Vila, Javier; Zaera, Ramón; Fernández-Sáez, José (2016). Axisymmetric free vibration of closed thin spherical nanoshell with bending effects. *Journal of Vibration and Control*, 22(18), pp.: 3789–3806.

DOI: https://doi.org/10.1177/1077546314565808

Journal of Vibration and Control

Axisymmetric free vibration of closed thin spherical nanoshell with bending effects

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Abstract
Nanoscale spheres have led to a growing interest in their potential applications in a wide range of technological fields. Hollow nanospheres can be modelled as closed spherical shells, the ratio of thickness to radius being determinant to ascertain the pertinence of considering only membrane forces, or both membrane forces and bending moments, in the analysis of their mechanical behaviour. The nonlocal elasticity theory has been widely used to analyse the mechanical behaviour of nanostructures. This paper investigates the free axisymmetric vibration of nanoscale spherical shells accounting for both types of internal forces, by extending the Kirchhoff-Love plate theory to Eringen nonlocal elasticity theory. The influence of coupled size and bending effects on frequencies and modal shapes is studied, revealing specific features that cannot be observed in an uncoupled analysis. This study could be useful in biomedical and bioengineering applications as well as in other fields related with the nanotechnology.

Keywords
nonlocal elasticity, sphere vibration, nanotechnology, natural frequencies, modal shapes, bending effects

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Introduction

Classical continuum solid mechanics theories have been widely used to solve fundamental problems in civil, mechanical and materials engineering, as well as in various fields of physics and life sciences. However, a defining characteristics of this framework is that it is a scale-free theory. Because of its constitutive equations structure, it lacks an internal length and, for this reason, it cannot predict any size effect. Nevertheless, materials used nowadays like composites, functionally graded materials, polycrystalline solids or granular materials, among others, all have inherent microstructures at different scales. Additionally, at high-frequency excitations, microstructural and size effects are observed in wave propagation in solids when the wavelength of a travelling signal becomes comparable with the scale of material heterogeneities (Gonella et al., 2011).

Moreover, from the discovery of fullerenes (Kroto et al., 1985) and carbon nanotubes (CNTs) (Iijima, 1991), technological applications which involve the use of systems which can be devised as micro or nanostructures have experienced an exponential growth, mainly in micro- or nano-electromechanical systems (MEMS or NEMS) (Martin, 1996), nanomachines (Drexler, 1992; Han et al., 1997; Fennimore et al., 2003; Bourlon et al., 2004), as well as in biotechnology and biomedical fields (Saji et al., 2010).

A main characteristic of these nanostructures is that their dimensions become comparable to the size of their material microstructure or the molecular distances, thus the size effects are significant regarding their mechanical behaviour, but they could not be captured by the standard elasticity and plasticity theories.

The above problems could be addressed using Molecular Dynamics (Wei and Srivastava, 2004; He et al., 2007; Tsai and Fang, 2007; Liew et al., 2007), but this approaches require a great computational effort, and other possibilities as generalized continuum models have been explored.

The first attempts to capture the effects of microstructure using the continuum equations of elasticity with additional higher-order derivatives, can be dated in the 19^{th} century with the works of Cauchy and Voigt, and in the first half of the 20^{th} century through the work of the Cosserat brothers. However, a major revival took place in the 1960s. From this time are the works of Mindlin and Tiersten (1962), Kröner (1963), Toupin (1963, 1964), Green and Rivlin (1964), Mindlin (1964, 1965) and Mindlin and Eshel (1968). Nevertheless, these theories were excessively complex, with too many parameters and equations. A modification of the above theory (classical couple stress theory) was proposed by Yang et al. (2002) (modified couple stress theory) and further developed by Park and Gao (2006, 2008) and Kahrobaiyan et al. (2012).

In the 1980s, Eringen derived, from his earlier integral nonlocal theories (Eringen, 1983), a simple stress-gradient formulation which contains a length scale parameter. In the early 1990s, Aifantis and co-workers suggested to extend the linear elastic constitutive relations with the Laplacian of the strain through a length scale parameter again (Aifantis, 1992; Altan and Aifantis, 1992; Ru and Aifantis, 1993). Askes and Gitman (2010) shown that an unification of both Eringen and Aifantis theories is possible. An overview on the historical development of some of theses theories, as well as its meaning and implementation can be found in the paper by Askes and Aifantis (2011).

Among the size-dependent continuum theories, the theory of nonlocal continuum mechanics initiated by Eringen (1972a,b, 1983) has been widely used to analyse many problems, such as wave propagation, dislocation, and crack singularities. From the pioneer work of Peddieson et al. (2003), this theory has been also used to solve problems involving nanostructures. Thus, the Eringen nonlocal theory of elasticity has been used to address the behaviour of beams (Xu, 2006; Lu, 2007; Wang et al., 2006, 2007; Reddy, 2007; Loya et al., 2009; Ke et al., 2012; Roostai and Haghpanahi, 2013; Eltaher et al., 2013), beams under rotation (Pradhan and Murmu, 2010; Murmu and Adhikari, 2010b; Narendar and Gopalakrishnan, 2011b; Aranda-Ruiz et al., 2012), rods (Sun and Zhang, 2003; Murmu and Pradhan, 2009b; Kiani, 2010; Narendar and Gopalakrishnan, 2010; Murmu and Adhikari, 2010a; Narendar, 2011; Aydogdu, 2009), plates (Ke et al., 2008; Murmu and Pradhan, 2009a; Murmu et al., 2011; Hosseini-Hashemi et al., 2013; Alibeigloo and Pasha Zanoosi, 2013; Murmu et al., 2013; Malekzadeh and Shojaee, 2013), cylindrical shells (Wang and Varadan, 2007; Wang and Wang, 2007; Hua et al., 2008), conical shells (Firouz-Abadi et al., 2011; Tsai and Fang, 2007; Liew et al., 2007), rings (Wang and

Duan, 2008; Moosavi et al., 2011) and particles (Ghavanloo and Fazelzadeh, 2013b), as well as carbon nanotubes (Fleck and Hutchinson, 1997; Zhou and Li, 2001; Sudak, 2003; Chen et al., 2004; Heireche et al., 2008; Murmu and Pradhan, 2009c; Narendar and Gopalakrishnan, 2011a; Ansari et al., 2013; Aydogdu, 2014; Ansari et al., 2014; Karami and Farid, 2013). A valuable implementation of nonlocal theories to wave propagation phenomena in nanostructures can be found in Gopalakrishnan and Narendar (2013).

On the other hand, the dynamics of the closed spherical shells (fluid-filled or empty) is a problem of technological importance in some modern industrial, biomedical, biological and other applications (Kessentini and Barchiesi, 2012). There exists an extensive bibliography dedicated to the analysis of buckling and vibrations of spherical shells from the point of view of the classical elasticity theory (Flugge, 1960; Baker, 1961; Kalnins, 1961; Prasad, 1964; Brush and Almroth, 1975; Soedel, 2005). The classical continuum framework has been also applied to the case of nanospheres (Kahn and Kim, 2001; Knoche and Kierfeld, 2011). In various modern biomedical and biological applications, spherical membrane structure (fluid-filled or empty) can be considered to model some micro/nanosized components, which are used as targeted drug delivery systems (Tachibana and Tachibana, 1999), biological cells (Wang and Wu, 2003), and some kind of viruses (Lidmar et al., 2003; Ru, 2009).

Concerning spherical shells, analyses that take into account size effects using nonlocal continuum theories are due to Ghavanloo and Fazelzadeh (2013a), and Zaera et al. (2013), who presented studies on the radial vibrations, and free axisymmetric vibrations, respectively, of a closed spherical shell. In these papers, the hypotheses of membrane behaviour were taken into account and then bending moments, shear efforts and radial normal stresses were neglected.

In this paper we present a detailed study of free axisymmetric vibrations of a closed spherical nanoshell using the Eringen nonlocal elasticity theory, considering not only membrane efforts, but also bending effects. The solution method proposed extends the procedure used by Zaera et al. (2013) for the thin shell case. The natural frequencies and modal shapes are obtained and the effects of the nonlocal parameter as well as the thickness of the shell are discussed.

General equations of Eringen elasticity theory

The fundamental field equations of isotropic nonlocal elasticity have the following form (Eringen, 1972b; Eringen and Edelen, 1972; Eringen, 2002)

$$\nabla \cdot \mathbf{t} + \mathbf{f} = \rho \ddot{\mathbf{u}} \tag{1}$$

$$\mathbf{t}(\mathbf{x}) = \int_{\Omega} \alpha \left(|\mathbf{x'} - \mathbf{x}|, \gamma\right) \boldsymbol{\sigma}(\mathbf{x'}) d\Omega(\mathbf{x'}) \tag{2}$$

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon} \tag{3}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla\left(\mathbf{u}^T\right)\right) \tag{4}$$

where $\mathbf{t(x)}$ and $\boldsymbol{\sigma}(\mathbf{x})$ are, respectively, the nonlocal and the classical local stress-tensors at any point $\mathbf{x}$ of the body. $\mathbf{C}$ is the fourth-order elasticity tensor, relating local stress-tensor to the infinitesimal strain tensor, $\boldsymbol{\varepsilon}$, and $\mathbf{u}$ is the displacement vector. In addition, $\mathbf{f}$ represents the external body forces vector, and ρ the mass density.

In Eq. (2) the kernel function $\alpha\left(|\mathbf{x'} - \mathbf{x}|, \gamma\right)$ represents the nonlocal modulus. Here, $|\mathbf{x'} - \mathbf{x}|$ is the Euclidean distance and γ is a material constant given by $\gamma = e_0 a/l$, that depends on internal and external characteristic lengths (a and l, respectively) through an adjusting constant e_0, dependent on each material. Alternatively, for a class of physically admissible kernel $\alpha\left(|\mathbf{x'} - \mathbf{x}|, \gamma\right)$, the nonlocal constitutive equations, Eq. (2), can be written in an equivalent differential form (Eringen, 1983) as

$$\mathbb{L}\mathbf{t} = \boldsymbol{\sigma} \tag{5}$$

where $\mathbb{L}$ is an operator given by

$$\mathbb{L} = \left(1 - \kappa^2 \nabla^2\right) \tag{6}$$

$\kappa = e_0 a$ being the length scale which takes into account the size effect on the response of nanostructures.

After using Eq. (5), the balance of linear momentum, Eq. (1), results in the following equation of motion

$$\nabla \cdot \boldsymbol{\sigma} + \mathbb{L}\mathbf{f} = \mathbb{L}\rho\ddot{\mathbf{u}} \tag{7}$$

Note that the displacement field of a nonlocal solid subjected to an external body force field $\mathbf{f}$ and an inertial body force $-\rho\ddot{\mathbf{u}}$ is the same as that of a classical solid subjected to the external force $\left(1 - \kappa^2\nabla^2\right)\mathbf{f}$ and an inertial body force $-\left(1 - \kappa^2\nabla^2\right)\rho\ddot{\mathbf{u}}$.

Then, the equations of motion can be obtained in terms of the displacements as

$$(\lambda_L + G)\,\nabla\,(\nabla\mathbf{u}) + G\nabla^2\mathbf{u} = \mathbb{L}\,(\rho\ddot{\mathbf{u}} - \mathbf{f}) \tag{8}$$

λ_L and G being the Lamé constants. The above relation constitutes the Navier equation of motion for Eringen nonlocal solids, which must be solved with the appropriate initial and boundary conditions applicable in each case.

Axisymmetric motion for the nonlocal spherical shell

The above theory will be used to study the axisymmetric free vibration of a nonlocal closed thin spherical shell.

Problem formulation

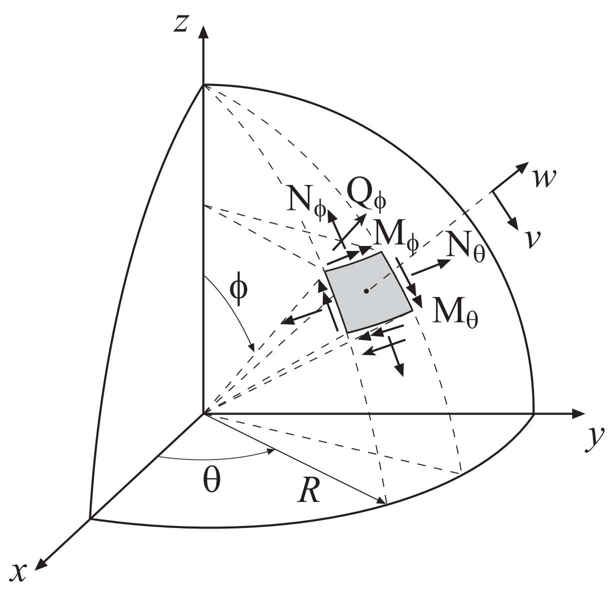

Fig. 1. Differential element of shell, internal forces and symmetric displacements in spherical coordinates.

In the formulation of the problem, the kinematic assumptions of the Kirchhoff-Love theory of plates have been considered:

- Straight lines normal to the mid-surface remain straight and normal to the mid-surface after deformation.

- The thickness of the plate does not change during deformation.
- Rotatory inertia is neglected in dynamic problems, which permits to establish a relation between shear and bending moments.

It is also assumed that the deformations are small enough for linear equations to adequately describe the motion. The sign convention for radial $w(\phi, t)$ and meridional $v(\phi, t)$ displacements, and for stress resultants and moments in meridional ($N_\phi(\phi, t)$, $M_\phi(\phi, t)$), circumferential ($N_\theta(\phi, t)$, $M_\theta(\phi, t)$), and out-of-plane ($Q_\phi(\phi, t)$) directions are defined in Fig. 1.

On behalf of simplicity in the notation, spatial and time dependence of w, v, N_ϕ, M_ϕ, N_θ, M_θ and Q_ϕ are not explicitly detailed from now on.

Soedel (2005) showed that conservation of linear momentum in radial direction, Eq. (9), meridional direction, Eq. (10), and conservation of angular momentum in circumferential direction, Eq. (11), lead to

$$-\frac{\partial \hat{Q}_\phi}{\partial \phi} - \cot\phi\ \hat{Q}_\phi - \hat{N}_\phi - \hat{N}_\theta = \rho R h \frac{\partial^2 w}{\partial t^2} \tag{9}$$

$$\frac{\partial \hat{N}_\phi}{\partial \phi} + \cot\phi \left(\hat{N}_\phi - \hat{N}_\theta\right) - \hat{Q}_\phi = \rho R h \frac{\partial^2 v}{\partial t^2} \tag{10}$$

$$\frac{\partial \hat{M}_\phi}{\partial \phi} + \cot\phi \left(\hat{M}_\phi - \hat{M}_\theta\right) - R\hat{Q}_\phi = 0 \tag{11}$$

where $\hat{Q}_\phi$, $\hat{N}_\phi$, $\hat{N}_\theta$, $\hat{M}_\phi$ and $\hat{M}_\theta$ are the stress resultants in the equivalent local problem. R and ρ are the sphere radius and mass density respectively. Eliminating $\hat{Q}_\phi$ from the first two equations by the use of the third one, two equations containing $\hat{N}_\phi$, $\hat{M}_\phi$, $\hat{N}_\theta$ and $\hat{M}_\theta$ are obtained. All these quantities can be expressed in terms of the displacements w and v, by using the following relations (Timoshenko, 1959)

$$\varepsilon_\phi = \frac{1}{R}\left(\frac{\partial v}{\partial \phi} + w\right), \quad \varepsilon_\theta = \frac{1}{R}\left(\cot\phi\ v + w\right) \tag{12}$$

$$\chi_\phi = -\frac{1}{R^2}\left(\frac{\partial v}{\partial \phi} - \frac{\partial^2 w}{\partial \phi^2}\right), \quad \chi_\theta = -\frac{\cot\phi}{R^2}\left(v - \frac{\partial w}{\partial \phi}\right) \tag{13}$$

where ε_ϕ, ε_θ are membrane strains in meridional and circumferential directions respectively, and χ_ϕ, χ_θ are changes of curvature of the meridian and in the plane perpendicular to the meridian.

Hence, forces and moments are expressed as a function of displacements

$$\hat{N}_\phi = \frac{D_N}{R}\left(\frac{\partial v}{\partial \phi} + w + \nu\left(\cot\phi\ v + w\right)\right) \tag{14}$$

$$\hat{N}_\theta = \frac{D_N}{R}\left(\cot\phi\ v + w + \nu\left(\frac{\partial v}{\partial \phi} + w\right)\right) \tag{15}$$

$$\hat{M}_\phi = -\frac{D_M}{R^2}\left(\frac{\partial v}{\partial \phi} - \frac{\partial^2 w}{\partial \phi^2} + \nu\cot\phi\left(v - \frac{\partial w}{\partial \phi}\right)\right) \tag{16}$$

$$\hat{M}_\theta = -\frac{D_M}{R^2}\left(\cot\phi\left(v - \frac{\partial w}{\partial \phi}\right) + \nu\left(\frac{\partial v}{\partial \phi} - \frac{\partial^2 w}{\partial \phi^2}\right)\right) \tag{17}$$

with

$$D_N = \frac{Eh}{(1-\nu^2)}\ ; \quad D_M = \frac{Eh^3}{12\,(1-\nu^2)} \tag{18}$$

E and ν are the Young modulus and Poisson ratio respectively.

The governing equations of motion for the nonlocal spherical shell can be directly derived from the classical formulation for local elasticity, as a result of the aforementioned analogy between the nonlocal solid subjected to a body force $\mathbf{f} - \rho\ddot{\mathbf{u}}$ and the equivalent local solid subjected to a body force $\left(1-\kappa^2\nabla^2\right)(\mathbf{f}-\rho\ddot{\mathbf{u}})$. Neglecting the external body force $\mathbf{f}$, in order to consider free vibrations, and using the proper differential operators in spherical coordinates, the nonlocal counterparts of Eqs. (9) and (10) may be expressed respectively as

$$\begin{aligned}&\frac{D_N}{R}(1+\nu)\left(-\frac{\partial v}{\partial\phi}-\cot\phi\, v-2w\right)+\frac{D_N}{R}\alpha\left[\frac{\partial^3 v}{\partial\phi^3}+2\cot\phi\frac{\partial^2 v}{\partial\phi^2}-\left(1+\nu+\cot^2\phi\right)\frac{\partial v}{\partial\phi}\right]+\\&\frac{D_N}{R}\alpha\left[\cot\phi\left(2-\nu+\cot^2\phi\right)v-\frac{\partial^4 w}{\partial\phi^4}-2\cot\phi\frac{\partial^3 w}{\partial\phi^3}+\left(1+\nu+\cot^2\phi\right)\frac{\partial^2 w}{\partial\phi^2}-\cot\phi\left(2-\nu+\cot^2\phi\right)\frac{\partial w}{\partial\phi}\right]=\\&\sqrt{12\alpha}R^2\rho\left[\frac{\partial^2 w}{\partial t^2}-\mu^2\left(\frac{\partial^4 w}{\partial t^2\partial\phi^2}+\cot\phi\frac{\partial^3 w}{\partial t^2\partial\phi}-2\frac{\partial^2 w}{\partial t^2}-2\frac{\partial^3 v}{\partial t^2\partial\phi}-2\cot\phi\frac{\partial^2 v}{\partial t^2}\right)\right]\end{aligned} \tag{19}$$

$$\begin{aligned}&\frac{D_N}{R}(1+\alpha)\left[\frac{\partial^2 v}{\partial\phi^2}+\cot\phi\frac{\partial v}{\partial\phi}-\left(\nu+\cot^2\phi\right)v\right]+\\&\frac{D_N}{R}(1+\nu)\frac{\partial w}{\partial\phi}-\frac{D_N}{R}\alpha\left[\frac{\partial^3 w}{\partial\phi^3}+\cot\phi\frac{\partial^2 w}{\partial\phi^2}-\left(\nu+\cot^2\phi\right)\frac{\partial w}{\partial\phi}\right]=\\&\sqrt{12\alpha}R^2\rho\left[\frac{\partial^2 v}{\partial t^2}-\mu^2\left(\frac{\partial^4 v}{\partial t^2\partial\phi^2}+\frac{\partial^3 v}{\partial t^2\partial\phi}\cot\phi-\frac{\partial^2 v}{\partial t^2}\frac{1}{\sin^2\phi}+2\frac{\partial^3 w}{\partial t^2\partial\phi}\right)\right]\end{aligned} \tag{20}$$

where the following nondimensional parameters have been used

$$\alpha = \frac{1}{12}\left(\frac{h}{R}\right)^2\ ; \quad \mu = \frac{\kappa}{R} \tag{21}$$

α is a parameter related to the ratio between the thickness and the radius of the sphere while μ is the nonlocal parameter that takes into account size effects.

Considering the following nondimensional quantities

$$\overline{v} = \frac{v}{R}\ ; \quad \overline{w} = \frac{w}{R}\ ; \quad \tau = \frac{c_s t}{R}\ ; \quad c_s = \sqrt{\frac{E}{\rho\,(1-\nu^2)}} \tag{22}$$

the equations of motion may be finally written in terms of dimensionless displacements as

$$\begin{aligned}&(1+\nu)\left(-\frac{\partial\overline{v}}{\partial\phi}-\cot\phi\,\overline{v}-2\overline{w}\right)+\alpha\left[\frac{\partial^3\overline{v}}{\partial\phi^3}+2\cot\phi\frac{\partial^2\overline{v}}{\partial\phi^2}-\left(1+\nu+\cot^2\phi\right)\frac{\partial\overline{v}}{\partial\phi}+\cot\phi\left(2-\nu+\cot^2\phi\right)\overline{v}\right]+\\&\alpha\left[-\frac{\partial^4\overline{w}}{\partial\phi^4}-2\cot\phi\frac{\partial^3\overline{w}}{\partial\phi^3}+\left(1+\nu+\cot^2\phi\right)\frac{\partial^2\overline{w}}{\partial\phi^2}-\cot\phi\left(2-\nu+\cot^2\phi\right)\frac{\partial\overline{w}}{\partial\phi}\right]\\&-\frac{\partial^2\overline{w}}{\partial\tau^2}+\mu^2\left(\frac{\partial^4\overline{w}}{\partial\tau^2\partial\phi^2}+\cot\phi\frac{\partial^3\overline{w}}{\partial\tau^2\partial\phi}-2\frac{\partial^2\overline{w}}{\partial\tau^2}-2\frac{\partial^3\overline{v}}{\partial\tau^2\partial\phi}-2\cot\phi\frac{\partial^2\overline{v}}{\partial\tau^2}\right)=0\end{aligned} \tag{23}$$

$$(1+\alpha)\left[\frac{\partial^2\overline{v}}{\partial\phi^2}+\cot\phi\frac{\partial\overline{v}}{\partial\phi}-\left(\nu+\cot^2\phi\right)\overline{v}\right]+(1+\nu)\frac{\partial\overline{w}}{\partial\phi}-\alpha\left[\frac{\partial^3\overline{w}}{\partial\phi^3}+\cot\phi\frac{\partial^2\overline{w}}{\partial\phi^2}-\left(\nu+\cot^2\phi\right)\frac{\partial\overline{w}}{\partial\phi}\right]$$
$$-\frac{\partial^2\overline{v}}{\partial\tau^2}+\mu^2\left(\frac{\partial^4\overline{v}}{\partial\tau^2\partial\phi^2}+\frac{\partial^3\overline{v}}{\partial\tau^2\partial\phi}\cot\phi-\frac{\partial^2\overline{v}}{\partial\tau^2}\frac{1}{\sin^2\phi}+2\frac{\partial^3\overline{w}}{\partial\tau^2\partial\phi}\right)=0 \quad (24)$$

These equations lead to the local formulation if the nonlocal parameter μ is set to zero. Similarly, if the ratio α between shell thickness and shell radius is set to zero, the nonlocal formulation considering exclusively membrane forces obtained previously by Zaera et al. (2013) is recovered.

Following Baker (1961), it is assumed that the shell is initially at rest in a deformed shape, according to the initial conditions

$$\overline{w}(\phi,0)=f(\phi)\ ;\quad \left.\frac{\partial\overline{w}}{\partial\tau}\right|_{\tau=0}=0 \quad (25)$$

$$\overline{v}(\phi,0)=g(\phi)\ ;\quad \left.\frac{\partial\overline{v}}{\partial\tau}\right|_{\tau=0}=0 \quad (26)$$

Eqs. (23, 24), and initial conditions (25, 26) determine the free vibrational movement of the nonlocal spherical shell provided that spatial functions are single valued over the whole domain.

Solving method

The differential equations to be solved are linear and homogeneous. Thus, in order to obtain the natural frequencies and modal shapes of the vibration of the shell, the method of separation of variables will be used. Let us assume

$$\overline{w}(\phi,\tau)=W(\phi)\,S(\tau) \quad (27)$$

$$\overline{v}(\phi,\tau)=V(\phi)\,Q(\tau) \quad (28)$$

For simplicity in notation, spatial or time dependence of variables W, V, S and Q are not specified in each equation, as it was done before. Then, Eqs. (23) and (24) become

$$\Gamma_{11}S+\Gamma_{12}Q=\Gamma_{13}\frac{d^2S}{d\tau^2}+\Gamma_{14}\frac{d^2Q}{d\tau^2} \quad (29)$$

$$\Gamma_{21}S+\Gamma_{22}Q=\Gamma_{23}\frac{d^2S}{d\tau^2}+\Gamma_{24}\frac{d^2Q}{d\tau^2} \quad (30)$$

with

$$\Gamma_{11}=-2(1+\nu)W+\alpha\left[-W^{(4)}-2\cot\phi W'''+\left(\nu+\csc^2\phi\right)W''-\cot\phi\left(1-\nu+\csc^2\phi\right)W'\right] \quad (31)$$

$$\Gamma_{12}=-\cot\phi\left[1+\nu-\alpha\left(1-\nu+\csc^2\phi\right)\right]V+\alpha\left(V'''+2\cot\phi V''\right)-\left[1+\nu+\alpha\left(\nu+\csc^2\phi\right)\right]V' \quad (32)$$

$$\Gamma_{13}=W-\mu^2\left(W''+\cot\phi W'-2W\right) \quad (33)$$

$$\Gamma_{14}=2\mu^2\left(\cot\phi V+V'\right) \quad (34)$$

$$\Gamma_{21}=-\alpha\left(W'''+\cot\phi W''\right)+\left[1+\nu-\alpha\left(1-\nu-\csc^2\phi\right)\right]W' \quad (35)$$

$$\Gamma_{22}=(1+\alpha)\left[\left(1-\nu-\csc^2\phi\right)V+V''+\cot\phi V'\right] \quad (36)$$

$$\Gamma_{23} = -2\mu^2 W' \tag{37}$$

$$\Gamma_{24} = V - \mu^2\left(V'' + \cot\phi V' - \csc^2\phi V\right) \tag{38}$$

Dividing Eq. (29) by Γ_{13} and Eq. (30) by Γ_{24} we get

$$\lambda_1 S + \lambda_2 Q = \frac{d^2 S}{d\tau^2} + \lambda_3 \frac{d^2 Q}{d\tau^2} \tag{39}$$

$$\lambda_4 S + \lambda_5 Q = \lambda_6 \frac{d^2 S}{d\tau^2} + \frac{d^2 Q}{d\tau^2} \tag{40}$$

with

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{\Gamma_{11}, \Gamma_{12}, \Gamma_{14}\}/\Gamma_{13} \tag{41}$$

$$\{\lambda_4, \lambda_5, \lambda_6\} = \{\Gamma_{21}, \Gamma_{22}, \Gamma_{23}\}/\Gamma_{24} \tag{42}$$

For Eqs. (39) and (40) to be separable, λ_i must be constant. The constants λ_1 to λ_6 and the functions W and V are now determined.

Determination of λ_i and functions W and V

For convenience, $x = \cos\phi$ is used in Eqs. (41) and (42).

- V and λ_5: The equation $\lambda_5 = \Gamma_{22}/\Gamma_{24}$ leads to the general Legendre differential equation

$$\left(1 - x^2\right)\frac{d^2 V}{dx^2} - 2x\frac{dV}{dx} + \left(n(n+1) - \frac{1}{1 - x^2}\right)V = 0 \tag{43}$$

  with

$$n(n+1) = \frac{(1+\alpha)(1-\nu) - \lambda_5}{1 + \alpha + \mu^2\lambda_5} \tag{44}$$

  which is satisfied by the Associated Legendre Functions. For meridional displacements V being finite and single-valued over the entire sphere, n should be an integer greater than zero, and the solution for V is given by the Associated Legendre Polynomials of the first kind (Condon-Shortley phase omitted, see Arfken et al. (2012))

$$V_n(x) = P_n^1(x) = \frac{\left(1 - x^2\right)^{1/2}}{2^n n!}\frac{d^{n+1}\left(x^2 - 1\right)^n}{dx^{n+1}}; \quad n = 1, 2, 3, ... \tag{45}$$

  Each solution V_n corresponds to the following value of λ_{5n}

$$\lambda_{5n} = \frac{(1+\alpha)(1 - \nu - n(n+1))}{1 + \mu^2 n(n+1)} \tag{46}$$

- W and λ_3: $V = P_n^1$ is substituted in equation $\lambda_3 = \Gamma_{14}/\Gamma_{13}$, leading to

$$\left(1 - x^2\right)\frac{d^2 W}{dx^2} - 2x\frac{dW}{dx} - \frac{\left(1 + 2\mu^2\right)}{\mu^2}W = \frac{2\left(1 - x^2\right)}{\lambda_3}\frac{d^2 P_n}{dx^2} - \frac{4x}{\lambda_3}\frac{dP_n}{dx} \tag{47}$$

where P_n are the Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n\left(x^2-1\right)^n}{dx^n}; \quad n = 0,1,2,3,... \tag{48}$$

and the following property of Legendre Polynomials has been used

$$P_n^1(\phi) = -P_n'(\phi) \tag{49}$$

A bounded solution of the corresponding homogeneous equation (Legendre equation) should satisfy the condition $-\left(1+2\mu^2\right)/\mu^2 = n(n+1)$ with n integer equal or greater than zero, which is not possible for real values of the nonlocal parameter μ. Therefore, only the trivial solution $W = 0$ is suitable for the homogeneous equation, and a particular solution is valid as general one for the complete equation. It may be easily verified that the Legendre polynomials P_n constitute a particular solution to the complete equation

$$\left(1-x^2\right)\frac{d^2P_n}{dx^2} - 2x\frac{dP_n}{dx} - \frac{\left(1+2\mu^2\right)\lambda_{3n}}{\mu^2\left(\lambda_3-2\right)}P_n = 0 \tag{50}$$

if

$$\lambda_{3n} = \frac{2\mu^2 n(n+1)}{1+\mu^2\left(2+n(n+1)\right)} \tag{51}$$

Therefore, displacement W is satisfied by the Legendre Polynomials

$$W_n(x) = P_n(x) \tag{52}$$

- λ_1: Eq. $\lambda_1 = \Gamma_{11}/\Gamma_{13}$ may be written as

$$\mathcal{S}_1\left(\mathcal{L}\left(W_n\right)\right) = 0 \tag{53}$$

where $\mathcal{L}$ is the Legendre operator

$$\mathcal{L} = \left(1-x^2\right)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + n(n+1) \tag{54}$$

and $\mathcal{S}_1$ is the following operator

$$\mathcal{S}_1 = \left(1-x^2\right)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + \frac{m_1}{n(n+1)} \tag{55}$$

In order to express $\lambda_1 = \Gamma_{11}/\Gamma_{13}$ as Eq. (53), m_1 and n shall verify the following equations

$$m_1 = \frac{2(1+\nu) + \left(1+2\mu^2\right)\lambda_1}{\alpha} \tag{56}$$

and

$$n(n+1) = \frac{1}{2}\left(1 + k_1 \pm \sqrt{(1+k_1)^2 - 4m_1}\right) \tag{57}$$

with

$$k_1 = \frac{-\alpha\nu - \mu^2\lambda_1}{\alpha} \tag{58}$$

Eq. (53) is satisfied by Legendre Polynomials if n is an integer equal or greater than zero, leading to the following values for λ_1

$$\lambda_{1n} = -\frac{2(1+\nu) + \alpha n(n+1)(-1+\nu+n(n+1))}{1+\mu^2(2+n(n+1))} \tag{59}$$

where λ_{1n} satisfies either the positive or the negative sign in Eq. (57), depending on the value of α, ν, μ and n.

- λ_2: considering Eq. (49), the relation $V_n(\phi) = -W'_n(\phi)$ and successive derivatives may be used to eliminate V_n, V'_n, V''_n and V'''_n in Eq. $\lambda_2 = \Gamma_{12}/\Gamma_{13}$, leading to

$$\mathcal{S}_2(\mathcal{L}(W_n)) = 0 \tag{60}$$

where $\mathcal{S}_2$ is the following operator

$$\mathcal{S}_2 = \left(1-x^2\right)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + \frac{m_2}{n(n+1)} \tag{61}$$

with

$$m_2 = \frac{\left(1+2\mu^2\right)\lambda_2}{\alpha} \tag{62}$$

$$n(n+1) = \frac{1}{2}\left(1+k_2 \pm \sqrt{(1+k_2)^2 - 4m_2}\right) \tag{63}$$

and

$$k_2 = \frac{-1-\nu(1+\alpha)-\mu^2\lambda_2}{\alpha} \tag{64}$$

As for λ_1, Eq. (60) is satisfied by Legendre Polynomials if n is an integer equal or greater than zero, leading to the following values for λ_2

$$\lambda_{2n} = -\frac{n(n+1)(1+\nu+\alpha(-1+\nu+n(n+1)))}{1+\mu^2(2+n(n+1))} \tag{65}$$

- λ_6: now using the relation $V_n(\phi) = -W'_n(\phi)$ in $\lambda_6 = \Gamma_{23}/\Gamma_{24}$ to eliminate W'_n, the general Legendre differential equation is obtained again.

$$\left(1-x^2\right)\frac{d^2V_n}{dx^2} - 2x\frac{dV_n}{dx} + \left(-\frac{1}{\mu^2} + \frac{2}{\lambda_6} - \frac{1}{1-x^2}\right)V_n = 0 \tag{66}$$

For V_n to be a finite and single-valued solution of this equation, the following λ_6 values are required

$$\lambda_{6n} = \frac{2\mu^2}{1+\mu^2 n(n+1)} \tag{67}$$

- λ_4: to obtain λ_4 from Eq. $\lambda_4 = \Gamma_{21}/\Gamma_{24}$, the relation $V_n(\phi) = -W'_n(\phi)$ and successive derivatives is used again to eliminate W'_n, W''_n and W'''_n, leading to the general Legendre differential equation in V_n. From this equation, the following condition is derived

$$\lambda_{4n} = \frac{-1 - \nu + \alpha\,(1 - \nu - n\,(n+1))}{1 + \mu^2 n\,(n+1)} \tag{68}$$

Solution for the time-dependent functions S_n and Q_n

By suitable combinations of Eqs. (39) and (40), they can be transformed into two uncoupled biquadratic differential equations

$$\frac{d^4 S_n}{d\tau^4} + \xi_n \frac{d^2 S_n}{d\tau^2} + \gamma_n S_n = 0 \tag{69}$$

$$\frac{d^4 Q_n}{d\tau^4} + \xi_n \frac{d^2 Q_n}{d\tau^2} + \gamma_n Q_n = 0 \tag{70}$$

Note that both equations have the same coefficients

$$\xi_n = \frac{-\lambda_{1n} + \lambda_{3n}\lambda_{4n} - \lambda_{5n} + \lambda_{2n}\lambda_{6n}}{1 - \lambda_{3n}\lambda_{6n}} \tag{71}$$

$$\gamma_n = \frac{-\lambda_{2n}\lambda_{4n} + \lambda_{1n}\lambda_{5n}}{1 - \lambda_{3n}\lambda_{6n}} \tag{72}$$

The solution of these linear differential equations can be written as

$$S_n(\tau) = A_n^S \cos(a_n\tau) + B_n^S \sin(a_n\tau) + C_n^S \cos(b_n\tau) + D_n^S \sin(b_n\tau) \tag{73}$$

$$Q_n(\tau) = A_n^Q \cos(a_n\tau) + B_n^Q \sin(a_n\tau) + C_n^Q \cos(b_n\tau) + D_n^Q \sin(b_n\tau) \tag{74}$$

where the natural frequencies are roots of the corresponding characteristic equation of the differential problem given by Eqs. (69) and (70)

$$a_n = \left(\frac{\xi_n + \sqrt{\xi_n^2 - 4\gamma_n}}{2}\right)^{1/2} \tag{75}$$

$$b_n = \left(\frac{\xi_n - \sqrt{\xi_n^2 - 4\gamma_n}}{2}\right)^{1/2} \tag{76}$$

As in the local elasticity case, there are two natural frequencies for each value of n: the upper branch a_n and the lower branch b_n. The 8 unknown amplitudes in Eqs. (73) and (74) are obtained, applying the corresponding initial conditions, in terms of the initial values of the functions S_n and Q_n and of their first, second and third derivatives

$$S_n(0) = S_n^0\,; \quad Q_n(0) = Q_n^0 \tag{77}$$

$$\left.\frac{dS_n}{d\tau}\right|_{\tau=0} = \left.\frac{dQ_n}{d\tau}\right|_{\tau=0} = 0 \tag{78}$$

$$\left.\frac{d^2 Q_n}{d\tau^2}\right|_{\tau=0} = \ddot{Q}_n^0 = p_n S_n^0 + q_n Q_n^0 \tag{79}$$

$$\left.\frac{d^2 S_n}{d\tau^2}\right|_{\tau=0} = \ddot{S}_n^0 = r_n S_n^0 + s_n Q_n^0 \tag{80}$$

and

$$\left.\frac{d^3S_n}{d\tau^3}\right|_{\tau=0} = \left.\frac{d^3Q_n}{d\tau^3}\right|_{\tau=0} = 0 \tag{81}$$

with

$$p_n = \frac{\lambda_{4n} - \lambda_{1n}\lambda_{6n}}{1 - \lambda_{3n}\lambda_{6n}} \tag{82}$$

$$q_n = \frac{\lambda_{5n} - \lambda_{2n}\lambda_{6n}}{1 - \lambda_{3n}\lambda_{6n}} \tag{83}$$

$$r_n = \frac{\lambda_{1n} - \lambda_{3n}\lambda_{4n}}{1 - \lambda_{3n}\lambda_{6n}} \tag{84}$$

$$s_n = \frac{\lambda_{2n} - \lambda_{3n}\lambda_{5n}}{1 - \lambda_{3n}\lambda_{6n}} \tag{85}$$

Initial conditions (77) and (78) correspond to Eqs. (25) and (26). Initial conditions (79) and (80) are obtained combining Eqs. (39) and (40), whereas initial conditions (81) are obtained by derivation of Eqs. (39) and (40). The solution of the algebraic system of equations given by the eight initial conditions, Eqs. (77) to (81), leads to

$$B_n^S = D_n^S = B_n^Q = D_n^Q = 0 \tag{86}$$

$$A_n^S = S_n^0 \, \frac{\ddot{S}_n^0/S_n^0 + b_n^2}{b_n^2 - a_n^2} \tag{87}$$

$$C_n^S = S_n^0 \, \frac{\ddot{S}_n^0/S_n^0 + a_n^2}{a_n^2 - b_n^2} \tag{88}$$

$$A_n^Q = Q_n^0 \, \frac{\ddot{Q}_n^0/Q_n^0 + b_n^2}{b_n^2 - a_n^2} \tag{89}$$

$$C_n^Q = Q_n^0 \, \frac{\ddot{Q}_n^0/Q_n^0 + a_n^2}{a_n^2 - b_n^2} \tag{90}$$

Taking into account conditions (79) and (80), and the relation $a_n^2 + b_n^2 = \xi_n$, the final expression for the time-dependent functions are

$$S_n(\tau) = S_n^0 \left[\frac{\beta_n - a_n^2}{b_n^2 - a_n^2} \cos(a_n\tau) + \frac{\beta_n - b_n^2}{a_n^2 - b_n^2} \cos(b_n\tau) \right] \tag{91}$$

$$Q_n(\tau) = Q_n^0 \left[\frac{\alpha_n - a_n^2}{b_n^2 - a_n^2} \cos(a_n\tau) + \frac{\alpha_n - b_n^2}{a_n^2 - b_n^2} \cos(b_n\tau) \right] \tag{92}$$

where

$$\beta_n = \frac{\ddot{S}_n^0}{S_n^0} + \xi_n \tag{93}$$

$$\alpha_n = \frac{\ddot{Q}_n^0}{Q_n^0} + \xi_n \tag{94}$$

Modal shapes

Modal shapes corresponding to upper branch frequencies a_n may be found by setting to zero the amplitudes corresponding to the lower branch frequency b_n in Eqs. (91) and (92)

$$\beta_n - {b_n}^2 = 0\ ; \quad \alpha_n - {b_n}^2 = 0 \tag{95}$$

therefore

$$\beta_n = \alpha_n \tag{96}$$

or

$$\frac{\ddot{S}_n^0}{S_n^0} = \frac{\ddot{Q}_n^0}{Q_n^0} \tag{97}$$

Combining the last condition with Eqs. (79) and (80) we get

$$p_n \left(\frac{S_n^0}{Q_n^0}\right)^2 + (q_n - r_n)\left(\frac{S_n^0}{Q_n^0}\right) - s_n = 0 \tag{98}$$

and solving this quadratic equation

$$\Psi_{a_n} = \left.\frac{S_n^0}{Q_n^0}\right|_{a_n} = \frac{r_n - q_n - \sqrt{(q_n - r_n)^2 + 4p_n s_n}}{2p_n} \tag{99}$$

where the negative sign before the radical is chosen to satisfy either of conditions (95). Finally, the temporal function corresponding to the frequency a_n is given by

$$S_n^{a_n}(\tau) = \left.S_n^0\right|_{a_n} \cos(a_n\tau)\ ; \quad Q_n^{a_n}(\tau) = \left.Q_n^0\right|_{a_n} \cos(a_n\tau) \tag{100}$$

Likewise, modal shapes corresponding to lower frequencies b_n are obtained by setting to zero the amplitudes corresponding to the upper branch frequencies in Eqs. (91) and (92)

$$\beta_n - {a_n}^2 = 0\ ; \quad \alpha_n - {a_n}^2 = 0 \tag{101}$$

leading to

$$\Psi_{b_n} = \left.\frac{S_n^0}{Q_n^0}\right|_{b_n} = \frac{r_n - q_n + \sqrt{(q_n - r_n)^2 + 4p_n s_n}}{2p_n} \tag{102}$$

where the positive sign before the radical is chosen to satisfy either of conditions (101). Therefore, the temporal function corresponding to the frequency b_n is given by

$$S_n^{b_n}(\tau) = \left.S_n^0\right|_{b_n} \cos(b_n\tau)\ ; \quad Q_n^{b_n}(\tau) = \left.Q_n^0\right|_{b_n} \cos(b_n\tau) \tag{103}$$

As expected, the solution of Soedel (2005) is recovered for $\mu = 0$, which implies absence of nonlocal effects. If $\alpha = 0$ is consider instead, the expressions of the nonlocal spherical membrane free vibrations (Zaera et al., 2013) are obtained.

Coupled influence of bending stiffness and nonlocal parameters

Frequencies

Eqs. (75) and (76) provide the natural frequencies of the free vibration of the shell. As in local elasticity analysis considering only membrane forces (Baker, 1961), b_0 leads to a spurious mode, and b_1 equals zero for every value of the nonlocal parameter thus forecasting a translational mode. Fig. 2 shows the upper branch frequencies of vibration for the first natural modes up to $n = 15$ and for different values of the thickness-to-radius ratio α and nonlocal parameter μ. Fig. 3 shows lower branch natural frequencies up to $n = 33$.

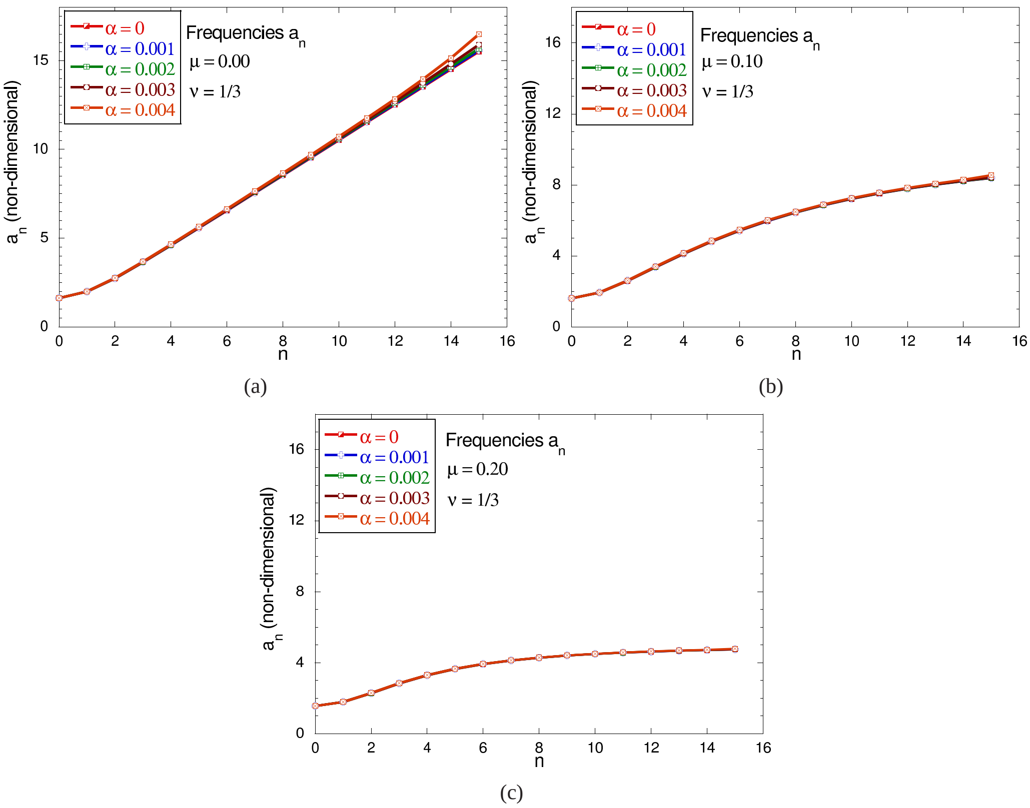

Fig. 2. Natural frequencies of the upper branch for different values of α and μ ($\nu = 1/3$). (a) $\mu = 0$. (b) $\mu = 0.10$. (c) $\mu = 0.20$.

It can be seen that natural frequencies visibly decrease for increasing values of the nonlocal parameter μ. Also, the frequencies of both upper and lower branches increase with increasing values of α. Nevertheless, while upper branch a_n is slightly affected by α, even small variations of α make significant changes in the lower branch frequencies b_n. To highlight the influence of α over b_n, the following analysis has been made. The local case setting $\mu = 0$ will be firstly considered. In absence of flexural stiffness ($\alpha = 0$), the non-dimensional frequency b_n tends to 1 for high values of n, as Baker (1961) showed. But in presence of α different than 0, even for very small values, lower frequency b_n tends asymptotically to $b_n = n + 1/2$ for high values of n. This is shown in Fig. 3 (a).

Now, Figs. 3 (b) and 3 (c) show the frequencies of the lower branch including nonlocal effects. Analysing this results, it is revealed that for $\alpha = 0$ (only membrane forces) and any value of μ different than 0, the lower natural frequency b_n tends to 0 for high modes (high values of n), instead of 1. When bending forces are also considered (coupled effect), the curves of non-dimensional natural frequencies b_n become increasing to tend asymptotically to $b_n = 1/\mu$ for higher values of n.

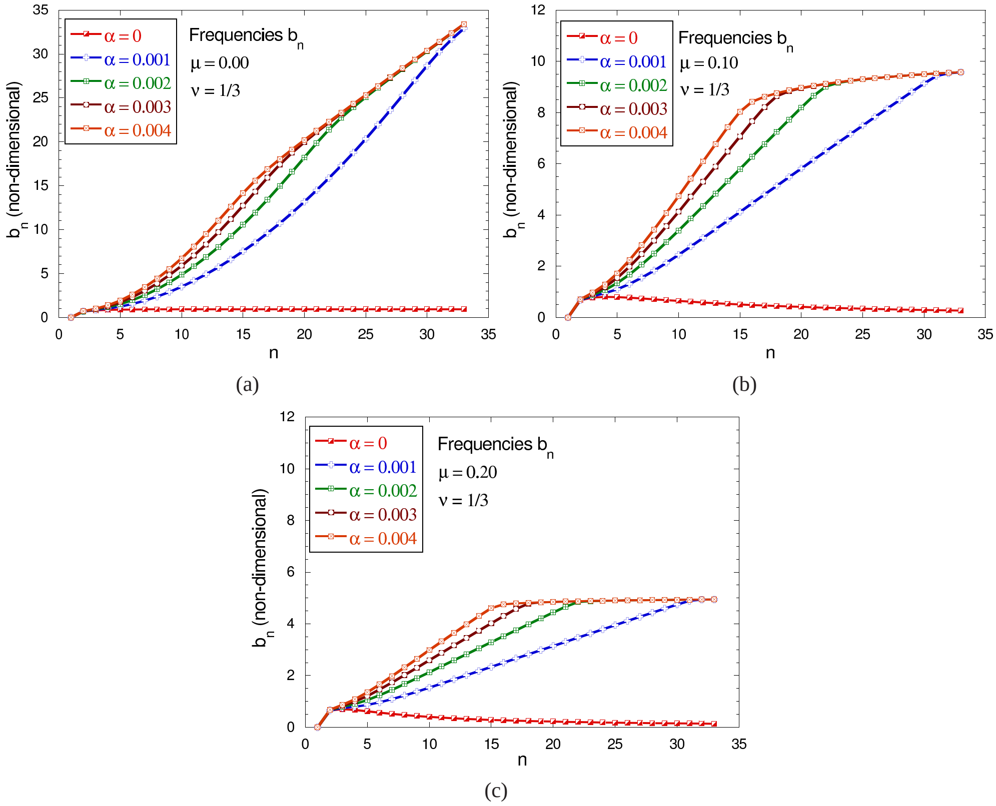

Fig. 3. Natural frequencies of the lower branch for different values of α and μ ($\nu = 1/3$). (a) $\mu = 0$. (b) $\mu = 0.10$. (c) $\mu = 0.20$.

Modal shapes

To illustrate the deformed shape of the vibrating shell, a meridional section of the sphere is plotted. In Figs. 4 and 5, $n = 2$ and $n = 5$ modes of vibration are shown.

In order to evaluate the influence of the non-local and the thickness-to-radius parameters on the radial to meridional displacement ratio, the factor Ψ_{a_n} (upper branch) is plotted in Fig. 6 and the factor Ψ_{b_n} (lower branch) is plotted in Fig. 7.

Due to the physical meaning of Ψ_{a_n} and Ψ_{b_n} ratios, they are going to be analysed in absolute value. It can be observed that for each value of n, the ratio Ψ_{a_n} increases as α increases. Also, the consideration of bending stiffness ($\alpha \neq 0$) induces the appearance of an absolute minimum in Ψ_{a_n} at a value n that depends on both α and μ. However, for each n, Ψ_{b_n} decreases in absolute value as α increases. In this case, bending stiffness induces the appearance of an absolute maximum in $|\Psi_{b_n}|$ at a value n that depends on both α and μ.

Analogous to the previous study of natural frequency b_n, an asymptotic analysis of Ψ_{b_n} has been done, showing that for any value of $\alpha \neq 0$ the ratio Ψ_{b_n} tends asymptotically to -1 for higher values of n.

In absence of flexural stiffness, Ψ_{b_n} is not influenced by nonlocal effects, meanwhile it is actually influenced by μ when considering $\alpha \neq 0$. It is worth to point out that this behaviour is a consequence of the coupled effect of μ and α over the ratio Ψ_{b_n}.

Discussion

The influences of the non-local parameter μ and the flexural stiffness parameter α on the vibration modes were shown in Figs. 2 to 5. A physical insight into these trends is presented below.

The natural frequencies of both lower and upper branches increase with α as a result of the higher value of the bending inertia. Here the thickness plays a role in the flexural stiffness of the shell, thus decreasing the period of vibrational modes.

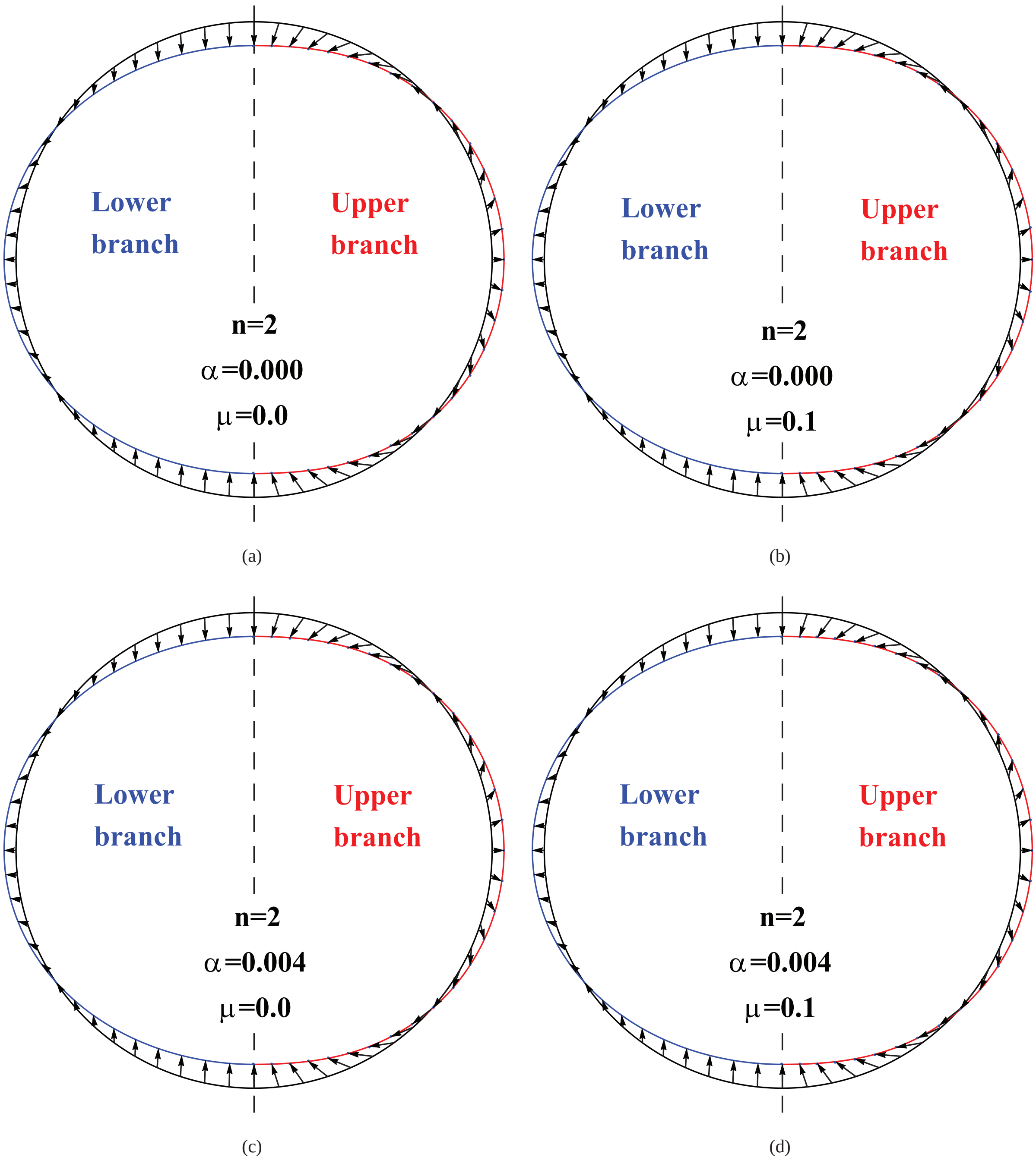

Fig. 4. Meridional section of modal shape for $n = 2$ and different values of μ and α.

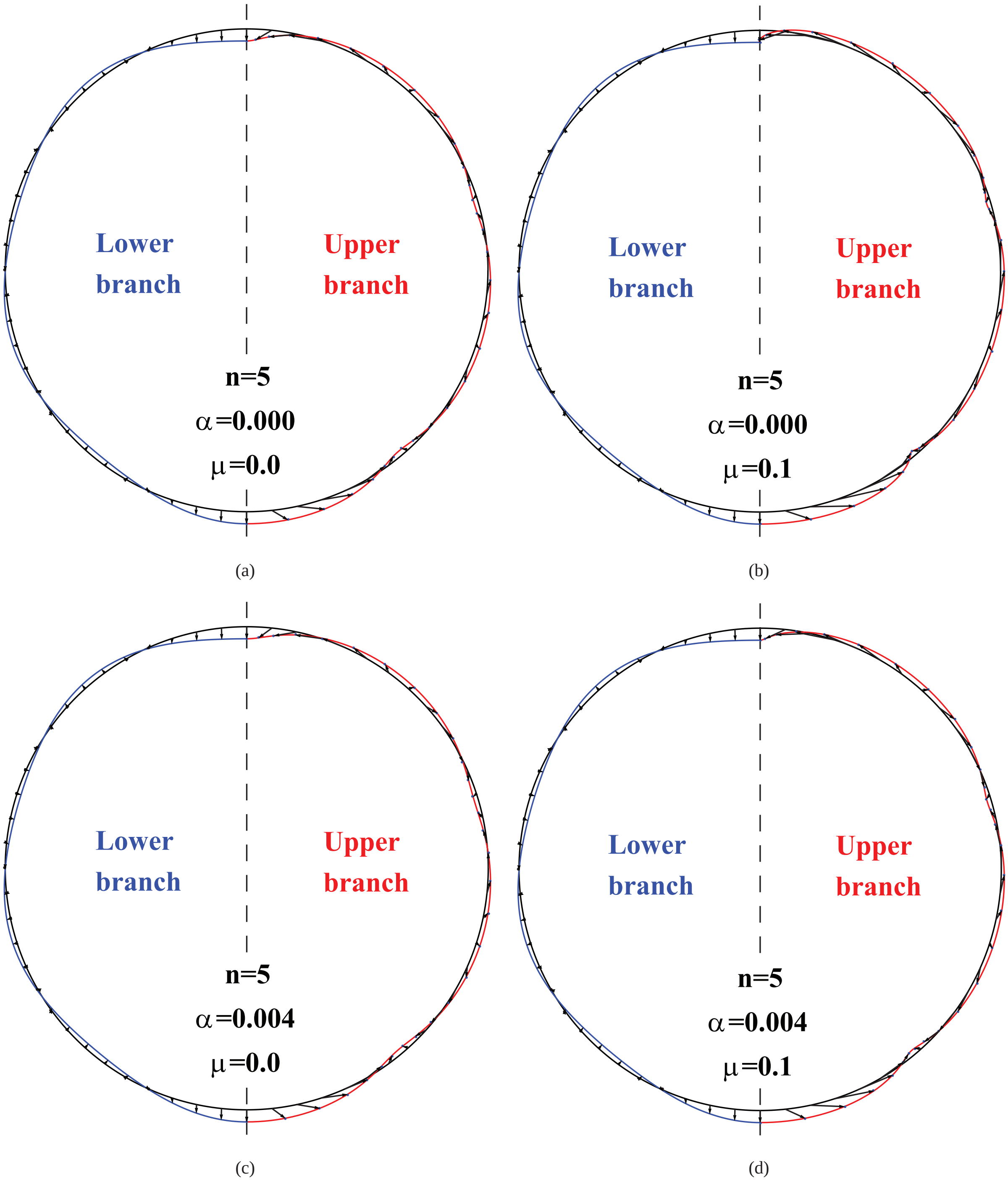

Fig. 5. Meridional section of modal shape for $n = 5$ and different values of μ and α.

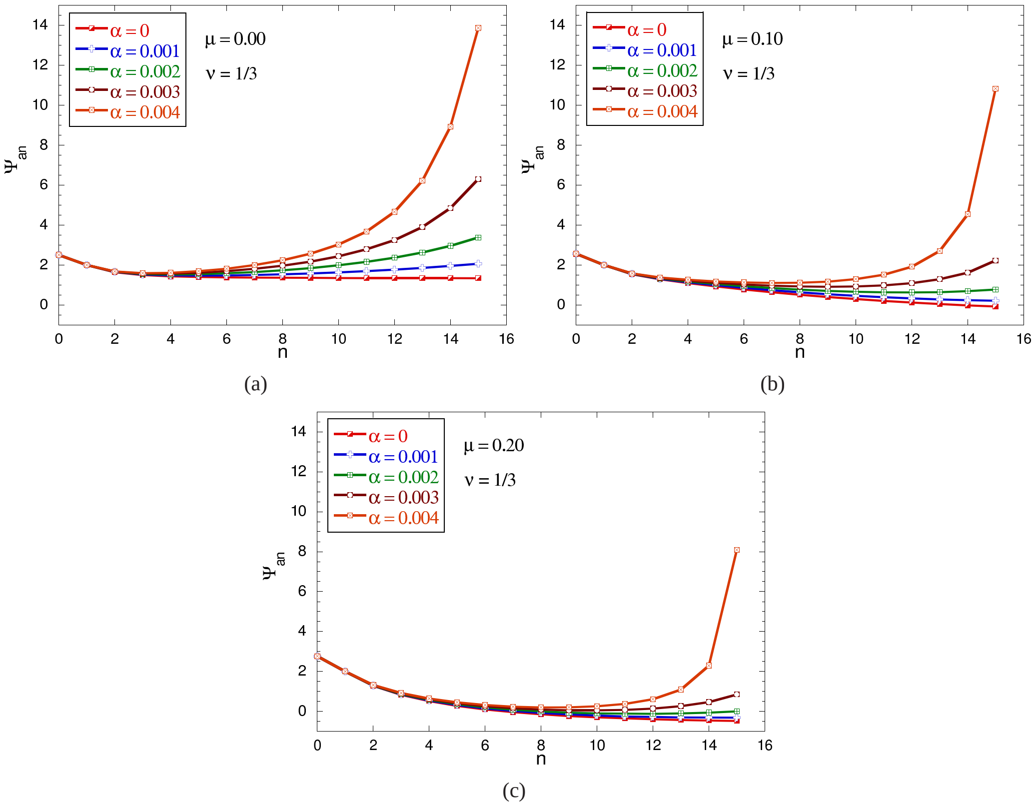

Fig. 6. Ψ_{a_n} in upper branch modes of vibration for different values of α and μ ($\nu = 1/3$). (a) $\mu = 0$. (b) $\mu = 0.10$. (c) $\mu = 0.20$.

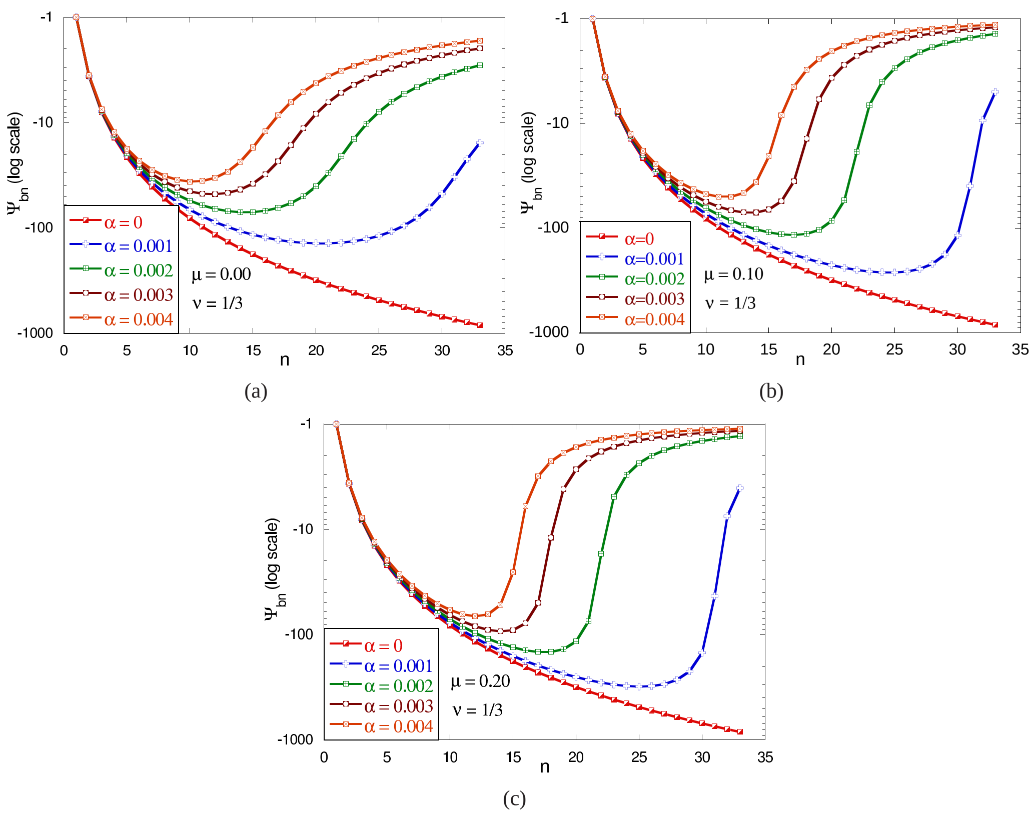

Fig. 7. Ψ_{b_n} in lower branch modes of vibration for different values of α and μ ($\nu = 1/3$). (a) $\mu = 0$. (b) $\mu = 0.10$. (c) $\mu = 0.20$.

Nevertheless, the influence of stiffness is significantly larger in the lower branch, and this is linked to the geometric features of the modal shapes. Fig. 8 shows the change in curvatures and the membrane strains along the whole domain $0 \leq \phi \leq \pi$, according to Eqs. (12) and (13). These figures correspond to a particular mode ($n = 5$) and to a particular value of the parameters being analysed ($\mu = 0.1$, $\alpha = 0.004$), depicting however the relative weight of stretching or bending deformation than can be observed in a general case. It can be seen that upper branch modes mainly induce membrane strains and, for this reason, the flexural stiffness does not play a relevant role in the frequency. On the contrary, the lower branch modes induce higher curvatures thus boosting the effect of bending on the vibration.

Regarding the influence of the non-local parameter μ, it can be observed in Figs. 2 to 5 that it is negligible over the first modes of vibration ($n \leq 3$) but it becomes relevant as n increases. To elaborate on this point, it is important to recall the influence of n on the deformation of the spherical shell. The integer n coincides with the number of semi-waves in the model shape. In other words, it is the number of times the deformed shape crosses over the undeformed one. As n increases the wave length of the vibration decreases and approaches the characteristic length of the microstructure, which is directly linked to the non-local parameter μ. High values of n are consequently more affected by the microstructure, and both modal shapes and frequencies are significantly influenced by μ. Conversely, for low values of n the wave length is large and the influence of microstructure (nonlocal parameter) is small.

In Figs. 2 and 3, the influence of μ over natural frequencies can be observed. It can be seen that, for all values of n, natural frequencies decrease as μ increases. This is a characteristic phenomenon in microstructured materials and has been widely studied (Brillouin, 1946).

In addition, the natural frequency of vibration grows unbounded for classical materials/theories ($\mu = 0$). Nevertheless, it has an asymptotic value for any $\mu \neq 0$, even for very small values. This implies the existence of a maximum frequency of vibration, and consequently forbidden frequencies (band gap). These frequencies cannot be excited in the sphere. This is a usual phenomenon of microstructured materials and discrete systems (Polyzos and Fotiadis, 2012; Fafalis et al., 2012). Therefore, the Eringen non-local model applied to this kind of nanoshells is able to predict the existence of typical band-gaps in microstructured solids, whereas classical theories cannot bring out this vibrational behavior.

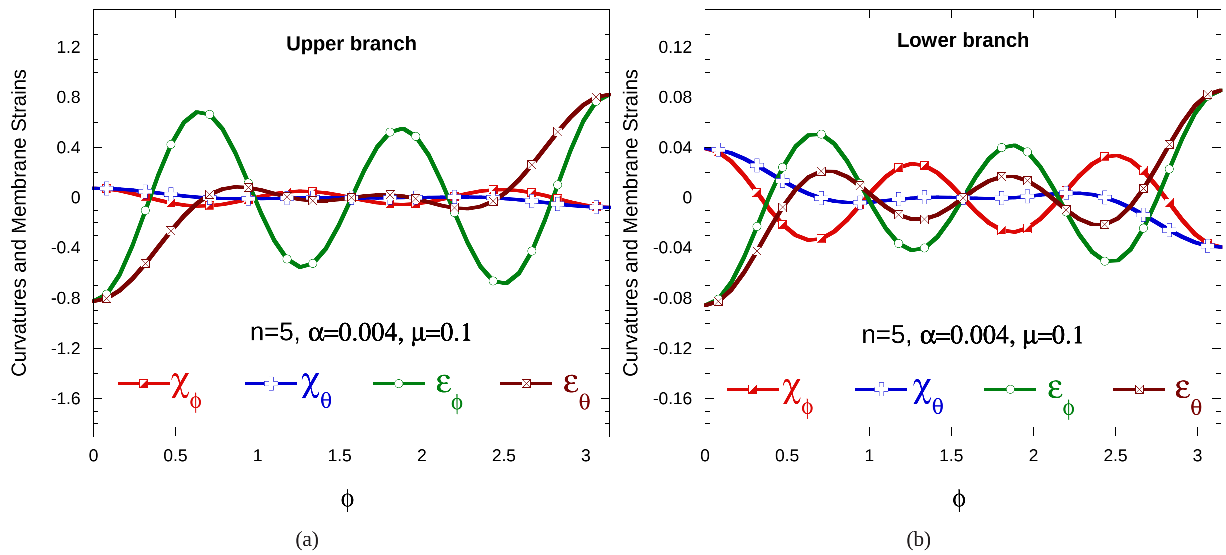

Fig. 8. Curvatures χ_ϕ, χ_θ and membrane strains $\varepsilon_\phi, \varepsilon_\theta$ in modal shape $n = 5$ ($\mu = 0.1$, $\alpha = 0.004$).

Conclusions

The paper presents a detailed study of free axisymmetric vibrations of a closed spherical nanoshell using the Eringen nonlocal elasticity theory. The analysis extends the hypotheses of thin shells –that were already considered by the authors, accounting only for membrane forces– to plate behaviour including bending effects. This modification permits to broaden the field of application to nanospheres with higher thickness-to-radius ratios. This study could be useful in biomedical and bioengineering applications as well as in other fields related with the nanotechnology.

The solution was obtained through the method of separation of variables, assuming products of both space-dependent and time-dependent functions, and closed-form expressions for the natural frequencies and modal shapes were derived in terms of parameters μ and α. Size effects are taken into account by nonlocal parameter μ, while α is the thickness-to-radius ratio parameter, which represents the importance of bending forces against membrane forces. This enables to study the coupling of both effects, which was not possible in previous developments.

The following conclusions can be established:

- The natural frequencies of the sphere increase with increasing values of α, whether or not the nonlocal effects are considered. Whereas the upper vibrational branch is slightly affected by α, the lower branch is strongly sensitive to slight changes in the thickness-to-radius ratio.
- The classical (local) solution considering bending effects predicts an unbounded growth of the lower branch frequencies, which tend asymptotically to $b_n = n + 1/2$. However the nonlocal effect drastically changes this behaviour, bounding the lower branch frequencies by a value that is inversely proportional to the nonlocal parameter μ.
- Bending effects alters the ratio of radial to meridional displacements in the modal shapes. For the upper branch, bending effects increase the ratio, thus leading to larger values of w relative to v. On the contrary, bending effect decrease the ratio for the lower branch (in absolute value) leading to larger values of v relative to w. The nonlocal parameter affects the rate of change of both Ψ_{a_n} and Ψ_{b_n} with n. Moreover, previous studies developed by the authors have shown that the nonlocal parameter does not influence the ratio of radial to meridional displacements for the lower branch if bending effects are ignored. Here it can be noticed that μ has a remarkable influence when $\alpha \neq 0$.
- The value of Ψ_{b_n} approaches asymptotically to -1 for large values of n, regardless the value of μ and α. Therefore, nonlocal effects and bending effects do not have any effect in the modal shapes of the lower branch at the highest frequencies.
- The nonlocal parameter does not play any role in the lower branch modal shapes when bending effects are neglected. Nevertheless, its effect becomes relevant when $\alpha \neq 0$.

Funding

This work was supported by the *Ministerio de Ciencia e Innovación de España* [grant numbers DPI/2011-24068, DPI/2011-23191].

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