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A Dynamic Model of Altruistically-Motivated Transfers

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Abstract

This paper studies a dynamic Markovian game of two inf nitely-lived altruistic agents without commitment. Players can save, consume and give transfers to each other. We identify a continuum of equilibria in which imperfectly-altruistic agents act as if they were a perfectly-altruistic dynasty which is less patient than the two agents themselves. In such equilibria, the poor agent receives transfers until both effectively pool their wealth and tragedy-of-the-commons-type ineff ciencies occur. We also provide a sharp characterization of strategic interactions in consumption and transfer behavior. This provides new insights relative to existing two-period models. It allows us to differentiate between the Samaritan's dilemma – e.g. a child runs down its assets ineff - ciently fast in anticipation of transfers – and what we refer to as the Prodigal-Son dilemma – e.g. parents do not leave an early bequest, anticipating a child's prof igate behavior.

Keywords: consumption-saving decisions; inter-vivos transfers; altruism; differential games.

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1 Introduction

Dynamic economies with altruistic agents are an important class of models, but the literature has so far restricted itself to studying rather special cases. Our research agenda aims to fill this gap by providing a tractable theory for the behavior of imperfectly-altruistic agents in a fully-dynamic setting without commitment. In particular, we hope to provide a building-block model that is flexible and stable enough to be used in larger settings, such as heterogeneous-agents models in macroeconomics (as already done in Barczyk, 2012), but potentially also in microeconomic models of the family and development.

We have found that in order to understand the fundamental workings and tensions in such frameworks (which so far seem to have obstructed progress in the literature) one has to begin by studying the simplest-possible setting. This is what we do in this paper: we study Markov-perfect equilibria in a deterministic environment inhabited by two infinitely-lived altruistic agents. Their only sources of income are a risk-free return on savings and voluntary transfers from the other player. These assumptions imply a homogeneous environment, which we exploit analytically.

We find a continuum of *tragedy-of-the-commons-type equilibria* with the following features. When the asset distribution is imbalanced, the poor player receives an increasing schedule of transfers that give her incentives to save herself out of poverty. Once the asset distribution is sufficiently balanced, players essentially pool their assets and tragedy-of-the-commons-type inefficiencies occur. Agents consume at inefficiently high rates out of the common pool unless both players are perfectly altruistic. This is a novel type of equilibrium (it depends crucially on the infinite-horizon assumption) and exists despite well-defined property rights. It can rationalize why even families with imperfectly-altruistic members and intact property rights may behave as if they were a perfectly-altruistic dynasty. However, this fictitious dynasty has a strictly higher discount rate than the individual household members.

Furthermore, we characterize dynamic incentives and distortions in consumption-savings decisions induced by strategic interactions between altruists. There are three new points we make with respect to the existing two-period models. First, in addition to the known possibility of over-consumption, agents under-consume in certain situations with respect to the efficient allocation; indeed, both under- and over-consumption are present in the class of equilibria we find. Second, the model predicts that inefficiencies occur long before transfers actually flow, a feature that the two-period models in the literature are necessarily silent upon. Third, we find that not only are the recipient's consumption-savings decisions distorted (a phenomenon known as the Samaritan's dilemma) but also the donor's.

Our analysis allows us to draw a sharp distinction between the Samaritan's dilemma, which in our setting is characterized by the *Party Theorem*, and what we call the *Prodigal-Son dilemma*. The latter says that under imperfect altruism no equilibrium exists in which a rich donor lifts a poor recipient out of poverty and both players are self-sufficient ever after. This may, for example, explain why in reality early bequests are rarely observed or why certain forms of development aid, e.g. large infusions of cash linked to a pledge that no additional aid will be granted ever after, are futile. The potential donor realizes that the recipient would squander the transfer, come back, and ask for more. The squandering of the transfer is the Samaritan's dilemma (the *party*) whereas not providing transfers in anticipation of this is the Prodigal-Son dilemma. Both rest importantly on the assumption of no-commitment.

On the technical side, we argue that it is useful to work in continuous time in dynamic-altruism models. Certain strategic interactions are of second order, which makes instantaneous best-response functions constant and eliminates multiplicity of equilibria in the stage games. Furthermore, our approach enables us to characterize equilibria by ordinary differential equations (ODEs) and a set of boundary conditions; the number of boundary conditions tells us if to expect no equilibrium, a finite number or a continuum of them for any given equilibrium type. Finally, we propose an equilibrium concept that allows us to study non-smooth equilibria in a differential game[1] with the possibility of measure-valued controls (mass transfers in our setting).

In the model there are two infinitely-lived altruistic agents. One-sided altruism, perfect altruism, and selfish preferences are nested. Players decide about consumption, savings in a riskless asset (subject to a no-borrowing constraint), and a non-negative transfer to the other agent. They are endowed with an initial stock of assets

[1]A *differential game* is a continuous-time game in which the law of motion of the state vector is determined by differential equations; strategies are usually restricted to be Markov. The standard solution concept is Markov-perfect equilibrium (also referred to as *feedback* or *closed-loop* solution), but other concepts such as Stackelberg or Markov equilibrium (*open-loop*) are also common. For an introduction to the theory of differential games for economists see the book by Dockner et al. (2000) and chapter 13 in Fudenberg & Tirole (1993).

but have no labor income. This assumption, together with homotheticity of preferences, allows us to exploit the homogeneity of the setting and reduce the dimensionality of the state space to one.

The literature on altruism has analyzed a host of static and two-period models for transfers in the tradition of Becker's (1974) seminal paper. The analysis of Lindbeck & Weibull (1988) focuses on the Samaritan's dilemma and highlights the existence of multiple equilibria in a two-period model. Among others, Bernheim & Stark (1988) and Bruce & Waldman (1990) study two-period models that emphasize the Samaritan's dilemma and its consequences for economic policy. But, restricting the analysis to two periods limits what can be learned about the dynamics of altruism and prevents us from using the model in more realistic settings.

In the macroeconomics literature, dynamic models often have to take a stance on how agents within a dynasty or family are connected. Two standard workhorse models highlight this: the infinitely-lived household is justified by the assumption that altruistic concerns connect subsequent generations as in Barro's (1974) seminal paper, whereas pure life-cycle overlapping-generations models are usually populated by households that act in complete isolation. While these two extremes are often convenient representations, there is a substantial literature which deems it important to employ a model which lies somewhere in-between. Abel (1987) studies under which conditions one of the two inter-generational transfer motives is operative that are necessary for Barro's (1974) neutrality result to hold. Laitner (1988) assesses the impact of a social-security system on capital accumulation in an overlapping-generations economy in which children and parents are imperfectly altruistic. However, while generations are allowed to interact strategically in his setting, they overlap for only one period. Altig & Davis (1988) study an array of inter- and intra-generational redistributive policies in an economy with altruistic agents who overlap for a large number of periods, but they assume commitment.

Furthermore, there are many computational studies in the macro literature in which altruistic agents overlap for more than one period. However, the authors make simplifying assumptions in order to circumvent the tensions that we analyze. Laitner (1992) provides a framework in which agents overlap for many periods, but agents are restricted to be perfectly altruistic. Fuster et al. (2007) build on this framework to study pension systems. Nishiyama (2002) studies a setting with imperfect altruistic households in which generations overlap for at most two periods, but rules out the possibility that transfers are used for saving. In Kaplan (2012), imperfectly-altruistic parents and children interact strategically, but parents are not allowed to save.

In the applied microeconomics literature, many studies build on the *collective model*, which is due to Chiappori (1988). The key assumption of the model is that the family can always coordinate on efficient allocations. In reality, one would expect efficient outcomes within the household if agents have the ability to commit to future allocations (say at the point of marriage). Mazzocco (2007) employs an extension of the collective model that nests the possibilities of commitment and no-commitment and strongly rejects the assumption of commitment in the data. In light of this evidence, many authors have pointed out that it is important to explore other, non-cooperative models for dynamic interaction between altruistic agents, which is what we do in this paper. This case is even stronger for *inter*-household interaction (e.g. between parents and adult children) than for *intra*-household interaction.

The remainder of the paper is structured as follows. Section 2 outlines the setting of the model and characterizes the set of Pareto-efficient allocations. Section 3 studies dynamic incentives and distortions in consumption-savings decisions induced by strategic interactions between imperfect altruists. Section 4 exploits the homogeneous environment by reducing the dimensionality of the state space to one which makes additional analysis feasible. Section 5 characterizes equilibria and presents our main results. Section 6 concludes and points out the way for future research.

2 Setting

2.1 Physical environment

Time t is continuous. There are two infinitely-lived agents in the economy. We will denote variables for the first agent, whom we will refer to as "she", as plain lower-case letters, e.g. c_t. Variables referring to the second agent, whom we will call "he", are denoted with prime-superscripts, e.g. c'_t. Both agents can hold a non-negative amount k_t in a riskless asset that pays a time-invariant rate of interest r.

In each instant of time, agents choose a consumption rate $c_t \geq 0$ and a non-negative transfer g_t to the other agent (g stands for "gift"). Transfers may either be of “f ow type”, meaning that she gives $g_{ft}\Delta t$ to him over a short time period Δt, or of “mass type”, e.g. she gives a large quantity $g_{mt} \in [0, k]$ to him instantaneously. We will denote transfer policies by a vector $g_t = (g_{ft}, g_{mt})$.

There is a no-borrowing constraint for both agents: we require $k_t \geq 0$.[2] This is the natural borrowing limit in this setting if we assume that agents cannot borrow against future transfers from the other player.

When one of the players is broke, the specif c protocol that we impose on transfers and consumption becomes important. We take the stand that players give transfers g_t and g'_t simultaneously in the beginning of inf nitesimal periods Δt, and that consumption c_t and c'_t is chosen simultaneously after this. We impose that a broke player cannot give transfers, i.e. $k_t = 0$ implies $g_t = 0$. Whenever $k_t > 0$, however, there is no upper bound on the rates $\{c_t, g_{ft}\}$.[3] In order to enforce feasibility of consumption when she is broke, we def ne her *realized consumption* as

$$c_t^* = \begin{cases} c_t & \text{if } k_t > 0 \text{ or } g'_{mt} > 0, \\ \min\left\{c_t, g'_{ft}\right\} & \text{otherwise.} \end{cases} \tag{1}$$

This says that she cannot eat more than he gives to her when broke, but she may announce plans to do so.

If transfers are of the f ow type, (k_t, k'_t) are continuous functions of time. The laws of motion are

$$\dot{k}_t = rk_t - c_t^* - g_{ft} + g'_{ft}, \tag{2}$$

$$\dot{k}'_t = rk'_t - c'^*_t - g'_{ft} + g_{ft}, \tag{3}$$

where dots denote the time-derivative of a variable. A mass transfer g_{mt} induces a jump of size g_{mt} in the trajectories, i.e. $\lim_{h \searrow 0} k_{t+h} = k_t + g_{mt}$; see Figure 1 for an example in which she provides a mass-point transfer of size $\frac{1}{2}$ to him. Allowing for mass transfers is reasonable – in the real world it is feasible that large amounts

[2]We will only state expressions for her from now on when it is understood that his case is mirror-symmetric; please consult our online appendix for the analogous equations for him.

[3]Note that in a continuous-time setting, an agent is never constrained in the choice of the f ow rate of consumption (or transfers) unless he is *directly* at the borrowing constraint. Given any small amount ε of assets, she can always choose an arbitrarily high consumption rate M for some short time interval $\Delta t < \varepsilon/M$.

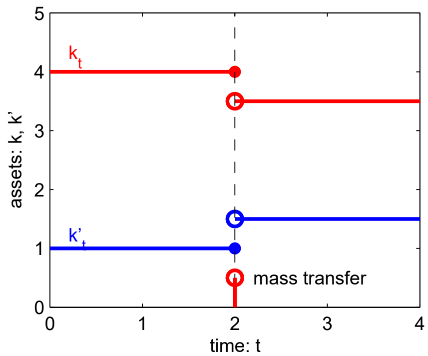

Figure 1: Illustration of mass transfer: $g_{m2} = 0.5$

of money change hands instantaneously –, and will turn out to be convenient for our analysis.[4]

Finally, her preferences are given by

$$v_0 = \int_0^\infty e^{-\rho t}[u(c_t^*) + \alpha u(c'^*_t)]dt, \tag{4}$$

where $\rho > 0$ is the discount rate and $0 \leq \alpha \leq 1$ is the parameter which measures the intensity of altruism.[5] He is a mirror-symmetric copy of hers, but might have a different altruism parameter $0 \leq \alpha' \leq 1$. His preferences are represented by

$$v'_0 = \int_0^\infty e^{-\rho t}[u(c'^*_t) + \alpha' u(c_t^*)]dt. \tag{5}$$

We assume that the agents have the same discount rate ρ; this is crucial for our analysis. We also assume that agents do not differ in form of the felicity function $u(\cdot)$, for which we choose the form $u(c) = \ln c$.[6]

[4]In purely mathematical terms, one could allow for an even larger class of transfers than the one we consider. To see this, define the cumulative transfer function $G(t)$ – analogous to a cdf in probability theory – as the integral over all transfers given on the time interval $[0, t]$. Technically, any function G belonging to the set A of weakly increasing right-continuous functions induces a valid transfer measure. The class of transfers we allow for is the set of functions B that are piecewise C^1 (continuously differentiable) and right-continuous. An example for a function $f : f \in A, f \notin B$ is the Cantor function (or Devil's Staircase), a classical example in real analysis. We thank Andrew Clausen and an anonymous referee for useful comments on this issue.

[5]With this separable formulation of altruistic preferences we are in line with the bulk of the literature.

[6]It is essential for our approach that u is homothetic. Many results still hold for power utility.

2.2 Equilibrium def nition

We restrict attention to Markov-perfect equilibria that are piecewise differentiable. The payoff-relevant state is obviously (k, k'). We look for a set of policies $\{c, (g_f, g_m)\}$, each mapping the state space $\mathbb{R}^2_+ = [0, \infty) \times [0, \infty)$ to $\mathbb{R}_+$, and value functions $\{v, v'\}$, mapping $\mathbb{R}^2_+$ to the extended real line $\overline{\mathbb{R}}$. We restrict attention to equilibria where these functions are smooth on subsets (*regions*) of the state space. We call a collection of value and policy functions $\{v, c, g; v', c', g'\}$ a *prof le* and def ne:

Def nition 1 (Piecewise-smooth prof les) *A prof le* $\{v, c, g; v', c', g'\}$ *is* piecewise-smooth *if it fulf lls the following conditions. Let* $\mathcal{K}_1, \ldots, \mathcal{K}_n$ *be open connected subsets (*regions*) of* $\mathbb{R}^2_+$ *such that* $\mathcal{K}_i \cap \mathcal{K}_j = \emptyset$ *for any* $i \neq j$ *and* $\bigcup_{i=1}^n \bar{\mathcal{K}}_i = \mathbb{R}^2_+$ *(where* $\bar{S}$ *denotes the closure of set* S*).*

1. (mass-transfer regions) g_m *is such that it def nes* mass-transfer regions*: suppose that* $g_m(\tilde{k}, \tilde{k}') > 0$ *for some* $(\tilde{k}, \tilde{k}') \in \mathcal{K}_i$ *(for some* i*); then this implies that* $g_m(k, k') = \min\{\tilde{g}_m : \tilde{g}_m > 0$ *and* $(k - \tilde{g}_m, k' + \tilde{g}_m) \notin \mathcal{K}_i\}$ *and* $g'_m(k, k') = 0$ *for all* $(k, k') \in \mathcal{K}_i$*.*

2. (smoothness) *Inside each region, the functions* $\{v, c, g_f; v', c', g'_f\}$ *are continuously differentiable.* v *is continuous on* $(0, \infty) \times [0, \infty)$ *if* $\alpha' = 0$*, and continuous on* $\mathbb{R}^2_+$ *otherwise. For any convergent sequence* $\{x_j\}_{j=1}^\infty \to x^*$ *such that* $x_j \in \mathcal{K}_i$ *for all* j *and* $x^* \notin \mathcal{K}_i$ *(for some* i*), the limit* $\lim_{j \to \infty} c(x_j)$ *exists.*

Note that we leave policies on the boundaries of regions unrestricted, i.e. it may be the case that $\lim_{j \to \infty} c(x_j) \neq c(x^*)$ for a convergent sequence $\{x_j\} \to x^*$ satisfying $x_j \in \mathcal{K}_i$ for all j and $x^* \notin \mathcal{K}_i$. This will be of importance especially for the case where one player is broke.[7] For the case that he is self sh, we must allow v to reach

[7] Our setting thus allows for a larger set of strategies than is usually considered in the differential-games literature. In this literature, strategies are usually restricted to be such that the law of motion is Lipschitz-continuous in order to ensure existence and uniqueness of the ordinary differential equation (ODE) for the state variable. As Fudenberg & Tirole (1993) point out in their textbook, this leads to the following inconsistency: when checking optimality of best responses, one typically uses Pontryagin's maximum principle. In order to do this, one needs to allow also for deviations that are only piecewise C^1, thus including functions that are not Lipschitz-continuous. Our equilibrium concept is a step towards resolving this issue: it is stated recursively and makes no reference to the path of (k, k'), nor the ODE associated with it. Furthermore, it allows us to deal with mass-type policies in a simple manner. See our online appendix for a discussion, e.g. how we interpret the path of the state in cases where the ODE does not satisfy the Lipschitz condition.

$-\infty$ at $k = 0$ and thus to be discontinuous at this point; after all she cannot expect transfers from him and we should expect $c(0, k') = 0$ for any k'. For the case $\alpha' > 0$, observe that an altruistic donor can always ensure positive consumption of the recipient by providing transfers. In this case we assume continuity of v on $\mathbb{R}^2_+$, thus ruling out the possibility $v(0, k') = -\infty$ and $c = 0$.[8]

Figure 2 shows an example of regions compatible with piecewise smoothness. He gives mass transfers in region $\mathcal{K}_1$, represented by arrows, making sure that she has at least half as much wealth as he has. In region $\mathcal{K}_2$ the wealth distribution is relatively balanced and so an intuitively appealing candidate could be that transfers are zero; f nally, she gives f ow transfers to him on $\mathcal{K}_3$ where he is poor relative to her.

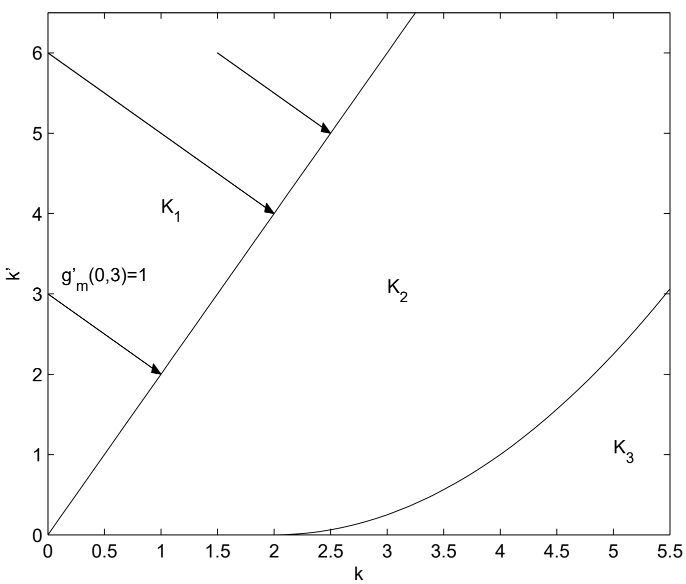

Figure 2: Example for regions

We now turn to the problem of f nding agents' best responses. Following the macroeconomic literature, we adopt a recursive notion of best-responding. This will allow us to deal with issues of non-smoothness and mass-type policies that we cannot

[8]Assuming continuity of v also means that we impose value-matching between smooth regions. We do not consider this a restrictive assumption. It is easy to show that value matching is a necessity if the state is controllable for an agent. We will see that controllability indeed holds, at least once we restrict attention to homogeneous strategies.

address using standard approaches from the differential-games literature.

First note that when his strategy is Markov, her best-response problem is a dynamic-programming problem, and her response will hence also be Markovian. Consider her situation at time t for a given state $x_t \equiv (k_t, k_t')$, and suppose that we are given the equilibrium value function v. Recall the timing protocol that specif es that transfers are given in the beginning of a short time interval Δt, and that consumption is chosen thereafter. Given his equilibrium strategy $\{c', g'\}$, *Bellman's principle* says that we can write her problem over a short horizon Δt as a *transfer stage*

$$v(x_t) = \max_{g \geq 0,\ g=0 \text{ if } k=0} \left\{ v^c\big(x_t; g, g'(x_t)\big) \right\}, \tag{6}$$

and an ensuing *consumption stage*

$$v^c\big(x_t; g, g'\big) = \max_{c \geq 0} \Big\{ u\big(c^*(c, k, g')\big)\Delta t + \alpha u\big(c'^*(c'^0(x_t; g, g'), k', g)\big)\Delta t + e^{-\rho\Delta t} v(x_{t+\Delta t}) \Big\},$$

$$\text{s.t.} \quad x_{t+\Delta t} = \begin{pmatrix} k_t - g_m + g_m' + \big[r k_t - c - g_f + g_f'\big]\Delta t \\ k_t' + g_m - g_m' + \big[r k_t' - c'(x_t; g, g') + g_f - g_f'\big]\Delta t \end{pmatrix}, \tag{7}$$

where realized consumption c^* and c'^* is def ned in (1). Due to the stage form, we introduce a *consumption-stage strategy* in the form of a function $c^0(x; g, g')$ which tells us her best response to any (g, g')-tuple. Its relationship to the equilibrium consumption function is $c(x) = c^0(x; g(x), g'(x))$. We will refer to the joint problem (6) and (7) as her Δt*-problem*. This problem is at the heart of our notion of best-responding.

We will now follow the approach that standard control theory takes to derive the Hamilton-Jacobi-Bellman equation (HJB): we take a f rst-order Taylor expansion of the term $e^{-\rho\Delta t} v(k_{t+\Delta t}, k'_{t+\Delta t})$ in Δt in equation (7) and then take limits as $\Delta t \to 0$. However, we make two adjustments. Since g_m and g_m' do not vanish as $\Delta t \to 0$, we approximate v around the point $x_m = (k - g_m + g_m', k' + g_m - g_m')$. Also, we take directional derivatives in the case that v has kinks.[9] The expansion is:

$$\begin{aligned} e^{-\rho\Delta t} v(k_{t+\Delta t}, k'_{t+\Delta t}) =& (1-\rho\Delta t) v(k_t - g_m + g'_m, k'_t + g_m - g'_m) + \\ & + \dot{x} \cdot \nabla_{\dot{x}} v(k - g_m + g'_m, k' + g_m - g'_m) + o(\Delta t), \end{aligned}$$

where $o(\Delta t)$ are terms of order lower than Δt, i.e. $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$, and where $\nabla_{\dot{x}} v(x)$ denotes the directional gradient of v at x when going into the direction implied by the vector $\dot{x}$.[10]

First, we see that using this expansion in (6) and (7) for any fixed policies as $\Delta t \to 0$ and taking the limit $\Delta t \to 0$ leads to the *order-0 requirement*[11]

$$v(x) = \max_{g_m \in [0,k]} v\big(k - g_m + g'_m(x), k' + g_m - g'_m(x)\big). \tag{8}$$

This requirement is only relevant for determining optimality of mass transfers. It becomes vacuous when both g_m and g'_m are zero, and it contains no information about optimality of the policies $\{c, g_f\}$ and c^0. We immediately see that this will imply that v is weakly decreasing in the direction $(-1, 1)$.

In order to find optimality conditions for the flow-type policies, we also have to keep terms of order Δt. Use again the Taylor expansion for $e^{-\rho\Delta t} v(x_{t+\Delta t})$ from (6) and (7), and observe that only mass transfers g_m can be optimal that satisfy (8) – after all, these are the highest-order terms. For such optimal mass transfers, we have $v(x) = v(x_m)$ by (8). Now use this equality, divide by Δt and take limits as $\Delta t \to 0$ to obtain the order-1 requirement, consisting of the transfer stage

$$\begin{aligned} \rho v(x) =& \max_{g \geq 0,\ g=0 \text{ if } k=0} \rho v^c(x; g, g'(x)) \qquad (9) \\ \text{s.t.} \qquad g_m =& \arg\max_{\tilde{g}_m \in [0,k]} v\big(k - \tilde{g}_m + g'_m(x), k' + \tilde{g}_m - g'_m(x)\big), \end{aligned}$$

[9] This directional approach to HJBs yields very stable results and is intimately related to viscosity solutions for HJBs, see Bardi & Capuzzo-Dolcetta (2008). Viscosity solutions are the agreed-upon solution concept for HJBs in the general (non-smooth) case in 1-agent settings. See our online appendix for a brief introduction to the viscosity concept and why it does not solve non-smoothness issues in differential games.

[10] We use the definition of the directional gradient that normalizes the length of $\dot{x}$. Formally, define $\nabla_z f(x) = \lim_{h \searrow 0} [f(x + hz) - f(x)]/(h|z|)$ for any vector z and positive h.

[11] We call this requirement "order-0" because it contains terms up to order $(\Delta t)^0$, whereas the "order-1 requirement" will contain terms up to order $(\Delta t)^1$.

and the consumption stage

$$\rho v^c(x; g, g') = \max_{c \geq 0} \Big\{ u\big(c^*(c, k, g')\big) + \alpha u\big(c'^*(c'^0(x; g, g'), k', g)\big) + \dot{x} \cdot \nabla_{\dot{x}} v(x_m) \Big\}, \tag{10}$$

where $$\dot{x} = \big(rk - c^*(c, k, g') - g_f + g'_f, rk' - c'^*(c'^0(x; g, g'), k', g) + g_f - g'_f\big),$$
$$x_m = \big(k - g_m + g'_m, k' + g_m - g'_m\big).$$

We see immediately that inside a region without mass transfers, i.e. for $x \in \mathcal{K}_i$ such that $g_m(x) = g'_m(x) = 0$, the best response in the consumption stage is independent of (g, g') since then transfers do not affect the gradient $\nabla_{\dot{x}} v$. We will also show later that when broke at $k = 0$, it is a dominant strategy to announce the unconstrained consumption plan, i.e. to set $c^0 = (u')^{-1}(v_k)$ for all (g, g'), where we denote $v_k = \partial v / \partial k$. So the distinction between c and c^0 will only matter on boundaries between regions and when mass transfers flow.

Our recursive equilibrium definition now is:

Definition 2 *A* (piecewise-smooth) Markov-perfect equilibrium (MPE) *is a piecewise smooth profile $\{v, c, g; v', c', g'\}$ and consumption-stage strategies $\{c^0, c'^0\}$ that satisfy mutual best-responding:*

1. $\{v; c, c^0, g\}$ *characterize a best response to $\{c'^0, g'\}$, i.e. for all $x \in \mathbb{R}^2_+$:*
   (a) (order-0 optimality) v *satisfies* (8)*, and $g_m(x)$ attains the maximum in* (8).
   (b) (order-1 optimality) v *satisfies* (9) *and* (10)*, and*
      i. $g(x)$ *attains the maximum in* (9).
      ii. $c^0(x; g, g')$ *attains the maximum in* (10) *for all* $(g, g') \geq 0$*, and* $c(x) = c^0(x; g(x), g'(x))$.
2. $\{v'; c', c'^0, g'\}$ *characterize a best response to $\{c^0, g\}$ in the sense laid out in point 1.*

It is worth to point out several features of this definition.

First, players' strategies are required to be optimal *for all* points in the state space, even if these are not reached given the initial condition (k_0, k'_0). As is well-known in the literature, Markov perfection thus implies subgame perfection.

Second, note that Definition 2 allows us to check optimality of any set of strategies that satisfy assumption 1, something that is not feasible given the standard approach in the literature. Discontinuities in his strategy induce a discontinuity in the state's law of motion in her best-response problem, which makes the standard approach break down.[12]

Third, we point out that checking optimality of mass transfers requires considering deviations from the equilibrium mass transfer in the first-order requirement (9), which may escape the reader's attention at first. To be precise, the definition says that it is not enough to check that $g_m(x)$ is order-0 optimal in (8) and that $g_f(x)$ is order-1 optimal in (9) given $g_m(x)$. We also need to make sure that no feasible policy $\{\tilde{g}_m, \tilde{g}_f\}$ with $\tilde{g}_m \neq g_m(x)$ gives better results in the order-1 requirement (9), always provided it does equally well up to order-0 terms in (8).

2.3 Pareto-optimal allocations

The set of Pareto-efficient allocations will serve as an important benchmark throughout our analysis. Consider a benevolent planner who places a weight η on her lifetime value and a weight $(1-\eta)$ on his. Given initial assets $K_0 = k_0 + k_0'$, the planner chooses a savings policy $K_t \geq 0$ and consumption policies c_t, c_t' for $0 \leq t < \infty$ to maximize

$$J^\eta = \eta \int_0^\infty e^{-\rho t}\big[u(c_t) + \alpha u(c_t')\big]dt + (1-\eta)\int_0^\infty e^{-\rho t}\big[u(c_t') + \alpha' u(c_t)\big]dt. \quad (11)$$

Varying $\eta \in [0,1]$ then yields all allocations on the Pareto frontier. Intra-temporally, the planner divides consumption between the two agents such that marginal utilities are proportional to each other, i.e.

$$u_c(c_t) = \frac{(1-\eta) + \alpha\eta}{\eta + \alpha'(1-\eta)} u_c(c_t') \qquad \forall t. \quad (12)$$

[12]Note that our equilibrium concept, in turn, requires the specification of value functions, which the standard approach does not. It may not be obvious how to construct value functions from policies for piecewise-smooth profiles that induce ODEs violating the Lipschitz condition. See our online appendix for an example of how we may construct $\{v, v'\}$ in such a case, and how our equilibrium concept still gives us reasonable equilibrium predictions and a satisfactory interpretation for the path of the state.

Inter-temporal optimality implies that the Euler equation from a standard (self sh) one-person consumption-savings problem must hold:

$$\frac{d}{dt}u_c(c_t) = (\rho - r)u_c(c_t) \qquad \forall t. \tag{13}$$

The closed-form solution for the consumption plans is then given by

$$c_{\eta t} = \rho P^*_\eta K_t, \qquad c'_{\eta t} = \rho(1 - P^*_\eta)K_t, \tag{14}$$
$$\text{where} \quad P^*_\eta = \frac{\eta + (1-\eta)\alpha'}{1 + \eta\alpha + (1-\eta)\alpha'}, \quad K_t = k_t + k'_t.$$

Intuitively, the sum of both agents' consumption, $c_t + c'_t = \rho K_t$, is what a single agent with assets K_t would optimally consume. Here, the planner splits the amount ρK_t between the two players, where the splitting rule depends on the weight η as well as the altruism parameters α and α'.

3 Understanding players' incentives

In this section we analyze the consumption, savings, and transfer incentives that players face. Throughout this section, we restrict attention to the interior of smooth regions without mass transfers.

3.1 Instantaneous best response functions

Inside smooth regions without mass transfers, the optimality criteria (8), (9) and (10) from the equilibrium def nition collapse to the following standard Hamilton-Jacobi-Bellman equation (HJB):

$$\begin{aligned}\rho v =& \alpha u(c') + (rk + g')v_k + (rk' - g' - c')v_{k'} + \\ &+ \max_{g \geq 0}\Big\{g\big[\underbrace{v_{k'} - v_k}_{\equiv \mu:\ \text{transfer motive}}\big]\Big\} + \max_{c\geq 0}\Big\{u(c) - cv_k\Big\},\end{aligned} \tag{15}$$

where subscripts denote partial derivatives, e.g. $v_k = \frac{\partial v}{\partial k}$. We suppress the dependence of the functions v etc. on (k, k'), and we omit the f-subscript from g_f throughout this section for better readability.[13] The first-order condition (FOC) for consumption is given by

$$u_c(c) = v_k. \tag{16}$$

Continuous time provides us here with a crucial simplification with respect to discrete time: his contemporaneous consumption decision c' does not affect her optimal choice c, nor does his transfer decision g'. In other words, her best-response function over a short amount of time is a constant. Computationally, this means that we can obtain her optimal consumption strategy given the value function in the end of a Δt-interval as in a standard consumption-savings problem – there is no need to calculate her best responses for each action of the other player.[14] Furthermore, constant best responses ensure existence and uniqueness of equilibrium in the stage games, i.e. the interactions of players over very short horizons Δt-games. This is not the case in discrete time; for example, Lindbeck & Weibull (1988) find multiple equilibria already in a two-period setting.

A second important simplification with respect to discrete time is that there is no interaction between the decision problems of the consumption and the transfer choices: the two max-operators for c and g are separate.

These simplifications with respect to discrete time arise because immediate strategic considerations are of second order.[15] It is important to stress, however, that this separability of contemporaneous actions is only valid over short horizons. Over time,

[13]The HJB is a partial differential equation (PDE) that imposes restrictions on v and its partial derivatives v_k and $v_{k'}$ for all points (k, k'). His problem is characterized by a mirror-symmetric HJB for his value function v'. See our online appendix for his equations.

[14]One could argue that this problem can be avoided in discrete time by having players move sequentially. However, this has the following disadvantages: first, assumptions on the timing protocol can influence the results. Second, discrete time periods are tantamount to assuming commitment over the period length. Note that the player who moves first cannot adjust his decision when observing the other's action, even though this might be in his best interest. The advantage of the continuous-time setting is that the timing protocol does not matter (at least for consumption) since agents can react infinitely fast – the planning horizon goes to zero.

[15]To see this, consider following first-order approximation:

$$v_k(k_{t+\Delta t}, k'_{t+\Delta t}) = v_k(k_t, k'_t) + \underbrace{v_{kk'}(k_t, k'_t)\big[rk'_t - c'_t - g'_t \ldots\big]\Delta t}_{\text{of second order}} + \ldots.$$

Clearly, as Δt becomes small, the changes in the marginal value of saving induced by his policies c'_t and g'_t over the planning period Δt become negligible. In the limit, v_k equals its current value and so her best response is unaffected by his actions.

his decisions do of course matter for her best response, as will become evident from the Euler equation in Subsection 3.3. Contemporaneously, however, $(v_k, v_{k'})$ are enough to encode all relevant information about the continuation of the game.

We now turn to the optimal transfer choice in (15), which is a linear optimization problem. The term $\mu \equiv (v_{k'} - v_k)$ is the marginal benef t of transfering a marginal unit of resources from her to him. We will refer to μ as her *transfer motive*. Whenever μ is negative, the optimal transfer is zero. If $\mu = 0$, any transfer f ow is consistent with optimality; in this case she is locally indifferent with regard to the distribution of assets between him and her. If μ was strictly larger than zero, she would want to choose g as large as possible. In our setting, she would indeed choose to give a mass transfer – which we had assumed not to be the case when writing down (15). This is only consistent, since equation (8) shows that it is actually impossible that $\mu > 0$ in the f rst place (we will later formalize these insights, see Proposition 2).

3.2 Over- and under-consumption

Players' joint consumption decisions generally lead to ineff ciencies because players do not fully internalize the effect of their actions on the other. We will now have a closer look at these ineff ciencies.

Consider the game at time t for a given state (k_t, k'_t), where $k_t > 0$, $k'_t > 0$ and no mass transfers are given. Suppose that after a short period Δt the equilibrium policies and the continuation values are given. How would the agents feel about different consumption tuples (c, c') over a short time interval Δt, assuming that afterwards the equilibrium strategies are played? We write today's value $v(k_t, k'_t)$ as a sum of f ow utility collected over Δt and a f rst-order approximation of the continuation value:

$$v(k_t, k'_t) = v(k_t, k'_t) - \rho v \Delta t + \max_c \left\{ u(c)\Delta t + \alpha u(c')\Delta t + v_k \dot{k} \Delta t + v_{k'} \dot{k}' \Delta t \right\} + o(\Delta t).$$

Def ne the term inside the curly bracket divided by Δt as

$$H(c, c') = u(c) + \alpha u(c') + v_k \dot{k} + v_{k'} \dot{k}'.$$

$H(c, c')$, the Hamiltonian, tells us how she feels about consumption tuples (c, c'), taking into account both f ow utility over Δt and their repercussions on the continuation

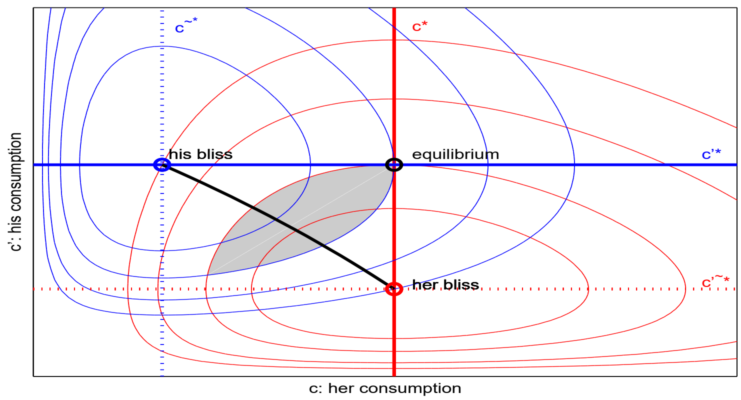

Figure 3: Equilibrium over Δt.

value of the game. We def ne his Hamiltonian $H'(c, c')$ equivalently.

Figure 3 plots the contours of H and H' as a function of (c, c'), where we f x some derivatives $v_k > v_{k'}$ and $v'_{k'} > v'_k$ for the continuation value (more on our choice for these follows below). The vertical solid line indicates $c^*(c') \equiv \arg\max_c H(c, c')$, which is her best-response function; the horizontal solid line is his best response $c'^*(c) \equiv \arg\max_{c'} H'(c, c')$. As discussed above, best responses are constant in the other player's contemporaneous action. The equilibrium Δt-allocation occurs at the intersection of the two best responses. The horizontal dotted line indicates $\tilde{c}'^*(c) \equiv \arg\max_{c'} H(c, c')$, which is the consumption rate that she would choose for him if she were allowed to do so; the vertical dotted line indicates his preferred consumption for her, $\tilde{c}^*(c') \equiv \arg\max_c H'(c, c')$.

An immediate observation is that the indifference curves of the two players are crossing each other, illustrating that they disagree about Δt-allocations. Only in the case of perfect altruism ($\alpha = \alpha' = 1$) will the indifference curves coincide. The crucial feature that Figure 3 highlights is the ineff ciency which generically arises in this environment. The shaded area emanating from the equilibrium allocation contains the allocations corresponding to Δt-Pareto-improvements; the curve connecting her and his bliss points indicates the Δt-Pareto-frontier.[16] Committing to any of the

[16]In order to trace out the Pareto frontier of the Δt-game, consider the problem of a planner who chooses (c, c') using a weight $\eta \in [0, 1]$ on her utility in the problem $\max_{c,c'} \{\eta H(c, c') + (1 - \eta) H'(c, c')\}$. Vary $\eta \in [0, 1]$ in this problem in order to trace out the Pareto frontier of the Δt-game, v.

policies over Δt would raise both players' welfare. However, note that both players would be tempted to break any such agreement and revert to the best-response strategy.

In the example of Figure 3, both players are over-consuming: it would be Pareto-improving if they coordinated on lower consumption rates. This is because both players would prefer the other to consume less, i.e. $\tilde{c}^* < c^*$ and $\tilde{c}'^* < c'^*$. However, it is also conceivable that one (or both) players are under-consuming. This would be the case if $\{v_k, v_{k'}; v'_k, v'_{k'}\}$ were such that $\tilde{c}^* > c^*$ and/or $\tilde{c}'^* > c'^*$. Indeed, in the class of equilibria we will analyze later (see Section 5.3) both over- and under-consumption occur in different regions of the state space.

We now study how over- and under-consumption are induced dynamically and how they are related to the altruism parameters (α, α') and the value functions' derivatives $(v_{k'}, v'_k)$.

3.3 Strategic interactions

In this subsection, we ask how his consumption and transfer strategies affect her consumption-savings behavior. This provides us with new insights on savings incentives in dynamic imperfect-altruism settings.

In order to understand the relevant trade-offs, consider her Euler equation inside a smooth region without mass transfers:[17]

$$\frac{d}{dt}\left[u_c(c_t)\right] = \underbrace{(\rho - r)u_c(c)}_{\text{standard = eff cient}} + \underbrace{\left[v_{k'} - \alpha u_c(c')\right]c'_k}_{\text{altruistic-strategic distortion}} + \underbrace{\left[v_{k'} - u_c(c)\right]g'_k}_{\text{transfer-induced incentives}} . \tag{17}$$

Here, c'_k denotes the partial derivative of his equilibrium consumption function c' with respect to k and g'_k is the partial derivative of the transfer function g' with respect to k; again, we drop the f-subscript for f ow transfers.

We gain intuition for the three terms on the right-hand side of (17) in a discrete-time version of her consumption-savings problem. There are three periods $t = 1, 2, 3$,

[17] In order to obtain her Euler equation, take the derivative of her HJB (15) with respect to k in a smooth region without mass transfers and use the fact that $\frac{d}{dt}u_c(c_t) = \frac{d}{dt}v_k = \dot{k}v_{kk} + \dot{k}'v_{kk'}$ by the FOC (16). The Euler equation (17) has to hold in the sense of a PDE inside all smooth regions, not only as an ODE on the equilibrium path. This is related to the equilibrium concept requiring subgame-perfection. We discuss this connection in a note in our online appendix.

	period 1	period 2	period 3
standard	$-u'(c_1)$	$+\beta R u'(c_2)$	
altruistic-strategic		$+\alpha\beta u'(c_2')\frac{\partial c_2'}{\partial k_2}$	$-\beta^2 R v_{k'}(k_3, k_3')\frac{\partial c_2'}{\partial k_2}$
transfer-induced		$+\beta u'(c_2)\frac{\partial g_2'}{\partial k_2}$	$-\beta^2 R v_{k'}(k_3, k_3')\frac{\partial g_2'}{\partial k_2}$

Table 1: Marginal costs and benefits in discrete-time Euler equation

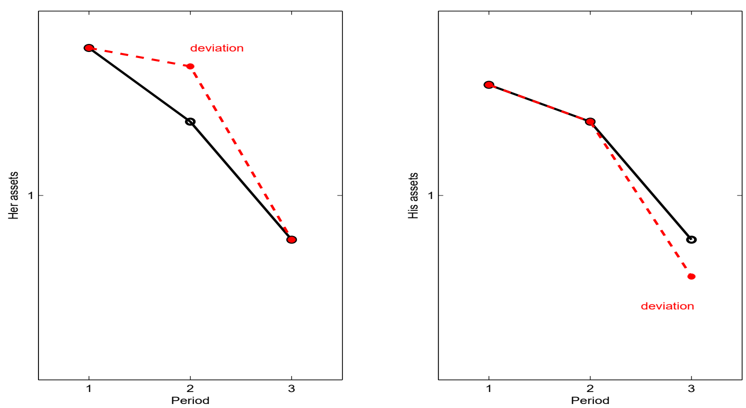

Figure 4: Discrete-time intuition for savings incentives

and she takes his strategy $\{c'(\cdot), g'(\cdot)\}$ and her continuation value $v(k_3, k_3')$ as given. Figure 4 shows the paths for assets (the solid lines) that result from the players' equilibrium strategies.

Consider now the following hypothetical deviation from her equilibrium strategy: she saves a marginal unit in period 1 and reverts back to the equilibrium level of assets in period 3 (the dotted line in the left panel of Figure 4). If her equilibrium behavior is optimal, then such a deviation is not profitable and its marginal effect on her criterion is zero. Table 1 accounts for all marginal effects this deviation has on her utility.

First, the two standard terms account for the usual consumption-savings trade-off: saving one unit more in period 1 costs marginal utility $u'(c_1)$, but yields $\beta R u'(c_2)$ in period 2 (β is the discount factor and R the gross interest rate). In continuous time, these two terms correspond to $(\rho - r)u_c(c)$, which coincides with the efficient growth rate of marginal utility, see equation (13). Thus the additional two terms on the right-hand side of (17) are distortions that make the agent stray from efficient behavior.

Second, the *altruistic-strategic distortion* is comprised of the terms in the second row in Table 1; they stem from his consumption response.

In period 2, his consumption will react to her higher asset level, which is captured by $\partial c_2'/\partial k_2$. Suppose for now that $\partial c_2'/\partial k_2 > 0$, which seems intuitive: if she has higher assets, he can count on receiving larger transfers from her in the future and/or is less likely to have to give transfers to her, so he can consume more today. Because his consumption increases, she realizes a gain $\alpha\beta u'(c_2)\partial c_2'/\partial k_2$ in period two; in continuous time this shows up as an immediate effect $-\alpha u_c(c')c_k'$. This constitutes an additional incentive to save, and, therefore, enters with the same sign as the interest rate R (or r) in the standard term.

However, his increased consumption comes at a cost: in period 2, he saves less and enters period 3 with lower assets (as shown by the dotted line in the right panel of Figure 4). Hence, the equilibrium path is left in period 3 and the economy goes to an equilibrium where he has $R\partial c_2'/\partial k_2$ less wealth. We expect $v_{k'}$ to be positive, so the term $-\beta^2 R v_{k'}(k_3, k_3')\partial c_2'/\partial k_2$ is negative and acts as a disincentive to save. In continuous time, the corresponding term $v_{k'}c_k'$ enters the Euler equation with the same sign as ρ, and thus discourages savings.

Which of the two terms forming part of the altruistic-strategic distortion dominates is directly related to his under- or over-consumption: the bracket $[v_{k'} - \alpha u_c(c')]$ in (17) is negative whenever he consumes less than she desires, i.e. if and only if $c' < \tilde{c}'^{*}$; it is positive otherwise. As long as $c_k' > 0$ and he is over-consuming, the altruistic-strategic consideration acts as a disincentive to save for her. She responds with front-loading consumption to his over-consumption. If he is under-consuming, the opposite is true: she saves more since this might induce him to consume more. In the class of equilibria we study later on both over- and under-consumption will occur.

The third row in Table 1 shows *transfer-induced incentives*. These two terms only come into play in regions where he gives f ow transfers, i.e. $g' > 0$. Let us suppose that he chooses transfers such that they reward thrift, i.e. $\partial g'/\partial k > 0$.

Since he rewards her for saving more she reaps a benef t $\beta u'(c_2)\partial g_2'/\partial k_2$ in period 2 (in continuous time: $u_c(c)g_k'$).[18] This acts as an incentive to save. However,

[18]Note that in order to revert back to the equilibrium level k_3 she immediately has to consume all additional transfers in period 2.

there is again a negative effect on his assets in period 3, as captured by $-\beta^2 R v_{k'}(k_3, k_3)\partial g'_2/\partial k_2$ (in continuous time: $v_{k'} g'_k$). It acts as a disincentive to save for her.

To see which of the two terms dominates, note that the bracket $[v_{k'} - u_c(c)]$ is equal to her transfer motive μ, see her FOC (16). Since we assumed that he is giving transfers to her, we expect that she strictly prefers not to give transfers to him, i.e. $\mu < 0$. This means that for the transfer regime under consideration $\mu g'_k$ is negative and thus his transfer schedule acts as an incentive to save for her, as was to be expected.

We now brief y consider the Euler equation (17) for the special cases of selfishness and perfect altruism. This will highlight how distortions disappear when altruism is either absent or perfect, and why we expect distortions to be stronger in the intermediate range of altruism.

Under self shness ($\alpha = \alpha' = 0$) we have $c'_k = g'_k = 0$; in equilibrium, he will have no reason to condition his behavior on her assets.[19] Then there are neither altruistic-strategic distortions nor transfer-induced incentives. We are left with the Euler equation from the one-agent world, which says that marginal utility grows at the eff cient rate $\rho - r$.

Under perfect altruism ($\alpha = \alpha' = 1$), in equilibrium we will have $v_{k'} = v_k = u_c(c)$: she values an additional unit of assets in his pocket ($v_{k'}$) as much as she values it in her own pocket (v_k). So the brackets $[v_{k'} - \alpha u_c(c')]$ and $[v_{k'} - u_c(c)]$ vanish and distortions are zero. This would manifest itself in the indifference curves in the Δt-equilibrium in Figure 3 lying on top of each other, indicating that agents are in perfect agreement.

From these two extreme cases we conjecture that distortions are strongest for intermediate levels of altruism. For values of α and α' close to zero, the behavioral responses c'_k and g'_k should go to zero. When α and α' approach one, instead, the brackets in (17) should go to zero since agents increasingly take into account the externalities of their behavior on the other.

[19] In Section 5.1, we will present formal propositions of the equilibrium under self shness and perfect altruism.

4 Exploiting homogeneity

What makes additional analysis feasible is the fact that the environment is homogeneous, i.e. both players have homothetic utility and income is proportional to assets. While there are currently two state variables, k and k', homogeneity reduces the dimensionality of the state space to one. As a result, the economy can take at most two directions from each point in the state space. This facilitates studying best responding in mass-transfer-type and non-smooth equilibria. Furthermore, the equations characterizing potential equilibria turn from partial differential equations (PDEs) into ordinary differential equations (ODEs). The number of boundary conditions for the ODEs will then provide crucial information on whether to expect zero, a f nite number, or a continuum of equilibria for each given equilibrium type. Such predictions would be a formidable task in a two-dimensional setting with PDEs.

We def ne the following mapping from pairs (k, k') to pairs (P, K):

$$P = \frac{k}{k + k'}, \qquad K = k + k'. \tag{18}$$

Thus $P \in [0, 1]$ is the fraction she owns out of the combined wealth $K \geq 0$ of both players. The bounds $[0, 1]$ on P are due to the no-borrowing constraints the agents face.[20]

We restrict attention to consumption and transfer policies that are such that all families, rich and poor, are "proportionally alike": given the same distribution of assets P, all families choose the same policies as a percentage of total assets K.

Def nition 3 (Homogeneous/K-linear strategy) *The strategy* $\{c, c^0; g_f, g_m\}$ *is* homogeneous *or* K-linear *if there exist functions* $\{C, C^0; G_f, G_m\}$ *such that*

$$c(k, k') = C(P)K, \qquad g_f(k, k') = G_f(P)K, \qquad g_m(k, k') = G_m(P)K,$$
$$c^0(k, k'; g, g') = C^0(P; g/K, g'/K)K \tag{19}$$

for all (k, k')*, where* $K = k + k'$ *and* $P = k/(k + k')$.

[20] Note that the mapping in (18) is ill-def ned for the point $k = k' = 0$. This is unproblematic for practical purposes, since agents' policies are trivially obtained in this situation ($c = c' = g = g' = 0$) and both agents must obtain utility of $-\infty$. The value function $\tilde{V}$ def ned below will indeed converge to this value as $K \to 0$.

We will again collect transfer functions in a vector $G = (G_f, G_m)$. Given that the other player uses K-linear policies, we will see shortly that it is optimal to respond with K-linear policies.

Feasibility of consumption when broke is now given by

$$C^*(C, P, G') = \begin{cases} \min\{C, G'_f\} & \text{if } P = 0 \text{ and } G'_m = 0, \\ C & \text{otherwise,} \end{cases}$$

and the laws of motion for P and K in the absence of mass transfers are

$$\dot{K} = \left[r - C^* - C'^*\right] K, \tag{20}$$

$$\dot{P} = -(1-P)C^* + PC'^* + \left[G'_f - G_f\right]. \tag{21}$$

A mass transfer induces a jump: $\lim_{h \searrow 0} P_{t+h} = P_t - G_{mt}$. We note that $\dot{P}$ is independent of K, which is intuitive since all families are proportionally alike.

In the state variables (k, k'), homogeneous strategies imply that regions are cones separated by rays emanating from the origin (such as region 1 in Figure 2). Since now policies only depend on P, regions are given by intervals $\mathcal{P}_i = (P_{i-1}, P_i)$, $i = 1, \ldots, n$, that cover the state space $\mathcal{P} = [0, 1]$. The numbers $0 = P_0 < P_1 < \cdots < P_n = 1$ represent the boundaries of regions. Piecewise smoothness obviously implies that the functions $\{C, G_f; C', G'_f\}$ are C^1 on each $\mathcal{P}_i$; they may be discontinuous at the boundaries P_i, however.[21] Mass transfers are such that she shoots the economy to the upper bound of an interval, i.e. $G_m(P) = P_i - P$ for all P that fall into a mass-transfer region $\mathcal{P}_i = (P_{i-1}, P_i)$.

We now turn to players' value functions. Let $\tilde{V}(P, K)$ be her value at (P, K). K-linear strategies imply that $\tilde{V}$ must be separable in P and K, and logarithmic in K:[22]

$$v(k, k') = \tilde{V}(P, K) = \frac{1+\alpha}{\rho} \left[\frac{r}{\rho} + \ln(K)\right] + V(P). \tag{22}$$

[21] Again, note that we allow for the case where policies on the kink differ from their right- and left-hand side limit, e.g. $\lim_{P \nearrow P_i} C(P) < C(P_i) < \lim_{P \searrow P_i} C(P)$.

[22] This and the other statements leading up to Proposition 1 are formally shown in the proof for Proposition 1 in the appendix.

The terms in K are thus known; they represent the *value of common assets* K to her. The function $V(\cdot)$ remains to be determined; it represents how she feels about the distribution of assets between him and her. Def nition 1 obviously implies that for any piecewise-smooth homogeneous prof le, V is continuous on $[0,1]$ and C^1 inside each $\mathcal{P}_i$.[23]

We will call the functions $\{V, C, G; V', C', G'\}$ associated with a homogeneous prof le $\{v, c, g; v', c', g'\}$ a *P-prof le*. We now derive the requirements for best-responding that P-prof les have to fulf ll. Using the functional form of $\tilde{V}$ from equation (22) in the order-0 requirement from equation (8), one obtains

$$V(P) = \max_{G_m \in [0,P]} V\big(P + G'_m(P) - G_m\big). \tag{23}$$

For order-1 optimality, we can use $c = CK$ and the laws of motion (20) and (21) to write the HJB in the new variables (P, K). Inside a smooth region without mass transfers, we have

$$\begin{aligned}\rho\tilde{V} = \max_{C \geq 0, G_f \geq 0} \Big\{ & \ln(CK) + \alpha \ln(C'K) + (r - C - C')K\tilde{V}_K + \\ & + \big[PC' - C(1-P) + G'_f - G_f\big]\tilde{V}_P \Big\}. \end{aligned} \tag{24}$$

We can now use the functional form of $\tilde{V}$ from (22) to eliminate terms in K to obtain the following HJB for smooth (no-mass-transfer) regions:

$$\begin{aligned}\rho V =& \alpha \ln C' - C'\frac{1+\alpha}{\rho} + PC'V_P + G'V_P + \\ & + \max_{C \geq 0}\left\{\ln C - C\frac{1+\alpha}{\rho} - C(1-P)V_P\right\} + \max_{G_f \geq 0}\{-G_f V_P\}. \end{aligned} \tag{25}$$

This is an ODE in P, which is an important simplif cation – recall that previously the HJB (15) was a PDE in the state variables (k, k').

When also taking into account mass transfers and non-smoothness, one can show

[23] For the case $\alpha' = 0$, V is only continuous on $(0, 1]$.

that the transfer stage in order-1 optimality from equation (9) becomes

$$\rho V(P) = \max_{G \geq 0,\ G=0 \text{ if } P=0} \rho V^c(P; G, G') \tag{26}$$
$$\text{s.t.} \quad G_m = \arg \max_{\tilde{G}_m \in [0,P]} V\big(P + G'_m(P) - \tilde{G}_m\big),$$

and that the consumption stage is represented by

$$\rho V^c(P; G, G') = \max_{C \geq 0} \left\{ \ln C^* + \alpha \ln C'^* - (C^* + C'^*)^{\frac{1+\alpha}{\rho}} + \dot{P} \nabla_{\dot{P}} V(P_m) \right\}, \tag{27}$$

$$\text{where} \quad C^* = C^*(C, P, G'), \qquad C'^* = C'^*\big(C'(P; G, G'), P, G\big),$$
$$P_m = P + G'_m - G_m, \qquad \dot{P} = -(1-P)C^* + PC'^* + G'_f - G_f.$$

The directional gradient is now simply either the right- or left-derivative:

$$\nabla_{\dot{P}} V(P) = \begin{cases} V_P^+(P) & \text{if } \dot{P} \geq 0, \\ V_P^-(P) & \text{if } \dot{P} < 0. \end{cases}$$

The following proposition formalizes these insights; it tells us that (23), (26) and (27) fully characterize best-responding in homogeneous equilibria.

Proposition 1 (Characterization of homogeneous MPE) *A piecewise-smooth, homogeneous prof le $a = \{v, c, g; v', c', g'\}$ with associated consumption-stage strategies $\{c^0, c'^0\}$ is a MPE if and only if there exists a P-*prof le *$A = \{V, C, G; V', C', G'\}$ with associated functions $\{C^0, C'^0\}$ that satisfy the following (for both players):*

1. *(*homogeneity*) $\{C, C^0, G\}$ are related to $\{c, c^0, g\}$ as given in* (19).

2. *(*piecewise-smoothness*) Let $\mathcal{P}_i = (P_{i-1}, P_i)$ for $0 = P_0 < P_1 < \cdots < P_n = 1$. The functions in A are C^1 on each $\mathcal{P}_i$. V is continuous on $(0, 1]$ if $\alpha' = 0$, and continuous on $[0, 1]$ otherwise. $G_m(\tilde{P}) > 0$ for some $\tilde{P} \in \mathcal{P}_i$ (for any i) implies $G_m(P) = P_i - P$ for all $P \in \mathcal{P}_i$. For any converging sequence $\{P_j\} \to P_i$ the limit $\lim_{j\to\infty} C(P_j)$ exists.*

3. *(*best-responding*) V and v are related as given in* (22)*. V satisf es* (23)*,* (26) *and* (27) *for each $P \in [0, 1]$, the maximum being attained by $G_m(P)$, $G_f(P)$*

and $C^0(P; G, G')$, respectively. Also, $C(P) = C^0(P; G(P), G'(P))$.

From (23) we immediately see that V is constant (i.e. $V_P = 0$) in regions where either she or he gives mass transfers, and that V cannot be decreasing in regions where $G_m = G'_m = 0$.

Proposition 2 (V monotone) *In any homogeneous MPE, V is weakly increasing in P (and V' is weakly decreasing in P).*

In Appendix A.1.2 we derive additional properties of homogeneous equilibria that will be useful in the further analysis. Most importantly, we show that equilibrium strategies are invariant in r (since income and substitution effects cancel out under log-utility) and linear in ρ. This enables us to restrict our analysis to the parameter tuple (α, α').

We will now have a closer look at well-behaved regions in order to gain more intuition. Invoking Proposition 2, we immediately see from (25) that f ow transfers are only possible if $V_P = 0$. The FOC for her consumption choice is

$$\frac{1}{C} = \frac{1+\alpha}{\rho} + (1-P)V_P, \quad \text{if} \quad P \in (0, 1]. \tag{28}$$

This equation says that she sets the marginal utility of consumption equal to the marginal value of savings. The marginal value of savings can be decomposed into two components: f rst, $(1+\alpha)/\rho = \tilde{V}_K K$ measures the (proportional) *marginal value of common assets*, i.e. the value obtained if K is increased by 1% while leaving the distribution P unchanged. Second, $-V_P = (v_{k'} - v_k)K = \mu K$ measures the (proportional) *transfer motive*: $-V_P$ is the value to her when 1% of total assets K are transfered to him while holding total assets K unchanged.

Finally, by taking the derivative of her HJB (25) in P and using the FOC (28) we obtain her Euler equation, which is again an ODE that holds inside smooth regions without mass transfers:[24]

$$\frac{d}{dt}V_P(t) = \big[PC' - (1-P)C - G + G'\big]V_{PP} =$$
$$= \big[\rho - C - C'\big]V_P + \left[\frac{1}{C} - \frac{\alpha}{C'} - V_P\right]C'_P - G'_P V_P. \tag{29}$$

5 Results

5.1 Special cases

As a benchmark for further analysis it is useful to consider two models that are respectively closely related to the special cases of self shness and perfect altruism.

First, there is a connection between a *self-suff ciency* (SS) model – in which the possibility of transfers is ruled out – and to players being self sh. Let us consider the SS strategies, i.e. the strategies players would choose in our setting if we imposed $G = G' = 0$. Players then face a standard savings problem and we have

$$C_{SS}(P) = \rho P, \qquad G_{SS}(P) = 0 = G'_{SS}(P), \qquad C'_{SS}(P) = \rho(1-P). \tag{30}$$

As shown by equation (14), these policies induce the eff cient allocation if the initial distribution $P_{t=0}$ lies in the range of P^*_η spanned by the planner's weight $\eta \in [0,1]$.[25]

Proposition 3 (Self-suff cient equilibrium) *Consider the SS strategies in* (30).

1. *The SS strategies can be sustained as an equilibrium only if* $\alpha = \alpha' = 0$.

2. *If* $\alpha = \alpha' = 0$, *then the SS strategies constitute the unique equilibrium, and this equilibrium is eff cient.*

For the proof, see the online appendix.

It is not surprising that the SS strategies are the unique equilibrium under self shness. But one may wonder here why they cannot be an equilibrium otherwise, say

[24] We note here that differentiability of C implies that V is twice differentiable. The term $G_P(P)V_P(P)$ vanishes in regions where $G = 0$ since $G_P = 0$ and in regions where $G > 0$ since $V_P = 0$; a similar argument shows that the term $G(P)V_{PP}(P)$ vanishes in both regions.

[25] The fact that SS policies cannot be supported as an equilibrium would be obvious if invoking lower-boundedness of V under altruism; note that we do not invoke this in our proof.

for the case that $\alpha = \alpha' > 0$ and when the initial wealth distribution is balanced, i.e. $P_{t=0} = 0.5$. In this situation it seems reasonable that players will never give transfers to each other. Given that they cannot expect transfers, one may think they should also consume as if self-suff cient, keeping the wealth distribution at $P_t = 0.5$ for all eternity and attaining an eff cient outcome. The f aw in this reasoning is the following: it only considers the equilibrium path, but does not entertain the question of what would happen if one of the players deviated from the equilibrium strategy. Subgame perfection imposes that threats be credible also off the equilibrium path; in the concept of Markov equilibrium this shows up in the requirement that the equilibrium strategies be optimal *for all* P, and not only for $P = 0.5$. But this is clearly not the case for the SS strategies as $P \to 0$. Since she is altruistic, she would not let him starve to death and give transfers at some point – it is simply not credible to rule out future transfers for an altruistic donor who cannot commit.

Second, we consider a *wealth-pooling* (WP) model without property rights. It is linked to agents being perfectly altruistic. We remove the restrictions $G \geq 0$ and $G' \geq 0$ from our setting, which amounts to players consuming out of a pooled asset stock $K \geq 0$. In our online appendix we show that the equilibrium policies of this game are

$$C_{WP} = \frac{\rho}{1+\alpha}, \qquad C'_{WP} = \frac{\rho}{1+\alpha'}. \tag{31}$$

When players are self sh, i.e. $\alpha = \alpha' = 0$, they consume at the same rate ρ as in the SS case, but now out of the common stock K. This is the tragedy of the commons, which is well-known from the resource-extraction literature (*f sh wars*). On the other hand, under perfect altruism (i.e. $\alpha = \alpha' = 1$) – and *only* in this case – the WP-policies induce the eff cient allocation. For imperfect altruism (i.e. $0 < \alpha + \alpha' < 2$) we have something in between: there is a tragedy of the commons, but it is alleviated by altruism. Players now take into account some of the externality they impose on the other, but extraction from the common stock is still ineff ciently high. Note also that the more-altruistic player will take into account to a larger extent the externalities she causes, and will consume less than the more-self sh player.

We now return to our original setting with property rights and ask if there exists an equilibrium where players pool their wealth (under any initial conditions).

Proposition 4 (Wealth-pooling equilibrium) *Consider the wealth-pooling (WP) consumption strategies: $C(P) = C_{WP}$ and $C'(P) = C'_{WP}$ for all P.*

1. *The WP consumption strategies can be sustained in a MPE only if $\alpha = \alpha' = 1$.*
2. *If $\alpha = \alpha' = 1$, then $C_{WP} = C'_{WP} = \rho/2$ is the unique consumption strategy that can be sustained in MPE. In this equilibrium:*
   (a) *Transfer strategies are indeterminate for $P \in (0, 1)$, but must make WP-consumption feasible on $P \in \{0, 1\}$, i.e. $G_m(0) > 0$ or $G_f(0) \geq \frac{\rho}{2}$.*
   (b) *The unique eff cient allocation is attained.*

For the proof, see the online appendix.

Since the WP allocation is the unique eff cient allocation under perfect altruism, point 2 of the proposition should come as no surprise. As for point 1, we will now see why it is impossible to support WP when altruism is imperfect. To do this it is again suff cient to f nd one state P of the game for which one of the players can do better by deviating from WP. Consider the situation where an imperfectly-altruistic player is left with the entire wealth, say $P = 1$ and $\alpha < 1$. In this case it turns out that she essentially becomes the "family dictator" and can implement her globally preferred allocation. Since she is less-than-perfectly-altruistic, this means providing less consumption to him than WP stipulates. So the equilibrium breaks down because it is optimal for her to wait for him to go broke and then spoon-feed him what she deems appropriate.

5.2 Characterization of regions

Our strategy to f nd equilibria under imperfect altruism is to f rst characterize different types of smooth candidate regions for $\mathcal{P}_1, \ldots, \mathcal{P}_n$. We then check which of these regions can be patched together into equilibria.

An exhaustive listing of the various types of candidate regions is as follows. No-transfer (NT) regions: none of the players gives transfers; f ow-transfer (FT) regions: transfers of the f ow type occur; mass-transfer (MT) regions: a mass transfer is given by one player. Furthermore, there are the following two important sub-types. Self-suff cient (SS) regions are a special kind of NT region, where policies are given by the

SS strategies in (30). Wealth-pooling (WP) regions are a special kind of FT-region, where consumption is given by the WP rates in (31) and transfers are indeterminate. We formally define and characterize the different types of regions in Appendix A.2. The key results are the following:

1. NT-regions are characterized by the altruistic-strategic distortions in the Euler equations discussed in Section 3.3. NT-regions that are not SS are left in finite time (for almost any initial condition). If a NT-region is of SS type, then the economy stays inside the region forever.

2. FT-regions are characterized by the transfer-induced incentives discussed in Section 3.3. FT-regions that are not WP are left in finite time (for almost any initial condition). If a FT-region is of WP type, then the economy may stay in this region forever, but need not do so.

3. MT-regions are always left immediately. Since the transfer recipient can count on having his account filled up constantly, he chooses the WP-consumption rate upon leaving the MT-region.

It turns out that the results on "transitoriness" paired with the sharp characterizations on SS and WP-regions provide us with the ODEs and boundary conditions that we need to find (and discard) equilibria.

5.3 Tragedy-of-the-commons-type equilibrium

We are now in a position to start our quest for equilibria. In the case of imperfect altruism, there are no smooth equilibria (i.e. equilibria that only consist of one region), see our online appendix for details. We now present the only type of equilibrium we found for imperfect altruism.

There exists a continuum of non-smooth equilibria, the sequence of regions being FT'-WP-FT. When the asset distribution is imbalanced, the poor player receives an increasing transfer schedule with incentives to save herself out of poverty. The economy always winds up in a WP-region in which players essentially pool their wealth.

Theorem 1 (Continuum of tragedy-of-the-commons-type equilibria) *If and only if $\alpha\alpha' > 0$ there exists a continuum of homogeneous MPE of the following type:*

1. *He gives transfers in a FT-region* $\mathcal{P}_{FT'} = [0, P_1)$. *We have* $C'(P) = C'_{WP}$ *and* $C(P) = C_{FT'} = const$ *for all* $P \in \mathcal{P}_{FT'}$.

2. *There is a WP-region* $\mathcal{P}_{WP} = [P_1, P_2]$. *Transfers are indeterminate, and* $C(P) = C_{WP}$, $C'(P) = C'_{WP}$ *for all* $P \in \mathcal{P}_{WP}$. *For all* $P_0 \in (0, 1)$, $P_t \in \mathcal{P}_{WP}$ *for all* $t \geq t_{WP}$ *for some* $t_{WP} < \infty$.[26]

3. *She gives transfers in a FT-region* $\mathcal{P}_{FT} = (P_2, 1]$. *We have* $C(P) = C_{WP}$ *and* $C'(P) = C'_{FT} = const$ *for all* $P \in \mathcal{P}_{FT}$.

$C_{FT'}$ *is the smallest solution to* $J_{WP}(C_{FT'}, \alpha') = J_{WP}(C_{WP}, \alpha')$, *and* C'_{FT} *is the smallest solution to* $J_{WP}(C'_{FT}, \alpha) = J_{WP}(C_{WP}, \alpha)$, *where* $J_{WP}(\tilde{C}, \tilde{\alpha}) = \tilde{\alpha} \ln \tilde{C} - \frac{1+\tilde{\alpha}}{\rho}\tilde{C}$. *We have* $P_1 \in (0, P_{max}(\alpha, \alpha')]$ *and* $P_2 \in [1 - P_{max}(\alpha', \alpha), 1)$, *where* $P_{max}(\cdot)$ *is given by equation* (A.12).

The formal construction of the equilibrium is given in the appendix. Note that the boundaries P_1 and P_2 may each be chosen from an interval, so there is double multiplicity of equilibria (in addition to transfers being indeterminate in the $\mathcal{P}_{WP}$-region). The equilibria within this class differ in consumption and transfer functions and in the allocations they induce for a given initial condition $P_{t=0}$. Equilibria with values of P_1 closer to zero (and P_2 closer to one) Pareto-dominate other equilibria, which will become clear from the ensuing discussion.

Figure 5 displays one such equilibrium. When the initial asset allocation is tilted in his favor, he provides her with f ow transfers (the blue dotted line in $\mathcal{P}_{FT'}$). In order to provide incentives to her to save herself out of poverty, transfers are increasing in her wealth share. The economy moves to the right, as the solid black line, which depicts $\dot{P}$, suggests. As the solid red line shows, her consumption in $\mathcal{P}_{FT'}$ is lower than the donor's consumption (the solid blue line). Once the asset distribution is suff ciently balanced, the players essentially pool their assets and play the WP strategies forever. This can be seen in region $\mathcal{P}_{WP}$ in the middle, where both players consume the same (which is the case since $\alpha = \alpha'$ in this example).

From the right panel in the f gure we see that his value function is f at throughout regions $\mathcal{P}_{FT'}$ and $\mathcal{P}_{WP}$, so that he is indifferent between these two regimes. Indeed,

[26] In fact, for all $P_1 < P_{max}(\alpha, \alpha')$ this is also true for $P_0 = 0$. Only if $P_1 = P_{max}(\alpha, \alpha')$ we have $\dot{P}_{|P=0} = 0$.

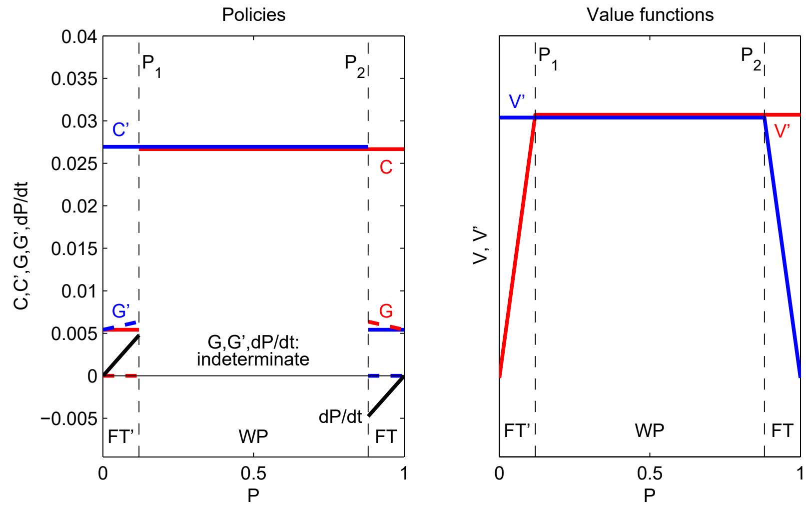

Figure 5: Tragedy-of-the-commons-type equilibrium: $\alpha = \alpha' = 0.5$, $\rho = 0.04$, $P_1 = P_{max}(0.5, 0.5)$, $P_2 = 1 - P_{max}(0.5, 0.5)$

she under-consumes on $\mathcal{P}_{FT'}$ by *exactly* as much as she over-consumes on $\mathcal{P}_{WP}$ in his eyes. So for him, there are no incentives to return to the $\mathcal{P}_{FT'}$ once the wealth-pooling regime is reached – he is indeed indifferent. As for her, she strictly prefers staying in $\mathcal{P}_{WP}$ to returning to $\mathcal{P}_{FT'}$, as her value function shows.

Why is there a continuum of such equilibria? The key point is that there is no boundary condition on policies at the points $P \in \{0, 1\}$ since the economy is moving away from there. This means that no information is fed from the margins into the equilibrium. Due to the lack of such boundary conditions, both P_1 and P_2 can be chosen on a continuum. Economically speaking, the entire WP-region is the unique steady state of the economy, and there are many transfer-incentive schemes that make players want to reach this region.

Also, one might wonder why the rich player is not able to implement his preferred allocation at $P = 0$. What would happen if he gave her exactly the amount that he desires her to consume, namely, $\alpha' C'_{WP}$? The answer is that he is not controlling her consumption because she is saving at $P = 0$. If he increased G', she would just save the additional amount provided, which would not make him better off. As for her, she chooses such a low consumption rate due to the prospect of WP in the future.

We stress that this class of equilibria can rationalize a wide range of transfer behavior. Transfers may be chosen to f ow everywhere in the state space, but it is also possible that they only f ow on tiny subsets of it. The latter can be engineered as follows: choose the boundaries P_1 and P_2 very close to 0 and 1, set $G = G' = 0$ on $\mathcal{P}_{WP}$ and ensure the WP-outcome by setting $G'_f(P_1) = C_{WP}$ and $G_f(P_2) = C'_{WP}$. Transfers then only f ow from a rich to an almost-broke agent, i.e. on $[0, P_1] \cup [P_2, 1]$, which is empirically plausible. When starting off with $P_0 < \frac{1}{2}$, she will end up impoverished at P_1. If $P_0 > \frac{1}{2}$, the reverse is true and almost all wealth ends up in her hands. This kind of impoverishment rings true for an agent who can count on an altruistic donor to f nance her consumption.

Finally, we observe that in the long run the outcome here is the same as in the perfectly-altruistic case: players pool their wealth. Indeed, when choosing P_1 and P_2 very close to 0 and 1, the WP stage can be reached arbitrarily fast. So this equilibrium can rationalize that a family composed of imperfectly-altruistic individuals behaves as if they were a perfectly-altruistic dynasty. This is despite individuals' property rights being intact. From the planner's problem, we see that this as-if dynasty has a discount rate $\rho_d \geq (\frac{1}{1+\alpha} + \frac{1}{1+\alpha'})\rho > \rho$ and is thus less patient than the individual agents are. The as-if dynasty planner assigns weight $\frac{1}{1+\alpha}$ to her and $\frac{1}{1+\alpha'}$ to him, i.e. he favors the less-altruistic agent.

This type of equilibrium has, to the best of our knowledge, not been found in the existing literature on altruism. In the $\mathcal{P}_{WP}$-region, an ineff ciency occurs that is akin to the tragedy of the commons, which was discussed in Section 5.1. But there is not only over-, but also *under*-consumption (by the poor agent in $\mathcal{P}_{FT}$), a feature not known from f nite-horizon settings.

5.4 Further results

In the class of equilibria we just presented, transfers enable the poor agent to escape from being broke. But one may conjecture that there exists an equilibrium where exactly the opposite occurs: the donor may delay transfers until the recipient is broke in order to have control over the recipient's consumption. This is in the spirit of the equilibria found by Lindbeck & Weibull (1988) in a two-period setting, where transfers f ow only in the second period.

In our setting, however, no equilibrium exists in which transfers are delayed in this way. The reason is that this equilibrium would have two steady states, 0 and 1.

Intuitively, it is not clear to which of the steady states the economy should converge for P close to $\frac{1}{2}$. This creates strong tensions: one can imagine that each agent wants to be the one who over-consumes and receives transfers in the end.

The technical reason that no such equilibrium exists is that there are two first-order ODEs for $\{C, C'\}$ on $(0, 1)$, but there are four boundary conditions on $C(0)$, $C'(0)$, $C(1)$, $C'(1)$ arising from the characterization of the constraint in Theorem 2 below. We may start solving these ODEs given the two boundary conditions at $P = 0$, but we cannot expect that $\{C, C'\}$ will satisfy the boundary conditions at $P = 1$. Indeed, we find numerically that no solution exists to this system of ODEs for any combination of (α, α').[27]

This tension, however, can be resolved if a shock is introduced to the law of motion for wealth: chance then decides where the economy ends up. This is in the spirit of mixed strategies but circumvents the technical difficulties that would come with this in a differential game. In our online appendix, we show that in this slightly modified setting there exists indeed an equilibrium in which transfers flow only to constrained recipients (see also Barczyk & Kredler, 2012, who embed altruistic agents into an incomplete-markets heterogeneous-agents economy and find that this type of equilibrium arises).

The following theorem tells us what occurs around the constraint in such an equilibrium. It is similar to the Samaritan's dilemma known from two-period models, but it also provides some new insights. We denote the limiting policies and the limiting law of motion by $C_{lim} = \lim_{P\to 0} C(P)$ and $\dot{P}_{lim}$, respectively, and define $\dot{P}_0 = \dot{P}_{|P=0}$. Then:

Theorem 2 (Party Theorem) *Suppose $\alpha' > 0$ and $\alpha + \alpha' < 2$. If a NT-region borders $P = 0$, then the requirements for homogeneous MPE imply the following:*

1. $\dot{P}_{lim} < 0$ *and* $\dot{P}_0 = 0$*: Her being broke is an absorbing state.*
2. $C(0) = \alpha' C'_{WP} = \frac{\alpha'\rho}{1+\alpha'}$*: When she is broke, his preferred allocation is played.*
3. $C_{lim} = \exp\left(\frac{1-\alpha\alpha'}{1+\alpha'}\right) C(0) > C(0)$*:* (Party) *On reaching $P = 0$, her consumption path has a downward jump.*

[27]For more detail see the online appendix. There we also show that no such equilibrium exist if we extend our quest to 3-region equilibria, where we restrict attention to symmetric equilibria under symmetric altruism ($\alpha = \alpha'$).

4. $V'_P(0) > 0$: *He strictly prefers her to be broke to her returning to be unconstrained.*

A striking feature, which at first glance seems at odds with optimizing behavior, is that the recipient's consumption path exhibits a downward discontinuity. The future recipient of transfers over-consumes relative to the efficient level, a phenomenon analogous to the Samaritan's dilemma[28] in a two-period model.

Theorem 2 offers the following three new (to the best of our knowledge) insights: First, the inefficiencies in consumption-savings decisions are not limited to the instant before receiving transfers, as is the case in the standard Samaritan's dilemma in a two-period model. The effects propagate further back in time due to the inefficiencies caused by altruistic-strategic distortions in the Euler equation (17). Second, in the time span prior to the actual transfer flow, there are also inefficiencies in the *donor's* consumption behavior, which is in contrast to the donor's Euler equation being the efficient one in the two-period model.[29] Third, we see that the party – i.e. the discontinuity of the recipient's consumption path – constitutes an inefficiency of a higher order than the inefficiencies occurring before. The inefficiencies before are characterized by consumption not growing at the efficient rate, but the path still being continuous.

Another plausible conjecture for an equilibrium seems to be that a rich donor lifts the recipient out of poverty with a mass transfer and both remain self-sufficient ever after. The following result tells us that such an equilibrium does not exist. As a matter of fact, there cannot be any equilibrium in which a mass transfer goes to a broke player.[30]

Theorem 3 (The Prodigal-Son Dilemma: no MT when broke) *There cannot be a mass transfer by him to her at $P = 0$ unless $\alpha = \alpha' = 1$.*

[28]The Samaritan's dilemma states the following in a two-period model: if an agent receives a transfer from an altruistic donor in period 2, then her consumption is inefficiently high in period 1, see Lindbeck & Weibull (1988).

[29]The donor's consumption being efficient in the two-period model corresponds to the donor's consumption plan being continuous at $P = 0$ (in contrast to the recipient's).

[30]In our online appendix, we show that lifting a poor recipient into self-sufficiency by *flow* transfers is not supported in equilibrium either. We have to resort to numerical techniques to establish this, however.

The intuition for the result is as in the prodigal-son parable[31]. He cannot commit to not-provide transfers after having made the initial mass transfer. She would then consume the transfer, come back and ask for more. Of course, if he had the ability to commit to not-give transfers, an equilibrium of this type could be supported, as Section 4.1 in the online appendix shows.

We conclude the discussion by noting that Bergstrom (1989) has a discussion on the prodigal son. However, the author uses this term interchangeably with the Samaritan's dilemma. Our framework clearly distinguishes between the two. The Samaritan's dilemma refers to the transfers that flow once the recipient is broke and the over-consumption that this induces on the side of the recipient; the Prodigal-Son dilemma refers to the donor's decision of not giving a large transfer in anticipation of this.

6 Conclusions

We have studied a parsimonious dynamic model of voluntary transfers with two-sided altruism. In the deterministic setting, the only class of equilibria that we find are tragedy-of-the-commons-type equilibria. We draw a distinction between the Samaritan's dilemma (parties) and the Prodigal-Son dilemma (the anticipation of parties). Equilibria in which transfers are delayed until the recipient is constrained and in which parties occur only exist when a shock is added to the setting.

We have restricted attention to Markov-perfect equilibria that are at least piecewise differentiable. This approach has allowed us to make significant headway drawing on dynamic-programming tools. However, as is well known, Markov-perfect equilibria are only a subset of the equilibria that exist when agents can use fully-contingent strategies. It is likely that tit-for-tat strategies can sustain better outcomes than the equilibria we find (but maybe also worse ones). The range of outcomes covered by such equilibria is potentially large since the continuous-time formulation

[31]The prodigal son is one of the most famous parables from the New Testament: a wealthy father has two sons. The younger one asks to be paid out his share of the estate to start a new life in another town. He goes away and squanders all the money. After a spell of living in poverty as a swineherd, he decides to return to his father and to become a servant at his estate. However, his father welcomes the lost son with great festivities, forgives him and re-instates him as an heir equal to the elder brother, who had stayed hard-working at the estate the entire time, and who is angry about the father's decision.

without shocks is an ideal setting for the folk theorem to apply. Identifying the full set of contingent equilibria in our setting would be of tremendous interest, but is also a daunting task, and is relegated to future research.

In a follow-up paper, Barczyk & Kredler (2012), we include idiosyncratic income risk into the present setting and extend the analysis to finite-horizon and overlapping-generations economies. We find that the tensions that we have identified here also surface in these more-complex settings. The reason is that for rich-enough families the present value of labor earnings becomes insignificant relative to capital income. For these families, the world looks very similar to the cake-eating problem we have studied here.

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A Appendix

A.1 General properties of homogeneous MPE

A.1.1 Proof of Proposition 1 (homogeneous equilibria)

Proof: We first prove point 1 in the proposition. If the strategies $s = \{c, c^0, g\}$ are homogeneous, Definition 3 immediately implies the existence of functions $S = \{C, C^0, G\}$ satisfying equation (19). In the opposite direction, from the functions in S we can obviously construct those in s.

We now turn to point 2. If a is piecewise smooth, this obviously implies the claims about the strategies in S in point 2. In the opposite direction, if some given functions in S satisfy the properties in point 1, then obviously they induce a piecewise-smooth profile a.

We finally prove the claims made about the value function in point 2 and point 3.

We first derive the functional form of $\tilde{V}$ that is claimed in equation (22). Her value given initial values (P_0, K_0) may be written as

$$\tilde{V}(P_0, K_0) = \int_0^\infty e^{-\rho t} \left[\ln\left(C(P_t)K_t\right) + \alpha \ln\left(C'(P_t)K_t\right)\right] dt, \tag{A.1}$$

where $\dot{P}_t$ and $\dot{K}_t$ are as given in equations (20) and (21). Since $\dot{P}_t$ is independent of K_t, we can solve forward the ODE for K_t:[32]

$$K_t = \exp\left(\int_0^t r - C(P_\tau) - C'(P_\tau) d\tau\right) K_0 = e^{rt} K_0 \exp\left(-\int_0^t C(P_\tau) + C'(P_\tau) d\tau\right).$$

This enables us to simplify $\tilde{V}$ to what is claimed in (22), where V is given by

$$V(P) \equiv \max_{C,G} \left\{ \int_0^\infty e^{-\rho t} \left[\ln(C_t) + \alpha \ln(C'_t) - (1+\alpha) \int_0^t [C(P_\tau) + C'(P_\tau)] d\tau \right] dt \right\}.$$

Given a piecewise-smooth v, inspection of equation (22) tells us that V must be continuous on $[0, 1]$ (on $(0, 1]$ for $\alpha' = 0$), and C^1 on each $\mathcal{P}_i$. In the opposite direction, take any V

[32] Even for policies that imply that the Lipschitz-condition for the ODE is not fulfilled at some points, any discretization scheme that cuts time into small increments must lead to the path $\{P_t\}$ being independent of K_0 and thus induce the proposed functional form.

that fulfills the properties in point 2 in the proposition and note that v as defined by (22) is piecewise-smooth. This concludes the proof of point 2.

As for point 3 in the proposition, we have already shown that v and V are related as given by (22). We can now use the functional form for $\tilde{V}$ from (22) in order-0 optimality (8) to see that (8) is equivalent to (23), and that G_m is optimal iff g_m is. Finally, substitute (10) into (9) and use the functional form for $\tilde{V}$ again to obtain

$$\rho\tilde{V}(P,K) = \max_{G}\left\{\max_{C}\left\{\ln(C^*K) + \alpha\ln(C'^*K) + \dot{K}\tfrac{1+\alpha}{\rho} + \dot{P}\nabla_{\dot{P}}\tilde{V}\right\}\right\},$$

where we note that directional gradients are not necessary in the K-direction by the shape of regions in (P,K) space identified in point 2. Using the laws of motion (20) and (21), it is then easy to verify that (9) and (10) are equivalent to (26) and (27), and that $\{c, c^0, g\}$ are order-1 optimal iff $\{C, C^0, G\}$ are. ■

A.1.2 Further global properties of homogeneous MPE

Proposition 5 (Equilibrium independent of r and linear in ρ) *Let $(\tilde{\rho},\tilde{r}) \neq (1,0)$ and fix (α,α'). Then $\{V, C, C^0, G_f, G_m; V', C', C'^0, G'_f, G'_m\}$ are a homogeneous MPE for $(\rho,r) = (1,0)$ if and only if $\{V^{\tilde{\rho}}, \tilde{\rho}C, \tilde{\rho}C^0, \tilde{\rho}G_f, G_m; V'^{\tilde{\rho}}, \tilde{\rho}C', \tilde{\rho}C'^0, \tilde{\rho}G'_f, G'_m\}$ are a MPE for $(\rho,r) = (\tilde{\rho},\tilde{r})$, where we define $V^{\tilde{\rho}} \equiv [(1+\alpha)\ln\tilde{\rho} + V]/\tilde{\rho}$ and $V'^{\tilde{\rho}} \equiv [(1+\alpha')\ln\tilde{\rho} + V']/\tilde{\rho}$.*

Proof: By the characterization of homogeneous MPE in Proposition 1, it is sufficient to show that $A = \{V, C, C^0, G\}$ fulfill best-responding as defined in equations (23), (26) and (27) for parameters $(\rho, r) = (1,0)$ if and only if $A^{\tilde{\rho}} = \{V^{\tilde{\rho}}, \tilde{\rho}C, \tilde{\rho}C^0, \tilde{\rho}G\}$ fulfills best-responding for arbitrary r and $\tilde{\rho} > 0$.

Since r shows up in none of (23), (26) and (27), it is easy to see that if $A^{\tilde{\rho}}$ fulfills best-responding it must do so for any r. So we may restrict attention to the case $r = 0$ from now on and only vary ρ.

We will now show that a value function V^1 fulfills best-responding for $\rho = 1$ if and only if the function

$$V^{\rho} = \frac{1+\alpha}{\rho}\ln\rho + \frac{V^1}{\rho} \tag{A.2}$$

fulfills best-responding for $\rho \neq 1$.

V^{ρ} is a linear, slope-preserving transformation of V^1 (since $\rho > 0$). $\{V^1, G_m\}$ thus satisfy order-0 optimally (23) if and only if $\{V^{\rho}, G_m\}$ satisfy order-0 optimality.

It remains to prove that A satisfies order-1 optimality as given by (26) and (27) if and only if A^{ρ} satisfies order-1 optimality. We will first focus on the smooth case, i.e. $P \in \mathcal{P}_i$

and $G_m(P) = G'_m(P) = 0$, so the two requirements collapse to an HJB, which we write down first for arbitrary ρ and then for $\rho = 1$.

$$\rho V^\rho = \max_{\tilde{C},\tilde{G}_f} \Big\{ \ln \tilde{C} + \alpha \ln(\rho C') - (\tilde{C} + \rho C')\frac{1+\alpha}{\rho} + \big[P\rho C' - (1-P)\tilde{C} + \rho G'_f - \tilde{G}_f\big] V_P^\rho \Big\},$$
$$V^1 = \max_{\tilde{C},\tilde{G}_f} \Big\{ \ln \tilde{C} + \alpha \ln C' - (\tilde{C} + C')(1+\alpha) + \big[PC' - (1-P)\tilde{C} + G'_f - \tilde{G}_f\big] V_P^1 \Big\}.$$

Suppose that V^1 fulfills the HJB. Then using V^ρ as given in (A.2) on the right-hand side of (A.2) shows that $C^\rho = \rho C^1$ and $G_t^\rho = \rho G_f^1$ are maximizing policies for this V^ρ. Substituting these policies into the right-hand side of (A.2) and simplifying shows that V^ρ as defined above indeed fulfills the HJB. Now, follow the exact same logic in the opposite direction. Suppose that some function V^ρ fulfills the HJB for $\rho \neq 1$. Then $V^1 = \rho V^\rho - (1+\alpha)\ln\rho$ fulfills the HJB for $\rho = 1$, the optimal policies being $C^1 = C^\rho/\rho$ and $G_f^1 = G_f^\rho/\rho$. The exact same arguments carry over to the non-smooth case.

The analogous statements obviously hold for his best-responding, which concludes the proof. ■

Lemma 1 (Smooth regions) *For any $P \in \mathcal{P}_i$ (for some i) with $G_m(P) = G'_m(P) = 0$, in any homogeneous MPE the HJB* (25)*, the FOC* (28) *and EE* (29) *hold and we have $C^0(P; G, G') = C(P)$.*

Proof: Our assumptions imply $P_m = P$ and $\nabla_{\dot{P}} V(P) = V_P(P)$ in (27). This immediately implies $C^0(P; G, G') = C(P)$ for any (G_f, G'_f). Substituting (27) into (26) then yields the HJB (25), from which we obtain the FOC (28) and by differentiation in P (recall that $P \in \mathcal{P}_i$) the EE (29). ■

A.1.3 Results when one player is broke

Lemma 2 (Consumption when broke) *For any $G' = (G'_m, G'_f)$ such that $G'_m = 0$,*

$$C^0(0; 0, G') = C_{unc} \equiv \left(\tfrac{1+\alpha}{\rho} + V_P^+(0)\right)^{-1} \tag{A.3}$$

is a best response in the consumption stage (27) *at $P = 0$, and leads to the same outcome C^* as any other best response.*

Proof: Define the Hamiltonian at $P = 0$ as $H_0(C) = \ln C - C(\frac{\rho}{1+\alpha} + V_P^+(0))$ for fixed $V_P^+(0) \geq 0$. Note that C_{unc} maximizes H_0. So if C_{unc} is feasible, i.e. $C_{unc} \leq G'_f(0)$, then clearly C_{unc} must also attain the maximum in the consumption stage (27). If

$C_{unc} > G'_f(0)$, then note that H_0 is increasing on $(0, G'_f(0))$ since it is a concave function with maximum at $C_{unc} > G'_f(0)$. We then see that any $\tilde{C} \geq G'_f(0)$ – among them C_{unc} – is optimal, and they all lead to the optimal outcome $C^* = G'_f(0)$. ■

Lemma 3 (Transfers when broke) *Consider any homogeneous MPE with $G'_m(0) = 0$. Then $C'(0) = \frac{\rho}{1+\alpha'}$, and the following hold:*

1. *If $C(0) > \frac{\alpha'\rho}{1+\alpha'}$, then $C^* = G'_f(0) = \frac{\alpha'\rho}{1+\alpha'}$, $\dot{P}_{|P=0} = 0$ and $V'(0) = V'_{\eta=0}$, i.e. his globally-preferred allocation is played.*
2. *If $C(0) \leq \frac{\alpha'\rho}{1+\alpha'}$, then $C^* = C(0) \leq G'(0)$, i.e. he does not restrict her consumption.*

Proof: We will first prove point 1 of the proposition. Since only $G = 0$ is feasible at $P = 0$ for her, his order-1 optimality from (26) and (27) collapses to

$$\rho V'(0) = \max_{C', G_f} \left\{ \ln C' - C' \frac{1+\alpha'}{\rho} + \underbrace{\alpha' \ln C^* - C^* \frac{1+\alpha'}{\rho}}_{\equiv J_0(G'_f))} + \underbrace{(G'_f - C^*) V'^{+}_{P}(0)}_{\equiv B_0(G'_f)} \right\},$$

where $C^* = \min\{C(0), G'_f\}$. We see that $B_0(G'_f) \leq 0$ (as defined above) for any G_f since $V'_P \leq 0$ by Proposition 2. If $C(0) > \frac{\alpha'\rho}{1+\alpha'}$, then $G'_f = \frac{\alpha'\rho}{1+\alpha'}$ uniquely maximizes J_0 and attains the maximum in $B_0(G)$. Since $C' = \frac{\rho}{1+\alpha'}$ is obviously the unique maximizer for his consumption, this proves that $(C', G'_f) = (\frac{\rho}{1+\alpha'}, \frac{\alpha'\rho}{1+\alpha'})$ dominates any other strategy if $G'_m(0) = 0$.

We now turn to point 2 of the proposition. If $C(0) \leq \frac{\alpha'\rho}{1+\alpha'}$, then J_0 is strictly increasing on $(0, C(0)]$, and B_0 is invariant on this range. So $\tilde{G}'_f = C(0)$ dominates any transfer $G'_f < C(0)$, from which the claim in point 2 follows. ■

A.1.4 Results on boundaries

Lemma 4 (Consumption at boundary) *Consider a region $\mathcal{P}_{i+1} = (P_i, P_{i+1})$ that is not of mass-transfer type. In any homogeneous MPE, $\dot{P}_{|P=P_i} > 0$ implies $C(P_i) = \lim_{P \searrow P_i} C(P)$. Similarly, $\dot{P}_{|P=P_{i+1}} < 0$ implies $C(P_{i+1}) = \lim_{P \nearrow P_{i+1}} C(P)$.*

Proof: Since $\dot{P}|_{P=P_i} > 0$, $V^+_P(P_i)$ is the relevant directional derivative at P_i in the consumption stage (27) given $(G(P_i), G'(P_i))$. Since $V^+_P(P_i) = \lim_{P \searrow P_i} V_P(P)$, the claim $\lim_{P \searrow P_i} C(P) = C(P_i)$ then follows from the FOC (28). If $P_i = 0$, note that $\dot{P} > 0$ implies that $G'(0) \geq C(0)$ by Lemma 3, so her consumption is unconstrained, i.e. it also fulfills the FOC (28) and the same argument holds. The same argument works at P_{i+1}. ■

Lemma 5 (WP maximizes Hamiltonian) *For any $P < 1$ and $V_P \geq 0$, the Hamiltonian $H_c(C, V_P) \equiv \ln C - C\frac{1+\alpha}{\rho} - (1-P)CV_P$ is uniquely maximized at $(C, V_P) = (\frac{\rho}{1+\alpha}, 0)$. For $P = 1$, $C = \frac{\rho}{1+\alpha}$ maximizes H_c for any V_P.*

Proof: We first note that clearly $C = 0$ can never be optimal, so we restrict attention to $C > 0$. For $V_P = 0$ and $P < 1$, we clearly have $H_c(\frac{\rho}{1+\alpha}, 0) > H_c(C, 0)$ for any $C \neq \frac{\rho}{1+\alpha}$. Since $\partial H_c/\partial V_P < 0$ for any fixed $C > 0$ and any $P < 1$, we have $H_c(\frac{\rho}{1+\alpha}, 0) \geq H_c(C, 0) \geq H_c(C, V_P)$ for any (C, V_P), one of the inequalities holding strictly if $(C, V_P) \neq (\frac{\rho}{1+\alpha}, 0)$. This proves the first part of the claim. The second part of the claim is obvious. ■

A.2 Characterizing regions

We classify regions into those where transfers are possible and where they are not. Since transfers are often indeterminate, we use the slope of the value function for our classification. The following is an exhaustive listing of region types:

Definition 4 (Region types: NT, FT, MT; SS, WP) *We classify a region $\mathcal{P}_i$ as*

1. no-transfer (NT) *if $V_P'(P) < 0 < V_P(P)$ for all $P \in \mathcal{P}_i$,*
2. flow-transfer (FT) *if $V_P(P) = 0$ or $V_P'(P) = 0$ (or both), and $G_m(P) = G_m'(P) = 0$ for all $P \in \mathcal{P}_i$,*
3. mass-transfer (MT) *if $G_m(P) > 0$ or $G_m'(P) > 0$ for some $P \in \mathcal{P}_i$.*

Furthermore, we define (i) a self-sufficient *(SS) region as a NT-region in which policies are equal to the SS policies in* (30)*, and (ii) a* wealth-pooling *(WP) region as a FT-region where $V_P(P) = V_P'(P) = 0$ for all $P \in \mathcal{P}_i$.*

We will see that in WP-regions players' consumption equals the WP rates from equation (31), hence the name.

A.2.1 No-transfer (NT) regions

Consider a NT-region $\mathcal{P}_{NT}$. By Lemma 1, the EE (29) holds everywhere on $\mathcal{P}_{NT}$. $V_P'(P) < 0 < V_P(0)$ implies that there are no transfers, hence all terms in G_f and G_f' vanish. We obtain a system of two ODEs for $\{C, C'\}$. In order to learn more about the properties of such regions, it will be useful to study steady states. We express the law of motion for P in terms of $c = CP, c' = C'(1-P)$ and obtain $\dot{P} = P(1-P)(c' - c)$. We see that $\dot{P} > 0$ if and only if $c' > c$, as is intuitive. At a steady state P^*, we must have $c' = c = \bar{c}$.

In our online appendix, we show that if $\bar{c} \neq \rho$ (i.e. the agents' policies are not SS) we have $c_P(P^*) < 0$ and $c'_P(P^*) > 0$. This implies that the dynamics in the NT-region with a steady state must be as depicted in Figure 6. These dynamics also imply that there cannot be a second steady state in $\mathcal{P}_{NT}$: if c and c' intersected again, by continuity of the consumption functions they would have to do so in a way violating the condition above.

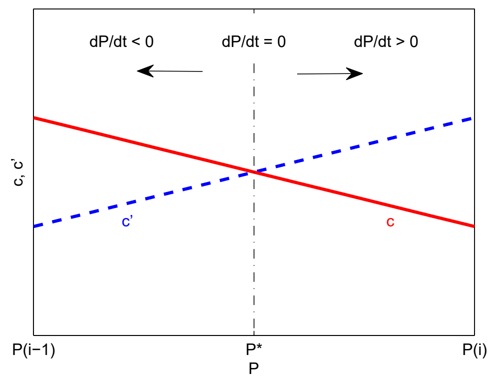

Figure 6: Dynamics in NT-region

Lemma 6 (NT-regime transitory) *Consider a no-transfer region that is not SS, in the sense that* $\{P \in \mathcal{P}_{NT} : c(P) = c'(P) = \rho\} = \emptyset$*. Then, for all but at most one point* $P^* \in \mathcal{P}_{NT}$*, the following holds:* $P_0 \in \mathcal{P}_{NT} \setminus P^*$ *implies* $P_t \notin \mathcal{P}_{NT}$ *for some* $t < \infty$*.*

In our online appendix we also treat the case $\bar{c} = \rho$; we show that also then the crossing of the functions c and c' must be such that the economy moves away from the steady state. But now, the functions c and c' may also stay on top of each other on an entire interval, which leads us to the special case of a SS-region.

A.2.2 Self-suff cient (SS) regions

Within a SS-region $\mathcal{P}_{SS}$, we have $\dot{P} = 0$ for all $P \in \mathcal{P}_{SS}$, as we can see from (21). This enables us to read off the value function from the HJB (25):

$$\rho V^{SS} = (1 + \alpha)(\ln \rho - 1) + \ln P + \alpha \ln(1 - P). \tag{A.4}$$

Taking the derivative in P gives us $\rho V_P^{SS} = \frac{1}{P} - \frac{\alpha}{1+P}$. Since $V_P \geq 0$ and $V'_P \leq 0$ by Proposition 2, we obtain the restriction $P \in [\frac{\alpha'}{1+\alpha'}, \frac{1}{1+\alpha}]$ for all $P \in \mathcal{P}_{SS}$. Thus, any SS-region has to be contained in this interval. The intuition for this result is very simple: if one player becomes too poor, the marginal utility of helping the other out becomes higher than the marginal utility of her own consumption. Observe that the interval corresponds to the region in the static altruism model with log-utility in which transfers are zero. It shrinks to zero as altruism increases, and extends over the entire state space as agents become more self sh. We summarize:

Lemma 7 (SS-regions) *For any SS-region $\mathcal{P}_{SS}$, we have $\mathcal{P}_{SS} \subseteq [\frac{\alpha'}{1+\alpha'}, \frac{1}{1+\alpha}]$. The value function is as given by equation* (A.4)*. Also, $P_t = P_0$ for all $t > 0$ whenever $P_0 \in \mathcal{P}_{SS}$.*

Given our results on NT and SS-regions, we can make an interesting observation regarding patched equilibria: if agents do not give transfers to each other from some initial point P_0 on, then the economy must end up in a SS-region at some point (almost always), since NT-regions are transitory.

A.2.3 Flow-transfer (FT) regions

Without loss of generality, consider a FT-region $\mathcal{P}_{FT}$ where $V_P = 0$. We thus have $G_f(P) \geq 0$ for all $P \in \mathcal{P}_{FT}$; he might or might not give f ow transfers (see below). By Lemma 1, we may work with the simplif ed HJB, FOC and EE. Since $V_P = 0$, her consumption throughout $\mathcal{P}_{FT}$ has to be $C = C_{WP}$ by her FOC (28). Furthermore, her HJB (25) implies that *his* consumption C' must also be constant on $\mathcal{P}_{FT}$. Indeed, if C' did vary, then so would the terms $\alpha \ln C' - C'[(1+\alpha)/\rho - PV_P]$, thus violating her HJB since all other terms of her HJB are constant in P. Now, his EE (29) becomes

$$V_P'(\rho - C_{WP} - C' + G_P) = 0, \tag{A.5}$$

where we drop the f-subscript from G_f for better readability. We can distinguish the following two cases: (i) $V_P' = 0$: we will treat this case separately as a WP-region, see below. (ii) $V_P' < 0$: his transfers are zero throughout $\mathcal{P}_{FT}$. If $V_P' < 0$, then we can infer that

$$G_P = C' - \frac{\alpha\rho}{1+\alpha}. \tag{A.6}$$

One may think about this equation in the following way: given some C' that she desires to implement, G_P is the transfer gradient needed to do this. Obviously, the more thrift she wants to induce (i.e. the lower C'), the lower G_P has to be, i.e. the more she must make transfers increasing in his wealth share $(1-P)$.

Finally, we will establish that FT-regions are transitory. Take the derivative of the law of motion $\dot{P}$ in P and use equation (A.6) and the fact that both $C = C_{WP}$ and C' are invariant in P to find that $\frac{d}{dP}\dot{P} = \rho > 0$, which means that $\mathcal{P}_{FT}$ must be left from all but at most one point.

Lemma 8 (FT-regions) *Consider a flow-transfer region $\mathcal{P}_{FT}$ which is not WP, i.e. $V_P(P) = 0$ and $V_P'(P) < 0$ for all $P \in \mathcal{P}_{FT}$. Then $G_f(P) \geq 0$, $G_f'(P) = 0$, $C(P) = C_{WP}$ and $C_P'(P) = 0$ for all $P \in \mathcal{P}_{FT}$. C' and G_f satisfy the ODE* (A.6) *on $\mathcal{P}_{FT}$. For all but at most one point $P^* \in \mathcal{P}_{FT}$, the following statement holds: $P_0 \in \mathcal{P}_{FT} \setminus P^*$ implies $P_t \notin \mathcal{P}_{FT}$ for some $t < \infty$.*

A.2.4 Wealth-pooling (WP) regions

Consider a WP-region $\mathcal{P}_{WP}$, i.e. $V_P(P) = V_P'(P) = 0$ for all $P \in \mathcal{P}_{WP}$. By the FOC (28), consumption policies are then constant and given by $C = C_{WP}$ and $C' = C_{WP}'$ for all $P \in \mathcal{P}_{WP}$. From the HJB (25) we see that both players' transfer schedules are indeterminate since they are locally indifferent with respect to the wealth distribution. This also implies that the dynamics $\dot{P}$ are not restricted in any way, so a WP-region may or may not be left eventually.[33]

The HJB (25) also tells us that the value functions within $\mathcal{P}_{WP}$ must be equal to those from the WP model in Section 5.1:

$$\rho V^{WP} = \ln C_{WP} + \alpha \ln C_{WP}' - (C_{WP} + C_{WP}')\frac{1+\alpha}{\rho} = const. \tag{A.7}$$

We will now see that this implies that all WP-regions must be connected in equilibrium. To see this, take two WP-regions. Observe that players' value functions must be equal to the constants V^{WP} and V'^{WP} in all WP-regions. Since the value functions are globally monotonic by Proposition 2, it must be that the value functions are also equal to V^{WP} and V'^{WP} on the convex hull of all WP-regions. So we may pool all WP-regions into one.

Furthermore, this WP-region cannot extend to the boundary $P = 1$ if $\alpha + \alpha' < 2$. Suppose this was the case. Then by continuity of V, we have $V'(P) = V'^{WP}$ for $P \in \mathcal{WP} \cup \{1\}$ and thus $V'^{-}_{P}(1) = 0$. By Lemma 2 he then chooses $C'(1) = C_{unc}' = C_{WP}'$ in the consumption stage. By Lemma 3, she should then deviate from WP in the transfer stage and implement her globally-preferred allocation. She can do this by setting $G(1) = \frac{\alpha}{1+\alpha} \leq \frac{1}{2} \leq \frac{1}{1+\alpha'} = C_{WP}'$, where one of the inequalities has to be strict since we assumed $\alpha + \alpha' < 2$. This would strictly dominate the value from WP and thus violate first-order optimality (26).

[33] There might actually also be mass-transfers inside an absorbing $\mathcal{P}_{WP}$; we will still call $\mathcal{P}_{WP}$ a WP-region for our purposes if the region is absorbing in this case.

Lemma 9 (WP-region) *Consider a WP-region $\mathcal{P}_{WP}$. Then $C(P) = C_{WP}$, $G(P)$ is indeterminate and $V(P)$ is given by equation* (A.7) *for all $P \in \mathcal{P}_{WP}$. WP-regions may be absorbing or transitory. In equilibrium, all WP-regions are connected. If $\alpha + \alpha' < 2$, then they cannot extend to the boundaries of the state space, i.e. $\{0, 1\} \cap \overline{\mathcal{P}_{WP}} = \emptyset$.*

A.2.5 Mass-transfer regions (MT)

The last type of region that remains to be characterized is the mass-transfer (MT) type. Let $\mathcal{P}_{MT} = (P_{i-1}, P_i)$ and $G'_m(P) = P_i - P$ for $P \in \mathcal{P}_{MT}$. From order-0 optimality (23) we immediately see that both players' value functions are f at on $\mathcal{P}_{MT}$. This implies that if the mass transfer was delayed (i.e. $G'_m = 0$), players would choose the WP-consumption policies in the consumption stage for all $P \in \mathcal{P}_{MT}$.

It turns out that when she can count on him f lling up her account at P_i, she will be profligate and set $C(P_i) = C_{WP}$ at this point. To see this, f rst note that piecewise-smoothness implies $G_m(P_i) = 0$ in equilibrium. We also have $G'_m(P_i) = 0$ – if not, we may pool the adjacent MT-region $[P_i, P_{i+1})$ into $\mathcal{P}_{MT}$. Now, if $V_P^+(P_i) = 0 = V_P^-(P_i)$, then $C(P_i) = C_{WP}$ follows since C_{WP} is then the unique maximizer in the consumption stage for any (G_f, G'_f). If $V_P^+(P_i) > 0$, then observe that given any $G'_f(P_i) \geq 0$, she may always choose G_f large enough in the transfer stage to ensure $\dot{P}_{|P=P_i} < 0$ in the consumption stage. This allows her to attain the unique maximum H^* in her Hamiltonian $H(C, V_P)$, which by Lemma 5 dominates any policy that leads to $\dot{P} > 0$ and thus $C(P_i) < C_{WP}$ (which follows from Lemma 4). Thus, $C(P_i) = C_{WP}$, and also $C^* = C'_{WP}$ at P_i since $P_i > 0$. We summarize:

Lemma 10 (MT-regions) *Consider a mass-transfer (MT) region $\mathcal{P}_{MT} = (P_{i-1}, P_i)$ with $G_m(P) = P_i - P$ for $P \in \mathcal{P}_{MT}$. Then $\mathcal{P}_{MT}$ is immediately left for any $P \in \mathcal{P}_{MT}$; we have $V(P) = V(P_i)$ and $V'(P) = V'(P_i)$ for all $P \in \mathcal{P}_{MT}$. For all $P \in \mathcal{P}_{WP}$, $C^0(P; G, G') = C_{WP}$ and ${C'}^0(P; G, G') = C'_{WP}$ for any (G, G') satisfying $G_m = G'_m = 0$. Furthermore, $C^*(P_i) = C(P_i) = C_{WP}$.*

A.3 Proofs for theorems

A.3.1 Proof for Theorem 1 (tragedy-of-the-commons-type equilibrium)

Proof: Throughout the proof, we restrict our analysis to the case $(r, \rho) = (0, 1)$; by Proposition 5 we can extend our result to any tuple (r, ρ).

We start with the if-part of the proposition.

Consider first the special case that $\alpha = \alpha' = 1$. The WP-equilibrium from Proposition 4 can be split into three WP-regions and transfers may be chosen such that they satisfy the properties of the proposed equilibrium. After all, WP-regions are a sub-type of FT, and transfers may be chosen in an arbitrary manner in WP-regions by Lemma 9.

We now turn to the case $\alpha + \alpha' < 2$. By lemmas 8 and 9, we have $C = C_{WP}$ on $\mathcal{P}_{WP} \cup \mathcal{P}_{FT}$ and $C' = C'_{WP}$ on $\mathcal{P}_{FT'} \cup \mathcal{P}_{WP}$, and transfers are indeterminate on $\mathcal{P}_{WP}$. It remains to pin down transfers and the recipients' consumption in the FT-regions.

First, we will determine her consumption $C_{FT'}$ on $\mathcal{P}_{FT'}$, which is invariant in P by Lemma 8. Define the function $J(C) = \alpha' \ln C - (1 + \alpha')C$. Note that J is concave, uniquely maximized at $\frac{\alpha'}{1+\alpha'}$ and $\lim_{C\to 0} J(C) = -\infty$. Value-matching and the HJB (25) imply that $C_{FT'}$ must solve

$$J(C_{FT'}) = \alpha' \ln C_{FT'} - (1 + \alpha')C_{FT'} = \alpha' \ln C_{WP} - (1 + \alpha')C_{WP} = J(C_{WP}). \quad \text{(A.8)}$$

One solution to this equation is obviously $C_{FT'} = C_{WP}$. However, by the FOC (28) this would imply that FT' is also a WP-region, which is impossible since $\alpha + \alpha' < 2$ by Lemma 9. We now argue that there is exactly one further solution to (A.8). Since C_{WP} lies on the decreasing part of J (since $C_{WP} = \frac{1}{1+\alpha} \geq \frac{1}{2} \geq \frac{\alpha'}{1+\alpha'} = \arg\max_C J(C)$, with one of the inequalities holding strictly since $\alpha + \alpha' < 2$), there must be exactly one further solution to (A.8) on the increasing part of J. We denote this second solution by $C_{FT'}(\alpha, \alpha')$ and note that she under-consumes in his eyes on $\mathcal{P}_{FT'}$:

$$C_{FT'}(\alpha, \alpha') < \tfrac{\alpha'}{1+\alpha'}. \quad \text{(A.9)}$$

To pin down transfers on $\mathcal{P}_{FT'}$, we now use her value-matching condition: $\lim_{P\nearrow P_1} \rho V(P) = \lim_{P\searrow P_1} \rho V(P)$. Define $L(C) = \ln C - (1 + \alpha)C$ and use again the HJB (25) to find

$$L(C_{FT'}) + \left[P_1 C'_{WP} - (1 - P_1)C_{FT'} + G'_1\right] V_P^-(P_1) = L(C_{WP}),$$

where we define $G'_1 = \lim_{P\to P_1} G'(P)$ and where $V_P^-(P_1)$ is the left-derivative of V at P_1. We can then solve for G'_1 as a function of parameters and P_1, where we eliminate V_P using the FOC (28):

$$G'_1(\alpha, \alpha'; P_1) = (1 - P_1)\frac{L(C_{WP}) - L(C_{FT'})}{\frac{1}{C_{FT'}} - (1 + \alpha)} + C_{FT'} - P_1(C'_{WP} + C_{FT'}). \quad \text{(A.10)}$$

We now introduce the function Q to facilitate notation:

$$Q(\alpha, \alpha') \equiv \frac{L(C_{WP}) - L(C_{FT'})}{\frac{1}{C_{FT'}} - (1+\alpha)} > 0. \tag{A.11}$$

Q is positive because the numerator in the fraction is always positive (recall that L is maximized at C_{WP}) and the denominator is strictly positive by the FOC (28) and the fact that $V_P^-(P_1) > 0$ (recall that we had ruled out before that $\mathcal{P}_{FT'}$ is WP if $\alpha + \alpha' < 2$).

Having found transfers at P_1, transfers G' on the remainder of $\mathcal{P}_{FT'}$ can now be backed out from the ODE (A.6) for transfers in the FT-region, which becomes $G'_P = \alpha' C'_{WP} - C_{FT'}$. We observe that $G'_P > 0$ by (A.9). We still have to ensure that (i) G' is such that the recipient's consumption plan is always feasible, i.e. $G'(0) \geq C(0)$, and (ii) that transfers are positive, i.e. $G'(P) \geq 0$ for all P. Since $G'_P > 0$, (i) and (ii) boil down to the condition $G'(0) \geq C_{FT'}$, which we may re-write as follows using the ODE (A.6) for G':

$$G'(0) = (1 - P_1)Q(\alpha, \alpha') + C_{FT'} - P_1 C'_{WP}(1 + \alpha') \geq C_{FT'}.$$

We see that we can always choose P_1 small enough for the inequality to hold, since $Q(\alpha, \alpha') > 0$ by (A.11). Using the fact that $(1 + \alpha')C'_{WP} = 1$, we can actually solve for the largest-possible P_1 that fulfills the inequality as

$$P_{max}(\alpha, \alpha') = \frac{Q(\alpha, \alpha')}{Q(\alpha, \alpha') + 1}, \tag{A.12}$$

where $Q(\cdot)$ is given by (A.11). This concludes the construction of the region $\mathcal{P}_{FT'}$; we may use the same arguments to construct $\mathcal{P}_{FT}$.

To conclude the equilibrium construction, observe that best-responding at the boundary P_1 is unproblematic. Since his value function is flat in both directions, $C'(P_1) = C'_{WP}$ is clearly optimal for him. He may choose $G'(P_1)$ in the transfer stage large enough such that $\dot{P} > 0$ is fulfilled in the consumption stage if she chooses C_{WP}. Note that for her, $C(P_1) = C_{WP}$ is optimal since it attains the global maximum of the Hamiltonian by Lemma 5 and thus dominates any lower consumption rate leading to $\dot{P} < 0$. Even if for some (G, G') in her consumption stage $C^0(P_1; G, G') = C_{FT'}$ is optimal, he is indifferent between any G' in his transfer stage since he will choose C'_{WP} anyway and $J(C_{WP}) = J(C_{FT'})$. Thus, a large $G'(P_1)$ that ensures $\dot{P} > 0$ is optimal. Agents are also best-responding at $P = 0$ by lemmas 2 and 3, which concludes the proof. ■

Figure 7 shows the function $P_{max}(\alpha, \alpha')$ defined in (A.12). We see that the largest range for P_1 that can be supported occurs when he is very altruistic and she is selfish. As is to be expected, the range of equilibria that can be supported becomes extremely small when the his altruism approaches zero.

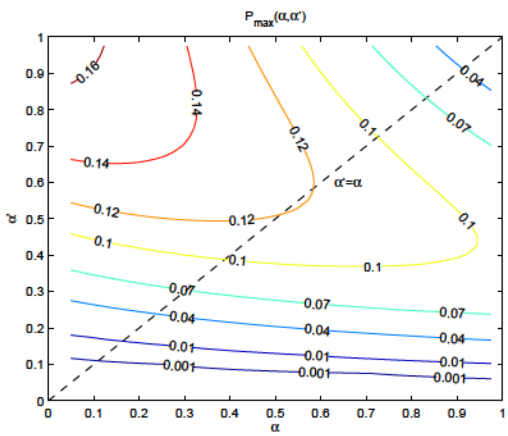

Figure 7: Maximal boundary in FT-WP-equilibrium

A noteworthy feature of this equilibrium is the following: Since transfers G' are linearly increasing in P on FT, we see from the law of motion for P in (22) that $\dot{P}$ linearly increases in P. This means that the economy is moving out of FT at increasing speed as she becomes richer. If $P_1 < P_{max}$, then the equilibrium is such that FT is left in finite time for any starting value P_0, even when $P_0 = 0$. For $P_1 = P_{max}$, however, the initial time spent in FT increases without bound as $P_0 \to 0$ and is indeed infinite when $P_0 = 0$. Then $\dot{P} = 0$ and the economy is stuck at $P = 0$ forever.

A.3.2 Proof for Theorem 2 (Party Theorem)

Proof: We begin by proving point 1 of the proposition. Since $\alpha' > 0$, V is continuous by Proposition 1. V is thus lower-bounded. This clearly implies that $C(P) \geq \epsilon$ for all P for some $\epsilon > 0$, which in turn implies $\dot{P}_{lim} \leq -\epsilon < 0$ by the law of motion (21) – recall that $G = G' = 0$ in the neighborhood of $P = 0$ since we assumed a NT-region $(0, P_1)$.

We now move on to point 2. By his FOC (28) his consumption must be continuous at zero and given by $C'_{lim} = C'(0) = C'_{WP}$. Denote her realized consumption at zero by $C_0^* = \min\{C(0), G'(0)\}$. We now state both agents' value-matching conditions at $P = 0$. Define the functions $H_{WP}(C) = \ln C - C\frac{1+\alpha}{\rho}$ and $J_{WP}(C) = \alpha' \ln C - C\frac{1+\alpha'}{\alpha'\rho}$ to write

$$H_{WP}(C_0^*) = H_{WP}(C_{lim}) + \dot{P}_{lim} V_P^+(0), \tag{A.13}$$

$$J_{WP}(C_0^*) = J_{WP}(C_{lim}) + \dot{P}_{lim} V'^+_P(0). \tag{A.14}$$

The functions H_{WP} and J_{WP} are depicted in Figure 8. The relevant domain for C is $[\epsilon, C_{WP})$, as is clear from her consumption FOC and the fact that $V_P^+(0) \geq 0$.

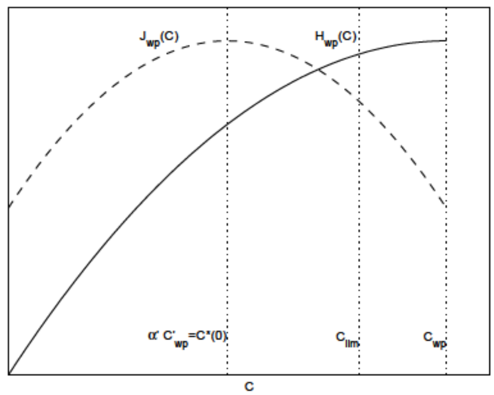

Figure 8: $H_{WP}(C)$ and $J_{WP}(C)$

We first rule out the case $C_0^* = C_{lim}$. By (A.13), this would imply $V_P^+(0) = 0$ since $\dot{P}_{lim} < 0$. But then $C_{lim} = C_{WP}$ by the FOC (28), and by Lemma 4 her strategy when broke also has to be $C(0) = C_{unc} = C_{WP}$. Since $C_{WP} = \frac{\rho}{1+\alpha} \geq \frac{\rho}{2} \geq \frac{\alpha'\rho}{1+\alpha'}$, with one of the inequalities being strict because $\alpha + \alpha' < 2$, by Lemma 3 he would respond to this with $G'(0) = \frac{\alpha'\rho}{1+\alpha'}$. Thus $C_0^* = G'(0) < C_{lim}$, a contradiction.

So it must be that $C_0^* \neq C_{lim}$. Since $\dot{P}_{lim} < 0$ and $V_P \geq 0$, we have $\dot{P}_{lim} V_P^+(0) \leq 0$. This implies by (A.13) that $C_0^* < C_{lim}$ since H_{WP} is strictly increasing, which is what we term a *party* before going broke. Since $V_P' \leq 0$ we have $\dot{P}_{lim} V'^+_P(0) \geq 0$, which in turn implies $J_{WP}(C_0^*) \geq J_{WP}(C_{lim})$. Since $C_0^* < C_{lim}$, this tells us that C_{lim} must be on the decreasing part of J, i.e. $C_{lim} > \alpha' C'_{WP}$ (see Figure 8), so she is over-consuming in his eyes before going broke. Then, again by Lemma 2 her consumption strategy when broke is $C(0) = C_{lim} > \alpha' C'_{WP}$. By Lemma 3 this implies $C_0^* = G'(0) = \alpha' C'_{WP}$ and $\dot{P}_0 = 0$, which concludes the proof for points 1 and 2 in the proposition.

Now, return to her value-matching (A.13) and write $\dot{P}_{lim}$ in terms of C_{lim} and C'_{WP} using the law of motion (21). Then replace $V_P^+(0)$ using the FOC (28) for C_{lim}, and solve for C_{lim} to find the closed-form expression given in point 3 of the proposition. Finally, we see from his value-matching (A.14) that $V'^+_P(0) < 0$ since $J_{WP}(C_0^*) = J_{WP}(\alpha' C'_{WP}) > J_{WP}(C_{lim})$, which proves point 4 of the proposition. ■

A.3.3 Proof for Theorem 3 (the Prodigal-Son Dilemma)

Proof: By way of contradiction, suppose that he gave a mass transfer in the region $[0, P_1)$. If he delayed the mass transfer at $P = 0$ and set $G'_m = 0$ in the transfer stage, then she would set $C = C_{unc} = C_{WP}$ by Lemma 2 (since $V_P^+(0) = 0$ for mass-transfer regions by Lemma 10). He can then implement his globally-preferred allocation by setting $(G'_m, G'_f) = (0, \frac{\alpha'\rho}{1+\alpha})$ at $P = 0$, thus restricting her consumption since $G'_f = \frac{\alpha'\rho}{1+\alpha} \leq \frac{\rho}{2} \leq \frac{1}{1+\alpha} = C_{WP}$, where one of the inequalities is strict since $\alpha + \alpha' < 2$. For this policy we see in the consumption stage that

$$\rho V^c\big(0; 0, (0, \tfrac{\alpha'\rho}{1+\alpha'})\big) = \max_{C,C'} \{ \underbrace{\ln C' + \alpha' \ln C - (C' + C)\tfrac{1+\alpha'}{\rho}}_{\equiv H_{tot}(C,C')} \}.$$

We will now see that this policy strictly dominates handing out the mass transfer at $P = 0$. By Lemma 10 on MT-regions, her consumption just after having received a mass transfer must be $C(P_1) = C_{WP} = \frac{\rho}{1+\alpha}$. Using the following two points in his order-1 problem (26) and (27) shows that delaying the mass transfer is indeed prof table: (i) $H_{tot}(C_{WP}, C') < H_{tot}(\frac{\alpha'\rho}{1+\alpha'}, C'_{WP})$, since $(C, C') = (\frac{\alpha'\rho}{1+\alpha'}, C'_{WP})$ is the unique maximizer of H_{tot} as def ned in the above equation and $C_{WP} \neq \frac{\alpha'\rho}{1+\alpha'}$, as shown before. (ii) Since ${V'}_P^+(P_1) \leq 0$ by Proposition 2 and ${V'}_P^-(P_1) = 0$ by Lemma 9, we have $\dot{P}\nabla_{\dot{P}}V'(P_1) \leq 0$. ■