(y,\beta,\pi)]$$
$$g''(y) = \boldsymbol{E}\Big[X_T^{\pi+y\beta}\Psi_T^2(y,\beta,\pi)\left(U''\left(X_T^{\pi+y\beta}\right)X_T^{\pi+y\beta}+U'\left(X_T^{\pi+y\beta}\right)\right) \\ +U'\left(X_T^{\pi+y\beta}\right)X_T^{\pi+y\beta}\frac{d}{dy}\Psi_T(y,\beta,\pi)\Big] \ ,$$
where Ψ is defined in (4.1).

Proof. The result directly follows from the conditions given in Assumption 4.1. □

Theorem 4.4. *With the previous set-up, if π is a local maximum for the problem* (3.7)*, with $\mathbb{E} \in \{\mathbb{F}, \mathbb{G}\}$, then the process $M^\pi = (M_t^\pi, 0 \le t \le T)$, defined as,*
$$M_t^\pi := \Psi_t(0,1,\pi) = \int_0^t\left(\mu_s-\rho_s-\pi_s\sigma_s^2-\int_{\mathbb{R}_0}\frac{\pi_s\theta_s^2(z)}{1+\pi_s\theta_s(z)}\nu(dz)\right)ds \\ +\int_0^t\sigma_s d^-W_s+\int_0^t\int_{\mathbb{R}_0}\frac{\theta_s(z)}{1+\pi_s\theta_s(z)}\tilde{N}(d^-s,dz)+\int_0^t\frac{\kappa_s}{1+\pi_s\kappa_s}dH_s \qquad (4.2)$$
has the martingale property under $(\boldsymbol{Q}, \mathbb{E})$ with
$$d\boldsymbol{Q} := F_T^\pi d\boldsymbol{P}, \quad F_T^\pi := \frac{U'(X_T^\pi)X_T^\pi}{\boldsymbol{E}[U'(X_T^\pi)X_T^\pi]} \ . \qquad (4.3)$$

Proof. The measure $\boldsymbol{Q}$ in (4.3) is well-defined thanks to Assumption 4.1(2). If π is a local maximum, for all bounded $\beta \in \mathcal{A}(\mathbb{E})$ we have, using the notation of the previous lemma, $g'(0) = 0$, i.e.,

$$0 = \boldsymbol{E}[U'(X_T^\pi) X_T^\pi \, \Psi_T(0, \beta, \pi)] \ .$$

Let $\beta_s = \xi_t \mathbb{1}_{\{t<s\leq t+h\}}$, where ξ_t is a bounded $\mathcal{E}_t$-measurable random variable, then,

$$\begin{aligned} 0 = \boldsymbol{E}\Bigg[\xi_t F_T^\pi \Big\{ \int_t^{t+h} \left(\mu_s - \rho_s - \pi_s \sigma_s^2 - \int_{\mathbb{R}_0} \frac{\pi_s \theta_s^2(z)}{1 + \pi_s \theta_s(z)} \nu(dz) \right) ds \\ + \int_t^{t+h} \sigma_s d^- W_s + \int_t^{t+h} \int_{\mathbb{R}_0} \frac{\theta_s(z)}{1 + \pi_s \theta_s(z)} \tilde{N}(d^- z, ds) + \int_t^{t+h} \frac{\kappa_s}{1 + \pi_s \kappa_s} dH_s \Big\} \Bigg] \ . \end{aligned} \tag{4.4}$$

As the previous expectation holds true for every $\mathcal{E}_t$-measurable bounded random variable, we conclude

$$0 = \boldsymbol{E}[F_T^\pi (M_{t+h}^\pi - M_t^\pi) | \mathcal{E}_t] = \boldsymbol{E}_{\boldsymbol{Q}}[M_{t+h}^\pi - M_t^\pi | \mathcal{E}_t] \ .$$

Moreover, by the Assumption 4.1(3) we conclude that

$$\boldsymbol{E}_{\boldsymbol{Q}}[|M_T^\pi|] = \boldsymbol{E}[F_T^\pi |\Psi_t(0, 1, \pi)|] < +\infty \ ,$$

and the result for every $t \in [0, T]$ follows by applying the Jensen inequality,

$$\boldsymbol{E}_{\boldsymbol{Q}}[|M_t^\pi|] = \boldsymbol{E}_{\boldsymbol{Q}}[|\boldsymbol{E}_{\boldsymbol{Q}}[M_T^\pi | \mathcal{E}_t]|] \leq \boldsymbol{E}_{\boldsymbol{Q}}[\boldsymbol{E}_{\boldsymbol{Q}}[|M_T^\pi| | \mathcal{E}_t]] = \boldsymbol{E}_{\boldsymbol{Q}}[|M_T^\pi|] < +\infty \ .$$

□

In particular, if we consider the optimization problem with $\mathbb{E} = \mathbb{G}$, then the process M^π is a martingale, because every term included is $\mathbb{G}$-adapted.

Remark 4.5. *Let* $F_t^\pi := \boldsymbol{E}\left[\frac{d\boldsymbol{Q}}{d\boldsymbol{P}} | \mathcal{G}_t\right]$ *and* $Z_t^\pi := \boldsymbol{E}\left[\frac{d\boldsymbol{P}}{d\boldsymbol{Q}} | \mathcal{G}_t\right]$, *by the Girsanov Theorem we know that if* π *is a local maximum, then the process* $(F^\pi M^\pi)$ *is a martingale in* $(\boldsymbol{P}, \mathbb{G})$*. Moreover, according to Theorem 37 of [45], the process*

$$M_t^\pi - \int_0^t \frac{1}{Z_{s-}^\pi} d\langle M_s^\pi, Z_s^\pi \rangle \ , \quad 0 \leq t \leq T \ ,$$

is also a $(\boldsymbol{P}, \mathbb{G})$*-martingale, where* $\langle M^\pi, Z^\pi \rangle$ *is the unique* $\mathbb{G}$*-predictable and increasing process such that* $(M^\pi Z^\pi - \langle M^\pi, Z^\pi \rangle)$ *is a* $(\boldsymbol{P}, \mathbb{G})$*-martingale.*

Remark 4.6. *Theorem 4.4 extends the results of Theorem 3.3 of [13] as it considers also the interval* $[[\tau \wedge T, T]]$*. However in [13] also the converse is proved, that is, under the concavity assumption on the function* g*, if* M^π *has the martingale property under* $(\boldsymbol{Q}, \mathbb{E})$*, then* π *is a local maximum for the problem* (3.7)*.*

Proposition 4.7. *Suppose* π *is a local maximum for the problem* (3.7) *with the information flow* $\mathbb{G}$*, then the following two processes are a* $(\boldsymbol{Q}, \mathbb{G})$*-semimartingales,*

$$\int_0^t \sigma_s d^- W_s, \ \int_0^t \int_{\mathbb{R}_0} \frac{\theta_s(z)}{1 + \pi_s \theta_s(z)} \tilde{N}(d^- s, dz) \ , \ 0 \leq t \leq T \ .$$

Proof. In the definition of the process $M^\pi = (M_t^\pi, 0 \leq t \leq T)$ given by (4.2), we can rewrite the default term by applying the Lemma 2.1 as the sum of a $(\boldsymbol{Q}, \mathbb{G})$-local martingale and a bounded variation predictable process,

$$\int_0^t \frac{\kappa_s}{1 + \pi_s \kappa_s} dH_s = \int_0^t \frac{\kappa_s}{1 + \pi_s \kappa_s} d\widetilde{J}_s + \int_0^t \frac{\kappa_s}{1 + \pi_s \kappa_s} \frac{d\langle J_s, F_s^\pi \rangle}{F_{s-}^\pi}$$

$$+ \int_0^t \frac{\kappa_s}{1+\pi_s\kappa_s} \frac{\mathbb{1}_{\{\tau>s\}}}{Z_{s-}} dA_s^\tau \ ,$$

where $\widetilde{J}_s = J_s - \dfrac{d\langle J_s, F_s^\pi\rangle}{F_{s-}^\pi}$. We conclude that the sum process

$$\begin{aligned}\int_0^t \sigma_s d^- W_s + \int_0^t \int_{\mathbb{R}_0} \frac{\theta_s(z)}{1+\pi_s\theta_s(z)} \tilde{N}(d^-s, dz) = M_t^\pi - \int_0^t \frac{\kappa_s}{1+\pi_s\kappa_s} d\widetilde{J}_s \\ - \int_0^t \left(\mu_s - \rho_s - \pi_s\sigma_s^2 - \int_{\mathbb{R}_0} \frac{\pi_s\theta_s^2(z)}{1+\pi_s\theta_s(z)} \nu(dz) \right) ds \\ - \int_0^t \frac{\kappa_s}{1+\pi_s\kappa_s} \frac{d\langle J_s, F_s^\pi\rangle}{F_{s-}^\pi} - \int_0^t \frac{\kappa_s}{1+\pi_s\kappa_s} \frac{\mathbb{1}_{\{\tau>s\}}}{Z_{s-}} dA_s^\tau\end{aligned}$$

is a special $(\boldsymbol{Q}, \mathbb{G})$-semimartingale because is the sum of a $(\boldsymbol{Q}, \mathbb{G})$-local martingale and a predictable bounded variation process. By using the the semimartingale decomposition in the continuous and pure discontinuous part, we conclude that there exist $\alpha^{(i)}$, $\gamma^{(i)}$ with $i \in \{1, 2, 3\}$ such that

$$\widehat{M}_t = \int_0^t \sigma_s d^- W_s + \int_0^t \sigma_s \alpha_s^{(1)} ds + \int_0^t \sigma_s \alpha_s^{(2)} \frac{d\langle J_s, F_s^\pi\rangle}{F_{s-}^\pi} + \int_0^{t\wedge\tau} \sigma_s \alpha_s^{(3)} \frac{dA_s^\tau}{Z_{s-}} \tag{4.5a}$$

$$\widehat{J}_t = \int_0^t \int_{\mathbb{R}_0} \frac{\theta_s(z)}{1+\pi_s\theta_s(z)} \tilde{N}(d^-s, dz) + \int_0^t \gamma_s^{(1)} ds + \int_0^t \gamma_s^{(2)} \frac{d\langle J_s, F_s^\pi\rangle}{F_{s-}^\pi} + \int_0^{t\wedge\tau} \gamma_s^{(3)} \frac{dA_s^\tau}{Z_{s-}} \tag{4.5b}$$

are $(\boldsymbol{Q}, \mathbb{G})$-local martingales and $\alpha^{(i)}, \gamma^{(i)}$ are the unique $\mathbb{G}$-adapted processes satisfying

$$\begin{aligned}&\int_0^t \left(\sigma_s\alpha_s^{(1)} + \gamma_s^{(1)}\right) ds + \int_0^t \left(\sigma_s\alpha_s^{(2)} + \gamma_s^{(2)}\right) \frac{d\langle J_s, F_s^\pi\rangle}{F_{s-}^\pi} + \int_0^{t\wedge\tau} \left(\sigma_s\alpha_s^{(3)} + \gamma_s^{(3)}\right) \frac{dA_s^\tau}{Z_{s-}} \\ &= \int_0^t \left(\mu_s - \rho_s - \pi_s\sigma_s^2 - \int_{\mathbb{R}_0} \frac{\pi_s\theta_s^2(z)}{1+\pi_s\theta_s(z)} \nu(dz) \right) ds + \int_0^t \frac{\kappa_s}{1+\pi_s\kappa_s} \left(\frac{d\langle J_s, F_s^\pi\rangle}{F_{s-}^\pi} + \frac{\mathbb{1}_{\{\tau>s\}}}{Z_{s-}} dA_s^\tau \right) .\end{aligned} \tag{4.6}$$

□

The previous result can be extended to W and $\tilde{N}$, as shown in the following corollary.

Corollary 4.8. *Suppose π is a local maximum for the problem* (3.7) *with the information flow $\mathbb{G}$, then the following two processes are a $(\boldsymbol{Q}, \mathbb{G})$-semimartingales,*

$$W_t, \ \int_0^t \int_{\mathbb{R}_0} z \ \tilde{N}(ds, dz) \ , \ 0 \le t \le T \ .$$

Proof. The reasoning is analogous to Theorem 15 of [12]. We consider the $(\boldsymbol{Q}, \mathbb{G})$-local martingale $\widehat{M}$ defined in (4.5a), so the result of a Itô integral is still a local martingale. In particular,

$$\int_0^t \frac{1}{\sigma_s} d\widehat{M}_s = W_t + \int_0^t \alpha_s^{(1)} ds + \int_0^t \alpha_s^{(2)} \frac{d\langle J_s, F_s^\pi\rangle}{F_{s-}^\pi} + \int_0^{t\wedge\tau} \alpha_s^{(3)} \frac{dA_s^\tau}{Z_{s-}} \ .$$

The same argument works with the compensated Poisson random measure. □

Remark 4.9. *Using Proposition 4.4 of [4], we know that the compensated Poisson measure is quasi-left continuous on $\{\tau > s\}$, so its compensator is continuous before default. From this fact we conclude that*

$$\begin{aligned}0 =& \mathbb{1}_{\{\tau>s\}} \alpha_s^{(2)} \frac{\Delta\langle J_s, F_s^\pi\rangle}{F_{s-}^\pi} + \mathbb{1}_{\{\tau>s\}} \alpha_s^{(3)} \frac{\Delta A_s^\tau}{Z_{s-}} \\ 0 =& \mathbb{1}_{\{\tau>s\}} \gamma_s^{(2)} \frac{\Delta\langle J_s, F_s^\pi\rangle}{F_{s-}^\pi} + \mathbb{1}_{\{\tau>s\}} \gamma_s^{(3)} \frac{\Delta A_s^\tau}{Z_{s-}} \ .\end{aligned}$$

Using Theorem 1.8 of [27] and the fact that $\mathcal{F}_t \subset \mathcal{G}_t$, we know that there exists a unique predictable compensator $\nu^{\mathbb{G}}(dt, dz)$ for the Poisson random measure $N(dt, dz)$. However, the Corollary 4.8 is stronger because we claim that this compensator is of finite variation. In the following, to short the notation, we will use $\nu^{\mathbb{G}}(dt, dz)$ in order to refer it. In the next corollary we rewrite the above processes under the measure $\boldsymbol{P}$.

Corollary 4.10. *Suppose π is a local maximum for the problem* (3.7) *with the information flow $\mathbb{G}$, then the following two processes are a $(\boldsymbol{P}, \mathbb{G})$-local martingale and semimartingale, respectively.*

$$\widehat{W}_t := W_t - A_t^W \ ,$$

$$\int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz) - \int_0^t \frac{1}{Z_s^\pi} \langle \int_0^s \int_{\mathbb{R}_0} z\tilde{N}(du, dz), Z_{s-}^\pi \rangle \ , \quad 0 \le t \le T \ ,$$

where A^W is a $\mathbb{G}$-continuous predictable process.

Proof. It follows by the previous result and the Girsanov theorem for local-martingales, see Theorem 35 on page 132 and its corollaries in [45] for details. □

Once we have proved that, under the Assumption 4.1 and the existence of a local maximum of the optimization problem, the enlargement of filtration $\mathbb{G} \supset \mathbb{F}$ preserves the semimartingale property for W and $\tilde{N}$ with respect to the measure $\boldsymbol{P}$, then we can apply the Lemmas A.1 and B.5 in order to conclude that the involved forward integrals reduce to the classical Itô integrals for semimartingale processes. We summarize the result in the next theorem in which the risky asset S is driven by a $(\boldsymbol{P}, \mathbb{G})$-local martingale and a $(\boldsymbol{P}, \mathbb{G})$-semimartingale.

Theorem 4.11. *Suppose π is a local maximum for the utility problem* (3.7) *with the information flow $\mathbb{G}$, then, the asset's dynamics satisfy the following SDEs under $(\boldsymbol{P}, \mathbb{G})$.*

$$\frac{dD_t}{D_t} = \rho_t dt \tag{4.7a}$$

$$\frac{dS_t}{S_{t-}} = \mu_t dt + \sigma_t dA_t^W + \int_{\mathbb{R}_0} \theta_t(z) \left(\nu^{\mathbb{G}} - \nu\right)(dt, dz) + \sigma_t d\widehat{W}_t + \int_{\mathbb{R}_0} \theta_t(z)\widehat{N}(dt, dz) + \kappa_t dH_t \tag{4.7b}$$

where the integrals are well-defined in the classical Itô sense because $\widehat{W}$ and $\widehat{N} := N - \nu^{\mathbb{G}}$ are $(\boldsymbol{P}, \mathbb{G})$-semimartigales. Moreover, we rewrite the dynamics of X^π as follows

$$\begin{aligned}\frac{dX_t^\pi}{X_{t-}^\pi} =& (1 - \pi_t)\rho_t dt + \pi_t \left(\mu_t dt + \sigma_t dA_t^W + \int_{\mathbb{R}_0} \theta_t(z) \left(\nu^{\mathbb{G}} - \nu\right)(dt, dz)\right) \\ &+ \pi_t \left(\sigma_t d\widehat{W}_t + \int_{\mathbb{R}_0} \theta_t(z)\widehat{N}(dt, dz) + \kappa_t dH_t\right) \ .\end{aligned} \tag{4.8}$$

Remark 4.12. *Note that for $U(x) = \ln(x)$ the measure $\boldsymbol{Q}$ coincides with $\boldsymbol{P}$, then a similar version of the previous theorem can be achieved only by assuming the existence of a local maximum for the logarithmic utility problem as it was motivated in Subsections 3.2 and 3.3.*

In particular, the following statement solves the optimal portfolio problem by using the compensators instead of the Malliavin derivatives. We decompose the compensators as the absolutely continuous, the singular and the purely discontinuous parts as follows,

$$dA_s^\tau = \lambda_s ds + dA_s^{\tau,\perp} + \Delta A_s^\tau \tag{4.9}$$

$$d\langle J_s, F_s^\pi \rangle = d\langle J_s, F_s^\pi \rangle^{ac} + d\langle J_s, F_s^\pi \rangle^\perp + \Delta \langle J_s, F_s^\pi \rangle \ . \tag{4.10}$$

As the jumps of N are totally inaccessible, we conclude that $\nu^{\mathbb{G}}$ is continuous. In the last decompositions we indicate the possibility of jumps.

Theorem 4.13. *Suppose π is a local maximum for the optimization problem* (3.7)*, then it satisfies the following equations,*

$$\begin{aligned} 0 =& \left(\sigma_s \alpha_s^{(2)} + \gamma_s^{(2)} - \frac{\kappa_s}{1+\pi_s\kappa_s}\right) \frac{d\langle J_s, F_s^\pi\rangle^{ac}}{F_{s-}^\pi} + \left(\sigma_s \alpha_s^{(3)} + \gamma_s^{(3)} - \frac{\kappa_s}{1+\pi_s\kappa_s}\right) \frac{\mathbb{1}_{\{\tau>s\}}}{Z_{s-}}\lambda_s \\ &- \left(\mu_s - \rho_s - \pi_s\sigma_s^2 - \int_{\mathbb{R}_0} \frac{\pi_s\theta_s^2(z)}{1+\pi_s\theta_s(z)}\nu(dz)\right) + \sigma_s\alpha_s^{(1)} + \gamma_s^{(1)} \end{aligned} \tag{4.11}$$

$$0 = \left(\sigma_s \alpha_s^{(2)} + \gamma_s^{(2)} - \frac{\kappa_s}{1+\pi_s\kappa_s}\right) \frac{d\langle J_s, F_s^\pi\rangle^{\perp}}{F_{s-}^\pi} + \left(\sigma_s \alpha_s^{(3)} + \gamma_s^{(3)} - \frac{\kappa_s}{1+\pi_s\kappa_s}\right) \frac{\mathbb{1}_{\{\tau>s\}}}{Z_{s-}} dA_s^{\tau,\perp}. \tag{4.12}$$

$$0 = \frac{\Delta\langle J_s, F_s^\pi\rangle}{F_{s-}^\pi} + \frac{\mathbb{1}_{\{\tau>s\}}}{Z_{s-}}\Delta A_s^\tau \ . \tag{4.13}$$

Moreover, if (4.11)*,* (4.12) *and* (4.13) *hold true, where $\alpha^{(i)}, \gamma^{(i)}$ satisfy* (4.5)*, and*

$$xU''(x) + U'(x) \le 0 \ , \quad \forall x > 0$$

then π is a local maximum for the optimization problem (3.7).

Proof. It follows by (4.6) and by splitting in absolutely continuous and singular parts via (4.9) and (4.10). The condition (4.13) is simplified using Remark 4.9.

Moreover, by assuming the conditions (4.11), (4.12) and (4.13), we recover the first order condition for optimality and we need to evaluate the second derivative. Note that $\frac{d}{dy}\Psi_T(y,\beta,\pi)$ is computed as the right-hand side of the following equation

$$\begin{aligned} 0 >& - \int_0^T \beta_s^2\sigma_s^2 ds - \int_0^T \int_{\mathbb{R}_0} \frac{\beta_s^2\theta_s^2(z)}{(1+(\pi_s+y\beta_s)\theta_s(z))^2}\nu(dz)ds \\ &- \int_0^T \int_{\mathbb{R}_0} \frac{\beta_s^2\theta_s^2(z)}{(1+(\pi_s+y\beta_s)\theta_s(z))^2}\tilde{N}(ds,dz) - \int_0^T \frac{\beta_s^2\kappa_s^2}{(1+(\pi_s+y\beta_s)\kappa_s)^2} dH_s \ . \end{aligned}$$

Therefore, by the Lemma 4.3

$$\begin{aligned} g''(0) =& \boldsymbol{E}\left[U''\left(X_T^\pi\right)(X_T^\pi)^2\Psi_T^2(0,\beta,\pi)\right] \\ &+ \boldsymbol{E}\left[U'\left(X_T^\pi\right)X_T^\pi\left(\Psi_T^2(0,\beta,\pi) + \frac{d}{dy}\Psi_T(0,\beta,\pi)\right)\right] < 0 \ , \end{aligned}$$

for all bounded $\beta \in \mathcal{A}(\mathbb{G})$ if $xU''(x) + U'(x) \le 0$, and $x > 0$. □

Note that the utility function U appears in the equations (4.11), (4.12) and (4.13) by the processes $\alpha^{(i)}, \gamma^{(i)}$, because they are induced by the change of measure from $\boldsymbol{Q}$, which is anticipating by the definition given in (4.3). In the next result we show how (4.11), (4.12) and (4.13) are simplified in case of $U(x) = \ln x$ obtaining also the intensity hypothesis as in Theorem 3.23.

Corollary 4.14. *There exists a local maximum portfolio π for the optimization problem* (3.7)*, with $U(x) = \ln x$, if and only if it satisfies the condition*

$$\begin{aligned} 0 =& \left(\sigma_s \alpha_s^{(3)} + \gamma_s^{(3)} - \frac{\kappa_s}{1+\pi_s\kappa_s}\right) \frac{\mathbb{1}_{\{\tau>s\}}}{Z_{s-}}\lambda_s \\ &- \left(\mu_s - \rho_s - \pi_s\sigma_s^2 - \int_{\mathbb{R}_0} \frac{\pi_s\theta_s^2(z)}{1+\pi_s\theta_s(z)}\nu(dz)\right) + \sigma_s\alpha_s^{(1)} + \gamma_s^{(1)} \ . \end{aligned} \tag{4.14}$$

and the intensity hypothesis (2.4) *holds true on τ, with intensity λ.*

Proof. Under logarithmic utility, the process M^π is a $(\boldsymbol{P}, \mathbb{G})$-martingale. We define

$$Y_t = \int_0^t \left(\mu_s - \rho_s - \pi_s \sigma_s^2 - \int_{\mathbb{R}_0} \frac{\pi_s \theta_s^2(z)}{1 + \pi_s \theta_s(z)} \nu(dz) \right) ds + \boldsymbol{E}\left[\int_0^t D_{s^+} \sigma_s ds \right] + \boldsymbol{E}\left[\int_0^t \int_{\mathbb{R}_0} D_{s^+,z} \frac{\theta_s(z)}{1 + \pi_s \theta_s(z)} \nu(dz) ds \right] + \int_0^t \frac{\kappa_s}{1 + \pi_s \kappa_s} dA_s^\tau \ , \ 0 \le t \le T \ ,$$

which is also a $(\boldsymbol{P}, \mathbb{G})$-martingale, because

$$\boldsymbol{E}[Y_{t+h} - Y_t | \mathcal{G}_t] = \boldsymbol{E}[M_{t+h}^\pi - M_t^\pi | \mathcal{G}_t] = 0$$

where we have used the Theorems 3.15 and 3.19, and Y is finite variation process, so we conclude that it is a null process. From the decomposition in its absolutely continuous and singular part we conclude that

$$0 = \int_0^t \frac{\kappa_s}{1 + \pi_s \kappa_s} dA_s^{\tau,\perp} \ , \quad 0 \le t \le T \ ,$$

and then $A^{\tau,\perp} = 0$ because $\kappa_s \neq 0$. We finally simply the conditions (4.11), (4.12) and (4.13) into (4.14). □

Remark 4.15. *We can achieve a more explicit result under the assumption of Hunt processes. Indeed, if the risky asset S is a Hunt process, the agent plays with the price filtration $\mathbb{S}$ and τ is a totally inaccessible $\mathbb{S}$-stopping time. By applying [28] we conclude that the default process H has an absolutely continuous compensator λ with respect to the Lebesgue measure. In particular, the necessary and sufficient condition for a local maximum strategy is*

$$0 = \mu_s - \rho_s - \pi_s \sigma_s^2 - \int_{\mathbb{R}_0} \frac{\pi_s \theta_s^2(z)}{1 + \pi_s \theta_s(z)} \nu(dz) + \frac{\kappa_s}{1 + \pi_s \kappa_s} \frac{\mathbb{1}_{\{\tau > s\}}}{Z_{s-}} \lambda_s \ .$$

In the following example we consider the same random time as in Example 3.26. In this case, we apply the sufficient approach developed in Section 4 and we show how this is connected with the computations in terms of Malliavin derivative of Section 3.

Example 4.16 (Example 3.26 revisited)**.** *We consider the random time $\tau = \sup_t \{W_t = W_T/2\}$. In Theorem 4.13 of [1], the compensator of W in $\mathbb{G}$ is computed as the right-hand side of the following equation,*

$$\alpha_s^{(1)} + \alpha_s^{(3)} \frac{dA_s^\tau}{Z_s} = - \mathbb{1}_{\{\tau > s\}} \sqrt{\frac{2}{\pi}} \frac{sgn(W_s)}{Z_s} \frac{W_s^2}{\sqrt{(T-s)^3}} \exp\left(-\frac{W_s^2}{2(T-s)} \right) + \mathbb{1}_{\{\tau \le s\}} \left(\frac{W_T - W_s}{T-s} - \frac{1}{\varphi_s} \frac{W_T}{(T-s)} \right) \ ,$$

where $\varphi_s = 1 - \exp\left(\frac{2W_s W_T - W_T^2}{2(T-s)} \right)$. By assuming that there is no Poisson jumps in the market, i.e., $\theta = 0$, the optimality condition under logarithmic utility is given by

$$0 = \left(\sigma_s \alpha_s^{(3)} - \frac{\kappa_s}{1 + \pi_s \kappa_s} \right) \frac{\mathbb{1}_{\{\tau > s\}}}{Z_{s-}} dA_s^\tau - (\mu_s - \rho_s) + \pi_s \sigma_s^2 + \sigma_s \alpha_s^{(1)} \ . \tag{4.15}$$

Note that, the condition before default is the same that we got in (3.46)*, so both approaches match. The optimality condition after default is reduced to*

$$0 = -(\mu_s - \rho_s) + \pi_s \sigma_s^2 + \sigma_s \left(\frac{W_T - W_s}{T-s} - \frac{1}{\varphi_s} \frac{W_T}{(T-s)} \right) \ .$$

Remark 4.17. *Under the assumption of the existence of a local maximum, we derive the existence of the compensators for* W *and* $\tilde{N}$ *and an explicit condition for this local maximum. In [25] it is mentioned that whenever* τ *is an Honest time, the process*

$$\widehat{W}_t := W_t - \int_0^{\tau\wedge t} \frac{\frac{d}{ds}\langle \widehat{A}_s, W_s\rangle}{Z_{s-}} ds + \int_{\tau\wedge t}^{T} \frac{\frac{d}{ds}\langle \widehat{A}_s, W_s\rangle}{1 - Z_{s-}} ds \ ,$$

is a $(\boldsymbol{P}, \mathbb{G})$*-Brownian motion, where* $\widehat{A}$ *is the unique* $\mathbb{G}$*-predictable and increasing process such that* $Z_t + \widehat{A}_t$ *is a* $(\boldsymbol{P}, \mathbb{G})$*-martingale. A similar result is obtained in [18] for those random times which satisfy the density hypotesis.*

5 Conclusion

In this paper, we extend the solution of the classical optimal portfolio problem in a defaultable framework for the case when the default time is not a stopping time in the natural filtration of the structural components of the processes modeling the asset. If a certain bankruptcy happens, the price of the risky asset will undergo a jump in its price process but the agent is still able to invest in. We succeed in identifying the essential hypothesis for the existence of a local maximum strategy under logarithmic utility and information flow $\mathbb{G}$ as the intensity one, see (2.4). In particular, we drop the density assumption so-called Jacod's hypothesis. When we deal with a general utility, we assume the existence of such local maximum and we deduce the semimartingale decomposition of the Brownian motion and the compensated Poisson random measure in the enlarged filtration $\mathbb{G}$. The problem is solved by splitting it with two subproblems namely before and after default occurrence. We include many examples to show how to apply the methodology in specific cases, being the main one the analysis of the time of occurrence of the maximum of the Brownian motion. To solve these examples we apply the interplay between the forward integral, the Skorohod integral and the Malliavin derivative in the anticipating calculus that allows to compute the expressions of the semimartingale decomposition on the enlarged filtration of the martingale process on the natural one.

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References

[1] A. Aksamit. *Random times, enlargement of filtration and arbitrages.* Theses, Université d'Evry-Val d'Essonne, June 2014.

[2] A. Aksamit and M. Jeanblanc. *Enlargement of Filtration with Finance in View.* Springer International Publishing, 2017.

[3] J. Amendinger, D. Becherer, and M. Schweizer. A monetary value for initial information in portfolio optimization. *Finance and Stochastics*, 7(1):29–46, 2003.

[4] E. Bandini, F. Confortola, and P. D. Tella. Progressively enlargement of filtrations and control problems for step processes. *Available at ArXiv:2112.12884*, 2021.

[5] N. Bellamy. Wealth optimization in an incomplete market driven by a jump-diffusion process. *Journal of Mathematical Economics*, 35(2):259 – 287, 2001.

[6] C. Bender. An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter. *Stochastic Processes and their Applications*, 104(1):81–106, 2003.

[7] H.-P. Bermin. A general approach to hedging options: Applications to barrier and partial barrier options. *Mathematical Finance*, 12(3):199–218, 2002.

[8] F. Biagini and B. Øksendal. A general stochastic calculus approach to insider trading. *Applied Mathematics and Optimization*, 52(2):167–181, 2005.

[9] F. Biagini and B. Øksendal. Forward integrals and an Itô formula for fractional Brownian motion. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 11(02):157–177, 2008.

[10] I. Cooper and M. Martin. Default risk and derivative products. *Applied Mathematical Finance*, 3(1):53–70, 1996.

[11] C. Dellacherie. *Capacités et processus stochastiques.* Springer, 1972.

[12] G. Di Nunno, T. Meyer-Brandis, B. Øksendal, and F. Proske. Optimal portfolio for an insider in a market driven by Lévy processes. *Quantitative Finance*, 6(1):83–94, 2006.

[13] G. Di Nunno and S. Sjursen. Information and optimal investment in defaultable assets. *International Journal of Theoretical and Applied Finance*, 17, 2014.

[14] G. Di Nunno, B. Øksendal, and F. Proske. White noise analysis for Lévy processes. *Journal of Functional Analysis*, 206(1):109–148, 2004.

[15] G. Di Nunno, B. Øksendal, and F. Proske. *Malliavin Calculus for Lévy Processes with Applications to Finance.* Universitext. Springer, 1st ed. 2009. corr. 2nd printing edition, 2009.

[16] P. Di Tella. On the weak representation property in progressively enlarged filtrations with an application in exponential utility maximization. *Stochastic Processes and their Applications*, 130(2):760–784, 2020.

[17] D. Duffie and D. Lando. Term structures of credit spreads with incomplete accounting information. *Econometrica*, 69(3):633–664, 2001.

[18] N. El Karoui, M. Jeanblanc, and Y. Jiao. What happens after a default: The conditional density approach. *Stochastic Processes and their Applications*, 120(7):1011 – 1032, 2010.

[19] R. Elliott, M. Jeanblanc, and M. Yor. On models of default risk. *Mathematical Finance*, 10(2):179–195, 2000.

[20] C. Fontana, M. Jeanblanc, and S. Song. On arbitrages arising with honest times. *Finance and Stochastics*, 18(3):515–543, Apr. 2014.

[21] C. Fontana and T. Schmidt. General dynamic term structures under default risk. *Stochastic Processes and their Applications*, 128(10):3353–3386, 2018.

[22] C. Hillairet and Y. Jiao. Information asymmetry in pricing of credit derivatives. *International Journal of Theoretical and Applied Finance*, 14(05):611–633, 2011.

[23] C. Hillairet and Y. Jiao. Portfolio optimization with insider's initial information and counterparty risk. *Finance and Stochastics*, 19(1):109–134, 2015.

[24] H. Holden, B. Øksendal, J. Ubøe, and T. Zhang. *Stochastic Partial Differential Equations: A Modeling, White Noise Functional Approach*. Universitext. Springer-Verlag New York, 2 edition, 2010.

[25] P. Imkeller. Random times at which insiders can have free lunches. *Stochastics and Stochastic Reports*, 74(1-2):465–487, 2002.

[26] T. Ishikawa and S. Robertson. Optimal investment and pricing in the presence of defaults. *Mathematical Finance*, 30(2):577–620, 2020.

[27] J. Jacod and A. Shiryaev. *Limit Theorems for Stochastic Processes*. Grundlehren der mathematischen Wissenschaften 288. Springer-Verlag Berlin Heidelberg, 2 edition, 2003.

[28] S. Janson, S. M'baye, and P. Protter. Absolutely continuous compensators. *International Journal of Theoretical and Applied Finance*, 14(3):335–351, 2011.

[29] M. Jeanblanc, M. Yor, and M. Chesney. *Mathematical Methods for Financial Markets*. Springer Finance. Springer London, 2009.

[30] T. Jeulin. *Semi-Martingales et Grossissement d'une Filtration*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1980.

[31] Y. Jiao, I. Kharroubi, and H. Pham. Optimal investment under multiple defaults risk: A BSDE-decomposition approach. *The Annals of Applied Probability*, 23(2):455–491, 2013.

[32] Y. Jiao and H. Pham. Optimal investment with counterparty risk: a default-density model approach. *Finance and Stochastics*, 15(4):725–753, 2011.

[33] I. Kharroubi, T. Lim, and A. Ngoupeyou. Mean-variance hedging on uncertain time horizon in a market with a jump. *Applied Mathematics and Optimization*, 68:413–444, 2013.

[34] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab.*, 9(3):904–950, 1999.

[35] L. Li and M. Rutkowski. Progressive enlargements of filtrations with pseudo-honest times. *The Annals of Applied Probability*, 24(4):1509 – 1553, 2014.

[36] T. Lim and M.-C. Quenez. Portfolio optimization in a default model under full/partial information. *Probability in the Engineering and Informational Sciences*, 29(4):565–587, 2015.

[37] R. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51(3):247–257, 1969.

[38] R. Merton. On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance*, 29(2):449–470, 1974.

[39] J. Mikusiński. *The Bochner Integral*, volume 77. Pure and Applied Mathematics, 1978.

[40] D. Neal. Commutation of variation and dual projection. *Proceedings of the American Mathematical Society*, 123(5):1591–1595, 1995.

[41] D. Nualart. *The Malliavin Calculus and Related Topics*. Probability and Its Applications. Springer Berlin Heidelberg, 1995.

[42] J. Obłój and M. Yor. *On Local Martingale and its Supremum:Harmonic Functions and beyond*, pages 517–533. Springer Berlin Heidelberg, Berlin, Heidelberg, 2006.

[43] G. Peskir and A. Shiryaev. *Optimal Stopping and Free-Boundary Problems*. Lectures in Mathematics. ETH Zürich. Birkhäuser Basel, 1 edition, 2006.

[44] N. Privault. *Stochastic analysis in discrete and continuous settings. With normal martingales*, volume 1982. Springer-Verlag Berlin Heidelberg, 01 2009.

[45] P. E. Protter. *Stochastic Integration and Differential Equations*. Springer Berlin Heidelberg, 2005.

[46] C. Tudor. An anticipating calculus for square integrable pure jump Lévy processes. *Random Operators and Stochastic Equations*, 15(1):1–14, 2007.

A Anticipating calculus in the white noise framework: Brownian motion

Lemma A.1. *Let $\mathbb{G}$ be a given filtration such that $\mathbb{F}^W \subset \mathbb{G}$ holds true. Suppose that $W = (W_t, 0 \leq t \leq T)$ is a semimartingale with respect to $\mathbb{G}$. Let φ be a $\mathbb{G}$-predictable process and the integral $\int_0^T \varphi_t dW_t$ exists as a classical Itô integral. Then φ is forward integrable with respect to W and*

$$\int_0^T \varphi_t d^- W_t = \int_0^T \varphi_t dW_t \ .$$

Proof. We refer the proof of Lemma 8.9 in [15]. □

Proof of Lemma 3.14. We consider the representation of F and $W_t - W_s$ with respect to the orthogonal basis $\{H_\alpha\}$

$$F = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha \ , \quad W_t - W_s = \sum_{k=1}^{\infty} \left(\int_s^t e_k(y) dy \right) H_{\varepsilon(k)} \ .$$

By definition of Wick product we get,

$$\begin{aligned} F \diamond (W_t - W_s) =& \sum_{\alpha,k} a_\alpha \left(\int_s^t e_k(y) dy \right) H_{\alpha+\varepsilon(k)} \\ =& \sum_{\alpha,k} a_\alpha \left(\int_s^t e_k(y) dy \right) h_{\alpha_k+1}(\theta_k) \prod_{j \neq k} h_{\alpha_j}(\theta_j) \end{aligned}$$

$$\begin{aligned}
&=\sum_{\alpha,k} a_\alpha \left(\int_s^t e_k(y)dy\right)(\theta_k h_{\alpha_k}(\theta_k)-\alpha_k h_{\alpha_k-1}(\theta_k))\prod_{j\neq k} h_{\alpha_j}(\theta_j)\\
&=\sum_{\alpha,k} a_\alpha \left(\int_s^t e_k(y)dy\right)\theta_k H_\alpha - \sum_{\alpha,k} a_\alpha\alpha_k \left(\int_s^t e_k(y)dy\right) h_{\alpha_k-1}(\theta_k)\prod_{j\neq k} h_{\alpha_j}(\theta_j)\\
&=\sum_{\alpha,k} a_\alpha \left(\int_s^t e_k(y)dy\right) H_{\varepsilon(k)}H_\alpha - \sum_{\alpha,k} a_\alpha\alpha_k \left(\int_s^t e_k(y)dy\right) H_{\alpha-\varepsilon(k)}\\
&=F\cdot(W_t-W_s)-\int_s^t D_yF dy\ ,
\end{aligned}$$

where we have used the Definition 3.8, the Equation (3.16), the fact $H_{\varepsilon(k)} = h_1(\theta_h) = \theta_k$ and the following recurrence relationship of the Hermite polynomials,

$$h_{n+1}(x) = xh_n(x) - nh_{n-1}(x)\ ,$$

with $n = \alpha_k$ and $x = \theta_k$. □

Proof of Theorem 3.15. We follow the lines of Theorem 8.18 in [15], by considering the convergence in the space $(\mathcal{S})^*$ via Lemma 3.14, and Theorem 3.7 in [9] and allowing $D_{\cdot^+}Y_\cdot \in (\mathcal{S})^*$.

$$\begin{aligned}
\int_0^T Y_t d^- W_t &= \lim_{\epsilon\to 0}\int_0^T Y_t \frac{W_{t+\epsilon}-W_t}{\epsilon}dt\\
&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\left(Y_t\diamond(W_{t+\epsilon}-W_t)+\int_t^{t+\epsilon} D_sY_t ds\right)dt\\
&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\left(\int_t^{t+\epsilon} Y_t\diamond \mathbb{W}_s ds+\int_t^{t+\epsilon} D_sY_t ds\right)dt\\
&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int_0^T\left(\int_{s-\epsilon}^s Y_t dt\right)\diamond \mathbb{W}_s ds+\int_0^T\int_{s-\epsilon}^s D_sY_t dt ds\right)\\
&=\int_0^T Y_s\diamond \mathbb{W}_s ds+\int_0^T D_{u^+}Y_u du\ ,
\end{aligned}$$

where in the third equality we applied the Example 2.5.13 of [24] in order to conclude

$$Y_t\diamond(W_{t+\epsilon}-W_t)=\int_t^{t+\epsilon} Y_t\diamond\mathbb{W}_s ds\ .$$

The Fubini Theorem for the Bochner integration is guaranteed by Theorem 2, page 93, of [39]. In the fourth equality, as $Y\in L^2(dt\times \boldsymbol{P},\mathbb{G})$ we applied the following convergence in $L^2(\boldsymbol{P})\subset(\mathcal{S})^*$,

$$Y_s^\epsilon := \frac{1}{\epsilon}\int_{s-\epsilon}^s Y_t dt \to Y_s \text{ if } \epsilon\to 0\ ,$$

see the proof of Theorem 8.18 in [15]. In the fifth equality, we first applied Theorem 5.24 of the mentioned reference in order to get

$$\int_0^T Y_s^\epsilon\diamond\mathbb{W}_s ds \to \int_0^T Y_s\diamond\mathbb{W}_s ds \text{ if } \epsilon\to 0$$

in $(\mathcal{S})^*$. Then, we use Lemma 6.7 in [15] to conclude that

$$\int_0^T \left\|D_t\left(\frac{1}{\epsilon}\int_{u-\epsilon}^u Y_s ds\right)\right\|_{-\hat{q}}^2 dt \le \left\|\frac{1}{\epsilon}\int_{u-\epsilon}^u Y_s ds\right\|_{-q}^2 \le \|Y_u\|_{L^2(\boldsymbol{P})}^2\ ,$$

$\hat{q}\ge 2q+\frac{1}{\ln 2}$, $q\in\mathbb{Z}^+$ and $u\in[\epsilon,T]$ and the last convergence is stated in $(\mathcal{S})_{-\hat{q}}\subset(\mathcal{S})^*$. □

B Anticipating calculus in the white noise framework: Poisson random measure

Let $f \in L^2((dt \times \nu)^n)$, we introduce the following functional

$$J_n^N(f) := \int_0^T \int_{\mathbb{R}_0} \dots \int_0^{t_2} \int_{\mathbb{R}_0} f(t_1, z_1, \dots, t_n, z_n) \tilde{N}(dt_1, dz_1) \dots \tilde{N}(dt_n, dz_n) \; .$$

In the case f is symmetric, we define $I_n^N(f) := n! J_n^N(f)$. We consider the symmetrization $sym(f)$ or $\widehat{f}$ of the deterministic function f as

$$sym(f) := \widehat{f}(t_1, z_1, \dots, t_n, z_n) := \frac{1}{n!} \sum_{\sigma \in S_n} f(t_{\sigma(1)}, z_{\sigma(1)}, \dots, t_{\sigma(n)}, z_{\sigma(n)}) \; ,$$

and we extend the definition $I_n^N(f) := n! J_n^N(\widehat{f})$ for any $f \in L^2((dt \times \nu)^n)$. We consider $\{l_m\}_m$ the orthogonalization of the family $\{1, z, z^2, ...\}$ and let's define

$$p_{j+1}(z) := z l_j(z) \left(\int_{\mathbb{R}_0} l_j^2(u) u^2 \nu(du) \right)^{-1/2} , \quad (z, j) \in \mathbb{R}_0 \times \mathbb{Z}^+ \tag{B.1}$$

$$\delta_{z(i,j)}(t, z) := e_i(t) p_j(z) \; , \quad (t, z) \in [0, T] \times \mathbb{R}_0 \; , \quad (i, j) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \; , \tag{B.2}$$

where $z(i, j) := j + (i + j - 2)(i + j - 1)/2$ is a bijective map and $\{e_i\}_i$ are the Hermite functions. For any $\alpha \in \mathcal{J}$ satisfying $\alpha = (\alpha_1, \dots, \alpha_j)$ with $|\alpha| := \alpha_1 + \dots + \alpha_j = m$, we define

$$\delta^{\widehat{\otimes}\alpha}((t_1, z_1), \dots, (t_m, z_m)) := \delta_1^{\widehat{\otimes}\alpha_1} \widehat{\otimes} \dots \widehat{\otimes} \delta_j^{\widehat{\otimes}\alpha_j}((t_1, z_1), \dots, (t_m, z_m)) \; ,$$

where $\delta^{\widehat{\otimes}\alpha}$ denotes the symmetrized tensor product. Finally we define

$$K_\alpha := I_{|\alpha|}^N \left(\delta^{\widehat{\otimes}\alpha}((t_1, z_1), \dots, (t_m, z_m)) \right) \; . \tag{B.3}$$

The family $\{K_\alpha\}_{\alpha \in \mathcal{J}}$ is an orthogonal basis for the space $L^2(\boldsymbol{P})$ in the sense of the following theorem.

Theorem B.1. *Let X be a $\mathcal{F}_T^N$-measurable random variable in $L^2(\boldsymbol{P})$, then there exists a unique sequence $\{a_\alpha\} \subset \mathbb{R}$ such that*

$$X = \sum_{\alpha \in \mathcal{J}} a_\alpha K_\alpha \; .$$

and the norm can be computed as $\|X\|_{L^2(\boldsymbol{P})}^2 = \sum \alpha! a_\alpha^2$.

Definition B.2.
- *Let $f = \sum_{\alpha \in \mathcal{J}} a_\alpha K_\alpha \in L^2(\boldsymbol{P})$ be a random variable, we say that f belongs to the Hida test function Hilbert space $(\mathcal{S})_k$, for $k \in \mathbb{R}$, if*

$$\|f\|_k^2 := \sum_{\alpha \in \mathcal{J}} \alpha! \, a_\alpha^2 \prod_{j=1}^m (2j)^{k\alpha_j} < \infty \; .$$

  We define the Hida test function space $(\mathcal{S}) = \bigcap_{k \in \mathbb{R}} (\mathcal{S})_k$ equipped with the projective topology.

- *Let* $F = \sum_{\alpha\in\mathcal{J}} a_\alpha K_\alpha$ *be a formal sum, we say that* F *belongs to the Hida distribution Hilbert space* $(\mathcal{S})_{-q}$*, for* $q \in \mathbb{R}$*, if*

$$\|F\|_{-q}^2 := \sum_{\alpha\in\mathcal{J}} \frac{\alpha!\, a_\alpha^2}{\prod_{j=1}^m (2j)^{q\alpha_j}} < \infty \ .$$

We define the Hida distribution space $(\mathcal{S})^* = \bigcup_{q\in\mathbb{R}} (\mathcal{S})_{-q}$ *equipped with the inductive topology, i.e, convergence is studied with* $\|\cdot\|_q$ *for some* $q \in \mathbb{R}$.

The generalized expectation and the generalized conditional expectation are defined analogously to the Brownian case.

Definition B.3. *The white noise process* $\tilde{\mathbb{N}}(t,z)$ *of* $\tilde{N}(dt,dz)$ *is defined by the expansion*

$$\tilde{\mathbb{N}}(t,z) := \sum_{i,j\in\mathbb{Z}^+} e_i(t)p_j(z)K_{\varepsilon(i,j)} \ , \tag{B.4}$$

where $\varepsilon(i,j) := \varepsilon(z(i,j))$.

As before, it can be proved that $\tilde{\mathbb{N}}(t,z)$ is well-defined as an object in $(\mathcal{S})^*$ and it satisfies

$$\tilde{\mathbb{N}}(t,z) = \frac{\tilde{N}(dt,dz)}{dt \times \nu(dz)} \ .$$

Definition B.4. *Let* $F = \sum_{\alpha\in\mathcal{J}} a_\alpha K_\alpha$*,* $G = \sum_{\beta\in\mathcal{J}} b_\beta K_\beta \in (\mathcal{S})^*$*, then we define the Wick product as*

$$F \diamond G := \sum_{\alpha,\beta\in\mathcal{J}} a_\alpha b_\beta K_{\alpha+\beta} \ .$$

Lemma B.5. *Let* $\mathbb{G}$ *be a given filtration such that* $\mathbb{F}^N \subset \mathbb{G}$*. Suppose that* $\int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds,dz)$*, with* $0 \le t \le T$*, is a semimartingale with respect to* $\mathbb{G}$*. Let* $\theta = \theta_t(z)$*, with* $(t,z) \in [0,T]\times\mathbb{R}_0$*, be a* $\mathbb{G}$*-predictable process and the integral* $\int_0^T \int_{\mathbb{R}_0} \theta_t(z)\tilde{N}(dt,dz)$ *exists as a classical Itô integral. Then* θ *is forward integrable with respect to* $\tilde{N}$ *and*

$$\int_0^T \int_{\mathbb{R}_0} \theta_t(z)\tilde{N}\left(d^-t,dz\right) = \int_0^T \int_{\mathbb{R}_0} \theta_t(z)\tilde{N}(dt,dz) \ .$$

Before giving the proof of Lemma B.7, we introduce some notions of the *contraction kernel operation* that it will appear in the next statement. See the beginning of the Section 3 of [46] for more details. Let $f \in L^2((dt\times\nu)^n)$ and $g \in L^2((dt\times\nu))$, then we define

$$\left(f *_1^0 g\right)(t_1,z_1,\dots,t_m,z_m) := f\left(t_1,z_1,\dots,t_m,z_m\right) g(t_1,z_1)$$
$$\left(f *_1^1 g\right)(t_1,z_1,\dots,t_m,z_m) := \int_0^T \int_{\mathbb{R}_0} f\left(t_1,z_1,\dots,t_m,z_m\right) g(t_1,z_1)\nu(dz_1)dt_1 \ .$$

Sometimes, if the variables $(t_1,z_1,\dots,t_m,z_m)$ are clear we will omit them. In the next lemma, we will compute the symmetrization of some kernel operations that we will need.

Lemma B.6. *Let* $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathcal{J}$ *with* $|\alpha| = m$ *and we consider the functions* $f = \delta^{\widehat{\otimes}\alpha}$ *and* $g = \mathbb{1}_{\{(u,z)\in\Lambda\}}$*, with* $\Lambda := [s,t]\times U_m$*. Then,*

$$sym\left(f *_1^0 g\right) = \sum_{i,j} \frac{\alpha_{z(i,j)}}{|\alpha|}\left(\delta^{\widehat{\otimes}(\alpha-\varepsilon(i,j))}\widehat{\otimes}\, e_i p_j\right)$$
$$sym\left(f *_1^1 g\right) = \sum_{i,j} \frac{\alpha_{z(i,j)}}{|\alpha|} \int_\Lambda e_i(t_1)p_j(z_1)\nu(z_1)dt_1\, I_{|\alpha|-1}\left(\delta^{\widehat{\otimes}(\alpha-\varepsilon(i,j))}\right)$$

Proof.

$$
\begin{aligned}
sym\left(f *_1^0 g\right) &= sym\left(\delta^{\widehat{\otimes}\alpha}(t_1,z_1,\dots,t_m,z_m)\,\mathbb{1}_{\{(t_1,z_1)\in\Lambda\}}\right)\\
&= \sum_{k=1}^{m}\frac{1}{|\alpha|}\delta^{\widehat{\otimes}\alpha}(t_1,z_1,\dots,t_m,z_m)\,\mathbb{1}_{\{(t_k,z_k)\in\Lambda\}}\\
&= \sum_{i,j}\frac{\alpha_{z(i,j)}}{|\alpha|}\left(\delta^{\widehat{\otimes}(\alpha-\varepsilon(i,j))}(t_2,z_2,\dots,t_m,z_m)\,\widehat{\otimes}\left(e_i(t_1)p_j(z_1)\mathbb{1}_{\{(t_1,z_1)\in\Lambda\}}\right)\right)\ ,
\end{aligned}
$$

where the first step follows by the definition of the symmetrization and the fact that $\delta^{\widehat{\otimes}\alpha}$ is indeed a symmetric function. The second step follows by the definition of α and the fact that each $\delta_{z(i,j)}$ appears $\alpha_{z(i,j)}$ times. Although we fix (t_1,z_1) in the last expression, by the symmetric property, we can write any pair (t_k,z_k).

$$
\begin{aligned}
f *_1^1 g &= \int_0^T\int_{\mathbb{R}_0}\delta^{\widehat{\otimes}\alpha}(t_1,z_1,\dots,t_m,z_m)\,\mathbb{1}_{\{(t_1,z_1)\in\Lambda\}}\,\nu(dz_1)dt_1\\
&= \int_\Lambda \delta^{\widehat{\otimes}\alpha}(t_1,z_1,\dots,t_m,z_m)\ \nu(dz_1)dt_1\\
&= \sum_{i,j}\frac{\alpha_{z(i,j)}}{|\alpha|}\int_\Lambda\left(\delta^{\widehat{\otimes}\alpha-\varepsilon(i,j)}(t_2,z_2,\dots,t_m,z_m)\otimes e_i(t_1)p_j(z_1)\right)\nu(dz_1)dt_1\\
&= \sum_{i,j}\frac{\alpha_{z(i,j)}}{|\alpha|}\int_\Lambda e_i(t_1)p_j(z_1)\nu(dz_1)dt_1\ I_{|\alpha|-1}\left(\delta^{\widehat{\otimes}\alpha-\varepsilon(i,j)}\right)\ ,
\end{aligned}
$$

where we have developed the symmetrization in the variables (t_1,z_1) in order to compute the integral in the set Λ. Note that $f *_1^1 g = sym(f *_1^1 g)$ in this particular case. □

Lemma B.7. *Let $F\in L^2(\boldsymbol{P})$ be a $\mathcal{F}_T^N$-measurable random variable such that $D_{t,z}F$ is $(\nu\times dt)$-integrable in $(\mathcal{S})^*$ and let $U_m\subset\mathbb{R}_0$ be a compact set, then*

$$
\begin{aligned}
F\cdot(\tilde N(t,U_m)-\tilde N(s,U_m)) = &\int_s^t\int_{U_m}(F+D_{u,z}F)\diamond\tilde{\mathbb{N}}(u,z)\,\nu(dz)du\\
&+\int_s^t\int_{U_m}D_{u,z}F\,\nu(dz)du\ . \qquad\text{(B.5)}
\end{aligned}
$$

Proof. We consider the representation of F with respect to the orthogonal basis $\{K_\alpha\}$ and $\tilde N(t,U_m)-\tilde N(s,U_m)$ as an one-dimensional integral:

$$
F=\sum_{\alpha\in\mathcal{J}}a_\alpha K_\alpha=\sum_{\alpha\in\mathcal{J}}a_\alpha I^N_{|\alpha|}(\delta^{\widehat{\otimes}\alpha})\ ,\quad \tilde N(t,U_m)-\tilde N(s,U_m)=I_1^N\left(\mathbb{1}_{\{(u,z)\in\Lambda\}}\right)\ ,
$$

where we short the notation by defining $\Lambda := [s,t]\times U_m$. We consider also the conditional expectation of the chaos K_α as $K_\alpha^{(t)} := \boldsymbol{E}[K_\alpha|\mathcal{F}_t^N]$. We compute the following usual product as they are $L^2(\boldsymbol{P})$-random variables:

$$
F\cdot(\tilde N(t,U_m)-\tilde N(s,U_m))=\sum_\alpha a_\alpha K_\alpha I_1^N(\mathbb{1}_{\{(u,z)\in\Lambda\}})\ .
$$

We apply the following product formula appearing in Theorem 3.1 of [46] for the Poisson case:

$$
K_\alpha I_1^N(\mathbb{1}_{\{(u,z)\in\Lambda\}}) = I^N_{|\alpha|+1}\left(\delta^{\widehat{\otimes}\alpha}\widehat{\otimes}\mathbb{1}_{\{(u,z)\in\Lambda\}}\right)+|\alpha|I^N_{|\alpha|}\left(sym\left(\delta^{\widehat{\otimes}\alpha}*_1^0\mathbb{1}_{\{(u,z)\in\Lambda\}}\right)\right)
$$

$$+ |\alpha| I^N_{|\alpha|-1}\left(sym\left(\delta^{\widehat{\otimes}\alpha} *_1^1 \mathbb{1}_{\{(u,z)\in\Lambda\}}\right)\right) \ .$$

By definition of Wick product it follows

$$\sum_\alpha a_\alpha I_{|\alpha|+1}\left(\delta^{\widehat{\otimes}\alpha}\widehat{\otimes}\mathbb{1}_{(u,z)\in\Lambda}\right) = F \diamond (\tilde{N}(t,U_m) - \tilde{N}(s,U_m)) = \int_\Lambda F \diamond \tilde{\mathbb{N}}(u,z)\,\nu(dz)du \ ,$$

where we have used the Wick product according to Remark 3.12 of [14]. For the second and the third terms we apply the Lemma B.6. In particular,

$$\begin{aligned}
&\sum_\alpha a_\alpha |\alpha| I_{|\alpha|}\left(\delta^{\widehat{\otimes}\alpha} *_1^0 \mathbb{1}_{\{(u,z)\in\Lambda\}}\right) = \sum_{\alpha,i,j} a_\alpha \alpha_{z(i,j)} I_{|\alpha|}\left(\delta^{\widehat{\otimes}(\alpha-\varepsilon(i,j))}\widehat{\otimes}\left(e_i p_j \mathbb{1}_{\{\cdot\in\Lambda\}}\right)\right)\\
&= \sum_{\alpha,i,j} a_\alpha \alpha_{z(i,j)} \int_\Lambda e_i(u)p_j(z)\left(K_{\alpha-\varepsilon(i,j)} - K^{(u)}_{\alpha-\varepsilon(i,j)} + K^{(u)}_{\alpha-\varepsilon(i,j)}\right)\tilde{N}(du,dz)\\
&= \sum_{\alpha,i,j} a_\alpha \alpha_{z(i,j)} \int_\Lambda e_i(u)p_j(z)\left(K_{\alpha-\varepsilon(i,j)} - K^{(u)}_{\alpha-\varepsilon(i,j)}\right)\diamond \tilde{\mathbb{N}}(u,z)\,\nu(dz)du\\
&\quad + \sum_{\alpha,i,j} a_\alpha \alpha_{z(i,j)} \int_\Lambda e_i(u)p_j(z) K^{(u)}_{\alpha-\varepsilon(i,j)} \diamond \tilde{\mathbb{N}}(u,z)\,\nu(dz)du\\
&= \int_\Lambda \left(D_{u,z}F - \boldsymbol{E}[D_{u,z}F|\mathcal{F}^N_u]\right)\diamond\tilde{\mathbb{N}}(u,z)\,\nu(dz)du + \int_\Lambda \boldsymbol{E}[D_{u,z}F|\mathcal{F}^N_u]\diamond\tilde{\mathbb{N}}(u,z)\,\nu(dz)du\\
&= \int_\Lambda D_{u,z}F\diamond\tilde{\mathbb{N}}(u,z)\,\nu(dz)du
\end{aligned}$$

where we have split the term $K_{\alpha-\varepsilon(i,j)}$ in order to get the Skorohod integral. For the third one, we get

$$\begin{aligned}
\sum_\alpha a_\alpha |\alpha| I_{|\alpha|-1}\left(\delta^{\widehat{\otimes}\alpha} *_1^1 \mathbb{1}_{\{(u,z)\in\Lambda\}}\right) &= \sum_{\alpha,i,j} a_\alpha \alpha_{z(i,j)} \int_\Lambda e_i(u)p_j(z)\nu(dz)du\, I_{|\alpha|-1}\left(\delta^{\widehat{\otimes}(\alpha-\varepsilon(i,j))}\right)\\
&= \int_\Lambda D_{u,z}F\,\nu(dz)du \ .
\end{aligned}$$

□