Working Paper 96-05
Statistics and Econometrics Series 01
January 1996

Departamento de Estadística y Econometría
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (341) 624-9849

MULTICOINTEGRATION AND PRESENT VALUE RELATIONS

Tom Engsted, Jesús Gonzalo and Niels Haldrup*

Abstract

It is well-known that if the forcing variable of a present value (PV) model is an integrated process, then the model will give rise to a particular cointegrating restriction. In this paper we demostrate that if the PV relation is *exact*, such that no additive error term appears in the specification, then te variables will be *multicointegrated* such that the cumlation of cointegration errors at one level of cointegration will cointegrate with the forcing variable. Multicointegration thus delivers a statistical property of the data that is necessary, though not sufficient, for this class of models to be valid. Estimation and inference of the model are discussed and it is shown that, provided the PV relation is exact, the discount factor of the model can be estimated with a rate of convergence that is faster than the usual super-consistent rate characterising estimators in the cointegration literature. Finally, the paper is completed with two empirical analyses of PV models using term structure data and farmland data, respectively.

Key Words

Present value realtions; cointegration; multicointegration; term structure; farmland prices.

*Engsted, Department of Information Science, Aarhus School of Business, KK-8210 Aarhus V. Denmark; Gonzalo, Departamento de Estadística y Econometría, Universidad Carlos III de Madrid; Haldrup, Department of Economics. Aarhus University, DK-8000 Aurhus C. Denmark.
We would like to thank Jesper Lund for helpful comments and suggestions.

1. Introduction

In economics and finance there is a long tradition for using present value models to describe economic behaviour. The basic set up can usually be described as follows. Let Y_t and X_t be an endogenous and a forcing variable, respectively. Then, fundamentally, Y_t is determined as the present discounted value of expected future values of the X's. Formally we have that

$$Y_t=\theta(1-\beta)\sum_{i=0}^{\infty}\beta^i E_t X_{t+i}+v_t, \tag{1.1}$$

where θ is a factor of proportionality and β is the constant discount factor ($\beta=(1+r)^{-1}$) with r being the rate of return. E_t is the mathematical expectation at time t conditional on the information set Ω_t which as a minimum includes $\{X_{t-j}, j\geq 0\}$. Finally, the additive component v_t is an error term measuring stochastic discrepancies from the linear relationship. In a wide number of empirical and theoretical studies it is a maintained assumption that this term be absent. This is a very critical assumption and refers to the class of models dubbed by Hansen and Sargent (1991) as *exact* linear rational expectations (ELRE) models. In such models "*there is an exact linear relation across forecasts of future values of one set of variables and current and past values of some other set of variables*", (Hansen and Sargent (1991), p. 45). The distinction between *exact* and *non-exact* models will be crucial for the results derived in this paper.

Many economic models fall within the class of models described in (1.1). For instance, the model may describe the expectations theory of the term structure where Y_t is the long-term yield and X_t is the short-term yield, see e.g. Campbell and Shiller (1987, 1991), Sargent (1979), and Shiller (1979). The model could also describe the pricing of stocks, where Y_t is then the stock price and X_t dividend payments, see e.g. Campbell and Shiller (1987), and West (1987, 1988a). With some modifications the model may also represent the permanent income theory of consumption, compare e.g. Campbell (1987), Campbell and Deaton (1989), Flavin (1981, 1993), Deaton (1987) and West (1988b), and also generalisations to the accomodation of adjustment costs in linear quadratic models is a possibility as for instance in labour and money demand relations, see Sargent (1978), Kennan (1979), Dolado et al. (1991), Gregory et al. (1993) and Engsted and Haldrup (1994, 1995). Finally, the possibility of time-varying discount rates can be accomodated by making a log-linearization of the PV-relation, see e.g. Campbell and Shiller (1989) and Timmerman (1995).

An important implication of present value models is that the variables Y_t and X_t necessarily have to be cointegrated in the sense of Engle and Granger (1987), required, of course, that the individual variables can be characterised as integrated processes, see e.g. Campbell and Shiller (1987). However, cointegration is not a sufficient but only a necessary condition for the model to be valid; further parametric restrictions need to be satisfied and in Campbell and Shiller (1987) these further requirements are provided with associated test procedures. Engsted and Haldrup (1994) generalize their results to the linear quadratic adjustment cost model.

The purpose of the present paper is to demonstrate that a deeper kind of cointegration will occur if the present value model is of the exact type. In fact, we show that in this case the integral of equilibrium errors at one level of cointegration (the integral of the spread between Y_t and X_t, for instance), will cointegrate with the level of X_t. A reparametrization of the model will then lead to cointegrating relations involving variables of integration order higher than one (polynomial cointegration). Basically this characteristic of the data has the property of *multicointegrated* time series as initially conceived by Granger and Lee (1989, 1990). The statistical concept of multicointegration thus delivers a necessary (though not sufficient) condition for the model to be an *exact* linear rational expectations model.

Since the exact model is so prevalent in empirical work[1], testing for multicointegration provides a useful and important additional empirical check of this kind of models. In addition, the multicointegration framework provides a method to obtain an estimate of the discount factor β which has very favourable statistical properties. It is well-known that when Y_t and X_t are cointegrated I(1) series, a super-consistent estimate of β can be obtained by regressing Y_t on X_t. However, when the present value model is of the exact type we show that estimates with even faster rates of convergence can be achieved by including *integrals* of Y_t and X_t in the regression model.

The plan of the rest of the paper is the following. In the next section the distinction between exact and non-exact rational expectations models is explained in depth and the cointegration possibilities that may occur are derived. It is emphasized that in order for the multicointegration result to apply it is important that *either* the econometrician knows *all* the variables used by agents in forming expectations about the future, *or* that the model is formulated such that it includes an observable variable which, under the hypothesis that the exact rational expectations model is true, summarizes all the information used by agents. The subsequent section addresses estimation and testing of the multicointegrated model. Finally, empirical examples are

[1] For instance the variance bounds tests and tests of cross-equation restrictions presented in Campbell (1987), Campbell and Shiller (1987), West (1987, 1988), and Engsted and Haldrup (1994), *inter alia*, are derived for the exact model.

presented with reference to the term structure of interest rates and the present value model of farmland prices. The final section concludes with suggestions for future extensions and developments.

2. The cointegration implications of present value models.

Consider the present value (PV) model (1.1) which can be rearranged such that

$$S_t \equiv Y_t - \theta X_t - \theta \sum_{i=1}^{\infty} \beta^i E_t \Delta X_{t+i} + v_t \tag{2.1}$$

where S_t is frequently referred to as the spread between Y_t and X_t. The expression states that if X_t is integrated of order one, I(1), then the right hand side of (2.1) is stationary and integrated of order zero, I(0), requiring, naturally, that $\beta<1$ and that v_t is itself a stationary I(0) component. Consequently, the left hand side of (2.1) will also be I(0) so that Y_t and X_t will cointegrate with cointegration vector (1, $-\theta$). Using Engle and Granger's (1987) notation we thus have that $Y_t, X_t \sim CI(1,1)$.

In order to derive the multicointegration properties of the model it will be natural, at least to make the subsequent arguments more intelligible, to consider also the situation where X_t and Y_t are I(2). In this case the first difference of (2.1) can be written as

$$\Delta Y_t - \theta \Delta X_t - \theta \sum_{i=1}^{\infty} \beta^i E_{t-1} \Delta^2 X_{t+i} + \theta \sum_{i=1}^{\infty} \beta^i \Delta E_t \Delta X_{t+i} + v_t - v_{t-1} \tag{2.2}$$

where the first term on the right hand side is now I(0) by definition, whilst the second term is the expectational forecast revision of ΔX_t, which is an innovation under the assumption of rational expectations. Hence the first differenced variables ΔY_t, and ΔX_t are cointegrated as before. However, polynomial cointegration, see e.g. Engle and Yoo (1991) and Haldrup and Salmon (1995), is another possibility that may occur for an I(2) system. The PV relation (2.1) can also be written in the following way:

$$Y_t - \theta X_t - \frac{\theta\beta}{1-\beta} \Delta X_t - \frac{\theta}{1-\beta} \sum_{i=1}^{\infty} \beta^i E_t \Delta^2 X_{t+i} + v_t, \tag{2.3}$$

and hence Y_t, X_t and ΔX_t are fully cointegrated in the sense that a particular linear combination is integrated of order zero. Equations such as (2.3) arise naturally in the context of e.g. Cagan's model of hyperinflation, where money (X_t) and prices (Y_t) are I(2) processes, see Engsted (1993). Generally polynomial cointegration may occur for systems with higher order integrated variables. We will now demonstrate that even for I(1) systems a similar relationship will appear when the model is an ELRE model.

DEFINITION. MULTICOINTEGRATION. *Assume that Y_t and X_t are cointegrated time series of order CI(1,1) such that Y_t-θX_t=S_t is stationary. If the integral I(1)-variable $\Delta^{-1}S_t=\Sigma_{j=1}^{t}S_j$ cointegrates with X_t such that a parameter κ exists whereby $\Delta^{-1}S_t$-κX_t is also a stationary relation, then Y_t and X_t are said to be multicointegrated. In this case the cointegrating relationship amongst the variables can be written as*

$$\Delta^{-1}S_t-\kappa X_t-\Delta^{-1}Y_t-\theta\Delta^{-1}X_t-\kappa X_t \sim I(0)$$

where $\Delta^{-1}Y_t$ and $\Delta^{-1}X_t$ are now I(2) variables.

The above definition follows from Granger and Lee (1989, 1990) who used the notion of multicointegration to analyze inventory data. Observe that although the natural economic variables are integrated of order one, the single series can be transformed by integral operations and hence yielding a higher level of cointegration which involves variables integrated of an order higher than one. Potentially the property can be iterated further if, in addition, the cumulated cointegration errors from the multicointegrated relation cointegrate with the X_t variable. When this is the case the relation implicitly involves variables integrated of order higher than two.

It is interesting to observe how multicointegration amongst a set of variables has similarities to the standard (univariate) unit root model. Multicointegrated time series and the unit root model both have the property that the regressors are linear combinations of the errors driving the model. For instance, in the unit root model $y_t=y_{t-1}+\varepsilon_t$ and thus $y_{t-1}=\Sigma_{j=1}^{t-1}\varepsilon_j+y_0$, where y_0 is the initial value. So essentially y_t cointegrates with the cumulated historic errors. Similarly, the multicointegration model can be expressed as $\Sigma_{j=1}^{t}S_j=\kappa\ X_t$ + *error*, with $Y_t=\theta X_t+S_t$, so the cumulated errors cointegrate with the X_t variable and hence also with Y_t. Park (1992) has termed this class of models *singular* cointegrated models.

We are now able to show that an exact linear rational expectations model will imply multicointegration.

PROPOSITION 2.1. *Let Y_t and X_t satisfy the exact linear present value relation*

$$Y_t-\theta(1-\beta)\sum_{i=0}^{\infty}\beta^i E_t X_{t+i}.$$

Then Y_t and X_t are multicointegrated time series.

PROOF. Assume as the benchmark the non-exact model (1.1) or alternatively the model (2.1). We shall later assume that v_t=0. Following Campbell and Shiller (1987) we can define a limited information set $H_t=\{\Delta X_{t-j}, S_{t-j}, j\geq 0\}$, observable to the econometrician, and the following pth order VAR model:

$$\begin{bmatrix}\Delta X_t \\ S_t\end{bmatrix} = \begin{bmatrix}a(L) & b(L) \\ c(L) & d(L)\end{bmatrix}\begin{bmatrix}\Delta X_{t-1} \\ S_{t-1}\end{bmatrix} + \begin{bmatrix}e_{1t} \\ e_{2t}\end{bmatrix} \tag{2.4}$$

where $a(L)$, $b(L)$, $c(L)$ and $d(L)$ are lag polynomials in the lag-operator L. This way of writing the VAR model follows directly from Bewley's (1979) representation of a cointegrated VAR model. Equation (2.4) can be written more comprehensively in first order companion form as $Z_t=AZ_{t-1}+e_t$, where Z_t is the vector $[\Delta X_t, S_t, \Delta X_{t-1}, S_{t-1},\ldots, \Delta X_{t-p+1}, S_{t-p+1}]$ and with A being the corresponding companion matrix of VAR parameters. If we define g as the vector that selects the first equation of the VAR, forecasts of future ΔX conditional on H_t can be generated as $E_t(\Delta X_{t+i}|H_t)=gA^iZ_t$. By projecting each side of equation (2.1) onto H_t, and noting that the left hand side is unchanged since S_t is already in H_t, we obtain

$$S_t=\theta g\beta A(I-\beta A)^{-1}Z_t+\mu_t = A_1(L)\Delta X_t+A_2(L)S_t+\mu_t \tag{2.5}$$

where $\mu_t=E_t(v_t|H_t)$ and where $A_1(L)$ and $A_2(L)$ are lag-polynomials with coefficients that are complicated functions of the model parameters. From this we can solve with respect to S_t thus yielding

$$S_t=\frac{A_1(L)}{1-A_2(L)}\Delta X_t+\frac{1}{1-A_2(L)}\mu_t = B_1(L)\Delta X_t+B_2(L)\mu_t \tag{2.6}$$

where $B_1(L)$, $B_2(L)$ are rational lag-polynomials. Now, since any polynomial can be arranged such that for instance $B_1(L)=B_1(1)+\Delta B_1^*(L)$, each side of (2.6) can be transformed by the integral operator.Hence we obtain

$$\Delta^{-1}S_t=B_1(1)X_t+B_1^*(L)\Delta X_t+B_2(L)\Delta^{-1}\mu_t\ . \tag{2.7}$$

From this expression it can be seen that, in general, the integral of cointegration errors (the accumulated spread) $\Delta^{-1}S_t=\Delta^{-1}Y_t-\theta\Delta^{-1}X_t$ should not cointegrate with X_t because $\Delta^{-1}\mu_t$ is I(1), except when μ_t is non-invertible. Notwithstanding, if we have an *exact* model such that $v_t=0$ for all values of t implying that the last term in (2.7) is absent, then $\Delta^{-1}S_t$ and X_t will in fact cointegrate with cointegrating vector $(1, -B_1(1))$ which is a complicated, highly non-linear function of θ, β, and the VAR parameters. This proves that Y_t and X_t are multicointegrated when the model is exact.

□

Notice that since multicointegration implies operations with the lag-operator, the notion may also be referred to as polynomial cointegration. The notion of multicointegration is relevant in relation to I(2) systems since the integral of cointegration errors implicitly implies the cumulation of I(1) variables. One can thus redefine variables in an appropriate way such that essentially the model becomes of the type given in (2.3). It is interesting to observe, however, that the multicointegration result of exact present value models goes far beyond just one extra level of cointegration as illustrated above. Notice simply, that by considering the factorisation $B_1^*(L)=B_1^*(1)+\Delta B_1^{**}(L)$ one more level of cointegration appears by cumulating both sides of (2.7) once more given that $\mu_t=0$. This procedure can be repeated, and generally, if $B_1(L)$ is a finite p'th order AR polynomial, p such integral operations can be conducted yielding finally a perfectly cointegrated relationship between the variables. However, since $B_1(L)$ in most cases is a rational polynomial, c.f. the definition given in (2.6), integral operations can, at least in principle, be repeated infinitely. In practice, the presence of small measurement errors is likely to make cointegration at levels higher than two inconceivable since such errors become increasingly important when the variables are being integrated.

From the above derivations it is clear that it is very important to include S_t in the limited information set H_t in order for the multicointegration result to prevail: If, for example, H_t only includes current and lagged ΔX_t, and agents base their expectations on a larger information set, then the projection of (2.1) onto H_t does not leave the left-hand side unchanged. Hence, the left-hand side of (2.5) would be S_t plus an error which when transformed by the integral operator, Δ^{-1}, would become I(1), and hence implying no multicointegration. The intuition behind the result, that by including S_t in H_t we get multicointegration even if the econometrician does not know all the variables that agents exploit in forecasting, is that under the exact model S_t is the optimal predictor of the present value of future ΔX_t as indicated in (2.1). Therefore, if agents use more information than is observed by the econometrician, then S_t will summarize that additional information under rational expectations.

In order to get a deeper understanding of the nature of the multicointegration result, let us look at a few special cases.

EXAMPLE 2.1. Assume first, that the true data generating process for the forcing variable X_t is a random walk with a possible drift, i.e. $\Delta X_t=\delta+\eta_t$, then the right-hand side of (2.1) is a constant since $v_t=0$ for all t:

$$S_t = Y_t - \theta X_t = \frac{\theta\delta}{1-\beta} \quad \text{and hence} \quad \Delta^{-1}S_t = \frac{\theta\delta}{1-\beta}t.$$

In this special case there is *perfect* cointegration between Y_t and X_t, and consequently S_t is not a stochastic variable, but becomes a constant, whereby its integral becomes a time trend. Naturally this component cannot cointegrate with X_t and there will thus be no multicointegration. However, the example strives for the importance of a possible time trend in the specification of integral errors. Perfect cointegration is not a very realistic situation, so consider the second simple, though slightly more general example.

EXAMPLE 2.2. Assume that ΔX_t is a first order autoregressive process: $\Delta X_t = \gamma \Delta X_{t-1} + u_t$. Now, since $\mu_t = 0$, (2.1) can be reexpressed as

$$S_t = Y_t - \theta X_t = \frac{\theta\beta\gamma}{1-\beta\gamma}\Delta X_t \quad \text{and hence} \quad \Delta^{-1}S_t = \frac{\theta\beta\gamma}{1-\beta\gamma}X_t \; .$$

As seen there is now cointegration, though not perfect cointegration, between Y_t and X_t. Moreover there is perfect cointegration between $\Delta^{-1}S_t$ and X_t since there is no error term in this relation. Overall there is thus perfect multicointegration amongst the variables.

Naturally these examples easily generalize, and if we include more than one lag of ΔX_t and S_t as explanatory variables in the VAR, a stationary term equal to $B_1(L)\Delta X_t$ will always appear in the relationship between $\Delta^{-1}S_t$ and X_t, and since $B_1(L)$ generally will be a rational polynomial, there is, as previously argued, "non-perfect" multicointegration. This is what we would expect to see in real data when the exact linear rational expectations model is valid.

3. Estimating and testing multicointegrated systems.

Since the multicointegration result of exact PV models explicitly involves the cumulation of cointegration errors at another level of cointegration, the implied consequences w.r.t. estimation and testing become non-trivial. On the face of it there are two possible procedures to test for multicointegration that seem possible: A two-step procedure and a one-step procedure.

The former procedure is based on the idea that first cointegration between Y_t and X_t is tested using standard cointegration techniques. If the series are accepted to be cointegrated, i.e. such that $\hat{S}_t$ is stationary, the regression residuals from this first step are cumulated as the new variable $\Delta^{-1}\hat{S}_t=\Sigma_{j=1}^{t}\hat{S}_j$. In the second step this variable is regressed on X_t and possible deterministics like an intercept and a trend in order to take account of non zero means of the series. Subsequently the integration order of the regression residuals from this second step regression is tested. If the errors are I(0) the series are multicointegrated. Although this procedure seems plausible in principle it appears to be less attractive in practice. As we show in Appendix 1, the limiting theory to test for multicointegration is complicated by the fact that the auxiliary regression is based on cumulated regression residuals from another regression. Standard methods to test for cointegration therefore become invalidated for this particular type of models, since essentially the asymptotics will be expressed in terms of functionals of a Brownian Bridge process rather than a Brownian Motion process as is normally the case. See Appendix 1 for details.

The one step procedure, on the other hand, simultaneously tests both levels of cointegration, and as we shall argue this procedure will have several favourable statistical properties compared to the two-step procedure. Consider an *integral* regression of the form

$$\Delta^{-1}Y_t=\alpha_0+\alpha_1 t+\theta\Delta^{-1}X_t+\beta X_t+u_t\ . \tag{3.1}$$

The inclusion of a time trend follows from the fact that if the single series Y_t and X_t have a non-zero mean, then the cumulated series will have a trend. For instance, if

$$X_t=\gamma_1+X_t^0\ ,\quad \Delta X_t^0=\varepsilon_t \tag{3.2}$$

where ε_t has a zero mean, then

$$\Delta^{-1}X_t=\gamma_0+\gamma_1 t+\Delta^{-1}X_t^0. \tag{3.3}$$

The idea is therefore to test whether the errors u_t from the integral regression (3.1) follow an I(0) process (the case of multicointegration), an I(1) process (the case of first level cointegration but no multicointegration) and finally the case of an I(2) process where there is no cointegration amongst the variables. Assume in the case of a multicointegrated ELRE model that the least squares regression

$$\Delta^{-1}Y_t=\hat{\alpha}_0+\hat{\alpha}_1 t+\hat{\theta}\Delta^{-1}X_t+\hat{\beta}X_t+\hat{u}_t \tag{3.4}$$

is conducted where the true process satisfies (3.1)-(3.3) with $u_t=B_1^*(L)\Delta X_t^0$ in accordance with (2.7). Then, provided $\Delta X_t^0=\varepsilon_t$ satisfies the invariance principle, the following proposition follows as a special case of Haldrup (1994) (Theorem 1, p. 160):

PROPOSITION 3.1.

$$G\begin{pmatrix} T^{1/2}(\hat{\alpha}_0-\alpha_0) \\ T^{3/2}(\hat{\alpha}_1-\alpha_1) \\ T^2(\hat{\theta}-\theta) \\ T(\hat{\beta}-\beta) \end{pmatrix} \overset{d}{\Rightarrow} \left(\int_0^1 Q^*(r)Q^*(r)'dr\right)^{-1}\left(\int_0^1 Q^*(r)dQ+(0,0,0,\nu_0)'\right)$$

where

$$Q^*(r)=\left(1,\ r,\ \omega^*\overline{Q(r)},\ \omega^*Q(r)\right)' \text{ with } \overline{Q(r)} = \int_0^r Q(s)ds$$

and $Q(r)$ is a standard Brownian motion process. We also have the definitions

$$G = \begin{pmatrix} 1 & 0 & \gamma_0 & \gamma_1 \\ 0 & 1 & \gamma_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\omega^*=B_1^*(1)^{-1}\omega\ , \quad \omega^2=\lim_{T\to\infty}E(T^{-1}(\textstyle\sum_1^T u_j)^2)\ , \text{ and}$$

$$\nu_0=(\omega^2-\omega_u^2)/2\ , \quad \text{with } \omega_u^2=\lim_{T\to\infty}E(T^{-1}(\textstyle\sum_1^T u_j^2))\ .$$

Notice that ω is the long run variance associated with the limiting process of the normalised u_t process.

PROOF. A complete proof is given in Haldrup (1994), so in order to see how the result follows from this reference we just need to find the limiting behaviour of the regressors given in the regression (3.4). Let ε_t satisfy the usual conditions for the invariance principle such that

$$X_T^0(r) = T^{-1/2}\sum_{i=1}^{[Tr]}\varepsilon_i \overset{d}{\Rightarrow} W(r).$$

where $W(r)$ is a Brownian Motion process. Then

$$T^{-1/2}\sum_{i=1}^{[Tr]}u_i=T^{-1/2}\sum_{i=1}^{[Tr]}B_1^*(1)\varepsilon_i+o_p(1) \overset{d}{\Rightarrow} B_1^*(1)W(r)\equiv\omega Q(r)$$

where $Q(r)$ is a standard Brownian motion on [0,1]. Similarly it follows that

$$X_t^0 \overset{d}{\Rightarrow} B_1^*(1)^{-1}\omega Q(r) \equiv \omega^*Q(r)$$
$$\Delta^{-1}X_T^0 \overset{d}{\Rightarrow} \omega^*\int_0^r Q(s)ds \equiv \omega^*\overline{Q(r)}$$

□

The interesting thing to note about Proposition 3.1 is that the distribution of the various regression coefficients are linked together in a particular way through the *G*-matrix, although the distributions generally are non-standard. More importantly, however, it can be seen that when the model exhibits multicointegration the parameter θ can be estimated at the "super-super" consistent rate, $O_p(T^2)$, when the regression is of an integral form. This is in contrast to a standard (non-integral) cointegration regression where θ is estimated at the super consistent rate, $O_p(T)$. The parameter on the second level of cointegration on the other hand is estimated super-consistently. Since the exact PV model implies that an infinity of integral operations (in principle) can be conducted, even faster rates of convergence can be obtained. However, in practice an integral regression of the form (3.4) is likely to suffice if measurement errors are present.

Testing of the null of no multicointegration can be directly advocated by a residual based test, such as the Dickey-Fuller cointegration test, applied to the regression residuals $\hat{u}_t$. In Haldrup (1994) the limiting distribution of the Dickey-Fuller cointegration *t*-ratio is reported for the I(2) model which characterises the present model in integral form and with the regression counterpart (3.4). The critical values for this case are also provided in this reference for the situation where a constant is included in the regression. In appendix 2 of this paper the critical calues are extended to the case where a trend has been included in the regression as well.

As seen the single step procedure will have certain statistical advantages compared to the two-step procedure described in the appendix. Firstly, the distributions concerning tests of the null that there is no multicointegration will be well-known for the single step procedure based on integral regression, and, secondly, provided the system does exhibit multicointegration, the first level cointegration parameter can be estimated more consistently compared to the situation where a two-step procedure is considered or any other procedure not taking advantage of an integral parametrization of the model.

In many situations the cointegration parameters (at least at the first level of cointegration) are given a priori by economic theory (see the next section). In such cases, naturally, integral regressions directly specify I(1) systems and the associated test procedure available to I(1) systems can be applied directly.

4. Applications.

4.1 The Term Structure of Interest Rates.

In this section we illustrate the methods described in the previous section using US and Danish term structure data. Campbell and Shiller (1991) test the exact version of the expectations hypothesis of the term structure of interest rates (EHT) using the zero-coupon bond yield data set from McCulloch (1990), which probably is one of the most complete, comprehensive and widely used data sets covering the US bond market. The data is monthly and covers the period 1952:1 - 1987:2 of a 1-month yield, a 3-months yield and a 10-year yield. Overall, Campbell and Shiller reject the exact version of the EHT using these data. In contrast, Engsted and Tanggaard (1995) using monthly Danish zero-coupon bond yields across the same maturities as McCulloch's data, see Engsted and Tanggaard (1994)), find quite strong support for the exact EHT in Denmark during the period 1976:1 - 1985:7. This implies that we should be able to find multicointegration in the Danish data, whereas we might or might not find such evidence for the US data. In figures 1 and 2 the US and Danish interest series are displayed.

Figure 1 about here

Figure 2 about here

The expectations hypothesis of the term structure of zero-coupon yields cannot be exactly represented by equation (1.1) since there is no discount factor and the time horizon is finite. A direct analogue to (1.1) is, however, to write the model in linearized form as

$$R_t^{(n)} = \frac{1}{n}\sum_{i=0}^{n-1} E_t R_{t+i}^{(1)} + v_t \tag{4.1}$$

where $R_t^{(n)}$ is the n-period (long) interest rate and $R_t^{(1)}$ is the 1-period (short) interest rate. From (4.1) the spread $S_t = R_t^{(n)} - R_t^{(1)}$ can be written as

$$S_t = \sum_{i=1}^{n-1} \frac{n-i}{n} E_t \Delta R_{t+i}^{(1)} + v_t \tag{4.2}$$

which is the counterpart of equation (2.1). Notice that in contrast to the discussion given in section 3 concerning estimation and testing procedures, the cointegration parameter θ is known in this case to equal one. Otherwise the multicointegration property follows directly as for the PV-model set up in section 2 when $v_t=0$ for all values of t.

All involved time series for both countries were tested to be I(1) processes and all bilateral spreads, S_t, of the series were found to be I(0). So the idea was next to consider cointegration regressions like

$$R_t^{(1)}=\alpha_0+\alpha_1 t+\alpha_2\Delta^{-1}S_t+u_t \tag{4.3}$$

or alternatively

$$\Delta^{-1}S_t=\alpha_0^*+\alpha_1^* t+\alpha_2^* R_t^{(1)}+u_t^* \tag{4.4}$$

and test the order of integration[2] of the regression error terms. Notice that the theory does not tell which variable to put on the left hand side (LHS) of the cointegration regression; hence we try both. Since S_t was given by theory in this case we only need to consider an I(1) system of variables. In table 1 the Dickey-Fuller cointegration tests are reported for each country and for two different cases. In each of these cases $R_t^{(1)}$ was taken to be the short yield, whilst the long yield, $R_t^{(n)}$, was either taken to be the 3-months yield or the 10-year yield. As seen, there is absolutely no evidence of multicointegration in the US data. This is consistent with the previous findings in Campbell and Shiller (1991) who strongly reject the exact version of the EHT. The results for Denmark are less clearcut. When $\Delta^{-1}S_t$ is used as left-hand side variable in the multicointegration regression, the results are similar to those for the US. However, if instead $R_t^{(1)}$ is used as left-hand side variable there is now evidence of multi-cointegration in the Danish data at both the long and short end of the term structure. This is consistent with the findings in Engsted and Tanggard (1995) who cannot reject the exact EHT on the Danish data.

Table 1 about here

4.2 The Present Value Model of Farmland Prices.

Recently the exact rational expectations version of the present value model has been used extensively to interpret the behaviour of land prices, see e.g. Falk (1991) and Tegene and Kuchler (1993). Let P_t denote the real price per acre of farmland, measured at the end of period t; R_t denotes the real rent paid during period t; and $\beta=(1+r)^{-1}$ is the discount factor where r denotes the (constant) capitalization rate. The present value model relating P_t to R_t can then be stated as

$$P_t=\sum_{i=1}^{\infty}\beta^i E_t R_{t+i}\,. \tag{4.5}$$

As in section 2 this model can be rewritten as

[2] Only tests for 2. order multicointegration were conducted, because small measurement errors due to spline-smoothing construction of yield spreads, and the linearization errors, may potentially become important when integrated.

$$P_t = \frac{1}{r}R_t + \sum_{i=1}^{\infty} \beta^i E_t \Delta R_{t+i} \qquad (4.6)$$

which is equivalent to (2.1) for v_t=0 for all t and with $\theta = r^{-1}$. Notice that in this case θ is unknown and must be estimated. A super-consistent estimate is obtained by regressing P_t on R_t. However, as we have seen, a super-super-consistent estimate can be obtained be regressing $\Delta^{-1}P_t$ ($\Delta^{-1}R_t$) on a constant, a trend, R_t (P_t), and $\Delta^{-1}R_t$ ($\Delta^{-1}P_t$), c.f. (3.1), provided there is multicointegration.

In using this approach we apply annual farmland and rent data from the Corn Belt agricultural region in the US. The data span the period 1921 to 1989 and are listed in Tegene and Kuchler (1994). The data is displayed in figure 3. Tegene and Kuchler (1993) have previously established that these data are I(1) and cointegrated, and have obtained super-consistent estimates of the yearly capitalization rate $r=\theta^{-1}$ ranging from 0.052 to 0.056.

Table 2 gives the results of running regression (3.1) on the same data. Since the regression contains a mixture of I(1) and I(2) variables as well as a constant and a trend, the critical values tabulated in Appendix 2 should be used in testing the hypothesis of no multicointegration. The 5% critical value is approximately -4.42, so as seen, whether we use $\Delta^{-1}P_t$ or $\Delta^{-1}R_t$ as dependent variable, the ADF tests are strongly significant which indicates that P_t and R_t are multicointegrated in the way predicted by the exact present value model. The super-super-consistent estimates of the yearly capitalization rate are both close to 0.053 which is economically quite reasonable.

Figure 3 about here

Table 2 about here

5. Conclusion.

In this paper we have demonstrated, that for *exact* linear rational expectations models, e.g. present value models with no error term added to the model, a deeper form of cointegration, multicointegration, will occur such that the integral of cointegration errors at one level of cointegration will cointegrate with other time series components of the model. Multicointegration thus delivers a statistical property of the data that is *necessary*, though not sufficient, for this class of models to be valid. The exact present value model has testable implications, but although simple examples may exist where standard cointegration procedures for I(1) systems can be adopted to test for multicointegration, this will generally not be the case. Various estimation and inference procedures were discussed and our general suggestion for empirical practice is to consider different levels of cointegration jointly through integral regressions where potentially the cointegration procedures for I(2) systems need to be used. In an analysis of US and Danish zero coupon bond yields the *exact* version of the expectations hypothesis of the term structure was examined. In support of previous analyses of the same data series we found some evidence that the Danish term structure data exhibit multicointegration which is in contrast to the US data. Similarly, the Present Value model of farmland prices was examined for data from the Corn Belt agricultural region in the US. Multicointegration amongst the variables was supported for this particular data set.

A natural generalisation of the model considered in this paper is to include the possibility of time-varying discount rates. This can be accomodated by considering for instance log-linearized representations of the present value model as suggested for instance by Campbell and Shiller (1989).

Appendix 1.

The properties of a two step procedure to test for multicointegration.

The first level of cointegration in the PV model reads

$$Y_t=\mu+\theta X_t+S_t \qquad t=1,2,\ldots\ldots,T \tag{A.1}$$

where we have assumed that only one forcing variable is present in the model. In most cases this seems to be the natural although generalizations can be easily made. To achieve notational simplicity we restrict X_t to be a scalar variable. As seen we have also included a constant in the model. Assume that this level of cointegration is estimated using the Engle and Granger (1987) procedure. It follows that an estimate of the spread variable S_t can be found as

$$\hat{S}_t=Y_t-\hat{\mu}-\hat{\theta}X_t=S_t-T^{-1}\sum_1^T S_t-(\hat{\theta}-\theta)(X_t-T^{-1}\sum_1^T X_t) \tag{A.2}$$

where it is a well known result that $\hat{\theta}\rightarrow\theta$ of order $O_p(T)$, whereby the last term will vanish of order $O_p(T^{-1/2})$ since X_t is integrated of order one and hence is $O_p(T^{1/2})$. With respect to the second level of cointegration - multicointegration - it applies theoretically that

$$\Delta^{-1}S_t=\alpha+\beta X_t+u_t \tag{A.3}$$

where $\beta=B_1(1)$ and $u_t=B_1^*(L)\Delta X_t$ is I(0) according to (2.7) for an ELRE model. To make the algebra simple we have assumed here that the spread variable has zero mean such that no trend will be present in (A.3). A natural sample analogue of (A.3) based on least squares estimates is to consider the regression residuals from (A.2) and conducting in the second step the regression

$$\Delta^{-1}\hat{S}_t=\sum_{j=1}^{t}\hat{S}_j=\hat{\alpha}+\beta X_t+\hat{u}_t \tag{A.4}$$

to see whether the regression residuals follow a stationary I(0) process. A problem arises in this case, though, since the generated regressand $\Delta^{-1}\hat{S}_t$ obviously will have I(1) characteristics, but it is bounded in the sense that the sum across all T observations of $\hat{S}_t$ will equal zero. This is directly implied by accumulating regression residuals from a least squares regression with an intercept. We have now the following Proposition, which states that under suitable conditions the limiting process of $\Delta^{-1}\hat{S}_t$, after appropriate normalization, will be a Brownian bridge (a tied down Brownian motion) defined on the unit interval [0,1], (see Billingsley 1968, p. 64). This means that the process behaves like a Brownian motion but it has the property that it starts at zero and also ends at zero.

PROPOSITION A.1 *Assume that the invariance principle is satisfied, see e.g. Park and Phillips* (1988), *such that for* S_t *and* $\Delta X_t=\varepsilon_t$ *given from* (A.1)

$$\Delta^{-1}S_T(r)=T^{-1/2}\Delta^{-1}S_{[Tr]}=T^{-1/2}\sum_{i=1}^{[Tr]} S_i \overset{d}{\Rightarrow} W_1(r)$$
$$X_T(r)=T^{-1/2}X_{[Tr]}=T^{-1/2}\sum_{i=1}^{[Tr]} \varepsilon_i \overset{d}{\Rightarrow} W_2(r)$$

where $W_1(r)$ *and* $W_2(r)$ *are Brownian motion processes and* $[Tr]$ *signifies the integer value of its argument. Then for* $T\to\infty$ *the regression residuals following from (A.2) satisfy*

$$T^{-1/2}\sum_{i=1}^{[Tr]} \hat{S}_i \overset{d}{\Rightarrow} V(r)$$

where $V(r)$ *is a Brownian bridge on* $[0,1]$ *with the property that* $V(0)=V(1)=0$.

PROOF. If follows from Park and Phillips (1988), Theorem 3.2 p. 475, that

$$T(\hat{\theta}-\theta) \overset{d}{\Rightarrow} \left(\int_0^1 W_2^*(r)^2 dr\right)^{-1}\left(\int_0^1 W_2^* dW_1+\delta_{21}\right)$$

where $\delta_{21}=\lim_{T\to\infty}T^{-1}(\Sigma_1^T E(\varepsilon_t S_t)+\Sigma_{t=2}^T\Sigma_{j=1}^{t-1}E(\varepsilon_t S_j))$ and $W_2^*(r)=W_2(r)-\int_0^1 W_2(r)dr$ is a demeaned Brownian motion. It also applies that

$$T^{-3/2}\sum_1^{[Tr]} (X_t-T^{-1}\sum_1^{[Tr]} X_t) \overset{d}{\Rightarrow} \int_0^r (W_2(r)-\int_0^1 W_2(r)dr)dr \quad \text{and}$$
$$T^{-3/2}\sum_1^{[Tr]} S_t \overset{d}{\Rightarrow} \int_0^r W_1(r)dr.$$

Hence we have from (A.2) and (A.4) that

$$\Delta^{-1}\hat{S}_T(r) = T^{-1/2}\sum_{i=1}^{[Tr]} S_i-T^{-3/2}[Tr]\sum_1^{[Tr]} S_t-T(\hat{\theta}-\theta)T^{-3/2}\sum_1^{[Tr]} (X_t-T^{-1}\sum_1^T X_t) \quad \Leftrightarrow$$
$$\Delta^{-1}\hat{S}_T(r) \overset{d}{\Rightarrow} W_2(r)-rW_2(1)$$
$$-\left(\int_0^1 W_2^*(r)^2 dr\right)^{-1}\left(\int_0^1 W_2^*(r)dW_1+\delta_{21}\right)\left(\int_0^r (W_2(r)-\int_0^1 W_2(r)dr)dr\right) \equiv V(r)\ .$$

By evaluating the asymptotic limit for $r=0$ and $r=1$, it can be easily verified that the conditions for the expression to be a Brownian bridge are satisfied. However, it is very different from the way a Brownian bridge is normally treated in the literature where X_t is assumed to be either a stationary process or possibly a vector of deterministic regressors, compare e.g. MacNeill (1978), and Ploberger and Krämer (1992a,b).

It should be noted that the Brownian motion processes $W_1(r)$ and $W_2(r)$ generally will be correlated. In the special case of multicointegration such that the generating mechanism is (A.3), then $W_1(r)=\beta W_2(r)$ with an appropriate redefinition of δ_{21}. □

The fact that $\Delta^{-1}\hat{S}_t$ will act as a Brownian Bridge in the limit rather than a Brownian Motion implies that Dickey-Fuller tests for cointegration, for instance, will have a different limiting distribution compared to normal settings. This will obviously give rise to size distortions of the tests. This is also verified in Monte Carlo experiments that can be obtained from the authors upon request.

Appendix 2.

Insert table A1 about here.

6. References

Bewley, R.A. 1979, The Direct Estimation of the Equilibrium Response in a Linear Model, *Economics Letters*, Vol. 3.

Billingsley, p., 1968, Convervence of Probability Measures, John Wiley & Sons.

Campbell, J. Y., 1987, Does saving anticipate declining labour income ? An alternative test of the permanent income hypothesis, *Econometrica*, 55, 1249-1273.

Campbell, J.Y. and A. Deaton, 1989, Why is consumption so smooth ?, *Review of Economic Studies*, 56, 357-74.

Campbell, J. Y. and R. J. Shiller, 1987, Cointegration and tests of present value models, *Journal of Political Economy*, 95, 1052-1088.

Campbell, J. Y. and R. J. Shiller, 1989, The dividend-price ratio and expectations of future dividends and discount factors, *The Review of Financial Studies, 1, 195-228.*

Campbell, J. Y. and R. J. Shiller, 1991, Yield spreads and interest rate movements: A bird's eye view, *Review of Economic Studies* 58, 495-514.

Deaton, A., 1987, Life-cycle models of consumption: Is the evidence consistent with the theory?, in: T.F. Bewley (ed.): Advances in Econometrics: Fifth WorldCongress, Vol.II, Cambridge University Press, 121-48.

Dolado, J., J.W. Galbraith, and A. Banerjee, 1991, Estimating intertemporal quadratic adjustment cost models with integrated series, *International Economic Review* 32, 919-36.

Engle, R. F. & C.W.J. Granger, 1987, Co-integration and error correction: Represen tation, estimation and testing, *Econometrica* 55, 2, 251-276.

Engle, R. F. & S. Yoo, 1991, Cointegrated economic time series: An overview with new results, in R.F. Engle and C.W.J. Granger (eds.), 1991, Long run economic relationships, Oxford University Press, 237-266.

Engsted, T., 1993, Cointegration and Cagan's model of hyperinflation under rational expectations, *Journal of Money, Credit, and Banking*, 25, 350-360.

Engsted, T. and N. Haldrup, 1994, The linear quadratic adjustment cost model and the demand for labour, *Journal of Applied Econometrics*, 9 (supplement), S145-S159.

Engsted, T. and N. Haldrup, 1995, Money demand, adjustment costs, and forward looking behaviour, forthcoming in *Journal of Policy Modeling*.

Engsted, T. and C. Tanggaard, 1994, A cointegration analysis of Danish zero-coupon bond yields, *Applied Financial Economics*, 24, 265-278.

Engsted, T. and C. Tanggaard, 1995, The predictive power of yield spreads for future interest rates: Evidence from the Danish term structure, *Scandinavian Journal of Economics*, 97, 145-159.

Falk, B., 1991, Formally testing the present value model of farmland prices, *American Journal of Agricultural Economics*, 73, 1-10.

Flavin, M.A., 1981, The adjustment of consumption to changing expectations about future income, *Journal of Political Economy*, 89, 979-1009.

Flavin, M.A., 1993, The excess smoothness of consumption: Identification and interpretation, *Review of Economic Studies*, 60, 651-666.

Granger, C. W. J., and T. Lee, 1989, Investigation of production, sales and inventory relations using multicointegration and non-symmetric error correction models, *Journal of Applied Econometrics*, 4, S145-S159.

Granger, C. W. J., and T. Lee, 1990, Multicointegration, in Advances in Econometrics, Vol. 8, ed. by G. F. Rhodes and T. B. Fomby, Greenwich, CT: JAI Press, 71-84.

Gregory, A.W., A.R. Pagan, and G.W. Smith, 1993, Estimating linear quadratic models with integrated processes. In: P.C.B. Phillips (ed.), *Models, Methods, and Applications in Econometrics*, Basil Blackwell, Oxford, 220-39.

Haldrup, N., 1994, The asymptotics of single-equation cointegration regressions with I(1) and I(2) variables, *Journal of Econometrics* 63, 153-81.

Haldrup, N., and M. Salmon, 1995, Representations of I(2) cointegrated systems and the Smith-McMillan form, manuscript, University of Aarhus.

Hansen, L.P., and T.J. Sargent, 1991, Exact linear rational expectations models: Specification and estimation. Published in Hansen, L.P. and T.J. Sargent (ed.):

Rational expectations econometrics, Westview Press, 1991, pp. 45-76.

Kennan, J., 1979, The estimation of partial adjustment models with rational expectations", *Econometrica* 47, 1441-1455.

MacNeill, I. B., 1978, Properties of sequences of partial sums of polynomial regression residuals with application to tests for change of regression at unknown times, *The Annals of Statistics*, 6, 422-433.

McCulloch, J.H., 1990, US term structure data, 1946-1987, p. 672-715, published in Appendix B, Shiller (1990).

Park, J. Y., 1992, Canonical Cointegration Regressions, *Econometrica*, 60, 119-143.

Park, J. Y., and P. C. B. Phillips, 1988, Statistical inference in regressions with integrated processes: Part 1, *Econometric Theory*, 4, 468-498.

Phillips, P. C. B. and S. Ouliaris, 1990, Asymptotic properties of residual based tests for cointegration, *Econometrica* 58, 165-193.

Ploberger, W., and W. Krämer, 1992, The CUSUM test with OLS residuals, *Econometrica*, 60, 271-286.

Ploberger, W., and W. Krämer, 1992, A trend-resistant test for structural change based on OLS-residuals, mimeo, Cowles Foundation, Yale University, New Haven.

Sargent, T.J., 1978, Estimation of dynamic labor demand schedules under rational expectations, *Journal of Political Economy*, 86, 1009-1044.

Sargent, T.J., 1979, A note on maximum likelihood estimation of the rational expectations model of the term structure, *Journal of Monetary Economics*, 5, 133-43.

Shiller, R.J., 1979, The volatility of long-term interest rates and expectations models of the term structure, *Journal of Political Economy*, 87, 1190-1219.

Shiller, R.J., 1990, The term structure of interest rates, in B. Friedman and F. H. Hahn (eds.), *Handbook of Monetary Economics*, Elsevier Science Publishers.

Tegene, A. and F. Kuchler, 1993, Evidence on the existence of speculative bubbles in farmland prices, *Journal of Real Estate Finance and Economics*, 6, 223-236.

Tegene, A. and F. Kuchler, 1994, Evaluating forecasting models of farmland prices, *International Journal of Forecasting*, 10, 65-80.

Timmerman, A., 1995, Cointegration tests of present value models with a time-varying discount factor, *Journal of Applied Econometrics*, 10, 17-32.

West, K.D., 1987, A specification test for speculative bubbles, *Quarterly Journal of Economics*, 103, 553-80.

West, K.D., 1988a, Dividend innovations and stock price volatility, *Econometrica*, 56, 37-61.

West, K.D., 1988b, The insensitivity of consumption to news about income, *Journal of Monetary Economics*, 21, 17-33.

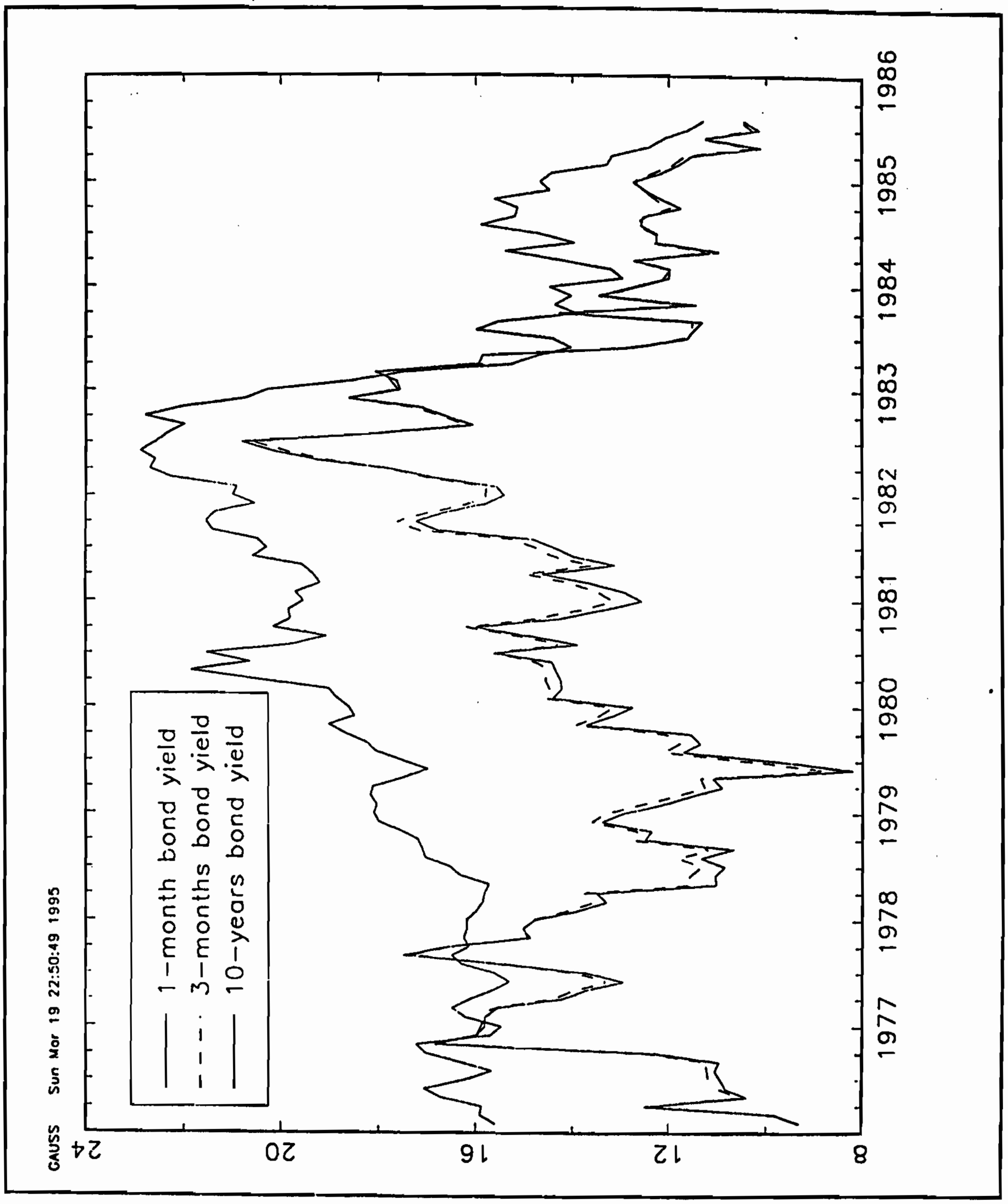

Figure 1. Danish zero coupon bond yields 1976.1 - 1985.7. Source: Engsted and Tanggaard (1994).

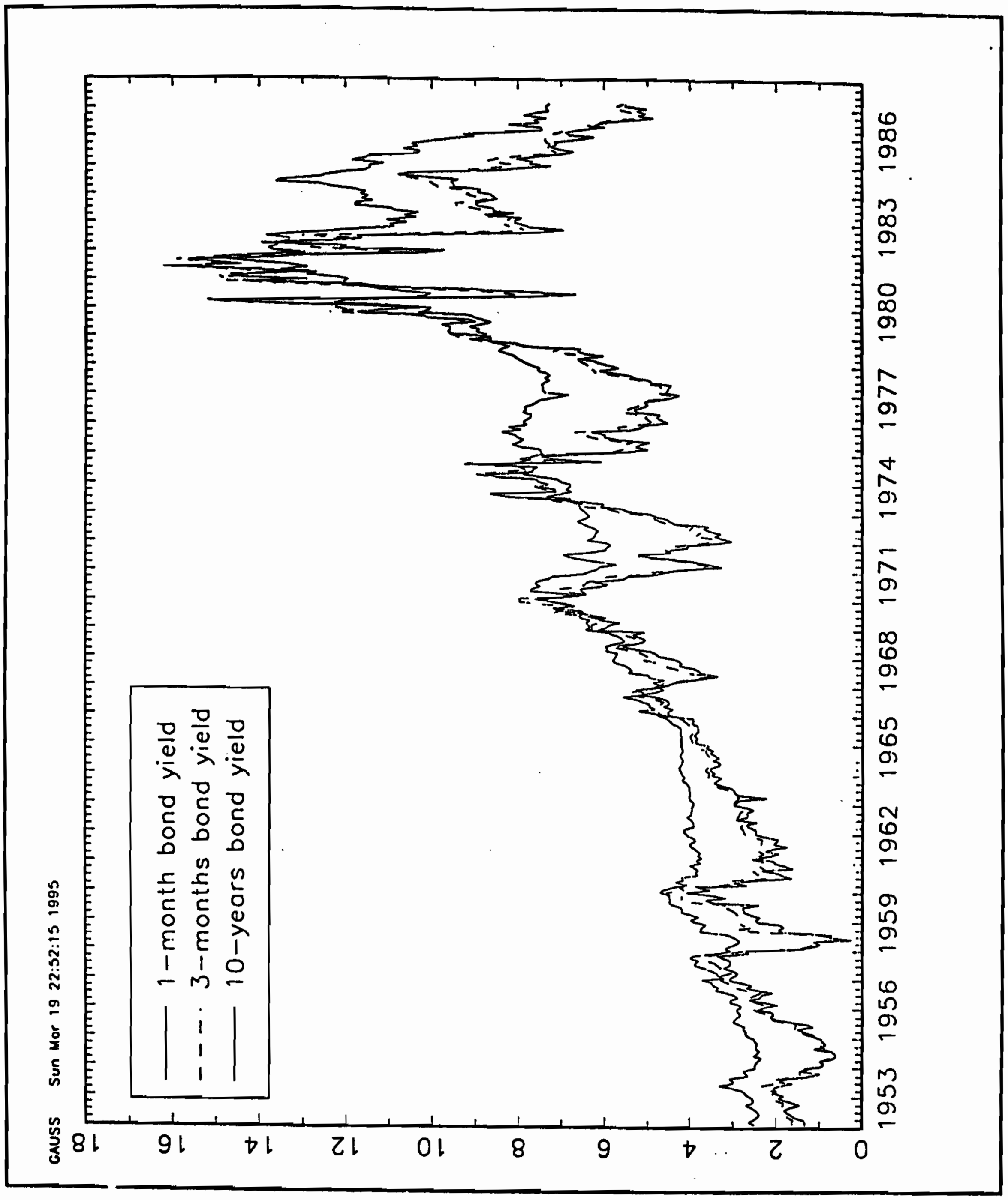

Figure 2. US zero coupon bond yields 1952.1 - 1987.2. Source: McCullough (1990).

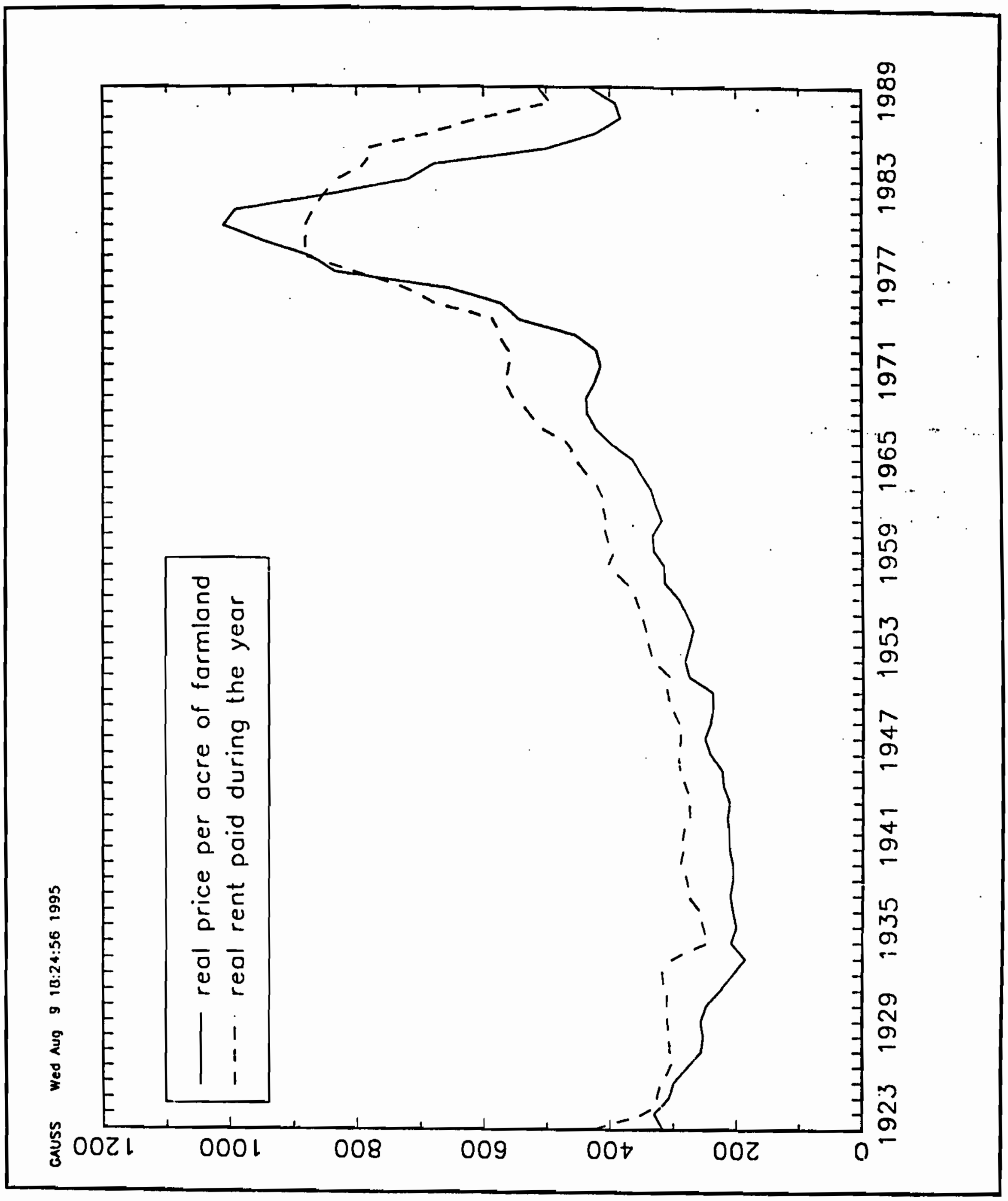

Figure 3. Farmland data for the Corn Belt agricultural region in the US 1921 - 1989. The real rent series has been rescaled to facilitate comparison. Source: Tegene and Kuchler (1994).

Table 1. Tests for multicointegration in the US and Danish term structure. Values of the augmented Dickey-Fuller cointegration *t*-ratio based upon the regressions (4.3) and (4.4), respectively. The entries *n*=3 months and *n*=10 years refer to the maturity of the long yield, $R_t^{(n)}$, and in both cases the short yield, $R_t^{(1)}$, was the 1-month yield whereby the spread variable was constructed as $S_t=R_t^{(n)}-R_t^{(1)}$, and $\Delta^{-1}S_t=\Sigma_{j=1}^{t}S_j$. For the US data the augmenting lags in the auxiliary regression were 1-8. For the Danish data no augmentation lags were needed. "*", "**" indicates significance on the 5% and 1% levels respectively, c.f. Phillips and Ouliaris (1990).

	US		*Denmark*	
LHS-variable	*n*=3 *months*	*n*=10 *years*	*n*=3 *months*	*n*=10 *years*
$\Delta^{-1}S_t$	-1.20	-1.97	-2.03	-2.28
$R_t^{(1)}$	-2.73	-3.34	-3.80*	-3.99**

Table 2. ADF tests for multicointegration in farmland prices and rents from the Corn Belt area. One augmenting lag was used in the auxiliary regression.

LHS variable	*Estimate of* $r=\theta^{-1}$	*ADF-t-ratio*
$\Delta^{-1}P_t$	.0535	-4.60*
$\Delta^{-1}R_t$	.0531	-5.09*

Table A1. Critical values for the cointegration ADF-test allowing for I(2) variables. An intercept plus a trend have been included in the cointegration regression. The indices m_1 and m_2 indicate the number of I(1) and I(2) variables on the right hand side of the cointegration regression. The left hand side variable is an I(2) variable. T indicates the sample size. It is further assumed that all the I(2) variables of the model cointegrate into an I(1) relation.

		Probability of a Smaller Value							
		$m_2=1$				$m_2=2$			
m_1	T	.01	.025	.05	.10	.01	.025	.05	.10
0	25	-5.21	-4.72	-4.29	-3.88	-5.81	-5.25	-4.83	-4.41
	50	-4.66	-4.33	-4.01	-3.67	-5.14	-4.77	-4.45	-4.10
	100	-4.55	-4.18	-3.90	-3.59	-4.93	-4.56	-4.31	-3.98
	250	-4.41	-4.08	-3.83	-3.51	-4.81	-4.49	-4.20	-3.91
	500	-4.33	-4.04	-3.78	-3.49	-4.75	-4.42	-4.14	-3.84
1	25	-5.60	-5.10	-4.71	-4.30	-6.24	-5.68	-5.21	-4.80
	50	-5.11	-4.70	-4.42	-4.08	-5.62	-5.22	-4.89	-4.51
	100	-4.85	-4.54	-4.26	-3.94	-5.23	-4.90	-4.62	-4.29
	250	-4.73	-4.43	-4.19	-3.89	-5.11	-4.77	-4.50	-4.20
	500	-4.73	-4.42	-4.15	-3.87	-5.05	-4.74	-4.48	-4.18
2	25	-6.09	-5.57	-5.14	-4.69	-6.70	-6.17	-5.70	-5.22
	50	-5.47	-5.07	-4.74	-4.38	-5.98	-5.53	-5.17	-4.79
	100	-5.21	-4.86	-4.58	-4.26	-5.59	-5.19	-4.93	-4.62
	250	-5.07	-4.79	-4.51	-4.20	-5.35	-5.07	-4.80	-4.51
	500	-5.00	-4.73	-4.48	-4.18	-5.34	-5.02	-4.75	-4.46
3	25	-6.47	-5.95	-5.53	-5.08	-7.19	-6.63	-6.08	-5.89
	50	-5.89	-5.43	-5.13	-4.76	-6.23	-5.81	-5.48	-5.12
	100	-5.52	-5.18	-4.91	-4.59	-5.97	-5.58	-5.25	-4.92
	250	-5.38	-5.05	-4.78	-4.74	-5.69	-5.37	-5.07	-4.80
	500	-5.34	-5.04	-4.78	-4.50	-5.67	-5.33	-5.06	-4.76
4	25	-6.95	-6.37	-5.90	-5.44	-7.61	-6.93	-6.43	-5.91
	50	-6.35	-5.85	-5.47	-5.10	-6.64	-6.18	-5.82	-5.41
	100	-5.86	-5.49	-5.20	-4.89	-6.09	-5.76	-5.50	-5.16
	250	-5.66	-5.35	-5.08	-4.77	-5.95	-5.61	-5.34	-5.04
	500	-5.63	-5.31	-5.06	-4.76	-5.92	-5.56	-5.29	-5.02

NOTE. The standard errors of the fractiles vary, but generally they lie in the interval {.01-.03}. The simulations were based upon 10000 replications. An intercept and a time trend were included in the cointegration regression. m_1 and m_2 denote the number of I(1)- and I(2)-regressors, respectively.