

# A canonical family of multiple orthogonal polynomials for Nikishin systems

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## Abstract

For any pair of compact intervals  $\Delta_1$  and  $\Delta_2$  of the real line such that  $\Delta_1 \cap \Delta_2 = \emptyset$  we obtain two pairs of absolutely continuous **probability** measures  $(\mu_1, \mu_2)$  and  $(\tau_1, \tau_2)$  supported on  $\Delta_1$  and  $\Delta_2$ , respectively, such that

- for appropriate constants  $C_1$  and  $C_2$ ,  $(\mu_1, \mu_2)$  is the Nikishin system generated by  $(\mu_1, C_1\tau_1)$  and  $(\tau_1, \tau_2)$  the Nikishin system generated by  $(\tau_1, C_2\mu_1)$ ,
- the polynomials of multiple orthogonality with respect to the Nikishin system  $(\mu_1, \mu_2)$  and indices  $\{\dots, (n, n), (n+1, n), \dots\}$  satisfy a recurrence relations with *constant* coefficients of period 2,
- $1/\widehat{\mu}_1(z)$  and  $1/\widehat{\mu}_2(z)$  are the functions which describe the ratio asymptotics of multiple orthogonal polynomials with respect to an arbitrary Nikishin system  $\mathcal{N}(\sigma_1, \sigma_2)$  verifying  $\text{supp}(\sigma_i) = \Delta_i$ , and  $\sigma'_i > 0$ ,  $i = 1, 2$ , almost everywhere on  $\Delta_i$ . Analogously,  $1/\widehat{\tau}_1(z)$  and  $1/\widehat{\tau}_2(z)$  give the ratio asymptotics for  $\mathcal{N}(\sigma_2, \sigma_1)$ .

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## 1 Introduction

Let  $\Sigma = (\sigma_1, \sigma_2)$  be a system of positive Borel measures such that, for each  $k = 1, 2$ ,  $\sigma_k$  is supported on a compact interval  $\Delta_k = [a_k, b_k] \subset \mathbb{R}$  (which does not reduce to a single point) and  $\Delta_1 \cap \Delta_2 = \emptyset$ . Let  $\mathcal{N}(\Sigma) = (s_1, s_2)$  be the Nikishin system generated by  $\Sigma$ . That is,

$$ds_1(x) = d\sigma_1(x), \quad ds_2(x) = \widehat{\sigma}_2(x)d\sigma_1(x).$$

For any measure  $\mu$ ,

$$\widehat{\mu}(z) = \int \frac{d\mu(x)}{z-x}.$$

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The general definition of a Nikishin system was given in [14] where the author called them MT systems. Such systems have attracted great attention in connection with the extension of the asymptotic theory of orthogonal polynomials and the convergence of simultaneous Hermite-Padé approximation.

For each multi-index  $\mathbf{n} = (n_1, n_2) \in \mathbf{I}$ , where

$$\mathbf{I} = \{(0, 0), (1, 0), (1, 1), \dots, (2, 1), \dots\},$$

we consider the *monic* multiple orthogonal polynomial  $Q_{\mathbf{n}}$  satisfying

$$\int x^\nu Q_{\mathbf{n}}(x) ds_k(x) = 0, \quad 0 \leq \nu \leq n_k - 1, \quad k = 1, 2.$$

In [14] it was shown that any multi-index  $(n_1, n_2) \in \mathbf{I}$  is normal. This means that  $\deg Q_{\mathbf{n}} = n_1 + n_2$  and is uniquely determined by the orthogonality conditions and has degree  $|\mathbf{n}| = n_1 + n_2$ . (Regarding normality, for more general classes of multi-indices and Nikishin systems, see also [5], [6], and [7]). Since for any  $\mathbf{n} \in \mathbf{I}$  there exist unique  $k \in \mathbb{Z}_+$  and  $i \in \{0, 1\}$  such that  $|\mathbf{n}| = 2k + i$ , for convenience we write  $Q_{2k+i} = Q_{\mathbf{n}}$ .

Consider the 3-sheeted Riemann surface

$$\mathcal{R} = \bigcup_{k=0}^{\infty} \mathcal{R}_k,$$

formed by the consecutive “glued” sheets

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus \Delta_1, \quad \mathcal{R}_1 := \overline{\mathbb{C}} \setminus (\Delta_1 \cup \Delta_2), \quad \mathcal{R}_2 := \overline{\mathbb{C}} \setminus \Delta_2,$$

where the upper and lower banks of the slits of two neighboring sheets are identified. Let  $F_i$ ,  $i = 1, 2$ , be the single valued rational function on  $\mathcal{R}$  whose divisor consists of a simple pole at  $\infty^{(0)}$  and a simple zero at  $\infty^{(i)}$  with the normalization

$$F_i(x) = x + \mathcal{O}(1), \quad x \rightarrow \infty^{(0)}, \quad (1)$$

and let  $\{F_{i,k}\}_{k=0}^2$  be the different branches of  $F_i$  corresponding to the different sheets  $\mathcal{R}_k$  of  $\mathcal{R}$ . When  $\text{supp}(\sigma_k) = \Delta_k$  and  $\sigma'_k > 0$  a.e. on  $\Delta_k$ ,  $k = 1, 2$ , in [2, Theorem 1.2] it was proved that

$$\lim_{n \rightarrow \infty} \frac{Q_{2n+i}(z)}{Q_{2n+i-1}(z)} = F_{i,0}(z), \quad i = 1, 2, \quad (2)$$

uniformly on compact subsets of  $\mathbb{C} \setminus \Delta_1$ . This result on ratio asymptotics extends E.A. Rakhmanov’s theorem given for standard orthogonal polynomials on the real line.

**Remarks.**

1. The limits  $F_{i,0}(z)$  are expressed in [2, Theorem 1.2] in a different way, but the equivalence between both expressions is an immediate consequence of Liouville's theorem and Cardano's formulas.
2. Ratio asymptotics of multiple orthogonal polynomials for Nikishin systems also hold when each  $\sigma_k$  is supported on  $\Delta_k \cup e_k$ , where  $e_k$  is a denumerable set without accumulation points in  $\mathbb{R} \setminus \Delta_k$ , and  $\sigma'_k > 0$  a.e.. This is proved in [13, Theorem 4.1].

It is well known and easy to verify that the polynomials  $Q_n(x)$  satisfy the recurrence relations

$$\begin{aligned} Q_{-2}(x) = Q_{-1}(x) = 0, \quad Q_0(x) = 1 \\ xQ_n(x) = Q_{n+1}(x) + \alpha_n^* Q_n(x) + \beta_n^* Q_{n-1}(x) + \gamma_n^* Q_{n-2}(x), \quad n = 0, 1, \dots \end{aligned} \quad (3)$$

for certain real constants  $\alpha_n^*, \beta_n^*, \gamma_n^*$ . Moreover, as proved in [3, Theorem 1.2], when the polynomials have ratio asymptotics as expressed in (2), the coefficients of the recurrence have limits with period 2. More precisely, there exist real constants  $\alpha_i, \beta_i, \gamma_i$  such that

$$\lim_{n \rightarrow \infty} \alpha_{2n+i}^* = \alpha_i, \quad \lim_{n \rightarrow \infty} \beta_{2n+i}^* = \beta_i, \quad \lim_{n \rightarrow \infty} \gamma_{2n+i}^* = \gamma_i, \quad i = 0, 1.$$

**Remark.** When  $\Delta_1 = [a, 0]$ ,  $\Delta_2 = [0, 1]$  and there is ratio asymptotics, the values of the constants  $\alpha_i, \beta_i, \gamma_i$ ,  $i = 0, 1$ , are given in [3, Section 4.5]. Notice that in this case  $\Delta_1 \cap \Delta_2 = \{0\} \neq \emptyset$  and is not considered in the present paper.

Here, we prove that

$$\left( \frac{1}{F_{1,0}(z)}, \frac{1}{F_{2,0}(z)} \right) = (\widehat{\mu}_1(z), \widehat{\mu}_2(z)), \quad \frac{\alpha_1 - \alpha_0}{\gamma_1} \left( \frac{F_{2,2}(z)}{F_{1,2}(z)}, F_{2,2}(z) \right) = (\widehat{\tau}_1(z), \widehat{\tau}_2(z)),$$

where  $\mu_k$  are probabilities supported on  $\Delta_1$ ,  $\tau_k$  are probabilities supported on  $\Delta_2$ , and

$$d\mu_2(x) = \frac{\gamma_0}{(\alpha_1 - \alpha_0)^2} \widehat{\tau}_1(x) d\mu_1(x), \quad d\tau_2(x) = \frac{-\gamma_0}{(\alpha_1 - \alpha_0)^2} \widehat{\mu}_1(x) d\tau_1(x).$$

Moreover, the sequence of polynomials  $(P_n)_{n=0}^\infty$ , defined by the recurrences

$$\begin{aligned} xP_{2n+i}(x) = P_{2n+i+1}(x) + \alpha_k P_{2n+i}(x) + \beta_k P_{2n+i-1}(x) + \gamma_k P_{2n+i-2}(x), \\ i = 0, 1, \quad n = 0, 1, \dots \end{aligned} \quad (4)$$

with initial conditions  $P_{-2} = P_{-1} = 0$ ,  $P_0 = 1$ , corresponds to the sequence of multiple orthogonality with respect to the Nikishin system  $(\mu_1, \mu_2) = \mathcal{N}(\mu_1, \frac{\gamma_0}{(\alpha_1 - \alpha_0)^2} \tau_1)$ . If the starting point was a Nikishin system on  $(\Delta_2, \Delta_1)$ , then  $\frac{1}{\widehat{\tau}_1(z)}$  and  $\frac{1}{\widehat{\tau}_2(z)}$  would give the ratio asymptotics for the corresponding multiple orthogonal polynomials.

Since the polynomials (4) are defined by recurrences with (periodic) constant coefficients they are the analogues of the second kind Chebyshev polynomials in standard orthogonality. Therefore, we call them Chebyshev-Nikishin multiple orthogonal polynomials, and Chebyshev-Nikishin measures the corresponding orthogonality measures.

In Section 2, we obtain the algebraic equations satisfied by the functions  $F_k$  in terms of the constants  $\alpha_i, \beta_i, \gamma_i$  as well as some relations between these constants. In Section 3, the measures are described and it is proved that they form a Nikishin system. Finally, in Section 4, we prove the orthogonality conditions for the sequence of polynomials  $(P_n)_{n=0}^\infty$ .

## 2 Algebraic equations

First of all, let us find the equations satisfied by the algebraic functions  $F_1$  and  $F_2$  in terms of the limits of the recurrence coefficients.

**Theorem 2.1.** *i) The coefficients of the recurrence relation (4) verify*

$$\beta_0 = \beta_1 := \beta, \quad \gamma_1 = \gamma_0 + \beta(\alpha_1 - \alpha_0). \quad (5)$$

*ii) The functions  $F_1, F_2$ , given in (1) satisfy  $F_1 - F_2 = \alpha_1 - \alpha_0$ .*

*iii) The algebraic equations for  $F_i$  are*

$$\begin{aligned} F_1^3 - (x + \alpha_1 - 2\alpha_0)F_1^2 + \{(\alpha_1 - \alpha_0)(x - \alpha_0) + \beta\}F_1 + \gamma_0 &= 0 \\ F_2^3 - (x + \alpha_0 - 2\alpha_1)F_2^2 + \{(\alpha_0 - \alpha_1)(x - \alpha_1) + \beta\}F_2 + \gamma_1 &= 0. \end{aligned}$$

*Proof.* Since  $F_1(z) - F_2(z)$  is a bounded function on the compact Riemann surface  $\mathcal{R}$  (it is sufficient to look at the divisor of both functions), by Liouville's theorem, this function is constant. Let  $C \neq 0$  be the constant such that  $F_1(z) - F_2(z) \equiv C$ . Then,  $C = -F_2(\infty^{(1)}) = F_1(\infty^{(2)})$  since  $F_k(\infty^{(k)}) = 0$ .

From the recurrence relation (3) we have

$$\begin{aligned} x \frac{Q_{2n+k}(x)}{Q_{2(n-1)+k}(x)} = \\ \frac{Q_{2n+k+1}(x)}{Q_{2(n-1)+k}(x)} + \alpha_{2n+k}^* \frac{Q_{2n+k}(x)}{Q_{2(n-1)+k}(x)} + \beta_{2n+k}^* \frac{Q_{2n+k-1}(x)}{Q_{2(n-1)+k}(x)} + \gamma_{2n+k}^*. \end{aligned}$$

Taking limits as  $n \rightarrow \infty$ ,  $k = 0, 1$ , using (2) we obtain algebraic expressions on the sheet  $\mathcal{R}^{(0)}$  which can be extended to the whole Riemann surface which allow us to write

$$\begin{aligned} z Z &= F_1(z) Z + \alpha_0 Z + \beta_0 F_1(z) + \gamma_0, \\ z Z &= F_2(z) Z + \alpha_1 Z + \beta_1 F_2(z) + \gamma_1, \quad z \in \mathcal{R}, \end{aligned} \quad (6)$$

where  $Z = F_1(z)F_2(z)$ . Then

$$\lim_{z \rightarrow \infty^{(k)}} (F_1(z) Z + \alpha_0 Z + \beta_0 F_1(z) + \gamma_0) = \lim_{z \rightarrow \infty^{(k)}} (F_2(z) Z + \alpha_1 Z + \beta_1 F_2(z) + \gamma_1)$$

and for  $k = 1$  and  $k = 2$  we get

$$\begin{aligned} \gamma_0 &= \beta_1 F_2(\infty^{(1)}) + \gamma_1 \\ \beta_0 F_1(\infty^{(2)}) + \gamma_0 &= \gamma_1. \end{aligned} \tag{7}$$

Then  $\gamma_0 = -\beta_1 C + \beta_0 C + \gamma_0$  and  $\beta_0 = \beta_1 := \beta$  follows. Using that  $F_1(z) - F_2(z) \equiv C$ , deleting one equation in (6) from the other, we deduce that

$$0 \equiv (C + (\alpha_0 - \alpha_1))Z + \beta C + \gamma_0 - \gamma_1.$$

Evaluating at  $\infty^{(1)}$  (or  $\infty^{(2)}$ ), it follows that  $\beta C + \gamma_0 - \gamma_1 = 0$ . Therefore,  $F_1(z) - F_2(z) = \alpha_1 - \alpha_0 = C$  and  $\gamma_1 = \gamma_0 + \beta(\alpha_1 - \alpha_0)$ . Finally, the equations of  $F_1(z)$  and  $F_2(z)$  can be obtained after writing  $F_2$  in terms of  $F_1$  or  $F_1$  in terms of  $F_2$  in (6).  $\square$

**Remark.** In [11, Theorem 3.1] a system of equations is given which allows to derive the algebraic equations satisfied by  $F_1, F_2$ , in terms of the endpoints of the intervals  $\Delta_1, \Delta_2$ . Combining those results with ours you can obtain the values  $\gamma_1, \gamma_2, \beta, \alpha_1, \alpha_2$  in terms of the endpoints of the intervals.

Now, consider the functions of second type

$$\Phi_n(x) = \int \frac{Q_n(t)}{x-t} d\sigma_1(t), \quad \Psi_n(x) = \int \frac{\Phi_n(t)}{x-t} d\sigma_2(t).$$

In [9, Proposition 1] it is proved that for the multi-index  $(n_1, n_2) \in \mathbf{I}$  and  $n = n_1 + n_2$ ,

$$\int x^\nu \Phi_n(x) d\sigma_2(x) = 0, \quad \nu = 0, 1, \dots, n_2 - 1.$$

Consequently,  $\Phi_n(x)$  has at least  $n_2$  zeros on  $\Delta_2$ . Moreover, it is shown in [9, Proposition 3] that  $\Phi_n(x)$  has exactly  $n_2$  zeros in  $\mathbb{C} \setminus \Delta_1$ , they are all simple, and lie in the interior of  $\Delta_2$ . Let  $Q_{n,2}(x)$  be the monic polynomial defined by the zeros of  $\Phi_n(x)$ . They also prove in [9, Proposition 2] that  $Q_n(x)$  and  $Q_{n,2}(x)$  satisfy full orthogonality relations with respect to certain varying measure. More precisely,

$$\begin{aligned} \int x^\nu Q_n(x) \frac{d\sigma_1(x)}{Q_{n,2}(x)} &= 0, \quad 0 \leq \nu \leq n-1, \\ \int x^\nu Q_{n,2}(x) \int \frac{Q_n^2(t)}{x-t} \frac{d\sigma_1(t)}{Q_{n,2}(t)} \frac{d\sigma_2(x)}{Q_n(x)} &= 0, \quad 0 \leq \nu \leq n_2 - 1. \end{aligned}$$

When  $\text{supp}(\sigma_k) = \Delta_k$  and  $\sigma'_k > 0$  a.e. on  $\Delta_k$ ,  $k = 1, 2$ , in [2, Theorem 4.1] it is proved that the polynomials  $Q_{n,2}$  also have ratio asymptotics; that is, there exist functions  $R_1(z)$  and  $R_2(z)$  such that

$$\lim_{n \rightarrow \infty} \frac{Q_{2n+i,2}(z)}{Q_{2n+i-1,2}(z)} = R_i(z), \quad i = 0, 1,$$

uniformly on compact sets of  $\mathbb{C} \setminus \Delta_2$ . Moreover, if we denote  $\widehat{Q}_n(x) = \kappa_n Q_n(x)$ ,  $\kappa_n > 0$ , the orthonormal polynomial with respect to  $\frac{d\sigma_1(x)}{|Q_{n,2}(x)|}$  and  $\widehat{Q}_{n,2}(x) = \kappa_{n,2} Q_{n,2}$ ,  $\kappa_{n,2} > 0$ , the orthonormal polynomial with respect to the varying measure  $\int \frac{\widehat{Q}_n^2(t)}{|x-t|} \frac{d\sigma_1(t)}{|Q_{n,2}(t)|} \frac{d\sigma_2(x)}{|Q_n(x)|}$ , in [2, Corollary 4.1] the ratio asymptotics of  $\widehat{Q}_n(x)$  and  $\widehat{Q}_{n,2}(x)$  is deduced. In particular, this gives that the following limits exist and are different from zero

$$\lim_{n \rightarrow \infty} \frac{\kappa_{2n+i}}{\kappa_{2n+i-1}} := \kappa_i, \quad \lim_{n \rightarrow \infty} \frac{\kappa_{2n+i,2}}{\kappa_{2n+i-1,2}} := \kappa_{i,2}, \quad i = 0, 1.$$

Then, using the orthogonality relations satisfied by the polynomials, we have

$$\begin{aligned} \frac{\Phi_{2n+i}(x)}{\Phi_{2n+i-1}(x)} &= \frac{\int \frac{Q_{2n+i}(t)}{x-t} d\sigma_1(t)}{\int \frac{Q_{2n+i-1}(t)}{x-t} d\sigma_1(t)} = \frac{Q_{2n+i,2}(x)}{Q_{2n+i-1,2}(x)} \frac{Q_{2n+i-1}(x)}{Q_{2n+i}(x)} \frac{\int \frac{Q_{2n+i}^2(t)}{x-t} \frac{d\sigma_1(t)}{Q_{2n+i,2}(t)}}{\int \frac{Q_{2n+i-1}^2(t)}{x-t} \frac{d\sigma_1(t)}{Q_{2n+i-1,2}(t)}} \quad (8) \\ &= \frac{Q_{2n+i,2}(x)}{Q_{2n+i-1,2}(x)} \frac{Q_{2n+i-1}(x)}{Q_{2n+i}(x)} \frac{\kappa_{2n+i-1}^2}{\kappa_{2n+i}^2} \frac{\int \frac{\widehat{Q}_{2n+i}^2(t)}{|x-t|} \frac{d\sigma_1(t)}{|Q_{2n+i,2}(t)|}}{\int \frac{\widehat{Q}_{2n+i-1}^2(t)}{|x-t|} \frac{d\sigma_1(t)}{|Q_{2n+i-1,2}(t)|}} c_i(\Delta_1, \Delta_2) \end{aligned}$$

where  $c_i(\Delta_1, \Delta_2)$  is 1 or  $-1$  depending on  $i$  and on the relative position of  $\Delta_1$  and  $\Delta_2$ . In [10, Corollary 3], it is proved that the measures  $\widehat{Q}_n^2(t) \frac{d\sigma_1(t)}{|Q_{n,2}(t)|}$  converge weakly to the Chebyshev measure of the interval  $\Delta_1$ . Consequently, using (8) and the results mentioned above, we find that there exist functions  $A_i(x)$  such that

$$\lim_{n \rightarrow \infty} \frac{\Phi_{2n+i}(x)}{\Phi_{2n+i-1}(x)} = A_i(x), \quad i = 0, 1,$$

uniformly on compact subsets of  $\mathbb{C} \setminus (\Delta_1 \cup \Delta_2)$ .

Analogously, one proves that there exist functions  $B_i(x)$  such that

$$\lim_{n \rightarrow \infty} \frac{\Psi_{2n+i}(x)}{\Psi_{2n+i-1}(x)} = B_i(x), \quad i = 0, 1,$$

uniformly on compact subsets of  $\mathbb{C} \setminus \Delta_2$ .

On the other hand, from the orthogonality relations satisfied by  $Q_n$  with respect to  $\sigma_1$  and of  $\Phi_n(x)$  with respect to  $\sigma_2$  it follows that

$$x\Phi_n(x) = \int \frac{xQ_n(t)}{x-t} d\sigma_1(t) = \int \frac{tQ_n(t)}{x-t} d\sigma_1(t), \quad n \geq 1,$$

and

$$x\Psi_n(x) = \int \frac{x\Phi_n(t)}{x-t} d\sigma_2(t) = \int \frac{t\Phi_n(t)}{x-t} d\sigma_2(t), \quad n \geq 2.$$

Consequently, the functions  $\Phi_n, n \geq 1$  and  $\Psi_n, n \geq 2$ , satisfy the same recurrence relations as the polynomials  $Q_n, n \geq 0$ .

**Theorem 2.2.** *The functions of second kind satisfy*

$$\lim_{n \rightarrow \infty} \frac{\Phi_{2n+i}(z)}{\Phi_{2n+i-1}(z)} = F_{i,1}(z), \quad \lim_{n \rightarrow \infty} \frac{\Psi_{2n+i}(z)}{\Psi_{2n+i-1}(z)} = F_{i,2}(z), \quad i = 1, 2,$$

and the convergence is uniform on compact subsets of  $\mathbb{C} \setminus (\Delta_1 \cup \Delta_2)$  and  $\mathbb{C} \setminus \Delta_2$ , respectively.

*Proof.* Since  $\Phi_n(x), n \geq 1$ , satisfy the same recurrence relations as  $Q_n(x)$ , we may deduce as in Theorem 2.1 the equations (6) for  $A_0$  and  $A_1$ , obtaining

$$\frac{\beta A_1 + \gamma_0}{x - \alpha_0 - A_1} = A_0 A_1 = \frac{\beta A_0 + \gamma_1}{x - \alpha_1 - A_0}. \quad (9)$$

Therefore,

$$(\beta A_1 + \gamma_0)(x - \alpha_1 - A_0) = (\beta A_0 + \gamma_1)(x - \alpha_0 - A_1)$$

or, equivalently,

$$(\beta(x - \alpha_1) + \gamma_1)A_1 - (\beta(x - \alpha_0) + \gamma_0)A_0 = \gamma_1(x - \alpha_0) - \gamma_0(x - \alpha_1).$$

Taking into account that  $\gamma_1 = \gamma_0 + \beta(\alpha_1 - \alpha_0)$ , the last equality leads to

$$(\beta(x - \alpha_0) + \gamma_0)(A_1 - A_0) = (\alpha_1 - \alpha_0)(\beta(x - \alpha_0) + \gamma_0).$$

Should  $\beta = \gamma_0 = 0$ , on account of (6), we would have  $F_1 = x - \alpha_0$  and  $F_2 = x - \alpha_1$  which is impossible. Therefore,  $A_1 - A_0 = \alpha_1 - \alpha_0$ . Substituting  $A_0$  in terms of  $A_1$  in the first equality of (9) and  $A_1$  in terms of  $A_0$  in the second equality, we see that  $A_1$  and  $A_0$  satisfy the same algebraic equations as  $F_1$  and  $F_2$ , respectively. From (8), it is easy to deduce that  $A_0(\infty) \in \mathbb{C} \setminus \{0\}$  and  $A_1(\infty) = 0$ . Therefore,  $A_1 = F_{1,1}$  and  $A_0 = F_{2,1}$ . Analogously, one proves that  $B_1 = F_{1,2}$  and  $B_0 = F_{2,2}$ .  $\square$

Dividing  $xQ_n(x)$  by  $Q_n(x)$  in the recurrence relation and taking limits, one sees that

$$F_{1,0}(z) = z - \alpha_0 + \mathcal{O}(1/z), \quad F_{2,0}(z) = z - \alpha_1 + \mathcal{O}(1/z) \quad z \rightarrow \infty.$$

Using that  $F_1 - F_2 = \alpha_1 - \alpha_0$ ,  $F_{1,1}(\infty) = F_{2,2}(\infty) = 0$ , and that from the algebraic equations  $\prod_{k=0}^2 F_{1,k}(z) = -\gamma_0$ ,  $\prod_{k=0}^2 F_{2,k}(z) = -\gamma_1$ , one immediately obtains that

$$\begin{aligned} F_{1,1}(z) &= \frac{-\gamma_0}{\alpha_1 - \alpha_0} \frac{1}{z} + \mathcal{O}(1/z^2), & F_{1,2}(z) &= \alpha_1 - \alpha_0 + \mathcal{O}(1/z), \\ F_{2,1}(z) &= \alpha_0 - \alpha_1 + \mathcal{O}(1/z), & F_{2,2}(z) &= \frac{\gamma_1}{\alpha_1 - \alpha_0} \frac{1}{z} + \mathcal{O}(1/z^2). \end{aligned} \quad (10)$$

### 3 Chebyshev-Nikishin measures

From the proof of [2, Theorem 2.1] we know that the analytic functions  $Q_n(x)$  and  $\Phi_n(x)$  satisfy

$$\begin{aligned} Q'_{n+1}(x)Q_n(x) - Q_{n+1}(x)Q'_n(x) &\neq 0, \quad x \in \Delta_1, \\ \Phi'_{n+1}(x)\Phi_n(x) - \Phi_{n+1}(x)\Phi'_n(x) &\neq 0, \quad x \in \Delta_2. \end{aligned}$$

This allowed the authors to show that the zeros of the polynomials  $Q_n$  and  $Q_{n,2}$  interlace those of  $Q_{n+1}$  and  $Q_{n+1,2}$ . An immediate consequence is that in the decomposition in simple fractions of  $\frac{Q_n(x)}{Q_{n+1}(x)}$  and  $\frac{Q_{n+1,2}(x)}{Q_{n,2}(x)}$  all the residues have the same sign (recall that the zeros of  $Q_n$  and  $Q_{n,2}$  are simple). Therefore, since we are considering monic polynomials, and there is ratio asymptotics, there exist probability measures  $\mu_1, \mu_2$  supported on  $\Delta_1$ , and  $\tau_1, \tau_2$  supported on  $\Delta_2$ , such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{Q_{2n}(z)}{Q_{2n+1}(z)} &= \widehat{\mu}_1(z), & \lim_{n \rightarrow \infty} \frac{Q_{2n-1}(z)}{Q_{2n}(z)} &= \widehat{\mu}_2(z), \\ \lim_{n \rightarrow \infty} \frac{Q_{2n+1,2}(z)}{Q_{2n,2}(z)} &= 1 + C\widehat{\tau}_1(z), & \lim_{n \rightarrow \infty} \frac{Q_{2n-1,2}(z)}{Q_{2n,2}(z)} &= \widehat{\tau}_2(z), \end{aligned}$$

for some real constant  $C$ .

**Lemma 3.1.** *The constant  $C$  in the definition of  $\widehat{\tau}_1(z)$  is equal to  $\frac{-\gamma_1}{(\alpha_1 - \alpha_0)^2}$  and the measures satisfy the relations*

$$\begin{aligned} i) \quad &\widehat{\mu}_2(z) - \widehat{\mu}_1(z) = (\alpha_1 - \alpha_0) \widehat{\mu}_1(z) \widehat{\mu}_2(z), \\ ii) \quad &\widehat{\mu}_2(z) - \widehat{\tau}_2(z) = \frac{\gamma_0}{(\alpha_1 - \alpha_0)^2} \widehat{\mu}_1(z) \widehat{\tau}_1(z), \\ iii) \quad &\widehat{\tau}_2(z) - \widehat{\tau}_1(z) = \frac{\gamma_1}{(\alpha_1 - \alpha_0)^2} \widehat{\tau}_1(z) \widehat{\tau}_2(z). \end{aligned}$$

*Proof.* From (8), (10), Theorem 2.2, and the fact that the measures are probabilistic, it is clear that

$$\begin{aligned} F_{1,0}(z) &= \frac{1}{\widehat{\mu}_1(z)}, & F_{1,1}(z) &= \frac{\gamma_0}{\alpha_0 - \alpha_1} (1 + C\widehat{\tau}_1(z)) \widehat{\mu}_1(z), & F_{1,2}(z) &= \frac{\alpha_1 - \alpha_0}{1 + C\widehat{\tau}_1(z)}, \\ F_{2,0}(z) &= \frac{1}{\widehat{\mu}_2(z)}, & F_{2,1}(z) &= (\alpha_0 - \alpha_1) \frac{\widehat{\mu}_2(z)}{\widehat{\tau}_2(z)}, & F_{2,2}(z) &= \frac{\gamma_1}{\alpha_1 - \alpha_0} \widehat{\tau}_2(z). \end{aligned}$$

Using these relations, the equality  $F_{1,0}(z) - F_{2,0}(z) = \alpha_1 - \alpha_0$  immediately gives *i*). The expression  $F_{1,2}(z) - F_{2,2}(z) = \alpha_1 - \alpha_0$  means that

$$\frac{\alpha_1 - \alpha_0}{1 + C\widehat{\tau}_1(z)} - \frac{\gamma_1}{\alpha_1 - \alpha_0} \widehat{\tau}_2(z) = \alpha_1 - \alpha_0$$



which is equivalent to  $-\gamma_1(1 + C\widehat{\tau}_1(z))\widehat{\tau}_2(z) = (\alpha_1 - \alpha_0)^2 C\widehat{\tau}_1(z)$ . Due to the behavior at  $\infty$ , this implies that  $C = \frac{-\gamma_1}{(\alpha_1 - \alpha_0)^2}$ . Consequently,  $(1 + C\widehat{\tau}_1(z))\widehat{\tau}_2(z) = \widehat{\tau}_1(z)$  which is *iii*).

Now, from  $F_{1,1}(z) - F_{2,1}(z) = \alpha_1 - \alpha_0$ , we deduce that

$$\frac{\gamma_0}{\alpha_0 - \alpha_1}(1 + C\widehat{\tau}_1(z))\widehat{\mu}_1(z) - (\alpha_0 - \alpha_1)\frac{\widehat{\mu}_2(z)}{\widehat{\tau}_2(z)} = \alpha_1 - \alpha_0$$

and using that  $(1 + C\widehat{\tau}_1(z))\widehat{\tau}_2(z) = \widehat{\tau}_1(z)$  we obtain *ii*.  $\square$

From the existing relations between the Cauchy transform of the measures  $\mu_k$ ,  $\tau_k$ , and the algebraic functions  $F_1$  and  $F_2$ , it is clear that these measures are absolutely continuous. We denote by  $\mu'_k$  and  $\tau'_k$  their Radon-Nikodym derivatives. We express by  $\widehat{\mu}_k^\pm$  and  $\widehat{\tau}_k^\pm$  the boundary values of these functions on  $\Delta_1^\pm$  and  $\Delta_2^\pm$  respectively.

The measures  $\mu_1$  and  $\tau_1$  define the Nikishin systems  $(\mu_1, \mu_2)$ ,  $(\tau_1, \tau_2)$ . In the same way that  $(\mu_1, \mu_2)$  gives the ratio asymptotics of polynomials defined by a Nikishin system on  $\Delta_1$  in which the second generating measure is supported on  $\Delta_2$ ,  $(\tau_1, \tau_2)$  gives the ratio asymptotics of polynomials defined by a Nikishin system on  $\Delta_2$  for which the second measure of the generating system is supported on  $\Delta_1$ . This is the next result.

**Theorem 3.1.** *The measures  $\mu_k$  and  $\tau_k$ ,  $k = 1, 2$ , satisfy the following relations*

$$d\mu_2(x) = \frac{\gamma_0}{(\alpha_1 - \alpha_0)^2} \widehat{\tau}_1(x) d\mu_1(x), \quad d\tau_2(x) = \frac{-\gamma_0}{(\alpha_1 - \alpha_0)^2} \widehat{\mu}_1(x) d\tau_1(x).$$

Moreover,  $\widehat{\tau}_1(z)$  and  $\widehat{\tau}_2(z)$  are the functions which describe the asymptotic behavior of orthogonal polynomials of a Nikishin system on  $\Delta_2$  where the second generating measure is taken on  $\Delta_1$ .

*Proof.* Since  $\widehat{\mu}_1(z)\widehat{\tau}_1(z)$  vanishes at infinity, we can write

$$\widehat{\mu}_1(z)\widehat{\tau}_1(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\widehat{\mu}_1(\xi)\widehat{\tau}_1(\xi)}{z - \xi} d\xi$$

where  $\Gamma$  is a positively oriented closed simple curve with  $\Delta_1$  and  $\Delta_2$  in its interior and  $z$  in the exterior of  $\Gamma$ . For  $k = 1, 2$ , let  $\Gamma_k$  be any positively oriented closed simple curve surrounding  $\Delta_k$  which leaves  $\Delta_j$ ,  $k \neq j$ , in the exterior. Then

$$\widehat{\mu}_1(z)\widehat{\tau}_1(z) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\widehat{\mu}_1(\xi)\widehat{\tau}_1(\xi)}{z - \xi} d\xi + \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\widehat{\mu}_1(\xi)\widehat{\tau}_1(\xi)}{z - \xi} d\xi.$$

Taking limits when  $\Gamma_1$  and  $\Gamma_2$  shrink to  $\Delta_1$  and  $\Delta_2$  respectively, using the Sokhotski formula, we obtain

$$\widehat{\mu}_1(z)\widehat{\tau}_1(z) = \frac{1}{2\pi i} \int_{\Delta_1} \frac{(\widehat{\mu}_1^-(x) - \widehat{\mu}_1^+(x))\widehat{\tau}_1(x)}{z - x} dx + \frac{1}{2\pi i} \int_{\Delta_2} \frac{(\widehat{\tau}_1^-(x) - \widehat{\tau}_1^+(x))\widehat{\mu}_1(x)}{z - x} dx$$

$$= \int_{\Delta_1} \frac{\mu'_1(x)\widehat{\tau}_1(x)}{z-x} dx + \int_{\Delta_2} \frac{\tau'_1(x)\widehat{\mu}_1(x)}{z-x} dx$$

From *ii*) of Lemma 3.1, it follows that

$$\begin{aligned} & \int_{\Delta_1} \left( \frac{\gamma_0}{(\alpha_1 - \alpha_0)^2} \mu'_1(x)\widehat{\tau}_1(x) - \mu'_2(x) \right) \frac{dx}{z-x} \\ &= - \int_{\Delta_2} \left( \frac{\gamma_0}{(\alpha_1 - \alpha_0)^2} \widehat{\mu}_1(x)\tau'_1(x) + \tau'_2(x) \right) \frac{dx}{z-x}. \end{aligned}$$

Then, both integrals represent entire functions. Since they take the value zero at infinity, by Liouville's theorem, both integrals are identically equal to zero and we conclude that

$$d\mu_2(x) = \frac{\gamma_0}{(\alpha_1 - \alpha_0)^2} \widehat{\tau}_1(x) d\mu_1(x), \quad d\tau_2(x) = -\frac{\gamma_0}{(\alpha_1 - \alpha_0)^2} \widehat{\mu}_1(x) d\tau_1(x).$$

On the other hand, by Sokhotski's formula

$$\begin{aligned} \mu'_2(x) &= \frac{1}{2\pi i} \left( \frac{1}{F_{2,0}^-(x)} - \frac{1}{F_{2,0}^+(x)} \right) = \frac{1}{2\pi i} \left( \frac{F_{2,0}^+(x) - F_{2,0}^-(x)}{F_{2,0}^-(x)F_{2,0}^+(x)} \right) = \frac{1}{2\pi i} \frac{F_{2,0}^+(x) - F_{2,1}^+(x)}{F_{2,0}^+(x)F_{2,1}^+(x)} \\ &= \frac{1}{2\pi i} \frac{(F_{1,0}^+(x) - F_{1,1}^+(x))F_{2,2}(x)}{-\gamma_1}, \quad x \in \Delta_1. \end{aligned}$$

Analogously,

$$\mu'_1(x) = \frac{1}{2\pi i} \left( \frac{1}{F_{1,0}^-(x)} - \frac{1}{F_{1,0}^+(x)} \right) = \frac{1}{2\pi i} \frac{(F_{1,0}^+(x) - F_{1,1}^+(x))F_{1,2}(x)}{-\gamma_0}, \quad x \in \Delta_1.$$

Hence  $\frac{\mu'_1(x)}{\mu'_2(x)} = \frac{\gamma_1 F_{1,2}(x)}{\gamma_0 F_{2,2}(x)}$ ,  $x \in \Delta_1$ . But  $\mu'_2(x) = \frac{\gamma_0}{(\alpha_1 - \alpha_0)^2} \widehat{\tau}_1(x) \mu'_1(x)$  and, consequently,

$$\frac{1}{\widehat{\tau}_1(z)} = \frac{\gamma_1}{(\alpha_1 - \alpha_0)^2} \frac{F_{1,2}(z)}{F_{2,2}(z)}, \quad z \in \mathbb{C} \setminus \Delta_2.$$

Notice that  $\frac{F_1}{F_2} = 1 + \frac{\alpha_1 - \alpha_0}{F_2}$  lives on  $\mathcal{R}$  and has for divisor a simple pole at  $\infty^{(2)}$  and a simple zero at  $\infty^{(1)}$ . Except for a constant factor, these are the characteristics which define one of the functions which describes the ratio asymptotics of Nikishin polynomials on  $\Delta_2$  with the second generating measure on  $\Delta_1$ . So this function must be  $\frac{1}{\widehat{\tau}_1(z)}$ . We have analogous conclusions for  $\widehat{\tau}_2$  since

$$\frac{1}{\widehat{\tau}_2(z)} = \frac{\gamma_1}{\alpha_1 - \alpha_0} \frac{1}{F_{2,2}(z)}, \quad z \in \mathbb{C} \setminus \Delta_2.$$

With this we conclude the proof.  $\square$

## 4 Chebyshev-Nikishin polynomials

With the recurrence (3) of the Nikishin polynomials  $(Q_n)_{n=0}^\infty$  we can associate the matrices

$$A_n(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_{2n}^* & -\beta_{2n}^* & x - \gamma_{2n}^* \end{pmatrix}, \quad B_n(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_{2n+1}^* & -\beta_{2n+1}^* & x - \gamma_{2n+1}^* \end{pmatrix},$$

and we have

$$\begin{pmatrix} Q_{2n}(x) \\ Q_{2n+1}(x) \\ Q_{2n+2}(x) \end{pmatrix} = B_{n-1}(x)A_{n-1}(x) \begin{pmatrix} Q_{2(n-1)}(x) \\ Q_{2(n-1)+1}(x) \\ Q_{2(n-1)+2}(x) \end{pmatrix}.$$

Let  $A(x) = \lim_{n \rightarrow \infty} A_n(x)$  and  $B(x) = \lim_{n \rightarrow \infty} B_n(x)$ . Then

$$\begin{aligned} B(x)A(x) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\gamma_1 & -\beta & x - \alpha_1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\gamma_0 & -\beta & x - \alpha_0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ -\gamma_0 & -\beta & x - \alpha_0 \\ -\gamma_0(x - \alpha_1) & -\gamma_1 - \beta(x - \alpha_1) & -\beta + (x - \alpha_1)(x - \alpha_0) \end{pmatrix}. \end{aligned}$$

From the vector version of the Poincaré-Perron theorem (see, for example, [4, pag. 1750]),  $\lim_{n \rightarrow \infty} \frac{Q_{2n+2}(x)}{Q_{2n}(x)} = \lim_{n \rightarrow \infty} \frac{Q_{2n+1}(z)}{Q_{2n-1}(z)} = F_{1,0}(z)F_{2,0}(z)$  is some eigenvalue of the matrix  $B(x)A(x)$ . As a consequence,  $Z := F_1F_2$  satisfies the equation

$$\begin{aligned} 0 &= \det(B(z)A(z) - Z I) \\ &= Z^3 - \{(z - \alpha_0)(z - \alpha_1) - 2\beta\}Z^2 + \{\beta^2 + \gamma_1(z - \alpha_0) + \gamma_0(x - \alpha_1)\}Z - \gamma_0\gamma_1. \end{aligned}$$

Let  $(P_n)_{n=0}^\infty$  and  $(P_n^*)_{n=0}^\infty$  be the sequences of monic polynomials defined by the recurrences

$$\begin{aligned} xP_{2n}(x) &= P_{2n+1}(x) + \alpha_0P_{2n}(x) + \beta P_{2n-1}(x) + \gamma_0P_{2n-2}(x), \quad n = 0, 1, \dots \\ xP_{2n+1}(x) &= P_{2n+2}(x) + \alpha_1P_{2n+1}(x) + \beta P_{2n}(x) + \gamma_1P_{2n-1}(x), \quad n = 0, 1, \dots \\ xP_{2n}^*(x) &= P_{2n+1}^*(x) + \alpha_1P_{2n}^*(x) + \beta P_{2n-1}^*(x) + \gamma_1P_{2n-2}^*(x), \quad n = 0, 1, \dots \\ xP_{2n+1}^*(x) &= P_{2n+2}^*(x) + \alpha_0P_{2n+1}^*(x) + \beta P_{2n}^*(x) + \gamma_0P_{2n-1}^*(x), \quad n = 0, 1, \dots \end{aligned}$$

with initial conditions  $P_0(x) = P_0^*(x) = 1$ ,  $P_{-2}(x) = P_{-2}^*(x) = P_{-1}(x) = P_{-1}^*(x) = 0$ .

Then

$$\begin{pmatrix} P_{2n}(x) \\ P_{2n+1}(x) \\ P_{2n+2}(x) \end{pmatrix} = (B(x)A(x))^{n+1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} P_{2n}^*(x) \\ P_{2n+1}^*(x) \\ P_{2n+2}^*(x) \end{pmatrix} = (A(x)B(x))^{n+1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Using the relation  $\frac{\beta F_1 + \gamma_0}{x - F_1 - \alpha_0} = Z = \frac{\beta F_2 + \gamma_1}{x - F_2 - \alpha_1}$  deduced from (6) one can prove that  $(1, F_{1,k}, Z_k)$  is an eigenvector associated to the eigenvalue  $Z_k$  for  $B(x)A(x)$  and  $k = 0, 1, 2$ . Taking into account that  $A(x)B(x)$  has the same expression as  $B(x)A(x)$  but interchanging the indices 0 and 1 in the entries of the matrix, it follows that  $(1, F_{2,k}, Z_k)$ ,  $k = 0, 1, 2$ , are eigenvectors for  $A(x)B(x)$ . Then we have

$$B(x)A(x) = P(x)D(x)P^{-1}(x), \quad A(x)B(x) = Q(x)D(x)Q^{-1}(x)$$

with

$$P(x) = \begin{pmatrix} 1 & 1 & 1 \\ F_{1,0} & F_{1,1} & F_{1,2} \\ Z_0 & Z_1 & Z_2 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} 1 & 1 & 1 \\ F_{2,0} & F_{2,1} & F_{2,2} \\ Z_0 & Z_1 & Z_2 \end{pmatrix}, \quad D(x) = \begin{pmatrix} Z_0 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} P_{2n} \\ P_{2n+1} \\ P_{2n+2} \end{pmatrix} = PD^{n+1}P^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{1,3}Z_0^{n+1} + a_{2,3}Z_1^{n+1} + a_{3,3}Z_2^{n+1} \\ a_{1,3}F_0Z_0^{n+1} + a_{2,3}F_1Z_1^{n+1} + a_{3,3}F_2Z_2^{n+1} \\ a_{1,3}Z_0^{n+2} + a_{2,3}Z_1^{n+2} + a_{3,3}Z_2^{n+2} \end{pmatrix},$$

where  $a_{i,j}$  for  $i, j \in \{1, 2, 3\}$  are the entries of  $P^{-1}$ .

The matrices  $P$  and  $Q$  have the same determinant because

$$\begin{aligned} \det P &= \\ &= \begin{vmatrix} 1 & 1 & 1 \\ F_{1,0} & F_{1,1} & F_{1,2} \\ Z_0 & Z_1 & Z_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ F_{1,0} & F_{1,1} & F_{1,2} \\ F_{1,0}^2 & F_{1,1}^2 & F_{1,2}^2 \end{vmatrix} = \prod_{2 \geq j > k \geq 0} (F_{1,j} - F_{1,k}) = \prod_{2 \geq j > k \geq 0} (F_{2,j} - F_{2,k}) \\ &= \det Q. \end{aligned}$$

Moreover,

$$P^{-1} \det P = \begin{pmatrix} - & \begin{vmatrix} F_{1,1} & Z_1 \\ F_{1,2} & Z_2 \end{vmatrix} & - & \begin{vmatrix} 1 & Z_1 \\ 1 & Z_2 \end{vmatrix} & - & \begin{vmatrix} 1 & F_{1,1} \\ 1 & F_{1,2} \end{vmatrix} \\ - & \begin{vmatrix} F_{1,0} & Z_0 \\ F_{1,2} & Z_2 \end{vmatrix} & - & \begin{vmatrix} 1 & Z_0 \\ 1 & Z_2 \end{vmatrix} & - & \begin{vmatrix} 1 & F_{1,0} \\ 1 & F_{1,2} \end{vmatrix} \\ - & \begin{vmatrix} F_{1,0} & Z_0 \\ F_{1,1} & Z_1 \end{vmatrix} & - & \begin{vmatrix} 1 & Z_0 \\ 1 & Z_1 \end{vmatrix} & - & \begin{vmatrix} 1 & F_{1,0} \\ 1 & F_{1,1} \end{vmatrix} \end{pmatrix} \quad (11)$$

and it is clear that, if  $a_{k,j}$  are the coefficients of  $P^{-1}$  and  $b_{k,j}$  are the coefficients of  $Q^{-1}$ , then  $a_{k,j} = b_{k,j}$  for  $j = 2, 3$  and for all  $k$ , and  $b_{k,1} = a_{k,1} - (\alpha_1 - \alpha_0)a_{k,2}$ ,  $k = 1, 2, 3$ . As a consequence, we have the following representation for the polynomials,

$$\begin{aligned} P_{2n}(x) &= a_{1,3}Z_0^{n+1} + a_{2,3}Z_1^{n+1} + a_{3,3}Z_2^{n+1}, & P_{2n}^*(x) &= P_{2n}(x), \\ P_{2n+1}(x) &= a_{1,3}F_{1,0}Z_0^{n+1} + a_{2,3}F_{1,1}Z_1^{n+1} + a_{3,3}F_{1,2}Z_2^{n+1}, & (12) \\ P_{2n+1}^*(x) &= a_{1,3}F_{2,0}Z_0^{n+1} + a_{2,3}F_{2,1}Z_1^{n+1} + a_{3,3}F_{2,2}Z_2^{n+1}, \\ P_{2n+1}^*(x) &= P_{2n+1}(x) - (\alpha_1 - \alpha_0)P_{2n}(x). \end{aligned}$$

**Remark.** Since  $|Z_0| > |Z_1|$  and  $|Z_0| > |Z_2|$  in a neighborhood of infinity, formulas (12) mean that the polynomials  $P_{2n}$  and  $P_{2n+1}$  have strong asymptotics and they behave like  $Z_0^n = (F_{1,0}F_{2,0})^n$  in  $\mathbb{C} \setminus \Delta_1$ . This coincides with the general result on strong asymptotics given in [1] for multiple orthogonal polynomials associated with Nikishin systems defined by absolutely continuous measures in the Szegő class.

**Theorem 4.1.** *i)  $(P_n)_{n=0}^\infty$  is the sequence of monic multiple orthogonal polynomials associated with the Nikishin system  $(\mu_1, \mu_2)$  and the multi-indices of the form  $(n, n)$  and  $(n+1, n)$ .*

*ii)  $(P_n^*)_{n=0}^\infty$  is the sequence of monic multiple orthogonal polynomials associated with the Nikishin system  $(\mu_1, \mu_2)$  and the multi-indices of the form  $(n, n)$  and  $(n, n+1)$ .*

*Proof.* Let  $Z = F_1F_2$ . Recall that  $Z$  has a double pole at  $\infty^{(0)}$  and simple zeros at  $\infty^{(1)}$  and  $\infty^{(2)}$ . From (12), taking into account that  $\hat{\mu}_1(z) = \frac{1}{F_{1,0}(z)}$ ,  $\hat{\mu}_2(z) = \frac{1}{F_{2,0}(z)}$ , that the determinant of  $P$  has a double pole at infinity, and the expressions of  $a_{k,3}(z)$  given in (11), we can write the following identities:

$$\hat{\mu}_1(z)P_{2n}(z) = a_{1,3}(z)\frac{Z_0^{n+1}}{F_{1,0}} + a_{2,3}(z)\frac{Z_1^{n+1}}{F_{1,0}} + a_{3,3}(z)\frac{Z_2^{n+1}}{F_{1,0}}$$

$$\begin{aligned}
&= a_{1,3}(z)F_{2,0}Z_0^n + a_{2,3}(z)F_{2,1}Z_1^n + a_{3,3}(z)F_{2,2}Z_2^n + a_{2,3}(z)\left(\frac{Z_1}{F_{1,0}} - F_{2,1}\right)Z_1^n + \\
&\quad + a_{3,3}(z)\left(\frac{Z_2}{F_{1,0}} - F_{2,2}\right)Z_2^n = P_{2n-1}^*(z) + \mathcal{O}(1/z^{n+1}), \quad z \rightarrow \infty, \\
&\quad \widehat{\mu}_2(z)P_{2n}^*(z) = a_{1,3}(z)F_{1,0}Z_0^n + a_{2,3}(z)\frac{Z_1^{n+1}}{F_{2,0}} + a_{3,3}(z)\frac{Z_2^{n+1}}{F_{2,0}} \\
&\quad = P_{2n-1}(z) + \mathcal{O}(1/z^{n+1}), \quad z \rightarrow \infty, \\
&\quad \widehat{\mu}_1(z)P_{2n+1}(z) = a_{1,3}(z)Z_0^n + a_{2,3}(z)\frac{F_{1,1}Z_1^{n+1}}{F_{1,0}} + a_{3,3}(z)\frac{F_{1,2}Z_2^{n+1}}{F_{1,0}} \\
&\quad = P_{2n}(z) + a_{2,3}(z)\left(\frac{F_{1,1}}{F_{1,0}} - 1\right)Z_1^{n+1} + a_{3,3}(z)\left(\frac{F_{1,2}}{F_{1,0}} - 1\right)Z_2^{n+1} \\
&\quad = P_{2n}(z) + \mathcal{O}(1/z^{n+2}), \quad z \rightarrow \infty,
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\mu}_2(z)P_{2n+1}^*(z) &= a_{1,3}(z)Z_0^{n+1} + a_{2,3}(z)\frac{F_{2,1}Z_1^{n+1}}{F_{2,0}} + a_{3,3}(z)\frac{F_{2,2}Z_2^{n+1}}{F_{2,0}} \\
&= P_{2n}(z) + \mathcal{O}(1/z^{n+2}), \quad z \rightarrow \infty, .
\end{aligned}$$

If  $\Gamma$  is a simple closed Jordan curve that surrounds  $\Delta_1$ , using Cauchy's integral formula and Fubini's theorem, from these relations it follows that

$$\begin{aligned}
0 &= \frac{1}{2\pi i} \int_{\Gamma} z^{\nu} P_{2n}(z) \widehat{\mu}_1(z) dz = \int x^{\nu} P_{2n}(x) d\mu_1(x), \quad \nu = 0, \dots, n-1, \\
0 &= \frac{1}{2\pi i} \int_{\Gamma} z^{\nu} P_{2n}^*(z) \widehat{\mu}_2(z) dz = \int x^{\nu} P_{2n}^*(x) d\mu_2(x), \quad \nu = 0, \dots, n-1, \\
0 &= \frac{1}{2\pi i} \int_{\Gamma} z^{\nu} P_{2n+1}(z) \widehat{\mu}_1(z) dz = \int x^{\nu} P_{2n+1}(x) d\mu_1(x), \quad \nu = 0, \dots, n,
\end{aligned}$$

and

$$0 = \frac{1}{2\pi i} \int_{\Gamma} z^{\nu} P_{2n+1}^*(z) \widehat{\mu}_2(z) dz = \int x^{\nu} P_{2n+1}^*(x) d\mu_2(x), \quad \nu = 0, \dots, n,$$

Taking into consideration that  $P_{2n} = P_{2n}^*$  and  $P_{2n+1}^*(x) = P_{2n+1}(x) - (\alpha_1 - \alpha_0)P_{2n}(x)$  (see (12)), the rest of the orthogonality relations immediately follow and we are done.  $\square$

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