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Instituto para el Desarrollo Empresarial
Universidad Carlos III de Madrid
C/ Madrid, 126
28903 Getafe Madrid (Spain)
FAX +34 916249607

The newsvendor problem with convex risk

Jean Philippe Charron¹
Departamento de Financiación e Investigación Comercial
Universidad Autónoma de Madrid

Alejandro Balbás²
Departamento de Economía de la Empresa
Universidad Carlos III de Madrid

¹ Campus de Cantoblanco. Ctra. Colmenar, Km.15. 28049, Madrid (Spain).

² C/ Madrid, 126. 28903 Getafe (Madrid, Spain).

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Jean Philippe Charron* and Alejandro Balbás**

* Autonomous University of Madrid. Cantoblanco Campus. Ctra. Colmenar, Km.15. 28049 (Madrid, Spain). jp.charron@uam.es

** University Carlos III of Madrid. C/ Madrid, 126. 28903 Getafe (Madrid, Spain). alejandro.balbas@uc3m.es.

Abstract The newsvendor problem is a classical topic in Management Science and Operations Research. It deals with purchases and price strategies when a least one deadline is involved. In this paper we will assume that the decision is driven by an optimization problem involving both expected profits and risks. As a main novelty, risks will be given by a convex risk measure, including the usual utility functions. This approach will allow us to find necessary and sufficient optimality conditions under very general frameworks, since we will not need any specific assumption about the demand distribution.

Key words News vendor problem, Risk, Convex risk measure, Utility function, Saddle point conditions.

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1 Introduction

The newsvendor problem is a classical topic in Management Science and Operations Research (Xu et al., 2010, Xu and Lu, 2013,, etc.). It deals with purchase and price strategies when a least one deadline is involved, and it will not be possible to sell after that date. Sometimes, several maturities (or deadlines) are considered, in which case decisions may become multi-period and maybe dynamic. An interesting overview of the State of the Art may be found in Choi (2012).

The decision maker (*i.e.*, the newsvendor) usually faces an optimization problem involving some utility function or the usual (risk, profit) pair. In the second case, risk is usually measured by dispersions (variance, mainly) with respect to expected values, but more recently, risk measure beyond the variance have been considered. For instance, Gotoh and Takano (2007) use the Conditional Value at Risk (Rockafellar *et al*, 2006), and Choi and Ruszczyński (2008) and Choi *et al* (2011) deal with more general coherent risk measures in the sense of Artzner *et al* (1999).

In this paper we will propose to deal with convex risk measures (see, amongst others, Kupper and Svindland, 2011) for several reasons. Firstly, every coherent risk measure is convex, and therefore the analysis becomes more general. Secondly, the minimization of convex risk measures contains the maximization of

(concave) utility functions as a particular case, and therefore our analysis integrates the two classical approaches, *i.e.*, the approach related to utilities and the one related to the couple (risk, profit). Thirdly, coherent risk measures must be sub-additive, but several authors have shown how sub-additivity may become a very restrictive constraint (Dhaene *et al* 2008). Fourthly, in dynamic settings every coherent risk measure losses an important property called “time consistency”, while some convex risk measures preserve this consistency and therefore are more suitable in dynamic studies (Kupper and Schachermayer, 2009).

The paper outline is as follows. In Section 2 we will present the problem we are going to deal with as well as some properties related to the convex functions we will involve in the analysis. Section 3 will be devoted to present necessary and sufficient saddle point optimality conditions and Section 4 will conclude the paper. The appendix contains a rigorous study about several properties of convex functions that will often apply throughout the paper.

2 Main problem

The decision maker (newsvendor) deals with a specific product that he/she will have to sell within a time period $[0, T]$. He/she must decide both, the number of units $x \geq 0$ that he/she will order to the manufacturer,¹ and the unitary sale price p . Obviously, we will assume that $p \geq P$, $P > 0$ denoting the manufacturer unitary price. The random variable $D(p)$ will represent the market demand, and, as usual, we will assume that $D(p)$ depends of p , though our approach will be general and we will not impose any specific relationship between p and $D(p)$.

Suppose that $B(x, p)$ represents the (random) profit generated by the decision x (ordered units) and p (sale price). Obviously, if $0 \leq X$, $X \leq x$, represents the sold units,

$$B(x, p) = pX - Px \tag{1}$$

must hold. Furthermore,

$$X = \text{Min} \{D(p), x\} = D(p) - (D(p) - x)^+$$

is obvious,² and (1) leads to

$$B(x, p) = p \left(D(p) - (D(p) - x)^+ \right) - Px. \tag{2}$$

Consider a vector space of random variables Y and a real valued function ρ which can be interpreted as a risk measure. We will assume that ρ is a CWCS function (see the appendix for details),³ and therefore, it can be represented according to (9). The newsvendor will attempt to maximize $\mathbb{E}(B(x, p))$, where

¹If necessary, one could impose x to be an integer, $x = 0, 1, 2, \dots$, though, in general, we will not consider this constraint.

²As usual, $u^+ = \text{Max} \{u, 0\}$ for every $u \in \mathbf{R}$.

³It will be shown in the appendix that every CWCS function is convex. Moreover, we will also show that every convex function may be properly approximated by means of CWCS functions.

$B(x, p)$ is given by (2) and $\mathbf{E}(\cdot)$ denotes mathematical expectation, and to minimize the risk $\rho(B(x, p))$. In other words, he/she will deal with the optimization problem

$$\begin{cases} \text{Max} & \mathbf{E} \left(p \left(D(p) - (D(p) - x)^+ \right) - Px \right) \\ \text{Min} & \rho \left(p \left(D(p) - (D(p) - x)^+ \right) - Px \right) \\ & x \geq 0, p \geq P \end{cases} \quad (3)$$

(x, p) being the decision variable. Problem (3) is a classical Profit/Risk problem usual in Portfolio Selection, Optimal Reinsurance, Newsvendor Problems, etc. The main novelty is that we are involving a very general risk measure containing every coherent risk measure (Artzner *et al*, 1999), every expectation bounded risk measure (Rockafellar *et al*, 2006) and many convex risk measures (Kupper and Svindland, 2011, and see the appendix). Since (3) contains two objective functions, the decision maker will be looking for Pareto solutions, and most of them can be obtained by minimizing scalar problems $U\rho(B(x, p)) - \mathbf{E}(B(x, p))$, $U > 0$ denoting the “relative importance of the risk $\rho(B(x, p))$ with respect to the expected profit $\mathbf{E}(B(x, p))$ ” (Nakayama *et al*, 1985).⁴ Consequently, (3) may become

$$\begin{cases} \text{Min} & \begin{cases} U\rho \left(p \left(D(p) - (D(p) - x)^+ \right) - Px \right) \\ -\mathbf{E} \left(p \left(D(p) - (D(p) - x)^+ \right) - Px \right) \end{cases} \\ & x \geq 0, p \geq P \end{cases} \quad (4)$$

In order to deal with (4) we will follow the approach of Balbás *et al* (2010). Thus, bearing in mind the representation (9) of ρ , (4) is equivalent to

$$\text{Min } U\theta - \mathbf{E}(y) \begin{cases} \theta \geq \langle y, z \rangle + k, & \forall (z, k) \in \Delta \\ y = p \left(D(p) - (D(p) - x)^+ \right) - Px \\ x \geq 0, p \geq P \end{cases} \quad (5)$$

(θ, x, p) being the decision variable. Actually, these authors show that (x^*, p^*) solves (4) if and only if there exists θ^* such that (θ^*, x^*, p^*) solves (5) in which case the equality

$$\theta^* = \rho \left(p^* \left(D(p^*) - (D(p^*) - x^*)^+ \right) - Px^* \right)$$

holds. In other words, θ may be interpreted as the risk that the newsvendor is facing. Consequently, the decision variable (θ, x, p) represents

$$(\text{Risk}, \text{Selected_Demand}, \text{Selected_Price}).$$

⁴The higher the value of U , the higher the importance of the global risk.

3 Dual approach and saddle point optimality conditions

Problem (5) is much easier to study than the equivalent newsvendor Problem (4). The main reason is the simplification of the objective function, which contains the analytically complex function ρ if one deals with (4) but becomes simple and linear in (5). As a consequence, under appropriate assumptions the methodology of Balbás *et al* (2010) applies in order to solve (5). Accordingly, one can consider the dual function of (5), given by

$$\Psi(z, k) = \text{Min} \quad -\mathbf{E}(y) + U(\langle y, z \rangle + k) \begin{cases} y = p \left(D(p) - (D(p) - x)^+ \right) - Px \\ x \geq 0, p \geq P \end{cases}$$

and the dual problem of (5), given by

$$\begin{cases} \text{Max} & \Psi(z, k) \\ & (z, k) \in \Delta \end{cases} \quad (6)$$

Both (5) and (6) satisfy the standard properties and have the same optimal value. In other words, the inequality $\Psi(z, k) \leq U\theta - \mathbf{E}(y)$ holds for feasible solutions and $\Psi(z^*, k^*) \leq U\theta^* - \mathbf{E}(y^*)$ holds for the optimal values. Moreover, (6) is solvable.

The relationships above between (5) and (6) imply the existence of saddle point necessary and sufficient optimality conditions for both problems. If (x^*, p^*) is (4)-feasible, (z^*, k^*) is (6)-feasible and $y^* = p^* \left(D(p^*) - (D(p^*) - x^*)^+ \right) - Px^*$ then (x^*, p^*, z^*, k^*) is said to be a saddle point of (4) if

$$\begin{aligned} -\mathbf{E}(y^*) + U(\langle y^*, z \rangle + k) &\leq \\ -\mathbf{E}(y^*) + U(\langle y^*, z^* \rangle + k^*) &\leq \\ -\mathbf{E}(y) + U(\langle y, z^* \rangle + k^*) &\leq \end{aligned} \quad (7)$$

holds for every (4)-feasible (x, p) , every (6)-feasible (z, k) and every random variable $y = p \left(D(p) - (D(p) - x)^+ \right) - Px$. According to Balbás *et al* (2010), if (x^*, p^*) is (4)-feasible, (z^*, k^*) is (6)-feasible and the equality of random variables $y^* = p^* \left(D(p^*) - (D(p^*) - x^*)^+ \right) - Px^*$ holds, then (x^*, p^*) solves (4) and (z^*, k^*) solves (6) if and only if they are feasible and (x^*, p^*, z^*, k^*) satisfies the saddle point condition (7).

4 Conclusion

In this paper we propose a new methodology allowing us to solve the news vendor problem in a very general setting, since there are no specific assumptions about the distribution of the random demand. The methodology is inspired in previous findings of Financial Theory and applies when the newsvendor deals

with a convex risk measure in order to control the risk level of her/his decisions. The use of convex risk measures may be interesting because they are totally compatible with the maximization of a standard utility function.

5 Appendix (approximation of convex functions)

Let us prove the results used in this paper related the representation of convex functions. Thus, consider a Banach space Y and its dual Z . Represent by $\langle y, z \rangle$ the usual bilinear product. The weak topologies will be denoted by $\sigma(Y, Z)$ and $\sigma(Z, Y)$, respectively.⁵

First we will study the *CWCS* functions

Theorem A1. *Let $\rho : Y \rightarrow \mathbb{R}$ be an arbitrary function. The assertions below are equivalent:*

a) *There exists a convex and $\sigma(Z, Y)$ – compact set $\Delta \subset Z$ such that*

$$\rho(y) = \text{Max} \{ \langle y, z \rangle ; z \in \Delta \} \quad (8)$$

for every $y \in Y$.

b) ρ is continuous, sub-additive ($\rho(y_1 + y_2) \leq \rho(y_1) + \rho(y_2)$) and positive homogeneous ($\rho(\lambda y) = \lambda \rho(y)$ if $\lambda \geq 0$).

If so, Δ is the unique convex and $\sigma(Z, Y)$ – compact subset of Z satisfying (8).

Proof. This results was proved in Balbás *et al* (2013), among others. \square

Definition A2. *Let $\rho : Y \rightarrow \mathbb{R}$ be an arbitrary function. ρ is said to be CWCS (convex with compact sub-gradient) if there exists a convex set $\Delta \subset Z \times \mathbb{R}$, which is compact in the weak topology of $\sigma(Z \times \mathbb{R}, Y \times \mathbb{R})$, and such that*

$$\rho(y) = \text{Max} \{ \langle y, z \rangle + k ; (z, k) \in \Delta \} \quad (9)$$

holds for every $y \in Y$. An arbitrary real valued function $\rho : Y \rightarrow \mathbb{R}$ is said to be convex if

$$\rho(ty_1 + (1-t)y_2) \leq t\rho(y_1) + (1-t)\rho(y_2)$$

holds for every $y_1, y_2 \in Y$ and every $0 \leq t \leq 1$. It is straightforward to see that every CWCS function is convex. \square

Theorem A3. $\rho : Y \rightarrow \mathbb{R}$ is a CWCS function if and only if there exists another function $\varphi : Y \times \mathbb{R} \rightarrow \mathbb{R}$, continuous, sub-additive and positive homogeneous such that $\rho(y) = \varphi(y, 1)$ holds for every $y \in Y$. In particular, every CWCS function is continuous. Furthermore, if $\varphi : Y \times \mathbb{R} \rightarrow \mathbb{R}$ and $\psi : Y \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the condition above, then $\varphi(y, r) = \psi(y, r)$ holds for every $y \in Y$ and every $r \geq 0$.

⁵Details about Banach spaces and related results may be found in Luenberger (1969) or Kopp (1984).

Proof. Suppose that ρ is *CWCS*. Consider the Banach space $Y \times \mathbb{R}$, its dual $Z \times \mathbb{R}$ and the convex and $\sigma(Z \times \mathbb{R}, Y \times \mathbb{R})$ -compact set $\Delta \subset Z \times \mathbb{R}$. Define

$$\varphi(y, r) = \text{Max} \{ \langle y, z \rangle + rk; (z, k) \in \Delta \} \quad (10)$$

for every $(y, r) \in Y \times \mathbb{R}$. Theorem A1 guarantees that φ is continuous, sub-additive and positive homogeneous. Moreover (9) and (10) guarantee the fulfillment of the equality $\rho(y) = \varphi(y, 1)$.

Conversely, Theorem A1, the existence of φ and the equality $\rho(y) = \varphi(y, 1)$ imply that (9) holds, and therefore ρ is *CWCS*.

Lastly, suppose that $\psi : Y \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the same conditions as φ . If $r > 0$ then

$$\psi(y, r) = r\psi(y/r, 1) = r\rho(y/r) = r\varphi(y/r, 1) = \varphi(y, r).$$

If $r = 0$ then

$$\psi(y, 0) = \text{Lim}_{n \rightarrow \infty} \psi(y, 1/n) = \text{Lim}_{n \rightarrow \infty} \varphi(y, 1/n) = \varphi(y, 0).$$

□

Theorem A4. *If $\rho : Y \rightarrow \mathbb{R}$ is a CWCS function and $\Delta \subset Z \times \mathbb{R}$ and $\partial \subset Z \times \mathbb{R}$ are convex and $\sigma(Z \times \mathbb{R}, Y \times \mathbb{R})$ -compact sets satisfying (9), then $\Pi_Z(\Delta) = \Pi_Z(\partial)$ and*

$$\text{Max} \{ k; \exists z \in Z \text{ with } (z, k) \in \Delta \} = \text{Max} \{ k; \exists z \in Z \text{ with } (z, k) \in \partial \} \quad (11)$$

Π_Z being the natural projection of $Z \times \mathbb{R}$ over Z .

Proof. Equality (11) becomes obvious because (9) shows that both expressions equal $\rho(0)$. In order to prove $\Pi_Z(\Delta) = \Pi_Z(\partial)$, suppose that the equality is false, *i.e.*, suppose that $\Pi_Z(\Delta) \neq \Pi_Z(\partial)$. Define the functions

$$\varphi_\Delta(y, r) = \text{Max} \{ \langle y, z \rangle + rk; (z, k) \in \Delta \}$$

and

$$\varphi_\partial(y, r) = \text{Max} \{ \langle y, z \rangle + rk; (z, k) \in \partial \}$$

for every $(y, r) \in Y \times \mathbb{R}$, and the functions

$$\rho_\Delta(y) = \text{Max} \{ \langle y, z \rangle; z \in \Pi_Z(\Delta) \}$$

and

$$\rho_\partial(y) = \text{Max} \{ \langle y, z \rangle; z \in \Pi_Z(\partial) \}$$

for every $y \in Y$. Theorem A1 proves that these functions are continuous, sub-additive and positive homogeneous. The same Theorem implies the inequality $\rho_\Delta \neq \rho_\partial$ because $\Pi_Z(\Delta) \neq \Pi_Z(\partial)$. Since Δ and ∂ satisfy (9), Theorem A3 implies that $\varphi_\Delta(y, r) = \varphi_\partial(y, r)$ if $r \geq 0$. In particular,

$$\varphi_\Delta(y, 0) = \varphi_\partial(y, 0)$$

for every $y \in Y$, i.e.,

$$\text{Max} \{ \langle y, z \rangle; (z, k) \in \Delta \} = \text{Max} \{ \langle y, z \rangle; (z, k) \in \partial \}$$

for every $y \in Y$. The latter equality trivially leads to

$$\text{Max} \{ \langle y, z \rangle; z \in \Pi_Z(\Delta) \} = \text{Max} \{ \langle y, z \rangle; z \in \Pi_Z(\partial) \},$$

which contradicts $\rho_\Delta \neq \rho_\partial$. \square

Next, let us show that every convex and continuous function has a *CWCS* approximation.

Lemma A5. *If ρ is convex and continuous, and we consider the sub-gradient set*

$$\partial = \{ (z, k) \in Z \times \mathbb{R}; \langle y, z \rangle + k \leq \rho(y) \quad \forall y \in Y \} \quad (12)$$

then ∂ is convex and $\sigma(Z, Y)$ -closed (and therefore closed if $\sigma(Z, Y)$ is replaced by the norm topology), and

$$\rho(y) = \text{Sup} \{ \langle y, z \rangle + k; (z, k) \in \partial \} \quad (13)$$

holds for every $y \in Y$.

Proof. It is very easy to see that ∂ is convex if so is ρ , and ∂ is obviously $\sigma(Z, Y)$ -closed. The inequality $\rho(y) \geq \text{Sup} \{ \langle y, z \rangle + k; (z, k) \in \partial \}$ is obvious too, so let us prove the opposite inequality. Consider $y_0 \in Y$ and $\varepsilon > 0$, and it is sufficient to see that $\rho(y_0) - \varepsilon \leq \text{Sup} \{ \langle y, z \rangle + k; (z, k) \in \partial \}$. Obviously, $(y_0, \rho(y_0) - \varepsilon) \notin A$, where

$$A = \{ (y, r) \in Y \times \mathbb{R}; r \geq \rho(y) \}.$$

Since ρ is continuous, it is easy to see that A is closed, and A is convex because so is ρ . Therefore, the Hahn-Banach Separation Theorem proves the existence of $(z_0, k) \in Z \times \mathbb{R}$ non null and such that

$$\langle y, z_0 \rangle + rk \geq \langle y_0, z_0 \rangle + k(\rho(y_0) - \varepsilon)$$

for every $(y, r) \in A$. It is easy to see that $k \geq 0$ since otherwise the latter inequality could not hold if $r \rightarrow \infty$. Moreover, $k = 0$ is not possible either, since $\langle y, z_0 \rangle \geq \langle y_0, z_0 \rangle$ for every $y \in Y$ would lead to $z_0 = 0$, and we have that $(z_0, k) \neq (0, 0)$. Thus, taking z_0/k instead of z_0 , and denoting z_0 again, we have that

$$\langle y, z_0 \rangle + r \geq \langle y_0, z_0 \rangle + (\rho(y_0) - \varepsilon)$$

holds for every $(y, r) \in A$. In particular,

$$\langle y, z_0 \rangle + \rho(y) \geq \langle y_0, z_0 \rangle + (\rho(y_0) - \varepsilon)$$

holds for every $y \in Y$, and therefore

$$\rho(y) \geq -\langle y, z_0 \rangle + \langle y_0, z_0 \rangle + (\rho(y_0) - \varepsilon).$$

Hence,

$$(-z_0, \langle y_0, z_0 \rangle + \rho(y_0) - \varepsilon) \in \partial. \quad (14)$$

Furthermore, (14) implies that

$$\text{Sup} \{ \langle y_0, z \rangle + k; (z, k) \in \partial \} \geq -\langle y_0, z_0 \rangle + \langle y_0, z_0 \rangle + \rho(y_0) - \varepsilon = \rho(y_0) - \varepsilon.$$

□

A brief generalization

Lemma A6. *If U is an open convex subset of the Banach space Y , $\rho : U \rightarrow \mathbb{R}$ is convex and lower semi-continuous and*

$$\partial_U = \{ (z, k) \in Z \times \mathbb{R}; \langle y, z \rangle + k \leq \rho(y) \quad \forall y \in U \},$$

then ∂_U is $\sigma(Z \times \mathbb{R}, Y \times \mathbb{R})$ -closed and convex, and

$$\rho(y) = \text{Sup} \{ \langle y, z \rangle + k; (z, k) \in \partial_U \}$$

holds for every $y \in U$.

Proof. It is obvious that ∂_U is $\sigma(Z \times \mathbb{R}, Y \times \mathbb{R})$ -closed and convex and $\rho(y) \geq \text{Sup} \{ \langle y, z \rangle + k; (z, k) \in \partial_U \}$ holds for every $y \in U$. In order to proof the opposite inequality, consider $y_0 \in U$ and $\varepsilon > 0$.

Consider also

$$A = \{ (y, r) \in U \times \mathbb{R}; r \geq \rho(y) \}.$$

Since ρ is lower semi-continuous, there exists $\delta > 0$ such that $\|y - y_0\| \leq \delta$, $y \in U \implies \rho(y) > \rho(y_0) - \varepsilon/2$. We have that

$$A \cap (B_\delta \times (-\infty, \rho(y_0) - \varepsilon/2)) = \emptyset,$$

B_δ being the intersection of U and the open ball with center at y_0 and radius δ . Indeed, if $(y, r) \in A$ and $y \in B_\delta$ then $r \geq \rho(y) > \rho(y_0) - \varepsilon/2$. Therefore, the Hahn-Banach Separation Theorem implies the existence of $(z_0, k) \in Z \times \mathbb{R}$ non null and such that

$$\langle y, z_0 \rangle + rk \geq \langle y_0, z_0 \rangle + k(\rho(y_0) - \varepsilon)$$

for every $(y, r) \in A$. It is easy to see that $k \geq 0$ since otherwise the latter inequality could not hold if $r \rightarrow \infty$. Moreover, $k = 0$ is not possible either, since $\langle y, z_0 \rangle \geq \langle y_0, z_0 \rangle$ for every $y \in U$ would lead to $z_0 = 0$ because U is open, and we have that $(z_0, k) \neq (0, 0)$. Thus, taking z_0/k instead of z_0 , and denoting z_0 again, we have that

$$\langle y, z_0 \rangle + r \geq \langle y_0, z_0 \rangle + (\rho(y_0) - \varepsilon)$$

holds for every $(y, r) \in A$. In particular,

$$\langle y, z_0 \rangle + \rho(y) \geq \langle y_0, z_0 \rangle + (\rho(y_0) - \varepsilon)$$

holds for every $y \in U$, and therefore

$$\rho(y) \geq -\langle y, z_0 \rangle + \langle y_0, z_0 \rangle + (\rho(y_0) - \varepsilon).$$

Hence,

$$(-z_0, \langle y_0, z_0 \rangle + \rho(y_0) - \varepsilon) \in \partial_U. \quad (15)$$

Furthermore, (15) implies that

$$\text{Sup } \{ \langle y_0, z \rangle + k; (z, k) \in \partial_U \} \geq -\langle y_0, z_0 \rangle + \langle y_0, z_0 \rangle + \rho(y_0) - \varepsilon = \rho(y_0) - \varepsilon.$$

□

Theorem A7. *If $\rho : Y \rightarrow \mathbb{R}$ is convex and continuous then there exists an increasing sequence $(\rho_n)_{n \in \mathbf{N}}$ of CWCS functions such that*

$$\rho(y) = \text{Lim}_{n \rightarrow \infty} \rho_n(y)$$

holds for every $y \in Y$.

Proof. It trivially follows from (9) and (13). Indeed, denote by B_n the closed ball of Z with center at $z = 0$ and radius n , and define

$$\rho_n(y) = \text{Max } \{ \langle y, z \rangle + k; (z, k) \in (B_n \times [-n, n]) \cap \partial \},$$

where ∂ is given by (12). □

Lastly, let us provide a “Mean Value Theorem” used in this paper. Indeed, according to Balbás *et al* (2010), it is required in order to guarantee that (6) is the dual problem of (4) and (5).

Lemma A6. *If $\Delta \subset Z \times \mathbb{R}$ is convex and $\sigma(Z \times \mathbb{R}, Y \times \mathbb{R})$ -compact, and ν is a probability measure on the Borel σ -algebra of Δ endowed with the topology $\sigma(Z \times \mathbb{R}, Y \times \mathbb{R})$, then there exists a unique $(z_\nu, k_\nu) \in \Delta$ such that*

$$\int_{\Delta} (\langle y, z \rangle + rk) d\nu(z, k) = \langle y, z_\nu \rangle + rk_\nu$$

for every $(y, r) \in Y \times \mathbb{R}$. In particular, taking $r = 1$,

$$\int_{\Delta} (\langle y, z \rangle + k) d\nu(z, k) = \langle y, z_\nu \rangle + k_\nu$$

for every $y \in Y$.

Proof. If $(y, r) \in Y \times \mathbb{R}$ then define

$$L(y, r) = \int_{\Delta} (\langle y, z \rangle + rk) d\nu(z, k)$$

and we will have a linear and continuous function on $Y \times \mathbb{R}$ endowed with the strong topology (recall that every $\sigma(Z \times \mathbb{R}, Y \times \mathbb{R})$ -compact set is bounded due to the Banach-Steinhaus Theorem). Hence, since $Z \times \mathbb{R}$ is the dual of $Y \times \mathbb{R}$, there exists a unique $(z_\nu, k_\nu) \in Z \times \mathbb{R}$ such that $L(y, r) = \langle y, z_\nu \rangle + rk_\nu$ for every $(y, r) \in Y \times \mathbb{R}$. In order to prove that $(z_\nu, k_\nu) \in \Delta$, let us suppose that the property is false. The Hahn-Banach Theorem implies the existence of $(y_0, r_0) \in Y \times \mathbb{R}$ such that

$$\langle y_0, z_\nu \rangle + r_0 k_\nu > \langle y_0, z \rangle + r_0 k$$

for every $(z, k) \in \Delta$, which implies the contradiction

$$\langle y_0, z_\nu \rangle + r_0 k_\nu > \int_{\Delta} (\langle y_0, z \rangle + r_0 k) d\nu(z, k) = \langle y_0, z_\nu \rangle + r_0 k_\nu.$$

□

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