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“SEX-EQUAL” STABLE MATCHING

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Abstract

This paper presents a solution concept that minimizes envy between groups in a bilateral matching market. This concept is designed to select stable matchings that are not men or women optimal. The idea is to compute the total number of women preferred by the men to their woman mates and the total number of men preferred by women to their mates in that matching. The absolute value of the distance between these two numbers generates the stable matchings with less envy between groups. An algorithm is provided to compute them.

Keywords: Matching Markets, Fair Distribution, No-envy

JEL classification: C78, C63, D63

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1 Introduction

The purpose of this paper is to find a criterion which allows us to avoid extreme matchings and compute intermediate ones. This paper provides a solution concept and an algorithm to define and compute intermediate allocations in bilateral matching markets. The solution concept proposed is the The Sex-Equal Matching (*SEM*). This concept is based on the minimization of the envy difference between the agents on both sides of the market. Agents in this model are called women and men. The set of agents and their preferences constitute a marriage problem. Under very weak assumptions this solution concept coincides with the one proposed as an open problem by Gusfield and Irving (1989) (from now on G&I ,1989).

Theoretical work on two-sided matching markets has been traditionally focused on the study of extreme stable matchings (men and women optimal stable matchings). The structure of the stable matching set generates an unanimously optimal stable matching for women. Men consider this matching as their worse possible stable allocation. The opposite happens with respect to the men optimal stable matching. This is consequence of an underlining conflict of interest existent in this market (see Knuth ,1976, and Roth and Sotomayor ,1990,). Any change for one stable allocation to another will weakly improve the situation of agents on one side of the market at expenses of the other side. Women and men optimal stable matchings are unique and there are algorithms that make them easy to compute (see Gale and Shapley ,1962,).

Knuth (1976) following Stan Selkow presents the first algorithm that computes matchings other than the extreme ones. It computes the position that each agents's partner has

in her/his preference list and selects the stable matching that minimizes the sum across individuals. This solution concept is utilitarian in the sense that maximize the social surplus. However it does not balance agents' aspirations. Intermediate matchings are a way to avoid the conflict of interest present in extreme ones. A proper definition of what an intermediate matching is shall overcome this conflict.

An appealing concept of equity was proposed by Foley (1967) (see also Varian ,1974,). It is based in the combination of weak pareto optimality and no envy. Any stable matching is pareto optimal. Stability is important since an unstable matching can be objected by a group of agents and will not be enforceable. A no envy solution is a matching where no agent prefers other agent's partner to his\her own mate. Yet, it is easy to find examples of markets where this solution concept is empty as shown by Masarany and Gokturk (1991).

Since the conflict of interests presented in the model is among agents in each side of the market, it seems natural to focus on justice among groups rather than try to implement any individual idea of justice. This paper combines the concepts of stability and minimal envy between groups providing a criterion to identify and compute stable allocations that are intermediate. This is because no other stable matching provide less conflict of interests between groups. The set *SEM* balance agent's aspirations providing a reasonable criterion to avoid the conflict in the model.

The rest of the paper is organized as follows. Section 2 introduces the model and defines some basic concepts. Section 3 defines the Sex-Equal concept and relates it with other intermediate solution concepts. An example illustrates the major steps to compute the

SEM in Section 4. Section 5 formally describes an algorithm to compute the Sex-Equal Matching and the paper's main results. Conclusions are collected in Section 6.

2 The model, notation and definitions

In a bilateral market there are two finite disjoint sets. Let us denote as $W = \{w_1, w_2, \dots, w_m\}$ the set of women and $M = \{m_1, m_2, \dots, m_n\}$ the set of men. Each m_i 's preferences P_{m_i} are described by a linear order on $W \cup \{m_i\}$. Given two $w_j, w_h \in W$, the expression $w_j P_{m_i} w_h$ means that m_i prefers to be matched to w_j rather than w_h ; $m_i P_{m_i} w_h$ means that m_i prefers to stay single rather than being matched to w_h . Similarly, each woman w_j 's preferences P_{w_j} are described by a linear order on $M \cup \{w_j\}$. The marriage problem is fully described by a triplet (M, W, P) where P is a preference profile containing a full description the agents' preferences.

A **matching** is a function $\mu : M \cup W \rightarrow M \cup W$, such that:

1. $(\mu(m_i) \notin W \rightarrow \mu(m_i) = m_i), (\mu(w_j) \notin M \rightarrow \mu(w_j) = w_j)$, and
2. $(\mu(m_i) = w_j) \leftrightarrow (\mu(w_j) = m_i)$.

It is important that no agent or group of agents should be able to improve their allocation in a matching by negotiation with other agents or by an individual decision.

Definition 1 *A matching μ is blocked by an individual $\sigma \in (M, W)$ in (M, W, P) iff $\sigma P_\sigma \mu(\sigma)$. A matching that can not be blocked by any individual is called **individually rational**.*

Definition 2 A matching μ is *blocked by a pair* (m_i, w_j) in (M, W, P) iff $w_j P_{m_i} \mu(m_i)$ and $m_i P_{w_j} \mu(w_j)$.

Any individually rational matching that can not be blocked by pairs is said to be **stable**.

In this model, the notion of stability is equivalent to the usual definition of the core. Let Γ denote the set of all possible stable matchings in (M, W, P) . When all agents have strict preferences the set Γ has a global maximum and a global minimum according with the men or women's preferences. These matchings are the men optimal and women optimal stable matchings, denoted by μ^M and μ^W .

Let us introduce some notation to relate each agent with his/her possible partners. For any m_i , $s_\mu(m_i)$ denotes the first woman w_j on P_{m_i} such that w_j strictly prefers m_i to $\mu(w_j)$ (her partner in μ); $next_\mu(m_i)$ denotes the partner in μ of the woman $s_\mu(m_i)$. This notation allows to formally define the process of changes that leads from a stable matching to another.

Definition 3 Let $\rho = \{ (m_0, w_0), (m_1, w_1), \dots, (m_{r-1}, w_{r-1}) \}$ be an ordered list of pairs in a stable matching μ such that for each i ($0 \leq i \leq r-1$) m_{i+1} is the $next_\mu(m_i)$, where $i+1$ is taken modulo r , i.e. $m_r = m_0$ and $w_r = w_0$. Then ρ is called a **rotation (exposed)** in μ ; m_i (or w_j) is in a rotation ρ if there is a pair (m_i, w_j) in the ordered list defining ρ .

Let $\rho = \{ (m_0, w_0), (m_1, w_1), \dots, (m_{r-1}, w_{r-1}) \}$ be a rotation exposed in μ . Let $\mu \setminus \rho$ be the matching in which each man not in ρ stays matched to his partner in μ , and the match for each man m_i in ρ is $w_{i+1} = s_\mu(m_i)$. The transformation of μ to $\mu \setminus \rho$ is called the **elimination** of ρ from μ .

A rotation is not associated with a unique matching. It may be exposed more than once. In any stable matching other than μ^W there is at least one rotation exposed. Moreover the matching μ^M can be transformed to μ^W through a sequence of stable matchings, by successively finding and eliminating any exposed rotation in each successive matching.

Definition 4 *The stable matching μ is said to be an immediate predecessor of the stable matching μ' if there is a rotation ρ such that $\mu \setminus \rho = \mu'$.*

Definition 5 *A chain $C = \{ \mu^1, \dots, \mu^q \}$ in Γ is an ordered set of elements of Γ such that μ_i is an immediate predecessor of μ_{i+1} for each i , $1 \leq i \leq q - 1$.*

Definition 6 *A maximal chain in Γ is a chain that extends from the minimal element to the maximal element of the lattice, i.e., a maximal chain is a chain from μ^M to μ^W .*

Definition 7 *Let $\rho = \{ (m_0, w_0), (m_1, w_1), \dots, (m_{r-1}, w_{r-1}) \}$ be a rotation; ρ moves m_i down from w_i to w_{i+1} , and moves w_i up from m_i to m_{i-1} . If w is either w_i or is strictly between w_i and w_{i+1} in P_{m_i} , then ρ moves m_i below w . Similarly, ρ moves w_i above m if m is m_i or is strictly between m_i and m_{i-1} in P_{w_i} .*

For any man m_i and for any woman w_j in (M, W, P) there is at most one rotation that moves m_i down to w_j , and w_j up to m_i .

There are two precedence relations between the rotations: Suppose (m, w) is in a rotation ρ . (i) If ρ' is the (unique) rotation that moves m to w , then ρ' is a **type 1 predecessor** of

ρ . (ii) If ρ moves m below w , and $\rho' \neq \rho$ is the (unique) rotation that moves w above m , then ρ' is a **type 2 predecessor** of ρ .

Type 1 precedence relation implies that if a rotation ρ joins a partner and another rotation ρ' separates it, ρ must be eliminated before ρ' is exposed. Type 2 precedence relation implies that if a pair (m, w) must be formed (by the elimination of rotation ρ) to avoid that another rotation elimination ρ' becomes an unstable matching, ρ must be eliminated before ρ' appears exposed.

Let $\Pi(\Gamma)$ be the set of all the rotations in Γ . $\Pi(\Gamma)$ is ordered by the precedence relations¹ of type 1 and 2. The relation of precedence in $\Pi(\Gamma)$ is transitive and asymmetric.

3 The Sex-Equal matching

Stability and no envy are two concepts impossible to combine within the framework considered in this paper. Given the particular structure of the market seems natural to study the possibilities of equal treatment among groups in each side of the market rather than try to implement an individual idea of justice.

Let us compute the difference between the number of envy situations in each side of the market. Given a pair (m_i, w_j) , m_i envies the partners of all the women w such that $w P_{m_i} w_j$. Let $r_{m_i}(m_i, w_j)$ be the position that w_j has in P_{m_i} . The number of agents envied by m_i when matched with w_j , is $r_{m_i}(m_i, w_j) - 1$. Respectively w_j envies $r_{w_j}(w_j, m_i) - 1$ agents. Given matching μ , the difference between the total envy in each side on the market is the following:

¹ The precedence relations of type 1 and 2 are defined over rotations, not over matchings. The relation of immediate predecessor is defined over matchings (Definition 4).

$$\nu(\mu) = \left\| \left[n + \sum_{m_i \in M} (r_{m_i}(m_i, w_j)) \right] - \left[m + \sum_{w_j \in W} r_{w_j}(w_j, m_i) \right] \right\|. \quad (1)$$

The matching with the lowest difference in envy is the one that minimizes (1):

$$\arg \min_{\mu} \left\| (n - m) + \sum_{(m_i, w_j) \in (\mu)} (r_{m_i}(m_i, w_j) - r_{w_j}(w_j, m_i)) \right\|. \quad (2)$$

The value of $n - m$ depends on the number of agents in each side of the market. From now on we assume $n = m$ ².

It is also important that the *SEM* belongs to the core of the market. Stability is necessary to guarantee fairness. Any unstable matching, even if it is selected by a very appealing concept of fairness, can be blocked by a pair of agents.. The matching resulting of this block can be itself blocked by other pair of agents. There are not guaranties that this blocking process may end in a matching that satisfying none of the properties that were the reason to select an unstable matching in the first place.

Given that equal number of agents in each side and stability have been imposed the initial criterion coincides with the solution concept proposed by G&I (1989).

Definition 8 *A Sex-Equal Matching is a stable matching that minimizes the absolute value of the sum of the difference between the position that each partner has in each other preference list,*

$$SEM(\Gamma) = \arg \min_{\mu \in \Gamma} \left\| \sum_{(m_i, w_j) \in (\mu)} (r_{m_i}(m_i, w_j) - r_{w_j}(w_j, m_i)) \right\|. \quad (3)$$

Since the number of stable allocations is finite, *SEM* is always non-empty.

² This assumption can be easily removed by adding a constant to the value of the matchings depending on the structure of the market.

Let us compare the matchings selected by the *SEM* with the selection by other solution concepts, using Knuth's example.

Knuth's Example. (Roth and Sotomayor ,1990, page 37). Let $M = \{ m_1, m_2, m_3, m_4 \}$ and $W = \{ w_1, w_2, w_3, w_4 \}$. The preferences are as follows:

$$\begin{aligned} P_{m_1} : w_1, w_2, w_3, w_4, m_1, & P_{w_1} : m_4, m_3, m_2, m_1, w_1, \\ P_{m_2} : w_2, w_1, w_4, w_3, m_2, & P_{w_2} : m_3, m_4, m_1, m_2, w_2, \\ P_{m_3} : w_3, w_4, w_1, w_2, m_3, & P_{w_3} : m_2, m_1, m_4, m_3, w_3, \\ P_{m_4} : w_4, w_3, w_2, w_1, m_4, & P_{w_4} : m_1, m_2, m_3, m_4, w_4. \end{aligned}$$

There are ten stable matchings where w_1, w_2, w_3, w_4 are matched to, respectively,

$$\begin{array}{cccccc} m_1 & m_2 & m_3 & m_4 & \mu^1 & \\ m_2 & m_1 & m_3 & m_4 & \mu^2 & \\ m_1 & m_2 & m_4 & m_3 & \mu^3 & \\ m_2 & m_1 & m_4 & m_3 & \mu^4 & \\ m_3 & m_1 & m_4 & m_2 & \mu^5 & \\ m_2 & m_4 & m_1 & m_3 & \mu^6 & \\ m_3 & m_4 & m_1 & m_2 & \mu^7 & \\ m_4 & m_3 & m_1 & m_2 & \mu^8 & \\ m_3 & m_4 & m_2 & m_1 & \mu^9 & \\ m_4 & m_3 & m_2 & m_1 & \mu^{10} & \end{array} \quad (4)$$

For each one of these stable matching $\nu(\mu)$ can be computed,

$$\begin{aligned} \nu(\mu^1) &= & -12 \\ \nu(\mu^2) &= \nu(\mu^3) &= & -8 \\ \nu(\mu^4) &= & -4 \\ \nu(\mu^5) &= \nu(\mu^6) &= & 0 \\ \nu(\mu^7) &= & 4 \\ \nu(\mu^8) &= \nu(\mu^9) &= & 8 \\ \nu(\mu^{10}) &= & 12 \end{aligned} \quad (5)$$

Matchings μ^5 and μ^6 are *SEM* in this example. Both matchings are in the middle of the lattice formed by the ten matchings in this market.

Knuth's solution concept is unable to select any particular matching among all the stable ones. This is because the value of the sum across individuals of the difference between

among agents and their mates remains the same for all matchings. This inability to select intermediate matchings is shared by the randomized matching mechanism in Roth and Vande Vate (1990). The random mechanism can not achieve every stable matching, especially those in the middle that have less conflict of interest (see Ma ,1996, for a proof).

4 Computing the Set of SEM

In this section proceeds the techniques to compute the set of *SEM* are introduced by an example. Let $M = \{ m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8 \}$ and $W = \{ w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8 \}$. Consider the following preferences:

$$\begin{array}{ll}
 P_{m_1} : w_3, w_5, w_1, w_7, w_2, m_1, & P_{w_1} : m_3, m_2, m_8, m_1, m_5, w_1, \\
 P_{m_2} : w_6, w_2, w_8, w_1, m_2, & P_{w_2} : m_1, m_4, m_7, m_8, m_2, w_2, \\
 P_{m_3} : w_5, w_2, w_3, w_7, w_4, m_3, & P_{w_3} : m_4, m_5, m_8, m_3, m_1, w_3, \\
 P_{m_4} : w_1, w_4, w_8, w_3, m_4, & P_{w_4} : m_6, m_3, m_7, m_8, m_4, w_4, \\
 P_{m_5} : w_1, w_4, w_5, w_7, w_6, m_5, & P_{w_5} : m_2, m_6, m_8, m_5, m_3, w_5, \\
 P_{m_6} : w_3, w_6, w_8, w_5, m_6, & P_{w_6} : m_1, m_5, m_7, m_8, m_6, w_6, \\
 P_{m_7} : w_7, w_8, m_7, & P_{w_7} : m_8, m_7, w_7, \\
 P_{m_8} : w_8, w_1, w_3, w_5, w_6, w_7, m_8, & P_{w_8} : m_7, m_8, w_8,
 \end{array}$$

There are ten stable matchings where $w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8$ are matched respectively with the men denoted by the numbers in bold in the following picture:

[Insert Figure 1 around here]

In Figure 1 there are ten stable matchings where w_1 to w_8 are matched to, respectively, each one of the man represented by the numbers in bold. Above the numbers in bold is $r_{m_i}(m_i, \mu(m_i))$ and numbers below represents $r_{w_j}(\mu(w_j), w_j)$. Let us compute $\nu(\mu)$ for each matching.

$$\begin{aligned}
\nu(\mu^M) &= -23 & \nu(\mu^e) &= 5 \\
\nu(\mu^a) &= -14 & \nu(\mu^f) &= 4 \\
\nu(\mu^b) &= -5 & \nu(\mu^g) &= 5 \\
\nu(\mu^c) &= -4 & \nu(\mu^h) &= 14 \\
\nu(\mu^d) &= -5 & \nu(\mu^W) &= 22
\end{aligned} \tag{6}$$

Matchings μ^c and μ^f are the ones that form the *SEM*. The question is how to compute these matchings without computing and evaluating every stable matching. There are some regularities that can be useful in this task. The difference between the value of $\nu(\mu^b)$ and the value of $\nu(\mu^a)$ is the same as the difference between $\nu(\mu^e)$ and $\nu(\mu^d)$ and the same as the difference between $\nu(\mu^h)$ and $\nu(\mu^g)$. This difference is generated by the same rotation. There are five different rotations between the stable matchings in the example:

$$\begin{aligned}
\rho_0 &= \{ (m_1, w_3), (m_5, w_1), (m_3, w_5) \} \\
\rho_1 &= \{ (m_1, w_1), (m_2, w_2) \} \\
\rho_2 &= \{ (m_3, w_3), (m_4, w_4) \} \\
\rho_3 &= \{ (m_5, w_5), (m_6, w_6) \} \\
\rho_4 &= \{ (m_7, w_7), (m_8, w_8) \}
\end{aligned} \tag{7}$$

For instance, ρ_0 is exposed in μ^M and its elimination produces μ^a . The changes produced by the elimination of ρ_i involves always the same partners. The value of this change is associated to each rotation, i.e., $\alpha_i = \nu(\mu \setminus \rho_i) - \nu(\mu)$, and it is independent from the stable matching where the rotation is exposed.

$$\begin{aligned}
\alpha_0 &= 9 \\
\alpha_1 &= 9 \\
\alpha_2 &= 10 \\
\alpha_3 &= 9 \\
\alpha_4 &= 8
\end{aligned} \tag{8}$$

Given $\nu(\mu^a)$ and α_1 , $\nu(\mu^a \setminus \rho_1)$ can be computed without computing $\mu^b = \mu^a \setminus \rho_1$:

$$\nu(\mu^a \setminus \rho_1) = \nu(\mu^a) + \alpha_1 = -14 + 9 = \nu(\mu^b) = 5. \tag{9}$$

The value of α_i is always positive. Hence if $\nu(\mu^b) = \nu(\mu^a \setminus \rho_1)$, then $\nu(\mu^b) > \nu(\mu^a)$.

In order to find all the possible rotations we do not have to compute the whole set of stable matchings; it is enough to compute the ones along a maximal chain (see G&I, 1989, Theorems 3.2.1 and 3.2.2 for a proof).

To find a rotation exposed in a stable matching is simple. The men involved in a rotation exposed in a stable matching will be worse once the rotation is eliminate. All women involved will be better off. It is enough to go down $\mu(m_i)$ in P_{m_i} , for each m_i , looking for a woman that prefers m_i to her partner. If one woman is found, w_j , the process is repeated with $\mu(w_j)$ and so on. A rotation is found when cycle is completed and the last woman found was initially matched with m_i . This process is repeated until μ^W is reach. In μ^W there is not rotations exposed.

To find all *SEM* is enough to go through all the maximal chains by eliminating rotations, until we find a change of sign between μ and $\mu \setminus \rho$. Once we have found all the sign changes the *SEM* are in the matchings with the last positive or the first negative value in each chain. When there is no change in signs along the maximal chains it means that $\nu(\mu^M)$ and $\nu(\mu^W)$ have the same sign. In this case μ^M is the unique *SEM* if $\nu(\mu^M) \geq 0$, and μ^W is the unique *SEM* if $\nu(\mu^W) \leq 0$.

All the information about which rotations may be eliminated at each time is contained in agents' preferences. All rotations can be ordered by using type 1 and type 2 precedence relations. In the example ρ_0 is a type 1 predecessor of ρ_1, ρ_2 and ρ_3 , and that ρ_1, ρ_2, ρ_3 are type 2 predecessors of ρ_4 .

Let $\Omega_{i,\dots,j}$ be the value of the last matching in the chain $C = \{ \mu^M, \dots, \mu^t \}$ resulting of the elimination of rotations $\{ \rho_i, \dots, \rho_j \}$. With the information available, Ω can be easily computed:

$$\begin{aligned}
\Omega_0 &= \mu^M + \alpha_0 = -14 \\
\Omega_{01} &= \mu^M + \alpha_0 + \alpha_1 = -5 \\
\Omega_{02} &= \mu^M + \alpha_0 + \alpha_2 = -4 \\
\Omega_{03} &= \mu^M + \alpha_0 + \alpha_3 = -5 \\
\Omega_{012} &= \mu^M + \alpha_0 + \alpha_1 + \alpha_2 = 5 \\
\Omega_{013} &= \mu^M + \alpha_0 + \alpha_1 + \alpha_3 = 4 \\
\Omega_{023} &= \mu^M + \alpha_0 + \alpha_2 + \alpha_3 = 5
\end{aligned} \tag{10}$$

On (10) all the maximal chains have changed their signs. The *SEM* will be found along them. The *SEM* are either the last matching with a positive value or the first one with a negative value. The matchings with the smallest absolute value between them are the *SEM*.

$$\min [|\Omega_{01}|, |\Omega_{02}|, |\Omega_{03}|, |\Omega_{012}|, |\Omega_{013}|, |\Omega_{023}|] = 4 \tag{11}$$

In this case we have two *SEM*, represented by the sequences of rotations Ω_{02} and Ω_{013} . After the rotations are eliminated from μ^M , μ^c and μ^f are obtained.

This enumeration process becomes exponential if we compute stable allocation more than once. This possibility can be easily avoided. Rotations already exposed are recorded in Ω 's subindex. This record can be used to avoid exponentiality just by preventing from elimination each rotation numbered with a number smaller that the last rotation eliminated in a particular matching. For example, in Ω_{02} rotation ρ_1 can not be eliminated or in Ω_{03} rotations ρ_1 and ρ_2 are prevented from elimination. In such way no matching will be computed more than once.

5 The algorithm

In this section an algorithm is provided that summarizes the steps followed to compute the *SEM* in the previous section. After the presentation of the *SEM* algorithm the section ends with the paper's main results.

The SEM algorithm:

- **Step 1:** Compute $\nu(\mu^M)$.
 - * If $\nu(\mu^M) \geq 0$ the algorithm stops and μ^M is the unique *SEM*.
 - * If not, continue to the next step.

- **Step 2:** Compute $\nu(\mu^W)$.
 - * If $\nu(\mu^W) \leq 0$ the algorithm stops and μ^W is the unique *SEM*.
 - * If not, continue to the next step.

- **Step 3:** Determine the preorder $\Pi(\Gamma)$ of rotations in the set Γ .

In the previous example an intuitive way to compute $\Pi(\Gamma)$ has been presented. I refer the reader to G&I (1989) for an efficient way to do this computations³.

- **Step 4:** Evaluate each rotation.

- **Step 5:** Detect the change in signs at each maximal chain.

³ The Irving and Gusfield algorithm is based on the construction of a directed, acyclic graph $\Theta(\Gamma)$, called the rotation digraph, whose edges correspond to a subset of pairs on $\Pi(\Gamma)$. Irving and Gusfield goal when defining this graph is to speed the process to compute $\Pi(\Gamma)$, because the transitive closure of $\Theta(\Gamma)$ is $\Pi(\Gamma)$ and the digraph $\Theta(\Gamma)$ can be constructed from the preference list in a more efficient way. Gusfield and Irving formalized this algorithm and made this process work on $O(n^2)$ time.

- **Step 6:** Compute

$$\arg \min [|\Omega_{0\dots k}^-|, |\Omega_{0\dots k+1}^+|, \dots, |\Omega_{0\dots t}^-|, |\Omega_{0\dots t+1}^+|], \quad (12)$$

- **Step 7:** Compute the set of *SEM* by eliminating the rotations on the $\Omega_{0,\dots,j}^*$ that satisfies condition (12).

Once the $\Omega_{0,\dots,j}^*$ are selected the rotations involved can be eliminated from μ^M and the *SEM* computed.

There is only one rotation between every pair of stable matchings μ and $\mu \setminus \rho_i$ and this rotation has always the same elements by definition. This means that the α_i of each rotation ρ_i , is fixed with independence of the μ for which ρ_i is exposed.

Lemma 9 *The value of α_i is positive for every rotation ρ_i on $\Pi(\Gamma)$.*

Proof. In a rotation elimination only the pairs in ρ_i change partners. By the definition of rotation each elimination increase $r_m(m, w)$ and decrease $r_w(m, w)$. By definition of α_i its value is precisely the scope of the change in the difference between $r_m(m, w)$ and $r_w(m, w)$. Therefore, the value of α_i is positive. ■

A stable matching belongs to the *SEM* if $\nu(\mu)$ is zero. If not, our task is to find the set of stable matchings that have a value nearest to zero knowing that if $\nu(\mu^M) < 0$ and $\nu(\mu^W) > 0$, there must be a change of sign in the set of stable matchings. Clearly the situation where $\nu(\mu^W) < 0$, and $\nu(\mu^M) > 0$ is impossible by Lemma 9.

Theorem 10 [*G&I, 1989, Th. 2.2.4*] *Every rotation of Γ appears once on every maximal chain of Γ .*

Theorem 11 [G&I ,1989, Th. 2.2.5] *Every stable matching μ' can be generated by a sequence of rotation eliminations, starting from the μ^M stable matching, and every sequence leading to the same stable matching contains exactly the same rotations.*

Proposition 12 *If $\nu(\mu^M) \geq 0$ the μ^M is the unique SEM. If $\nu(\mu^W) \leq 0$ the μ^W is the unique SEM.*

Proof. This result is a direct consequence of the lattice structure of the set of stable matchings, Theorem 11 and Lemma 9. ■

The problem of finding a SEM has been reduced to the problem of finding the set of stable matchings for which the value of the difference between the concessions made by each side of the market is nearest to zero.

Theorem 13 *The SEM algorithm generates the set of Sex-Equal stable matchings.*

Proof. If $\nu(\mu^M)$ is positive or $\nu(\mu^W)$ is negative or any of them is zero the SEM is computed by the SEM algorithm in step 1 or step 2 and the proof follows from Proposition 12.

For the proof of the remaining cases I will proceed by contradiction. Let us suppose that the statement of the Theorem is not true and $\nu(\mu^M) < 0$ and $\nu(\mu^W) > 0$. Then there must be a maximal chain with a stable matching μ' , where $\nu(\mu') \in [|\Omega_{0\dots k}^-|, |\Omega_{0\dots k+1}^+|, \dots, |\Omega_{0\dots t}^-|, |\Omega_{0\dots t+1}^+|]$ and is smaller than the ones obtained by the SEM algorithm, i.e.,

$\nu(\mu^*) = \operatorname{argmin} [|\Omega_{0\dots k}^-|, |\Omega_{0\dots k+1}^+|, \dots, |\Omega_{0\dots t}^-|, |\Omega_{0\dots t+1}^+|]$. This is not possible because $\nu(\mu^*)$ is a minimum of the expression 12. A contradiction.

If μ' is not involved in the change of sign of its maximal chain, $\nu(\mu')$ can not be smaller than $\nu(\mu^*)$ because going through a maximal chain from μ^M to μ^W the $\nu(\mu_i)$ is always increasing (see proof of Lemma 9). So its absolute value must be larger than the matchings in the expression (12). A contradiction. ■

6 Conclusions

This paper presents an approach to approximate equitable allocation in matching markets. This approach takes into account the conflict of interest between groups intrinsic to these markets. The criterion proposed in the paper combines stability and envy considerations to propose a criterion to select fair matching within the set of stable allocations. An algorithm is proposed to compute the set of *SEM*. this algorithm intensively uses the structure of the set of stable allocations and works in polynomial time.

The main short comings of this solution concept are its multiplicity and the absence of any structure linking the different solutions in the set. This problem makes hard to select a particular *SEM* in base to an envy criterion when the set of *SEM* is not a singleton. The problem may be avoided by using an additional criterion to select a unique stable matching when the set of *SEM* it is not a singleton. However any of the possible elements of the *SEM* minimizes the conflict of interest in the market as much as possible given stability. In that sense the *SEM* are the most equitable allocation that can be provide in a marriage problem.

Both the concept and algorithm in this paper can be easily generalized to the more general college admissions problem.

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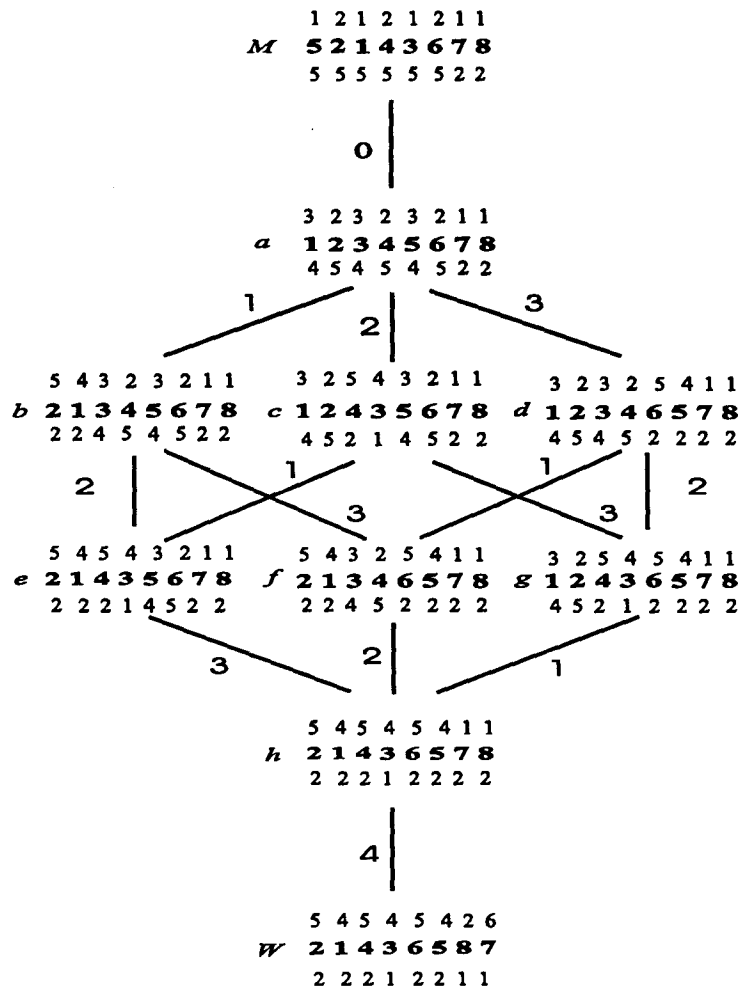


Figure 1: