Supplementary Material

Multidimensional Bargaining and Posted Prices

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The supplementary material has two sections: Section B shows that the function r_ϵ in the proof of Claim A.6 is convex. Section C contains the extension to a potentially unbounded set of feasible bundles.

B Convexity of r_ϵ

In this section, we show that the function $r_\epsilon \colon \mathbb{R}_+^{\bar{g}} \to \mathbb{R}_+$ defined in the proof of Claim A.6 is convex. Our argument rests on the observation that r_ϵ satisfies quasiconvexity, homogeneity of degree 1 and two auxiliary conditions. These four properties together imply that r_ϵ is convex.

Definition B.1. A function $f : \mathbb{R}_+^{\bar{g}} \to \mathbb{R}_+$ is

- **zero at zero** if $f(\mathbf{0}) = 0$;
- **positive except at zero** if for all $q \in \mathbb{R}_+^{\bar{g}} \setminus \{\mathbf{0}\}$, $f(q) > 0$;
- **homogeneous of degree 1** if for all $q \in \mathbb{R}_+^{\bar{g}}$ and $\lambda \in \mathbb{R}_{++}$, $f(\lambda q) = \lambda f(q)$;
- **quasiconvex** if for all $q, \hat{q} \in \mathbb{R}_+^{\bar{g}}$ and $\lambda \in (0,1)$, $f(\lambda q + (1-\lambda)\hat{q}) \le \max\{f(q), f(\hat{q})\}$;
- **convex** if for all $q, \hat{q} \in \mathbb{R}_+^{\bar{g}}$ and $\lambda \in (0,1)$, $f(\lambda q + (1-\lambda)\hat{q}) \le \lambda f(q) + (1-\lambda) f(\hat{q})$.

Lemma B.1. *Suppose $f : \mathbb{R}_+^{\bar{g}} \to \mathbb{R}_+$ is zero at zero, positive except at zero, homogeneous of degree 1 and quasiconvex. Then f is convex.*

Proof. The argument below largely follows the proof of Theorem 5 in Wilson (2008, pp. 6–7). Suppose $f : \mathbb{R}_+^{\bar{g}} \to \mathbb{R}_+$ is zero at zero, positive except at zero, homogeneous of degree 1 and quasiconvex. Consider any $q, \hat{q} \in \mathbb{R}_+^{\bar{g}}$ and $\lambda \in (0,1)$.

If $\hat{q} = \mathbf{0}$, then homogeneity of degree 1 and zero at zero imply that $f(\lambda q + (1-\lambda)\hat{q}) = f(\lambda q) = \lambda f(q) = \lambda f(q) + (1-\lambda) f(\hat{q})$, as desired. The same argument holds for $q = \mathbf{0}$.

Now suppose that $q, \hat{q} \in \mathbb{R}^{\bar{g}}_{+} \setminus \{\mathbf{0}\}$. Since f is positive except at zero, $f(q) > 0$ and $f(\hat{q}) > 0$. Hence, there exist $\mu, \hat{\mu} \in \mathbb{R}_{++}$ such that $\mu f(\hat{q}) = f(q)$ and $\hat{\mu} f(q) = f(\hat{q})$. Both equations together imply that $\mu\hat{\mu} = 1$.

By $f(q) = \mu f(\hat{q})$ and homogeneity of degree 1,

$$\begin{aligned}\lambda f(q) + (1-\lambda) f(\hat{q}) &= [\lambda\mu + (1-\lambda)] f(\hat{q}) \\ &= f\big([\lambda\mu + (1-\lambda)]\hat{q}\big).\end{aligned} \tag{B.1}$$

Analogously, by $f(\hat{q}) = \hat{\mu} f(q)$ and homogeneity of degree 1,

$$\begin{aligned}\lambda f(q) + (1-\lambda) f(\hat{q}) &= [\lambda + (1-\lambda)\hat{\mu}] f(q) \\ &= f\big([\lambda + (1-\lambda)\hat{\mu}]q\big).\end{aligned} \tag{B.2}$$

Define $\kappa \equiv \frac{\lambda}{\lambda + (1-\lambda)\hat{\mu}} \in (0,1)$. Since $\mu\hat{\mu} = 1$, it follows that $1 - \kappa = \frac{1-\lambda}{\lambda\mu + (1-\lambda)}$. Combined with (B.1), (B.2) and quasiconvexity, we conclude that

$$\begin{aligned}\lambda f(q) + (1-\lambda) f(\hat{q}) &= \max\Big\{ f\big([\lambda + (1-\lambda)\hat{\mu}]q\big), f\big([\lambda\mu + (1-\lambda)]\hat{q}\big) \Big\} \\ &\geq f\big(\kappa[\lambda + (1-\lambda)\hat{\mu}]q + (1-\kappa)[\lambda\mu + (1-\lambda)]\hat{q}\big) \\ &= f\big(\lambda q + (1-\lambda)\hat{q}\big). \qquad \square\end{aligned}$$

Fix $\epsilon \in \mathbb{R}_{+}$. Recall the definitions of $X_\epsilon \colon \mathbb{R}_{+} \rightrightarrows \mathbb{R}^{\bar{g}}_{+}$ and $r_\epsilon \colon \mathbb{R}^{\bar{g}}_{+} \to \mathbb{R}_{+}$:

$$\begin{aligned}&\forall \rho \in \mathbb{R}_{+}, \quad X_\epsilon(\rho) \equiv \Big\{ q \in \mathbb{R}^{\bar{g}}_{+} : \exists \lambda \in [0,1] \text{ s.t. } q \leq \rho\big[\lambda q_{l+1} + (1-\lambda)\hat{q} + \epsilon\mathbf{1}\big] \Big\}, \\ &\forall q \in \mathbb{R}^{\bar{g}}_{+}, \quad\ r_\epsilon(q) \equiv \min\big\{\rho \in \mathbb{R}_{+} : q \in X_\epsilon(\rho)\big\}.\end{aligned}$$

Note that for all $\rho \in \mathbb{R}_{+}$, $X_\epsilon(\rho)$ is a convex set. Moreover, for all $\rho, \hat{\rho} \in \mathbb{R}_{+}$ with $\rho \leq \hat{\rho}$, $X_\epsilon(\rho) \subseteq X_\epsilon(\hat{\rho})$.

Lemma B.2. *For all $\epsilon \in \mathbb{R}_{+}$, r_ϵ is zero at zero, positive except at zero, homogeneous of degree 1 and quasiconvex.*

Proof. Since $X_\epsilon(0) = \{\mathbf{0}\}$, r_ϵ is zero at zero and positive except at zero.

Next, we show that r_ϵ is homogeneous of degree 1. Consider any $q \in \mathbb{R}^{\bar{g}}_{+}$ and $\mu \in \mathbb{R}_{+}$. If $\mu = 0$, then zero at zero implies that $r_\epsilon(\mu q) = 0 = \mu r_\epsilon(q)$, as desired. Now suppose that $\mu > 0$. By definition of X_ϵ, $\mu q \in X_\epsilon(\rho) \iff q \in X_\epsilon(\rho/\mu)$. Thus, by definition of r_ϵ, $r_\epsilon(\mu q) = \rho \iff r_\epsilon(q) = \rho/\mu$, which implies that $r_\epsilon(\mu q) = \mu r_\epsilon(q)$.

Finally, we prove that r_ϵ is quasiconvex. Consider any $q, \hat{q} \in \mathbb{R}^{\bar{g}}_{+}$ and

$\mu \in (0,1)$. Without loss of generality, suppose that $r_\epsilon(q) \leq r_\epsilon(\hat{q})$. Then $X_\epsilon(r_\epsilon(q)) \subseteq X_\epsilon(r_\epsilon(\hat{q}))$, which implies that $q, \hat{q} \in X_\epsilon(r_\epsilon(\hat{q}))$. Since $X_\epsilon(r_\epsilon(\hat{q}))$ is convex, $\mu q + (1-\mu)\hat{q} \in X_\epsilon(r_\epsilon(\hat{q}))$. Hence, $r_\epsilon(\mu q + (1-\mu)\hat{q}) \leq r_\epsilon(\hat{q}) = \max\{r_\epsilon(q), r_\epsilon(\hat{q})\}$. □

Lemmas B.1 and B.2 imply that r_ϵ is convex.

C Unbounded feasible set

In this section, we explain the changes necessary to extend our analysis to the case where the set of feasible bundles, $\bar{Q}$, may not have an upper bound. That is, the only assumptions on $\bar{Q}$ are that $\bar{Q} \subseteq \mathbb{N}^{\bar{g}}$ and $\mathbf{0} \in \bar{Q}$.

C.1 Definition 8

The definition of the no-trade types must be adjusted as follows.

Definition 8*. A mechanism $(\varphi, \tau_s, \tau_b)$ has **no-trade types** if

(i) there exists $c_0 \in C$ such that for all $v \in V$, $\varphi(c_0, v) = \mathbf{0}$,

(ii) for all $\bar{q} \in \mathbb{N}^{\bar{g}}$, there exists $\bar{v}_0 \in V$ such that for all $c \in C$, $\varphi(c, \bar{v}_0) \in \{\mathbf{0}\} \cup \{q \in \mathbb{N}^{\bar{g}} : q \not\leq \bar{q}\}$.

While the definition of the seller's no-trade type c_0 is as before, the buyer's no-trade type $\bar{v}_0$ is now defined relative to the set $\{q \in \mathbb{N}^{\bar{g}} : q \leq \bar{q}\}$. That is, within this set, the only bundle $\bar{v}_0$ may end up with is the zero bundle.

C.2 Definition 9

The definition of a trade schedule must be amended by a joint condition on (Q, p_s, p_b).

Definition 9*. A **trade schedule** (Q, p_s, p_b) specifies three exogenous objects:

(i) a **set of tradable bundles** $Q \equiv \{q_0, q_1, \dots, q_n\} \subseteq \bar{Q}$, $n \in \mathbb{N} \cup \{\infty\}$, such that $q_0 = \mathbf{0}$ and $q_0 < q_1 < \cdots < q_n$,

(ii) a **price function for the seller** $p_s \colon Q \to \mathbb{R}$ which is strictly increasing and such that for all $k \in \{1, \ldots, n-1\}$ and $\gamma \in \mathbb{R}^{\bar{g}}_{++}$,

$$\frac{p_s(q_k) - p_s(q_{k-1})}{\gamma \cdot (q_k - q_{k-1})} \geq \frac{p_s(q_{k+1}) - p_s(q_k)}{\gamma \cdot (q_{k+1} - q_k)},$$

(iii) a **price function for the buyer** $p_b \colon Q \to \mathbb{R}$ which is strictly increasing and such that for all $k \in \{1, \ldots, n-1\}$ and $\gamma \in \mathbb{R}^{\bar{g}}_{++}$,

$$\frac{p_b(q_k) - p_b(q_{k-1})}{\gamma \cdot (q_k - q_{k-1})} \leq \frac{p_b(q_{k+1}) - p_b(q_k)}{\gamma \cdot (q_{k+1} - q_k)}.$$

Moreover, (Q, p_s, p_b) must satisfy the following condition: if $\#Q = \infty$, then

$$\forall \gamma \in \mathbb{R}^{\bar{g}}_{++}, \quad \lim_{k \to \infty} \frac{p_s(q_{k+1}) - p_s(q_k)}{\gamma \cdot (q_{k+1} - q_k)} = 0$$

or

$$\forall \gamma \in \mathbb{R}^{\bar{g}}_{++}, \quad \lim_{k \to \infty} \frac{p_b(q_{k+1}) - p_b(q_k)}{\gamma \cdot (q_{k+1} - q_k)} = \infty.$$

The novelty is the last sentence ("Moreover, ..."). It guarantees that $\mathrm{Opt}_s(c) \cup \mathrm{Opt}_b(v) \neq \emptyset$ for all $(c, v) \in C \times V$. That is, at least one of the two agents has an optimum bundle in Q. Otherwise, there exist preferences such that both agents want to trade infinitely much. In this case, a generalized posted-price mechanism is not well defined.

C.3 Definition 10

The definition of a generalized posted-price mechanisms must be adjusted to allow for the possibility than one of the two agents does not have any, or not a largest, optimum bundle in $\bar{Q}$. Let $\boldsymbol{\infty}$ denote a $\bar{g}$-dimensional vector of infinities. That is, $q < \boldsymbol{\infty}$ for all $q \in \mathbb{R}^{\bar{g}}_{+}$. Define $\underline{q}_s, \bar{q}_s \colon C \to Q \cup \{\boldsymbol{\infty}\}$ by

$$\underline{q}_s(c) \equiv \begin{cases} \min\{\mathrm{Opt}_s(c)\} & \text{if } \mathrm{Opt}_s(c) \neq \emptyset, \\ \boldsymbol{\infty} & \text{if } \mathrm{Opt}_s(c) = \emptyset, \end{cases}$$

$$\bar{q}_s(c) \equiv \begin{cases} \max\{\mathrm{Opt}_s(c)\} & \text{if } \exists \max\{\mathrm{Opt}_s(c)\}, \\ \boldsymbol{\infty} & \text{if } \nexists \max\{\mathrm{Opt}_s(c)\}. \end{cases}$$

Analogous definitions apply to $\underline{q}_b, \bar{q}_b \colon V \to Q \cup \{\boldsymbol{\infty}\}$.

Definition 10*. A mechanism $(\varphi, \tau_s, \tau_b)$ is a **generalized posted-price mechanism** if there exists a trade schedule (Q, p_s, p_b) as well as two functions $q_s \colon C \to Q \cup \{\infty\}$ and $q_b \colon V \to Q \cup \{\infty\}$ such that the following conditions hold for all $(c, v) \in C \times V$:

$$q_s(c) \in \begin{cases} \mathrm{Opt}_s(c) & \text{if } \exists \max\{\mathrm{Opt}_s(c)\}, \\ \mathrm{Opt}_s(c) \cup \{\infty\} & \text{if } \nexists \max\{\mathrm{Opt}_s(c)\}, \end{cases}$$

$$q_b(v) \in \begin{cases} \mathrm{Opt}_b(v) & \text{if } \exists \max\{\mathrm{Opt}_b(v)\}, \\ \mathrm{Opt}_b(v) \cup \{\infty\} & \text{if } \nexists \max\{\mathrm{Opt}_b(v)\}, \end{cases}$$

$$\varphi(c, v) \in \Big\{ q \in Q : \min\Big\{\min\big\{q_s(c), q_b(v)\big\}, \max\big\{\underline{q}_s(c), \underline{q}_b(v)\big\}\Big\} \leq q \leq \min\big\{q_s(c), q_b(v)\big\} \Big\},$$

$$\tau_s(c, v) = p_s\big(\varphi(c, v)\big),$$
$$\tau_b(c, v) = p_b\big(\varphi(c, v)\big).$$

C.4 Theorem

The statement of the Theorem and the proof of the "if" part remain unchanged.

C.5 Corollaries 1 and 2

Corollaries 1 and 2 remain valid. Note that the conditions stated in Corollary 1, combined with the last sentence of Definition 9*, imply that $\#Q < \infty$. That is, any incentive-compatible, individually rational and budget-balanced mechanism must have a finite range.

It is in the proof of Corollaries 1 and 2 where we are using the relative, rather than absolute, definition of the buyer's no-trade type (Definition 8*). If we maintained the original definition, the proof of Lemma A.1(iv) would not go through anymore. The issue lies in the last two sentences. They need to be adjusted as follows: "Hence, for all $\bar{q} \in \mathbb{N}^{\bar{g}}$, there exists $\bar{v}_0 \in V$ such that for all $\hat{q} \in \{q \in \bar{Q} : q \leq \bar{q}\} \setminus \{\mathbf{0}\}$, $\bar{v}_0(\hat{q}) < p_b(\hat{q})$. By IR_b, $\varphi(c, v_0) \in \{\mathbf{0}\} \cup \{q \in \bar{Q} : q \not\leq \bar{q}\}$ for all $c \in C$, so v_0 is a no-trade buyer type."

There is also a minor change in the proof Lemma A.2(iv). In the last paragraph, if $\bar{q}_s(c) = \infty$, then there does not exist $v \in V$ such that $\text{Opt}_b(v) = \{\bar{q}_s(c)\}$, due to the additional condition in Definition 9*. However, all we need for the proof to work is that $\underline{q}_b(v) > q_s(c)$; and such a v exists because $q_s(c) < \infty$.

C.6 Lemma 2

In the proof of Lemma 2, the definition of $\hat{c} \in C$ must be adjusted to guarantee that $\hat{c}(\hat{q}) - \hat{c}(q) < \min_{j\in\{1,2\}}\{c_j(\hat{q}) - c_j(q)\}$ for all $q \neq \hat{q}$. The issue with the original construction is that $\hat{c}$ is linear above $\hat{q}$. Hence, if c_1 or c_2 is sufficiently convex, then it will eventually grow faster than $\hat{c}$. This problem can be fixed by making $\hat{c}$ dependent on c_1 and c_2. In what follows, we explain how to do that.

Let $\mathbf{1}$ be a $\bar{g}$-dimensional vector of ones. For all $k \in \{0, 1, \dots\}$, define $\hat{q}_k \equiv \hat{q} + k\mathbf{1}$ and $\hat{Q}_k \equiv \{q \in \bar{Q} : q \leq \hat{q}_k\}$. Note that $\hat{q}_0 = \hat{q}$. Let $\epsilon_0 \in (0, \infty)$ be such that if $\hat{q} > \mathbf{0}$, then

$$\epsilon_0 < \frac{\min\big\{\min_{j\in\{1,2\}}\{c_j(\hat{q}) - c_j(q)\} : q \in \hat{Q}_0 \setminus \{\hat{q}\}\big\}}{\sum_{g\in\bar{G}} \hat{q}^g}.$$

For all $k \in \{1, 2, \dots\}$, define $\delta_k \in [0, \infty)$ by

$$\delta_k \equiv \epsilon_0 \sum_{g\in\bar{G}} \hat{q}^g + \max\big\{\max_{j\in\{1,2\}}\{c_j(q) - c_j(\hat{q})\} : q \in \hat{Q}_k\big\}.$$

Note that $\delta_1 \leq \delta_2 \leq \cdots$. For all $q \in \bar{Q}$, define $\hat{c} \in C$ by

$$\hat{c}(q) \equiv \sum_{g\in\bar{G}} \left[\epsilon_0 q^g + \sum_{k=1}^{\infty} \delta_k \max\big\{q^g - \hat{q}^g_{k-1}, 0\big\}\right].$$

We now verify that $\hat{c}(\hat{q}) - \hat{c}(q) < \min_{j\in\{1,2\}}\{c_j(\hat{q}) - c_j(q)\}$ for all $q \neq \hat{q}$. First, consider any $q < \hat{q}$. Note that $q \in \hat{Q}_0 \setminus \{\hat{q}\}$ and $\hat{q} > \mathbf{0}$. By definition of $\hat{c}$,

$$\hat{c}(\hat{q}) - \hat{c}(q) = \epsilon_0 \sum_{g\in\bar{G}} (\hat{q}^g - q^g) \leq \epsilon_0 \sum_{g\in\bar{G}} \hat{q}^g < \min_{j\in\{1,2\}} \{c_j(\hat{q}) - c_j(q)\}.$$

Now consider any $q \not\leq \hat{q}$. Note that $q > \mathbf{0}$. Define $l \equiv \min\{k \in \{1, 2, \dots\} :$

$q \in \hat{Q}_k\}$. Note that $q^g > \hat{q}^g_{l-1}$ for some $g \in \bar{G}$. By definition of $\hat{c}$,

$$
\begin{aligned}
&\hat{c}(q) - \hat{c}(\hat{q}) \\
&= \sum_{g\in\bar{G}} \left[\epsilon_0 q^g + \sum_{k=1}^{\infty} \delta_k \max\{q^g - \hat{q}^g_{k-1}, 0\} \right] - \sum_{g\in\bar{G}} \epsilon_0 \hat{q}^g \\
&\geq \epsilon_0 \sum_{g\in\bar{G}} (q^g - \hat{q}^g) + \delta_l \sum_{g\in\bar{G}} \max\{q^g - \hat{q}^g_{l-1}, 0\} \\
&\geq \epsilon_0 \sum_{g\in\bar{G}} (q^g - \hat{q}^g) + \delta_l \\
&\geq \epsilon_0 \sum_{g\in\bar{G}} (q^g - \hat{q}^g) + \epsilon_0 \sum_{g\in\bar{G}} \hat{q}^g + \max_{j\in\{1,2\}} \{c_j(q) - c_j(\hat{q})\} \\
&> \max_{j\in\{1,2\}} \{c_j(q) - c_j(\hat{q})\},
\end{aligned}
$$

which is equivalent to $\hat{c}(\hat{q}) - \hat{c}(q) < \min_{j\in\{1,2\}}\{c_j(q) - c_j(\hat{q})\}$.

C.7 Lemma 4

The only change in the statement of Lemma 4 is that $n \in \mathbb{N} \cup \{\infty\}$. In what follows, we explain how to modify the proof of Lemma 4.

The main difference to the original proof is that the definition of the buyer's no-trade type has changed (Definition 8*). The following result reflects this change but is otherwise identical to Lemma 4. Define $n \equiv \#Q - 1$. Note that $n = \infty$ if and only if Q is unbounded.

Lemma C.1. *There exists a labeling of bundles $\{q_0, \dots, q_n\} = Q$ such that $q_0 = \mathbf{0}$ and for all $k \in \{0, \dots, n\}$,*

(a) $q_k < q$ *for all* $q \in Q \setminus \{q_0, \dots, q_k\}$,

(b) *if* $k \neq 0$, *then* $p_s(q_{k-1}) < p_s(q)$ *for all* $q \in Q \setminus \{q_0, \dots, q_{k-1}\}$,

(c) *if* $k \neq 0$, *then* $p_b(q_{k-1}) < p_b(q)$ *for all* $q \in Q \setminus \{q_0, \dots, q_{k-1}\}$,

(d) $\exists c_k \in C$ *s.t.* $Q_b(c_k) = \{q_0, \dots, q_k\}$ *and* $q_0 \prec_{c_k} \cdots \prec_{c_k} q_k$,

(e)* $\forall \bar{q} \in \mathbb{N}^{\bar{g}}$ *s.t.* $q_k \leq \bar{q}$, $\exists \bar{v}_k \in V$ *s.t.* $Q_s(\bar{v}_k) \cap \{q \in Q : q \leq \bar{q}\} = \{q_0, \dots, q_k\}$ *and* $q_0 \prec_{\bar{v}_k} \cdots \prec_{\bar{v}_k} q_k$.

The only difference between Lemma C.1 and Lemma 4 is condition (e*). To obtain Lemma 4, (e*) must be replaced by

(e) $\exists v_k \in V$ s.t. $Q_s(v_k) = \{q_0, \dots, q_k\}$ and $q_0 \prec_{v_k} \cdots \prec_{v_k} q_k$.

We now show that Lemma C.1 implies (e). Hence, to establish Lemma 4, it suffices to prove Lemma C.1.

Proof: Lemma C.1 $\implies$ (e). Consider any $k \in \{0, \dots, n\}$. Define $v_k \in V$ as follows: for all $q \in \bar{Q}$,

$$v_k(q) \equiv \sum_{g \in \bar{G}} \left[\epsilon \left(1 - \frac{1}{1 + q^g} \right) + \frac{1}{\epsilon} \min\{q^g, q_k^g\} \right].$$

For sufficiently small $\epsilon > 0$, (a) and (c) imply that $q_0 \prec_{v_k} \cdots \prec_{v_k} q_k$ and $q_k \succ_{v_k} q_l$ for all $l \in \{k+1, \dots, n\}$. It remains to show that $Q_s(v_k) = \{q_0, \dots, q_k\}$.

We begin with $\{q_0, \dots, q_k\} \subseteq Q_s(v_k)$. Consider any $j \in \{0, \dots, k\}$. By (d), there exists $c_j \in C$ such that $Q_b(c_j) = \{q_0, \dots, q_j\}$. Since $\text{Opt}_b(v_k, Q_b(c_j)) = \{q_j\}$, Lemma 3 implies that $q_j \in Q_s(v_k)$.

Finally, we prove that $Q_s(v_k) \subseteq \{q_0, \dots, q_k\}$. By contradiction, suppose there exists $l \in \{k+1, \dots, n\}$ such that $q_l \in Q_s(v_k)$. Consider type $c_l \in C$ from (d), so $Q_b(c_l) = \{q_0, \dots, q_l\}$ and $q_0 \prec_{c_l} \cdots \prec_{c_l} q_l$. By Lemma 3, $\varphi(c_l, v_k) \in Q_b(c_l)$ and $\varphi(c_l, v_k) \in \text{Opt}_s(c_l, Q_s(v_k))$. Since $q_0 \prec_{c_l} \cdots \prec_{c_l} q_l$ and $q_l \in Q_s(v_k)$, it follows that $\varphi(c_l, v_k) = q_l$. Moreover, by (e*) with $\bar{q} = q_l$, there exists $\bar{v}_k \in V$ such that $Q_s(\bar{v}_k) \cap \{q \in Q : q \le q_l\} = \{q_0, \dots, q_k\}$. Repeating the previous argument, $\varphi(c_l, \bar{v}_k) = q_k$. But $q_k \succ_{v_k} q_l$, so incentive compatibility is violated for v_k. Therefore, $q_l \notin Q_s(v_k)$ for all $l \in \{k+1, \dots, n\}$. □

The proof of Lemma C.1 is similar to the original proof of Lemma 4. The relevant changes are explained in the following subsections.

C.7.1 Claim A.1

Type $\hat{c}$ must be adjusted to ensure that $\hat{c}(q) \gg \hat{c}(\hat{q})$ for all $q \not\le \hat{q}$. The construction is similar to the one from Section C.6. Since the type modification slightly changes the arguments, we provide the full proof of Claim A.1.

Consider any $\hat{q} \in Q \setminus \{q_0, \dots, q_l\}$. Let $\mathbf{1}$ be a $\bar{g}$-dimensional vector of ones. For all $k \in \{0, 1, \dots\}$, define $\hat{q}_k \equiv \hat{q} + k\mathbf{1}$ and $\hat{Q}_k \equiv \{q \in Q : q \le \hat{q}_k\}$.

Note that $\hat{q}_0 = \hat{q}$ and $\hat{Q}_0 \subseteq \hat{Q}_1 \subseteq \cdots$. Note also that $\hat{q}, q_0 \in \hat{Q}_0$ and $p_s(\hat{q}) > p_s(q_0)$. To see the later, consider any $\hat{v} \in V$ such that $\hat{q} \in Q_s(\hat{v})$. By

NTT, $q_0 \in Q_s(\hat{v})$. Since $\hat{q} > q_0$, $c(\hat{q}) > c(q_0)$ for all $c \in C$. If $p_s(\hat{q}) \leq p_s(q_0)$, then $\hat{q} \prec_c q_0$ and thus $\hat{q} \notin \mathrm{Opt}_s(c, Q_s(\hat{v}))$ for all $c \in C$. Lemma 3 then implies that $\hat{q} \notin Q_s(\hat{v})$, a contradiction. Hence, $p_s(\hat{q}) > p_s(q_0)$. This fact is important for the following definitions.

Let $\epsilon_0 \in (0, \infty)$ be such that

$$\epsilon_0 < \frac{\min\{p_s(q) - p_s(q') : q, q' \in \hat{Q}_0 \text{ and } p_s(q) > p_s(q')\}}{\sum_{g \in \bar{G}} \hat{q}^g}.$$

For all $k \in \{1, 2, \dots\}$, define $\delta_k \in (0, \infty)$ by

$$\delta_k \equiv \max\{p_s(q) - p_s(q_0) : q \in \hat{Q}_k\}.$$

Note that, since $\hat{Q}_1 \subseteq \hat{Q}_2 \subseteq \cdots$, $\delta_1 \leq \delta_2 \leq \cdots$. Define $\hat{c} \in C$ as follows: for all $q \in \bar{Q}$,

$$\hat{c}(q) \equiv \sum_{g \in \bar{G}} \left[\epsilon_0 q^g + \sum_{k=1}^{\infty} \delta_k \max\{q^g - \hat{q}^g_{k-1}, 0\} \right].$$

Proof of (i). Consider any $k \in \{1, \dots, l\}$. By (b), $p_s(q_{k-1}) < p_s(q_k)$. Moreover, by (a), $q_{k-1} < q_k < \hat{q}$. From the definition of $\hat{c}$, it follows that

$$\hat{c}(q_k) - \hat{c}(q_{k-1}) = \epsilon_0 \sum_{g \in \bar{G}} (q^g_k - q^g_{k-1}) < \epsilon_0 \sum_{g \in \bar{G}} \hat{q}^g < p_s(q_k) - p_s(q_{k-1}).$$

which implies that $q_{k-1} \prec_{\hat{c}} q_k$. Analogously, if $p_s(q_l) < p_s(\hat{q})$, then $q_l \prec_{\hat{c}} \hat{q}$.

Proof of (ii). Consider any $q \not\leq \hat{q}$. Note that $q > \mathbf{0}$. Define $l \equiv \min\{k \in \{1, 2, \dots\} : q \in \hat{Q}_k\}$. Note that $q^g > \hat{q}^g_{l-1}$ for some $g \in \bar{G}$. By definition of $\hat{c}$,

$$\begin{aligned}
\hat{c}(q) - \hat{c}(q_0) &= \sum_{g \in \bar{G}} \left[\epsilon_0 q^g + \sum_{k=1}^{\infty} \delta_k \max\{q^g - \hat{q}^g_{k-1}, 0\} \right] - 0 \\
&\geq \epsilon_0 \sum_{g \in \bar{G}} q^g + \delta_l \sum_{g \in \bar{G}} \max\{q^g - \hat{q}^g_{l-1}, 0\} \\
&\geq \epsilon_0 \sum_{g \in \bar{G}} q^g + \delta_l \\
&\geq \epsilon_0 \sum_{g \in \bar{G}} q^g + p_s(q) - p_s(q_0) \\
&> p_s(q) - p_s(q_0),
\end{aligned}$$

which implies that $q_0 \succ_{\hat{c}} q$.

Proof of (iii). Consider any $\hat{v} \in V$ such that $\hat{q} \in Q_s(\hat{v})$. For all $q \in Q_s(\hat{v}) \setminus \{\hat{q}\}$, we show that $q \prec_{\hat{c}} \hat{q}$. First, let $q < \hat{q}$. Note that $\hat{c}(q) < \hat{c}(\hat{q})$. If $p_s(q) \geq p_s(\hat{q})$, then $q \succ_c \hat{q}$ and thus $\hat{q} \notin \mathrm{Opt}_s(c, Q_s(\hat{v}))$ for all $c \in C$. Lemma 3 then implies that $\hat{q} \notin Q_s(\hat{v})$, a contradiction. Hence, $p_s(q) < p_s(\hat{q})$. From the definition of $\hat{c}$, it follows that

$$\hat{c}(\hat{q}) - \hat{c}(q) = \epsilon_0 \sum_{g \in \bar{G}} (\hat{q}^g - q^g) \leq \epsilon_0 \sum_{g \in \bar{G}} \hat{q}^g < p_s(\hat{q}) - p_s(q),$$

so $q \prec_{\hat{c}} \hat{q}$. Note that, therefore, $q_0 \prec_{\hat{c}} \hat{q}$. Finally, let $q \nleq \hat{q}$. From (ii), we know that $q \prec_{\hat{c}} q_0$. Since also $q_0 \prec_{\hat{c}} \hat{q}$, transitivity implies that $q \prec_{\hat{c}} \hat{q}$.

Proof of (iv). There are three steps. First, consider any $k \in \{0, \dots, l\}$. By (a), $q_k < \hat{q}$. Thus, by (e*) with $\bar{q} = \hat{q}$, there exists $\bar{v}_k \in V$ such that $Q_s(\bar{v}_k) \cap \{q \in Q : q \leq \hat{q}\} = \{q_0, \dots, q_k\}$. From (i) and (ii), it follows that $\mathrm{Opt}_s(\hat{c}, Q_s(\bar{v}_k)) = \{q_k\}$. Hence, by Lemma 3, $q_k \in Q_b(\hat{c})$.

Second, consider any $\hat{v} \in V$ such that $\hat{q} \in Q_s(\hat{v})$. By (iii), $\mathrm{Opt}_s(\hat{c}, Q_s(\hat{v})) = \{\hat{q}\}$ and thus $\hat{q} \in Q_b(\hat{c})$.

Third, by contradiction, suppose there exists $q \in Q_b(\hat{c})$ such that $q \nleq \hat{q}$. Then $\varphi(\hat{c}, v) = q$ for some $v \in V$. However, by (d), there exists $c_0 \in C$ such that $Q_b(c_0) = \{q_0\}$ and thus $\varphi(c_0, v) = q_0$. Since (ii) says that $q_0 \succ_{\hat{c}} q$, IC_s is violated for $\hat{c}$.

C.7.2 Claim A.3

Since (e) is replaced by (e*), the two sentences following "Moreover, ..." must be changed as follows: "Moreover, by (e*) with $\bar{q} = \hat{q}$, there exists $\bar{v}_l \in V$ such that $Q_s(\bar{v}_l) \cap \{q \in Q : q \leq \hat{q}\} = \{q_0, \dots, q_l\}$. From Claim A.1, parts (i) and (ii), it follows that $\mathrm{Opt}(\hat{c}, Q_s(\bar{v}_l)) = \{q_l\}$ and thus $\varphi(\hat{c}, \bar{v}_l) = q_l$."

C.7.3 Definition of q_{l+1}

If Q is unbounded, then bundle q_{l+1} as defined in the original proof may not exist. To circumvent this problem, we define q_{l+1} relative to an upper bound $\bar{q}$ (which can be arbitrary).

Fix any $\bar{q} \in \mathbb{N}^{\bar{g}}$. Let $q_{l+1} \in Q \setminus \{q_0, \dots, q_l\}$ be such that $q_{l+1} \leq \bar{q}$ and for all $q \in Q \setminus \{q_0, \dots, q_l\}$ with $q \leq \bar{q}$, either $p_b(q) > p_b(q_{l+1})$ or $[p_b(q) = p_b(q_{l+1})$ and $q \ngtr q_{l+1}]$. In words, among all bundles in Q larger than q_l but weakly below $\bar{q}$, q_{l+1} entails the lowest payment for the buyer. If there are multiple such bundles, none of them is larger than q_{l+1}. Note that, in the case that Q is bounded, we can choose $\bar{q}$ so large that for all $q \in Q$, $q \leq \bar{q}$. The resulting q_{l+1} coincides with the original definition from the main text.

C.7.4 Claim A.4

The statement of Claim A.4 must be slightly changed, reflecting the substitution of (e*) for (e).

Claim A.4*. There exists $\bar{v}_{l+1} \in V$ such that $Q_s(\bar{v}_{l+1}) \cap \{q \in Q : q \leq \bar{q}\} = \{q_0, \dots, q_{l+1}\}$ and $q_0 \prec_{\bar{v}_{l+1}} \cdots \prec_{\bar{v}_{l+1}} q_{l+1}$.

There are also three minor changes in the proof of Claim A.4*.

First, (A.7) must be replaced by

$$q_l \succ_{\bar{v}_{l+1}} \hat{q} \text{ for all } \hat{q} \in \{q \in Q : q \leq \bar{q}\} \setminus \{q_0, \dots, q_{l+1}\}. \tag{A.7*}$$

Second, the penultimate paragraph (starting with "Second...") must be modified as follows: "By Claim A.1 (with $\hat{q} = q_{l+1}$), there exists $c_{l+1} \in C$ such that $\{q_0, \dots, q_l, q_{l+1}\} \subseteq Q_b(c_{l+1})$ and for all $q \not\leq q_{l+1}$, $q \notin Q_b(c_{l+1})$. From (A.5), (A.6) and (A.7*), it follows that $\text{Opt}_b(\bar{v}_{l+1}, Q_b(c_{l+1})) = \{q_{l+1}\}$ and thus $q_{l+1} \in Q_s(\bar{v}_{l+1})$."

Third, in the last paragraph of the proof, the two sentences starting with "Moreover, ..." must be changed as follows: "Moreover, by (a), $\hat{q} > q_l$. Hence, by (e*) with $\bar{q} = \hat{q}$, there exists $\bar{v}_l \in V$ such that $Q_s(\bar{v}_l) \cap \{q \in Q : q \leq \hat{q}\} = \{q_0, \dots, q_l\}$. From Claim A.1, parts (i) and (ii), it follows that $\text{Opt}_s(\hat{c}, Q_s(\bar{v}_l)) = \{q_l\}$ and thus $\varphi(\hat{c}, \bar{v}_l) = q_l$."

C.7.5 Claim A.5

The main change in the proof of Claim A.5 concerns the construction of type c. The idea is similar to those of Sections C.6 and C.7.1.

Define $\hat{Q} \equiv \{q \in Q : q \leq \hat{q}\}$. Note that $\hat{Q}$ is finite and $q_l, \hat{q} \in \hat{Q}$. Moreover, by Claim A.2, $p_s(q_l) < p_s(\hat{q})$ and $p_s(q_l) < p_s(q_{l+1})$. Hence, there exists $\epsilon_0 \in (0, \infty)$ such that

$$\epsilon_0 < \min\left\{ \frac{\min\{p_s(q) - p_s(q') : q, q' \in \hat{Q} \text{ and } p_s(q) > p_s(q')\}}{\sum_{g \in \bar{G}} \hat{q}^g}, \right.$$

$$\left.\frac{\min\{p_s(\hat{q}), p_s(q_{l+1})\} - p_s(q_l)}{(1 + \sum_{g\in\bar{G}} \hat{q}^g) \sum_{g\in\bar{G}} q^g_{l+1}}\right\}.$$

Define also

$$\epsilon_1 \equiv \sum_{g\in\bar{G}} \epsilon_0 \hat{q}^g + \frac{\max\{p_s(q_{l+1}) - p_s(\hat{q}), 0\}}{\sum_{g\in\bar{G}}(q^g_{l+1} - \hat{q}^g)}.$$

Let $\mathbf{1}$ be a $\bar{g}$-dimensional vector of ones. For all $k \in \{0, 1, \dots\}$, define $\bar{q}_k \equiv q_{l+1} + k\mathbf{1}$, $\bar{Q}_k \equiv \{q \in Q : q \leq \bar{q}_k\}$ and

$$\delta_k \equiv \sum_{g\in\bar{G}} \left[\epsilon_0 q^g_{l+1} + \epsilon_1 (q^g_{l+1} - \hat{q}^g)\right] + \max\{p_s(q) - p_s(q_{l+1}) : q \in \bar{Q}_{k+1}\}.$$

Define $c \in C$ as follows: for all $q \in \bar{Q}$,

$$c(q) \equiv \sum_{g\in\bar{G}} \left[\epsilon_0 q^g + \epsilon_1 \max\{q^g - \hat{q}^g, 0\} + \sum_{k=0}^{\infty} \delta_k \max\{q^g - \bar{q}^g_k, 0\}\right].$$

Another, yet minor, change concerns the statement of (A.8). It must be amended as follows:

$$q_0 \prec_c \cdots \prec_c q_l \prec_c q_{l+1} \text{ and } q_{l+1} \succ_c q \text{ for all } q \not\leq q_{l+1}, \tag{A.8*}$$

The part after "and" is not in the original proof. The reason for this change is that, by Claim A.4*, there exists $\bar{v}_{l+1} \in V$ such that $Q_s(\bar{v}_{l+1}) \cap \{q \in Q : q \leq \bar{q}\} = \{q_0, \dots, q_{l+1}\}$. (A.8*) guarantees that $\text{Opt}_s(c, Q_s(\bar{v}_{l+1})) = \{q_{l+1}\}$ and thus $\varphi(c, \bar{v}_{l+1}) = q_{l+1}$.

The structure of the proof of (A.8*) and (A.9) remains unchanged. The specific arguments, however, must be adapted to the new definition of c.

Proof of (A.8*). There are three steps. First, for all $k \in \{1, \dots, l\}$, (a) and (b) imply that $q_{k-1} < q_k$ and $p_s(q_{k-1}) < p_s(q_k)$, respectively. Therefore,

$$c(q_k) - c(q_{k-1}) = \epsilon_0 \sum_{g\in\bar{G}} (q^g_k - q^g_{k-1}) < \epsilon_0 \sum_{g\in\bar{G}} \hat{q}^g < p_s(q_k) - p_s(q_{k-1}),$$

which implies that $q_{k-1} \prec_c q_k$. Second, $q_l \prec_c q_{l+1}$ because

$$c(q_{l+1}) - c(q_l)$$

$$
\begin{aligned}
&= \sum_{g\in\bar{G}} \Big[\epsilon_0(q^g_{l+1} - q^g_l) + \epsilon_1(q^g_{l+1} - \hat{q}^g)\Big] \\
&= \epsilon_0 \sum_{g\in\bar{G}} (q^g_{l+1} - q^g_l) + \epsilon_0\Big(\sum_{g\in\bar{G}} \hat{q}^g\Big) \sum_{g\in\bar{G}} (q^g_{l+1} - \hat{q}^g) \\
&\quad + \max\{p_s(q_{l+1}) - p_s(\hat{q}), 0\} \\
&< \epsilon_0\Big(1 + \sum_{g\in\bar{G}} \hat{q}^g\Big) \sum_{g\in\bar{G}} q^g_{l+1} + \max\{p_s(q_{l+1}) - p_s(\hat{q}), 0\} \\
&< \min\{p_s(\hat{q}), p_s(q_{l+1})\} - p_s(q_l) + \max\{p_s(q_{l+1}) - p_s(\hat{q}), 0\} \\
&= p_s(q_{l+1}) - p_s(q_l).
\end{aligned}
$$

Third, consider any $q \in Q$ with $q \nleq q_{l+1}$. Then there exists $k \in \{0, 1, \dots\}$ such that $\bar{q}_k \ngeq q \leq \bar{q}_{k+1}$. Hence,

$$
\begin{aligned}
c(q) &> \delta_k \sum_{g\in\bar{G}} \max\{q^g - \bar{q}^g_k, 0\} \geq \delta_k \\
&\geq \sum_{g\in\bar{G}} \big[\epsilon_0 q^g_{l+1} + \epsilon_1(q^g_{l+1} - \hat{q}^g)\big] + p_s(q) - p_s(q_{l+1}) \\
&= c(q_{l+1}) + p_s(q) - p_s(q_{l+1}),
\end{aligned}
$$

which implies that $q \prec_c q_{l+1}$.

Proof of (A.9). There are four steps. First, consider any $q < \hat{q}$. By definition of $\hat{q}$, $p_s(q) < p_s(\hat{q})$. Hence,

$$
c(\hat{q}) - c(q) = \epsilon_0 \sum_{g\in\bar{G}} (\hat{q}^g - q^g) \leq \epsilon_0 \sum_{g\in\bar{G}} \hat{q}^g < p_s(\hat{q}) - p_s(q),
$$

which implies that $q \prec_c \hat{q}$. Second, suppose that $\hat{q} \ngeq q < q_{l+1}$. By definition of $\hat{q}$, $p_s(q) \leq p_s(\hat{q})$. Moreover,

$$
c(q) = \sum_{g\in\bar{G}} \big[\epsilon_0 q^g + \epsilon_1 \max\{q^g - \hat{q}^g, 0\}\big] > \epsilon_1 \geq \epsilon_0 \sum_{g\in\bar{G}} \hat{q}^g = c(\hat{q}).
$$

Hence, $q \prec_c \hat{q}$. Third, $q_{l+1} \prec_c \hat{q}$ because

$$
\begin{aligned}
c(q_{l+1}) - c(\hat{q}) &= (\epsilon_0 + \epsilon_1) \sum_{g\in\bar{G}} (q^g_{l+1} - \hat{q}^g) \\
&= \epsilon_0\Big(1 + \sum_{g\in\bar{G}} \hat{q}^g\Big) \sum_{g\in\bar{G}} (q^g_{l+1} - \hat{q}^g) + \max\{p_s(q_{l+1}) - p_s(\hat{q}), 0\}
\end{aligned}
$$

$$> p_s(q_{l+1}) - p_s(\hat{q}).$$

Fourth, consider any $q \not\leq q_{l+1}$. From the proof of (A.8*), we know that $q \prec_c q_{l+1}$. Since also $q_{l+1} \prec_c \hat{q}$, it follows that $q \prec_c \hat{q}$.

C.7.6 Claim A.6

Since q_{l+1} is defined relative to $\bar{q}$, Claim A.6 must be adjusted.

Claim A.6*. For all $q \in Q \setminus \{q_0, \ldots, q_{l+1}\}$ such that $q \leq \bar{q}$, $q > q_{l+1}$.

The proof of Claim A.6* closely follows the original proof of Claim A.6. Before explaining the necessary changes, we show that Claim A.6* also holds when we drop the "such that $q \leq \bar{q}$" part.

By contradiction, suppose there exists $\hat{q} \in Q \setminus \{q_0, \ldots, q_{l+1}\}$ such that $\hat{q} \not> q_{l+1}$. By Claim A.6*, it must be that $\hat{q} \not\leq \bar{q}$. Let $\bar{q}' \in \mathbb{N}^{\bar{g}}$ be such that $\hat{q} \leq \bar{q}'$ and $q_{l+1} \leq \bar{q}'$. In analogy to q_{l+1}, let $q'_{l+1} \in Q \setminus \{q_0, \ldots, q_l\}$ be such that $q'_{l+1} \leq \bar{q}'$ and for all $q \in Q \setminus \{q_0, \ldots, q_l\}$ with $q \leq \bar{q}'$, either $p_b(q) > p_b(q'_{l+1})$ or $[p_b(q) = p_b(q'_{l+1})$ and $q \not> q'_{l+1}]$. Apply Claim A.6* to q'_{l+1} and $\bar{q}'$ instead of q_{l+1} and $\bar{q}$. Since $q_{l+1} \in Q \setminus \{q_0, \ldots, q_l, q'_{l+1}\}$ and $q_{l+1} \leq \bar{q}'$, we have that $q_{l+1} > q'_{l+1}$. The definition of q'_{l+1} then implies that $p_b(q_{l+1}) > p_b(q'_{l+1})$, which contradicts the definition of q_{l+1}. Hence, it must be that for all $q \in Q \setminus \{q_0, \ldots, q_{l+1}\}$, $q > q_{l+1}$.

Let us now turn to the proof of Claim A.6*. The structure follows the original proof of Claim A.6, but there are some adjustments. First, we must obviously add that $\hat{q} \leq \bar{q}$. Second, (A.11) must be replaced by

$$\begin{aligned} &q_0 \prec_c \cdots \prec_c q_l \prec_c q_{l+1} \prec_c \hat{q} \text{ and} \\ &q_{l+1} \succ_c q \text{ for all } q \in Q \setminus \{q_0, \ldots, q_{l+1}, \hat{q}\}. \end{aligned} \tag{A.11*}$$

The second line of (A.11*) differs from (A.11). Third, in the paragraph immediately after (A.10) and (A.11), the two sentences following "Moreover" must be changed as follows: "Moreover, by Claim A.4*, there exists $\bar{v}_{l+1} \in V$ such that $Q_s(\bar{v}_{l+1}) \cap \{q \in Q : q \leq \bar{q}\} = \{q_0, \ldots, q_{l+1}\}$. Since $\hat{q} \leq \bar{q}$, $\hat{q} \notin Q_s(\bar{v}_{l+1})$. From (A.11*), it follows that $\text{Opt}_s(c, Q_s(\bar{v}_{l+1})) = \{q_{l+1}\}$ and thus $\varphi(c, \bar{v}_{l+1}) = q_{l+1}$."

The major change concerns the construction of c that satisfies (A.11*). First, recall that $X_\epsilon \cap Q = \{q_0, \ldots, q_{l+1}, \hat{q}\}$ for sufficiently small $\epsilon > 0$.

Take any such ϵ and define $X \equiv X_\epsilon$. Next, note that $q_l < \hat{q}$. Moreover, by Claim A.2, $p_s(q_l) < p_s(\hat{q})$ and $p_s(q_l) < p_s(q_{l+1})$. Hence, there exists $\epsilon_0 \in (0, \infty)$ such that

$$\epsilon_0 < \min\left\{ \frac{\min\{p_s(q) - p_s(q') : q, q' \leq \hat{q} \text{ and } p_s(q) > p_s(q')\}}{\sum_{g \in \bar{G}} \hat{q}^g}, \right.$$
$$\left. \frac{\min\{p_s(\hat{q}), p_s(q_{l+1})\} - p_s(q_l)}{(1 + \sum_{g \in \bar{G}} \hat{q}^g) \sum_{g \in \bar{G}} q^g_{l+1}} \right\}.$$

Define also

$$\epsilon_1 \equiv \sum_{g \in \bar{G}} \epsilon_0 \hat{q}^g + \frac{\max\{p_s(q_{l+1}) - p_s(\hat{q}), 0\}}{\sum_{g \in \bar{G}} \max\{q^g_{l+1} - \hat{q}^g, 0\}}.$$

For all $q \in \mathbb{R}^{\bar{g}}_+$, $r(q) \equiv \min\{\rho \in \mathbb{R}_+ : q \in X(\rho)\}$. For all $k \in \{1, 2, \dots\}$,

$$\delta_k \equiv \frac{\begin{array}{c}\sum_{g \in \bar{G}} \left[\epsilon_0 q^g_{l+1} + \epsilon_1 \max\{q^g_{l+1} - \hat{q}^g, 0\}\right] \\ + \max\{p_s(q) - p_s(q_{l+1}) : q \in X(k+1)\}\end{array}}{\min\{r(q) - k : q \in \mathbb{N}^{\bar{g}} \text{ s.t. } q \in X(k+1) \setminus X(k)\}}.$$

For all $q \in \bar{Q}$,

$$c(q) \equiv \sum_{g \in \bar{G}} \left[\epsilon_0 q^g + \epsilon_1 \max\{q^g - \hat{q}^g, 0\}\right] + \sum_{k=1}^{\infty} \delta_k \max\{r(q) - k, 0\}.$$

Recall from Section B that r is a convex function, so $c \in C$.

The proof of (A.11*) has four steps. First, consider any $k \in \{1, \dots, l\}$. Note that, by (a), $q_{k-1} < q_k < \hat{q}$ and, by (b), $p_s(q_{k-1}) < p_s(q_k)$. Hence,

$$c(q_k) - c(q_{k-1}) = \epsilon_0 \sum_{g \in \bar{G}} (q^g_k - q^g_{k-1}) < \epsilon_0 \sum_{g \in \bar{G}} \hat{q}^g < p_s(q_k) - p_s(q_{k-1}),$$

which implies that $q_{k-1} \prec_c q_k$. Second, $q_l \prec_c q_{l+1}$ because

$$\begin{aligned} &c(q_{l+1}) - c(q_l) \\ &= \sum_{g \in \bar{G}} \left[\epsilon_0 (q^g_{l+1} - q^g_l) + \epsilon_1 \max\{q^g_{l+1} - \hat{q}^g, 0\}\right] \\ &= \epsilon_0 \sum_{g \in \bar{G}} (q^g_{l+1} - q^g_l) + \epsilon_0 \Big(\sum_{g \in \bar{G}} \hat{q}^g\Big) \sum_{g \in \bar{G}} \max\{q^g_{l+1} - \hat{q}^g, 0\} \end{aligned}$$

$$
\begin{aligned}
&\quad + \max\{p_s(q_{l+1}) - p_s(\hat{q}), 0\} \\
&< \epsilon_0 \Big(1 + \sum_{g \in \bar{G}} \hat{q}^g\Big) \sum_{g \in \bar{G}} q^g_{l+1} + \max\{p_s(q_{l+1}) - p_s(\hat{q}), 0\} \\
&< \min\{p_s(\hat{q}), p_s(q_{l+1})\} - p_s(q_l) + \max\{p_s(q_{l+1}) - p_s(\hat{q}), 0\} \\
&= p_s(q_{l+1}) - p_s(q_l).
\end{aligned}
$$

Third, $q_{l+1} \prec_c \hat{q}$ because

$$
\begin{aligned}
&c(q_{l+1}) - c(\hat{q}) \\
&= \sum_{g \in \bar{G}} \Big[\epsilon_0 (q^g_{l+1} - \hat{q}^g) + \epsilon_1 \max\{q^g_{l+1} - \hat{q}^g, 0\}\Big] \\
&= \epsilon_0 \sum_{g \in \bar{G}} (q^g_{l+1} - \hat{q}^g) + \epsilon_0 \Big(\sum_{g \in \bar{G}} \hat{q}^g\Big) \sum_{g \in \bar{G}} \max\{q^g_{l+1} - \hat{q}^g, 0\} \\
&\quad + \max\{p_s(q_{l+1}) - p_s(\hat{q}), 0\} \\
&> p_s(q_{l+1}) - p_s(\hat{q}).
\end{aligned}
$$

Fourth, consider any $q \in Q \setminus \{q_0, \dots, q_{l+1}, \hat{q}\}$. Note that $r(q) > 1$. Define $l \equiv \max\{k \in \{1, 2, \dots\} : r(q) > k\}$. Note that $q \in X(l+1) \setminus X(l)$. Thus,

$$
\begin{aligned}
c(q) &> \delta_l [r(q) - l] \\
&\geq \sum_{g \in \bar{G}} \big[\epsilon_0 q^g_{l+1} + \epsilon_1 \max\{q^g_{l+1} - \hat{q}^g, 0\}\big] + p_s(q) - p_s(q_{l+1}) \\
&= c(q_{l+1}) + p_s(q) - p_s(q_{l+1}),
\end{aligned}
$$

which implies that $q_{l+1} \succ_c q$.

C.8 Lemma 5

The statement of Lemma 5 must be adjusted to allow for the possibility that an agent's set of optimal bundles is empty.

Lemma 5* (IC+NB+NTT). There exist functions $q_s \colon C \to Q \cup \{\infty\}$ and $q_b \colon V \to Q \cup \{\infty\}$ such that for all $(c, v) \in C \times V$,

(i) $Q_b(c) = \{q_0, \dots, q_s(c)\}$ and

$$q_s(c) \in \begin{cases} \mathrm{Opt}_s(c) & \text{if } \exists \max\{\mathrm{Opt}_s(c)\}, \\ \mathrm{Opt}_s(c) \cup \{\infty\} & \text{if } \nexists \max\{\mathrm{Opt}_s(c)\}, \end{cases}$$

(ii) $Q_s(v) = \{q_0, \dots, q_b(v)\}$ and

$$q_b(v) \in \begin{cases} \mathrm{Opt}_b(v) & \text{if } \exists \max\{\mathrm{Opt}_b(v)\}, \\ \mathrm{Opt}_b(v) \cup \{\infty\} & \text{if } \nexists \max\{\mathrm{Opt}_b(v)\}. \end{cases}$$

The proof of Lemma 5* coincides with the proof of Lemma 5, except for the following modifications.

The definition of q_l must be adapted: If $\mathrm{Opt}_s(c) \neq \emptyset$, then there exists $v \in V$ such that $Q_s(v) \cap \mathrm{Opt}_s(c) \neq \emptyset$. IC_s requires that $\varphi(c, v) \in \mathrm{Opt}_s(c)$ and thus $Q_b(c) \cap \mathrm{Opt}_s(c) \neq \emptyset$. If $\max\{Q_b(c) \cap \mathrm{Opt}_s(c)\}$ exists, define $l \in \{0, \dots, n\}$ by $q_l \equiv \max\{Q_b(c) \cap \mathrm{Opt}_s(c)\}$. On the other hand, if $\max\{Q_b(c) \cap \mathrm{Opt}_s(c)\}$ does not exist or if $\mathrm{Opt}_s(c) = \emptyset$, define $l \equiv \infty$.

The proof of $Q_b(c) \subseteq \{q_0, \dots, q_l\}$ remains unchanged. Note that the inclusion holds trivially if $l = \infty$.

In the proof of $\{q_0, \dots, q_l\} \subseteq Q_b(c)$, the definition of $v \in V$ must be adjusted as follows: for all $q \in \bar{Q}$,

$$v(q) \equiv \sum_{g \in \bar{G}} \Bigg[\epsilon \left(1 - \frac{1}{1 + q^g}\right) + \frac{1}{\epsilon} \min\{q^g, q^g_{k-1}\} + \left(\frac{p_b(q_k) - p_b(q_{k-1})}{\sum_{g \in \bar{G}} (q^g_k - q^g_{k-1})} - \epsilon \right) \min\{q^g, q^g_k\} \Bigg].$$

For sufficiently small $\epsilon > 0$, $q_0 \prec_v \cdots \prec_v q_{k-1}$ and $q_k \succ_v q_l$ for all $l \in \{k+1, \dots\}$.[1] Moreover, $q_{k-1} \succ_v q_k$ because

$$\begin{aligned} & v(q_k) - v(q_{k-1}) \\ &= \epsilon \sum_{g \in \bar{G}} \left(\frac{1}{1 + q^g_{k-1}} - \frac{1}{1 + q^g_k} \right) + p_b(q_k) - p_b(q_{k-1}) - \epsilon \sum_{g \in \bar{G}} (q^g_k - q^g_{k-1}) \end{aligned}$$

[1]Note that for all $l \in \{k+1, \dots\}$, $v(q_l) < v(q_k) + \epsilon$. Moreover, by Lemma 4(iii), $p_b(q_k) < p_b(q_{k+1}) < \cdots$. Hence, if $\epsilon < p_b(q_{k+1}) - p_b(q_k)$, we have that $v(q_l) < v(q_k) + p_b(q_l) - p_b(q_k)$, so $q_l \prec_v q_k$.

$$= \epsilon \sum_{g \in \bar{G}} \left[\frac{q_k^g - q_{k-1}^g}{(1 + q_{k-1}^g)(1 + q_k^g)} - (q_k^g - q_{k-1}^g) \right] + p_b(q_k) - p_b(q_{k-1})$$

$$< p_b(q_k) - p_b(q_{k-1}).$$

C.9 Lemma 6

The only change in the proof of Lemma 6 concerns the third summand in the construction of $c \in C$: for all $q \in \bar{Q}$,

$$c(q) \equiv \epsilon(\gamma \cdot q) + (\delta - \epsilon) \max\{\gamma \cdot (q - q_{k-1}), 0\} + \sum_{l=k+1}^{\infty} \frac{p_b(q_{l+1}) - p_b(q_l)}{\gamma \cdot (q_{l+1} - q_l)} \max\{\gamma \cdot (q - q_l), 0\}.$$

This modification ensures that c grows sufficiently fast for the seller's utility to decrease from bundle q_{k+1} onward.

C.10 New lemma

If Q is unbounded, a seller with a very flat cost function or a buyer with a very steep value function may not have an optimal bundle. The following result presents necessary and sufficient conditions on the price functions under which the agents do not want to trade infinitely much.

Lemma 6.5 (IC+NB+NTT)**.** Suppose that $\#Q = \infty$.

(i) $\text{Opt}_s(c) \neq \emptyset$ for all $c \in C$ if and only if

$$\forall \gamma \in \mathbb{R}_{++}^{\bar{g}}, \quad \lim_{k \to \infty} \frac{p_s(q_{k+1}) - p_s(q_k)}{\gamma \cdot (q_{k+1} - q_k)} = 0. \tag{C.1}$$

(ii) $\text{Opt}_b(v) \neq \emptyset$ for all $v \in V$ if and only if

$$\forall \gamma \in \mathbb{R}_{++}^{\bar{g}}, \quad \lim_{k \to \infty} \frac{p_b(q_{k+1}) - p_b(q_k)}{\gamma \cdot (q_{k+1} - q_k)} = \infty. \tag{C.2}$$

Proof. We prove (i); the second part is analogous. Let us start with the "only if". Suppose (C.1) does not hold. Then there exists $\gamma \in \mathbb{R}_{++}^{\bar{g}}$ and $\pi_s \in (0, \infty)$ such that, for all $k \in \mathbb{N}$,

$$\frac{p_s(q_{k+1}) - p_s(q_k)}{\gamma \cdot (q_{k+1} - q_k)} \geq \pi_s.$$

Let $\epsilon \in (0, \pi_s)$ and define $c \in C$ by $c(q) = \epsilon(\gamma \cdot q)$ for all $q \in \bar{Q}$. Then, for all $k \in \mathbb{N}$,

$$c(q_{k+1}) - c(q_k) = \epsilon[\gamma \cdot (q_{k+1} - q_k)] < \pi_s[\gamma \cdot (q_{k+1} - q_k)] \leq p_s(q_{k+1}) - p_s(q_k),$$

which implies that $q_{k+1} \succ_c q_k$ for all $k \in \mathbb{N}$. Hence, $\text{Opt}_s(c) = \emptyset$.

Now we turn to the "if" part. Suppose (C.1) holds. Consider any $c \in C$. Let $\gamma_0 \in \mathbb{R}^{\bar{g}}_{++}$ be a subgradient of c at $\mathbf{0}$. Pick any $\delta \in (0, 1)$. By (C.1), there exists $k^* \in \mathbb{N}$ such that for all $k \geq k^*$, $\frac{p_s(q_{k+1}) - p_s(q_k)}{\gamma_0 \cdot (q_{k+1} - q_k)} \leq \delta$ and, equivalently, $p_s(q_{k+1}) \leq p_s(q_k) + \delta\gamma_0 \cdot (q_{k+1} - q_k)$. Applying this inequality recursively, we get that for all $l \geq k^*$,

$$p_s(q_l) \leq p_s(q_{k^*}) + \delta\gamma_0 \cdot (q_l - q_{k^*}).$$

Moreover, since c is convex and $c(\mathbf{0}) = 0$,

$$c(q_l) \geq c(\mathbf{0}) + \gamma_0 \cdot (q_l - \mathbf{0}) = \gamma_0 \cdot q_l.$$

Putting both inequalities together, we obtain that for all $l \geq k^*$,

$$p_s(q_l) - c(q_l) \leq p_s(q_{k^*}) - \delta\gamma_0 \cdot q_{k^*} - (1 - \delta)\gamma_0 \cdot q_l. \tag{C.3}$$

Recall that $1 - \delta > 0$ and $\gamma_0 \in \mathbb{R}^{\bar{g}}_{++}$. Moreover, by Lemma 4, $q_l < q_{l+1}$ for all $l \in \mathbb{N}$. Since $\#Q = \infty$, it follows that $\lim_{l \to \infty}(1 - \delta)\gamma_0 \cdot q_l = \infty$, so the right-hand side of (C.3) becomes arbitrarily small for large l. Hence, there exists $l^* \geq k^*$ such that for all $l \geq l^*$, $p_s(q_l) - c(q_l) < p_s(q_0)$ and thus $q_l \prec_c q_0$. Therefore, $\text{Opt}_s(c) \subseteq \{q_0, \ldots, q_{l^*+1}\}$. Since $\{q_0, \ldots, q_{l^*+1}\}$ is finite, it follows that $\text{Opt}_s(c) \neq \emptyset$. □

If $\text{Opt}_s(c) = \text{Opt}_b(v) = \emptyset$, then Lemma 5* implies that $Q_b(c) = Q_s(v) = \{q_0, q_1, \ldots\}$. Thus, by Lemma 3, $\varphi(c, v) \in \emptyset$, which is impossible. In light of Lemma 6.5, we conclude that (C.1) or (C.2) hold.

C.11 Lemma 7

If (C.1) or (C.2) hold, Lemma 6.5 implies that $\min\{q_s(c), q_b(v)\}$ exists for all $(c, v) \in C \times V$. Hence, the proof of Lemma 7 goes through unchanged.

References

Wilson, C. A. (2008, May 7). *Homogeneous functions*. Lecture notes, New York University, Department of Economics, Mathematics for Economists, V31.0006.