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\mathcal{K} -Formal Concept Analysis as linear algebra over idempotent semifields

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Abstract

We report on progress in characterizing \mathcal{K} -valued FCA in algebraic terms, where \mathcal{K} is an idempotent semifield. In this data mining-inspired approach, incidences are matrices and sets of objects and attributes are vectors. The algebraization allows us to write matrix-calculus formulae describing the polars and the fix-point equations for extents and intents. Adopting also the point of view of the theory of linear operators between vector spaces we explore the similarities and differences of the idempotent semimodules of extents and intents with the subspaces related to a linear operator in standard algebra. This allows us to shed some light into Formal Concept Analysis from the point of view of the theory of linear operators over idempotent semimodules.

In the opposite direction, we state the importance of FCA-related concepts for dual order homomorphisms of linear spaces over idempotent semifields, specially congruences, the lattices of extents, intents and formal concepts.

Key words: Generalised Formal Concept Analysis, concept lattice, neighborhood lattice, idempotent semiring, dioid, confusion matrix

1. Introduction

This paper explores the connection between a generalization in Formal Concept Analysis (FCA, [1]) and Linear Algebra over certain semirings, the idempotent semifields [2–4]. We will prove that when contexts have entries in an idempotent semifield, like the max-plus or the tropical algebra, concept lattices may be better understood in terms of functions between semivector spaces, that is vector spaces (or semimodules) over such semirings, and this sheds light onto Formal Concept Analysis as a whole.

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Reciprocally, we will show that complete lattices, in particular in the form of concept lattices, abound in the study of some transformations of idempotent semimodules, a result previously unknown. Perhaps, the idiosyncrasies of such spaces can be better understood under this new perspective.

Recall the fundamental results of *standard or binary FCA*

Definition 1. (Formal Concept Analysis, [1]) A *formal context* (G, M, I) [1] consists of a set of *objects* G , a set of *attributes* M and an *incidence* $I \subseteq G \times M$, a binary relation describing which objects show which attributes. For all sets of objects $A \subseteq G$, call *polar of extents* (with respect to (G, M, I)) the map $A_I^\uparrow = \{m \in M \mid \forall g \in A, gIm\}$. Similarly, for a set of attributes $B \subseteq M$ call *polar of intents*¹ the map $B_I^\downarrow = \{g \in G \mid \forall m \in B, gIm\}$. Pairs (A, B) such that $A_I^\uparrow = B$ and $B_I^\downarrow = A$ are the *formal concepts* of context (G, M, I) . Call $\mathfrak{B}(G, M, I)$ the set of all such concepts. If we define the order $(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 \Leftrightarrow B_1 \supseteq B_2$, then we can state the basic theorem of concept lattices.

Theorem 1.1 (Basic Theorem of FCA, [1]). *The set of concepts carries a complete lattice algebra $\underline{\mathfrak{B}}(G, M, I) = (\mathfrak{B}(G, M, I), \leq)$, with infimum and supremum given by*

$$\bigwedge_{t \in T} (A_t, B_t) = \left(\bigcap_{t \in T} A_t, \left(\bigcap_{t \in T} A_t \right)_I^\uparrow \right) \quad \bigvee_{t \in T} (A_t, B_t) = \left(\left(\bigcap_{t \in T} B_t \right)_I^\downarrow, \bigcap_{t \in T} B_t \right) \quad (1)$$

Conversely, a complete lattice V is isomorphic to $\underline{\mathfrak{B}}(G, M, I)$ if and only if there are mappings $\gamma : G \rightarrow V$ and $\mu : M \rightarrow V$ such that $\gamma(G)$ is join-dense in V , $\mu(M)$ is meet-dense in V , and gIm is equivalent to $\gamma(g) \leq \mu(m)$ for all $g \in G$ and all $m \in M$.

A generalization of Formal Concept Analysis was presented in [7] where incidences have values in a complete idempotent semifield \mathcal{K} , this being a complete idempotent semiring with a multiplicative group structure whose unit is distinct from top of the semiring. This extension was called \mathcal{K} -Formal Concept Analysis (\mathcal{K} -FCA) and later generalized to the other four types of Galois connections or adjunctions arising from a single \mathcal{K} -valued incidence [6, 8].

Note that idempotent semifields are clearly distinct from inclines—e.g. “fuzzy semirings”—where the top of the element and the unit are the same element. Likewise, idempotent semifields have well attested and distinct uses in morphological processing [9], dynamic programming [10], Markov chain decoding [11], Discrete Event Systems modeling and analysis [3], artificial neural networks [12]

¹The origin of this notation is, we believe, [5], but in [1] the polars were called the “derivational operators”. However, a single relation I has several pairs of derivational operators generating different kinds of Galois connections each [6, § 2.1.2]: in this paper the polars are those derivational operators that generate a Galois connection instead of other pairs of order morphisms.

and memories [13]—to mention but a few—what makes the results developed in this paper potentially useful to improve these or other engineering and mathematical applications.

This usefulness is enhanced because the extensions and intension in semifield-based FCA are related to the eigenvectors for certain matrices derived from the incidence, the *projector matrices* [14]. The authors later went on to develop the spectral theory [15, 16] but never reviewed their findings in two aspects:

- a. What is the actual relation between eigenvectors of the spectral projectors and extents or intents?
- b. What are the advantages of working in complete idempotent semifields?

In this paper we answer both of these questions, to wit:

- a. All procedures related to \mathcal{K} -Formal Concept Analysis take the form of matrix-vector equations in semimodules over semifields, including, the polars, the closure operators and finite generation of sub-semimodules.

This is possible due to the idempotent algebra analogues of vector spaces, the *complete idempotent semimodules*, sometimes called “idempotent semivector spaces”.

- (a) The system of extents and intents induced by the polars have a double semimodule structure.
- (b) Concept lattices over idempotent semifields are idempotent semimodules of a dual semifield.
- (c) The eigenvectors of the projectors generate the concepts by closure of their joins.
- b. We are able to provide, for the previously mentioned application fields, more tools for data processing that include lattices, formal concepts, etc. Yet such tools take the form of linear algebra techniques over idempotent semifields, instead of more complex order-theoretic procedures.

To address these issues, in Section 2 we first review the theory of idempotent semifields and semimodules over them with special attention to Galois connections, nuclear to FCA, spectra and congruences. In Section 3 we revisit \mathcal{K} -FCA with the focus on the linear algebra properties of the main conceptual abstractions of FCA. In particular, we prove that φ -concept lattices are actually pairs of idempotent semimodules, and that congruences play an important role in the characterization of such semimodules. We finish with a Discussion of issues and open avenues of research.

2. Methods and Materials

2.1. Idempotent semirings and semifields

A short systemization. A *semiring* is an algebra $\mathcal{S} = \langle S, \oplus, \otimes, \epsilon, e \rangle$ whose additive structure, $\langle S, \oplus, \epsilon \rangle$, is a commutative monoid and whose multiplicative

structure, $\langle S \setminus \{\epsilon\}, \otimes, e \rangle$, is a monoid with multiplication distributing over addition from right and left and with additive neutral element absorbing for \otimes , i.e. $\forall a \in S, \epsilon \otimes a = \epsilon$.

A *commutative semiring* is one whose multiplicative law is commutative. Note that semirings can be primitive or constructed from other algebras: although the base semiring is commutative, the construction may not as in the example below.

Example 1 (Square matrices semirings). Given a semiring \mathcal{S} , the semiring of square matrices of order n over \mathcal{S} is the structure $\mathcal{M}_n(\mathcal{S}) = \langle \mathcal{S}^{n \times n}, +, \times, 0, E \rangle$ where $+$ is the usual entry-wise addition, \times is usual matrix multiplication, 0 is the matrix whose entries are all the zero in \mathcal{S} and E that matrix whose entries are all zero except at the diagonal, where they are the unit in \mathcal{S} . Note that the operations for matrix addition and multiplication are those of the underlying semiring \mathcal{S} whence this construction is completely generic. \square

All basic semirings in this paper are commutative, except for the example above, which an important example of *non-commutative semiring*. Many more examples can be found in [17].

For the systematization of semirings the important operation seems to be addition [17, 18]. In particular, every semiring accepts a canonical preorder, $a \leq b$ if and only if there exists $c \in D$ with $a \oplus c = b$. A *dioid* is a semiring \mathcal{D} where this relation is actually an order that is *compatible* with multiplication and addition, e.g. if $a \leq b$ than for all $c \in S$ still holds $a \otimes c \leq b \otimes c$ and $a \oplus c \leq b \oplus c$. Dioids are zerosumfree, that is they have no non-null additive factors of zero. In this, dioids are as different as a semiring can be from a *ring*.

A semiring \mathcal{S} is *complete*, if for any index set I including the empty set, and any $\{a_i\}_{i \in I} \subseteq \mathcal{S}$ the (possibly infinite) summations $\sum_{i \in I} a_i$ are defined and the distributivity conditions: $(\sum_{i \in I} a_i) \otimes c = \sum_{i \in I} (a_i \otimes c)$ and $c \otimes (\sum_{i \in I} a_i) = \sum_{i \in I} (c \otimes a_i)$, are satisfied. Note that for $c = e$ the above demand that infinite sums have a result. Commutative complete dioids are already complete residuated lattices.

An *idempotent semiring* is a dioid whose addition is idempotent, examples of which are the following.

Example 2 (Idempotent semirings).

1. The *Boolean lattice* $\mathbb{B} = \langle \{0, 1\}, \vee, \wedge, 0, 1 \rangle$
2. All fuzzy semirings, e.g. $\langle [0, 1], \max, \min, 0, 1 \rangle$
3. The *min-plus algebra* $\mathbb{R}_{\min,+} = \langle \mathbb{R} \cup \{\infty\}, \min, +, \infty, 0 \rangle$
4. The *max-plus algebra* $\mathbb{R}_{\max,+} = \langle \mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0 \rangle$

Of the semirings above, only the boolean lattice and the fuzzy semirings are complete dioids, since the rest lack the greatest element in the order, the *top* \top .

A semiring is a *semifield* if there exists a multiplicative inverse for every element $a \in S$, except the null, notated as a^{-1} . Semifields are all entire, that is, they have no non-null factors of the zero element. Note that there is an infinite quantity of semifields [18, Ch. 8, § 4.4.3 and 4.4.4].

A semifield that is also a dioid is called a *positive semifield*, and these have all a natural order. If its addition is furthermore idempotent it is called an *idempotent semifield*.

Example 3 (Positive and idempotent semifields).

1. The non-negative rationals $\mathbb{Q}_{\geq 0}$ and non-negative reals $\mathbb{R}_{\geq 0}$ are two positive semifields.
2. The min-plus and max-plus algebras of Example 2 are idempotent semifields.
3. The *min-times algebra* $\mathbb{R}_{\min, \times} = \langle \mathbb{R} \cup \{\infty\}, \min, \times, \infty, 1 \rangle$ and the *max-times algebra* $\mathbb{R}_{\max, \times} = \langle \mathbb{R}, \max, \times, 0, 1 \rangle$ are also idempotent semifields. \square

Semifield completions. As noted above, positive semifields are incomplete in their natural order, lacking an adequate inverse for the bottom in the order, but there are procedures for *completing* such structures [6, Construction 1] and we will not differentiate between *complete or completed* structures. This construction actually obtains a pair of dually-ordered complete positive semifields from a single (incomplete) positive semifield. Its results are collected below as a theorem.

Theorem 2.1. *For every (incomplete) positive semifield $\mathcal{K} = \langle K, \oplus, \otimes, \cdot^{-1}, \perp, e \rangle$*

1. *There is a pair of completed semifields over $\overline{K} = K \cup \{\top\}$*

$$\overline{\mathcal{K}} = \langle K, \dot{\oplus}, \dot{\otimes}, \cdot^{-1}, \perp, e, \top \rangle \quad \overline{\mathcal{K}}^{-1} = \langle K, \dot{\oplus}, \dot{\otimes}, \cdot^{-1}, \top, e, \perp \rangle \quad (2)$$

where $\top = \perp^{-1}$ and $\perp = \top^{-1}$ by definition,

2. *In addition to the individual laws as positive semifields, we have the modular laws:*

$$(u \dot{\oplus} v) \dot{\otimes} (u \dot{\oplus} v) = u \dot{\otimes} v \quad (u \dot{\oplus} v) \dot{\otimes} (u \dot{\oplus} v) = u \dot{\otimes} v \quad (3)$$

the analogues of the De Morgan laws:

$$u \dot{\oplus} v = (u^{-1} \dot{\oplus} v^{-1})^{-1} \quad u \dot{\oplus} v = (u^{-1} \dot{\oplus} v^{-1})^{-1} \quad (4)$$

$$u \dot{\otimes} v = (u^{-1} \dot{\otimes} v^{-1})^{-1} \quad u \dot{\otimes} v = (u^{-1} \dot{\otimes} v^{-1})^{-1}$$

and the self-dual inequality in the natural order

$$u \dot{\otimes} (v \dot{\otimes} w) \preceq (u \dot{\otimes} v) \dot{\otimes} w. \quad (5)$$

3. Further, if \mathcal{K} is a positive dioid, then the inversion operation is a dual order isomorphism between the dual order structures $\bar{\mathcal{K}} = \langle K, \dot{\preceq} \rangle$ and $(\bar{\mathcal{K}})^{-1} = \langle K, \dot{\succ} \equiv \dot{\preceq}^\delta \rangle$ with the natural order of the original semifield a suborder of the first structure.

In fact, complete idempotent semifields $\bar{\mathcal{K}} = \langle K, \dot{\oplus}, \dot{\otimes}, \dot{\cdot}^{-1}, \perp, e, \top \rangle$, appear as enriched structures, the advantage of working with them being that meets can be expressed by means of joins and inversion as $a \dot{\oplus} b = (a^{-1} \dot{\oplus} b^{-1})^{-1}$.

Note, also, that in complete semifields $e \neq \top$ which distinguishes them from inclines, and also that the inverse for the null is prescribed as $\perp^{-1} = \top$. On a practical note, residuation in complete commutative idempotent semifields can be expressed in terms of inverses, and this extends to eigenspaces.

Example 4 (Complete idempotent semifields).

1. The “smallest” example of complete idempotent semifield is $\mathbb{B} \cong 2$, but it lacks a common neutral element for multiplications $\dot{\otimes}$ and $\dot{\otimes}$.
2. The next smallest is $\mathfrak{B} = \langle \{\perp, e, \top\}, \dot{\oplus}, \dot{\otimes}, \dot{\otimes}, \perp, e, \top \rangle$. 2 is embedded in \mathfrak{B} , and \mathfrak{B} is embedded in any bigger complete idempotent semifield.
3. The max-plus $\mathbb{R}_{\max,+}$ and min-plus $\mathbb{R}_{\min,+}$ semifields can be completed as:

The *complete min-plus semifield* $\bar{\mathbb{R}}_{\min,+} = \langle \mathbb{R} \cup \{-\infty, \infty\}, \min, \dot{+}, \dot{-}, \infty, 0 \rangle$
and the *complete max-plus semifield* $\bar{\mathbb{R}}_{\max,+} = \langle \mathbb{R} \cup \{-\infty, \infty\}, \max, \dot{+}, \dot{-}, -\infty, 0 \rangle$.

In this notation from [19], we have $\forall c, -\infty \dot{+} c = -\infty$ and $\infty \dot{+} c = \infty$, which solves several issues in dealing with the separately completed dioids. These two completions are inverses as semimodules $\bar{\mathbb{R}}_{\min,+} = \bar{\mathbb{R}}_{\max,+}^{-1}$ (see below), hence order-dual lattices. \square

Note that, although the completion procedure is applied here to idempotent semifields to obtain the completed $\bar{\mathbb{R}}_{\max,+}$ and $\bar{\mathbb{R}}_{\min,+}$, its applicability is much wider, e.g. as applied to the semifields of [18, Ch. 8, §4.4.3 and 4.4.4].

2.2. Idempotent semimodules

Let $\mathcal{D} = \langle D, +, \times, \epsilon_D, e_D \rangle$ be a commutative semiring. A \mathcal{D} -*semimodule* $\mathcal{X} = \langle X, \dot{\oplus}, \dot{\odot}, \epsilon_X \rangle$ is a commutative monoid $\langle X, \dot{\oplus}, \epsilon_X \rangle$ endowed with a scalar action $(\lambda, x) \mapsto \lambda \dot{\odot} x$ satisfying the following conditions for all $\lambda, \mu \in D$, $x, x' \in X$:

$$\begin{aligned} (\lambda \times \mu) \dot{\odot} x &= \lambda \dot{\odot} (\mu \dot{\odot} x) & \lambda \dot{\odot} (x \dot{\oplus} x') &= \lambda \dot{\odot} x \dot{\oplus} \lambda \dot{\odot} x' & (6) \\ (\lambda + \mu) \dot{\odot} x &= \lambda \dot{\odot} x \dot{\oplus} \mu \dot{\odot} x & \lambda \dot{\odot} \epsilon_X &= \epsilon_X = \epsilon_D \otimes x \\ e_D \dot{\odot} x &= x \end{aligned}$$

If \mathcal{D} is commutative, idempotent or complete, then \mathcal{X} is also commutative, idempotent or complete.

Example 5 (Semimodules of matrices over semirings). Rectangular matrices form a \mathcal{D} -semimodule $D^{g \times m}$ for given g, m . In this paper, we only use finite-dimensional semimodules where we can identify *right \mathcal{D} -semimodules* with column vectors, e.g. $\mathcal{X} \equiv \mathcal{D}^{g \times 1}$ and *left \mathcal{D} -semimodules* with row vectors.

To define concrete implementations of the matrix operations over any semiring or semifield \mathcal{S} only the concrete operations of the underlying semifield are needed, since they define the addition of matrices and their product with a scalar. \square

If $\mathcal{X} \subseteq \overline{\mathcal{K}}^{n \times 1}$ is a right semimodule over a complete idempotent semifield \mathcal{K} , three notions of “duals” may be distinguished:

- The *(pointwise) inverse* [2], $\mathcal{X}^{-1} \subseteq (\overline{\mathcal{K}}^{-1})^{n \times 1}$ is a right sub-semimodule of the inverse semifield $\overline{\mathcal{K}}^{-1}$ such that if $x \in X$, then $(x^{-1})_i = (x_i)^{-1}$. This duality is the order duality in partial orders and inverts the natural order: if $x \odot \lambda \leq z$ then $x^{-1} \dot{\odot} \lambda^{-1} \geq z^{-1}$.

$$(x \odot \lambda)^{-1} = x^{-1} \dot{\odot} \lambda^{-1} \quad (x_1 \oplus x_2)^{-1} = x_1^{-1} \dot{\oplus} x_2^{-1}$$

- The *transpose*, $\mathcal{X}^T \subseteq \overline{\mathcal{K}}^{1 \times n}$ is a *left* sub-semimodule of the same semifield such that if x is a right (column) vector, then x^T is a left (row) vector.

$$(x \odot \lambda)^T = \lambda \odot x^T \quad (x_1 \oplus x_2)^T = x_1^T \dot{\oplus} x_2^T$$

These two dualities commute and allow us the definition of the *(conjugate)* [2]², $\mathcal{X}^* \subseteq (\overline{\mathcal{K}}^{-1})^{1 \times n}$ is a *left* sub-semimodule of the inverse semifield $\overline{\mathcal{K}}^{-1}$ such that if $x \in X$, then $(x^{-1})_i = (x_i)^{-1}$, and x is a row vector. This duality at the same time inverts the natural order and the row-column nature of the semimodule.

$$(x \odot \lambda)^* = \lambda^{-1} \dot{\odot} x^* \quad (x_1 \oplus x_2)^* = x_1^* \dot{\oplus} x_2^*$$

In fact, we can choose any two of the previous three dualities as independent and have the other defined from it:

$$x^{-1} = (x^*)^T = (x^T)^* \quad x^T = (x^*)^{-1} = (x^{-1})^* \quad x^* = (x^{-1})^T = (x^T)^{-1}$$

For the matrices in Example 5 we also define an entry-wise inverse $(A^{-1})_{ij} = (A_{ij})^{-1}$, the transpose $(A^T)_{ij} = A_{ji}$ and a conjugate $A^* = (A^T)^{-1} = (A^{-1})^T$

- Using residuation [20], another dual can be defined [21]³: given a complete idempotent semifield $\overline{\mathcal{K}}$ and a right semimodule \mathcal{X} over it define

²When applied on $\overline{\mathbb{R}}_{\max,+}$ or $\overline{\mathbb{R}}_{\min,+}$ we call it the *Cuningham-Green conjugate*.

³The original name for this dual was “opposite” in [21], and so it was adopted in [6]. The authors in [21] now prefer to call this concept the “dual” [22] due to the order dualization of the construction, we surmise.

the (*residuation*) dual of \mathcal{X} as the left \mathcal{K} -semimodule \mathcal{X}^d with addition $x_1 \oplus^d x_2 = x_1 \wedge x_2$ related to the original addition \oplus (which is the join in the natural order) and action $(\lambda, x) \mapsto \lambda \odot^d x = x / \lambda$. Note that $(\mathcal{X}^d)^d = \mathcal{X}$ and that this dual commutes with inversion. One of the advantages of operating on completed idempotent semifields is that residuation can be expressed in terms of the original operations of the semimodule:

$$\lambda \odot^d x = x \dot{\otimes} \lambda^{-1} \qquad x_1 \oplus^d x_2 = x_1 \dot{\oplus} x_2 \qquad (7)$$

For complete idempotent semifields, the following matrix algebra equations are proven in [2, Ch.8]:

Proposition 2.2. *Let \mathcal{K} be an idempotent semifield, and $A \in \mathcal{K}^{m \times n}$. Then:*

1. $A \dot{\otimes} (A^* \dot{\otimes} A) = A \dot{\otimes} (A^* \dot{\otimes} A) = (A \dot{\otimes} A^*) \dot{\otimes} A = (A \dot{\otimes} A^*) \dot{\otimes} A = A$ and $A^* \dot{\otimes} (A \dot{\otimes} A^*) = A^* \dot{\otimes} (A \dot{\otimes} A^*) = (A^* \dot{\otimes} A) \dot{\otimes} A^* = (A^* \dot{\otimes} A) \dot{\otimes} A^* = A^*$.
2. Alternating $A - A^*$ products of 4 matrices can be shortened as in:

$$A^* \dot{\otimes} (A \dot{\otimes} (A^* \dot{\otimes} A)) = A^* \dot{\otimes} A = (A^* \dot{\otimes} A) \dot{\otimes} (A^* \dot{\otimes} A)$$

3. Alternating $A - A^*$ products of 3 matrices and another terminal, arbitrary matrix can be shortened as in:

$$A^* \dot{\otimes} (A \dot{\otimes} (A^* \dot{\otimes} M)) = A^* \dot{\otimes} M = (A^* \dot{\otimes} A) \dot{\otimes} (A^* \dot{\otimes} M)$$

4. The following inequalities apply:

$$\begin{aligned} A^* \dot{\otimes} (A \dot{\otimes} M) &\geq M & A^* \dot{\otimes} (A \dot{\otimes} M) &\leq M \\ A \dot{\otimes} (A^* \dot{\otimes} M) &\geq M & A \dot{\otimes} (A^* \dot{\otimes} M) &\leq M \end{aligned}$$

Consider a set of vectors $\mathcal{S} \subseteq \mathcal{X}$, then the *span* of \mathcal{S} (with respect to \mathcal{K} an idempotent semifield) $\langle \mathcal{S} \rangle_{\mathcal{K}} \subseteq \mathcal{X}$ is the subsemimodule of \mathcal{X} generated by linear combinations of finitely many such vectors. Note that a semimodule \mathcal{V} is *finitely generated* if there exists a finite set of vectors $\mathcal{S} \subseteq \mathcal{X}$ such that $\mathcal{V} = \langle \mathcal{S} \rangle_{\mathcal{K}}$. All such finitely generated subsemimodules of a \mathcal{K}^n are *closed*, both in the convex sense, that is, they include all of their limit points [23], and in the topological sense, in the Scott topology issued from the order properties of the idempotent semifield \mathcal{K} [24].

Sometimes we put together a matrix S whose columns are the vectors of \mathcal{S} and we write $\langle \mathcal{S} \rangle_{\mathcal{K}} = \langle S \rangle_{\mathcal{K}} = \{S \otimes z \mid z \in \mathcal{K}^{|\mathcal{S}|}\} = \text{Im}(S)$ the image of $S \in \mathcal{K}^{n \times p}$ when it is interpreted as a linear transformation $S : \mathcal{K}^p \rightarrow \mathcal{K}^n$.

2.3. Galois connections and adjunctions between idempotent semimodules

Most features of FCA are built on the notion of a Galois connection. The concepts here extracted follow the presentation in [6] focusing on Galois connections over idempotent semimodules. See [1, Ch.0] and [24, Ch.5, 7] for different presentations of the issue.

Let $\mathcal{P} = \langle P, \leq_{\mathcal{P}} \rangle$ and $\mathcal{Q} = \langle Q, \leq_{\mathcal{Q}} \rangle$ be partially ordered sets. We have:

- A map $f : P \rightarrow Q$ is *residuated* if inverse images of principal (order) ideals of Q under f are again principal ideals. Its *residual map* or simply *residual*, $f^{\#} : Q \rightarrow P$ is $f^{\#}(q) = \max\{p \in P \mid f(p) \leq_{\mathcal{Q}} q\}$.
- A map $g : Q \rightarrow P$ is *dually residuated* if the inverse images of principal dual (order) ideals under g are again dual ideals. Its *dual residual map* or simply *dual residual*, $g^{\flat} : P \rightarrow Q$ is $g^{\flat}(p) = \min\{q \in Q \mid p \leq_{\mathcal{P}} g(q)\}$.

This duality of concepts is fortunately simplified by the well-known fact that residual maps are dually residuated, while dual residual maps are residuated, hence we may maintain only the two notions of residuated maps and their residuals [3, Remark 4.53 and Theorem 4.56]. In fact, the two notions are so entwined that we give a name to them: an *adjoint pair of maps* (λ, ρ) is a pair $(\lambda : P \rightarrow Q, \rho : Q \rightarrow P)$ between two ordered sets such that $\forall p \in P, q \in Q, p \leq_{\mathcal{P}} \rho(q) \iff \lambda(p) \leq_{\mathcal{Q}} q$, equivalently, $p \leq_{\mathcal{P}} \rho(\lambda(p))$ and $\lambda(\rho(q)) \leq_{\mathcal{Q}} q$.

If the order relation is actually partial the *lower or left adjoint*, λ is uniquely determined by its *right or upper adjoint*, ρ , and conversely [25, §1.1]. The characterization theorem for adjoint maps [25, p. 7] states that (λ, ρ) are adjoint if and only if, λ is residuated with residual ρ , or equivalently, ρ is dually residuated with λ its dual residual.

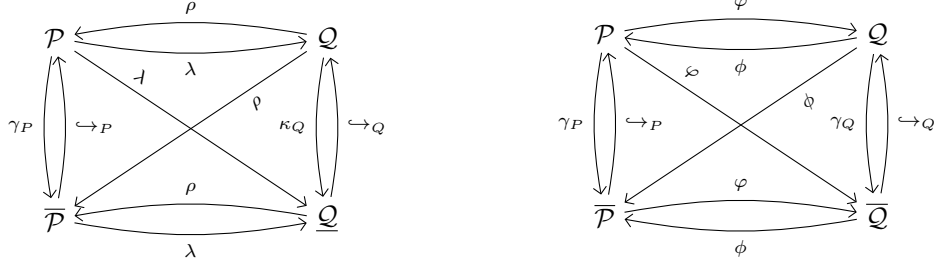
Now consider also their order duals $\mathcal{P}^{\text{d}} = \langle P, \geq_{\mathcal{P}} \rangle$ and $\mathcal{Q}^{\text{d}} = \langle Q, \geq_{\mathcal{Q}} \rangle$.

Definition 2. (Galois connections and adjunctions)

1. (λ, ρ) is a (*Galois adjunction (on the left)*) of two adjoints, and we write $(\lambda, \rho) : \mathcal{P} \rightleftarrows \mathcal{Q}$ iff: $\forall p \in P, q \in Q \quad \lambda(p) \leq_{\mathcal{Q}} q \iff p \leq_{\mathcal{P}} \rho(q)$, that is, the functions are covariant, and we say that λ is the *lower or left adjoint* while ρ is the *upper or right adjoint*.
2. (φ, ψ) is a *Galois connection (proper)* of two dual adjoints $(\varphi, \psi) : \mathcal{P} \rightleftarrows^{\text{d}} \mathcal{Q}$ iff: $\forall p \in P, q \in Q \quad \varphi(p) \geq_{\mathcal{Q}} q \iff p \leq_{\mathcal{P}} \psi(q)$, that is, both functions are contravariant. For that reason they are sometimes named *contravariant or symmetric adjunctions on the right*. Note that (ψ, φ) is also a Galois connection.

As a sort of graphical summary, we introduce the diagrams at the top of figure 1 as the pattern that carries the structures described in [25, §1.2]. We illustrate how to read it with the diagram at the top left, which has:

- A closure system, $\rho(Q) = \overline{P}$, the closure range of the right adjoint (see below).



(a) A Galois adjunction

(b) A Galois connection

Left Adjunction: $(\lambda, \rho) : \mathcal{P} \rightleftarrows \mathcal{Q}$	Galois connection: $(\varphi, \psi) : \mathcal{P} \rightleftarrows \mathcal{Q}$
$\forall p \in P, q \in Q, \lambda(p) \leq_Q q \Leftrightarrow p \leq_P \rho(q)$ $I_P \leq \rho \circ \lambda$ and $I_Q \geq \lambda \circ \rho$ $\lambda = \lambda \circ \rho \circ \lambda$ and $\rho = \rho \circ \lambda \circ \rho$ λ monotone, residuated ρ monotone, residual λ join-preserving, ρ meet-preserving	$\forall p \in P, q \in Q, \varphi(p) \geq_Q q \Leftrightarrow p \leq_P \psi(q)$ $I_P \leq \psi \circ \varphi$ and $I_Q \leq \varphi \circ \psi$ $\varphi = \varphi \circ \psi \circ \varphi$ and $\psi = \psi \circ \varphi \circ \psi$ φ antitone ψ antitone φ join-inverting, ψ join-inverting

(c) Properties of Galois connections and adjunction between posets \mathcal{P}, \mathcal{Q} .

Figure 1: Diagrams visually depicting the maps and structures involved in the adjunction on the left $(\lambda, \rho) : \mathcal{P} \rightleftarrows \mathcal{Q}$ (left) and Galois connection $(\varphi, \psi) : \mathcal{P} \rightleftarrows \mathcal{Q}$ (right) with a summary of their properties. Closure operators are denoted by γ_P, γ_Q , interior (kernel) operators by κ_P, κ_Q , closure systems by $\overline{\mathcal{P}}, \overline{\mathcal{Q}}$ and interior (kernel) systems by $\underline{\mathcal{P}}, \underline{\mathcal{Q}}$ (adapted from [6]).

- An interior system, $\lambda(P) = \underline{Q}$, the kernel range of the left adjoint (see below).
- A closure function (also, “closure operator”) $\gamma_P = \rho \circ \lambda \geq_P I_P$, from P to the closure range $\overline{P} = \rho(Q)$, with adjoint inclusion map \hookrightarrow_P , where I_P denotes the identity over P .
- A kernel function (also, “interior operator”, “kernel operator”) $\kappa_P = \lambda \circ \rho \leq_Q I_Q$, from Q to the range of $\underline{Q} = \lambda(P)$, with adjoint inclusion map \hookrightarrow_Q , where I_Q denotes the identity over Q .
- a *perfect adjunction* $(\tilde{\lambda}, \tilde{\rho}) : \overline{P} \rightleftarrows \underline{Q}$, that is, an order isomorphism between the closure and kernel ranges \overline{P} and \underline{Q} .

The different monotonicity conditions account for the different properties of the Galois adjunctions of figure 1.(a):

- if (λ, ρ) form a left adjunction, then λ is residuated, preserves existing least upper bounds (for lattices, *joins*) and ρ preserves existing greatest lower bounds (for lattices, *meets*).

- if (φ, ψ) form a Galois connection, then both φ and ψ invert existing least upper bounds (for lattices, they transform joins into meets).

Table 1.(c) summarizes the main properties of both types of connections.

Example 6. With the polars defined in Definition 1, $(\cdot \uparrow_I, \cdot \downarrow_I) : \mathcal{X} \equiv \mathcal{X} \bowtie \mathcal{Y}$ is the Galois connection in FCA, with $\mathcal{X} \equiv 2^G$ and $\mathcal{Y} \equiv 2^M$. \square

Also, [6] provides details on how to extend basic FCA with the different types of connections to provide different “flavors” of FCA—including some based in adjunctions—as well as extending it to incidences with values in an idempotent semifield.

Example 7. [6] When $\overline{\mathcal{K}}$ is a completed idempotent semifield and $\mathcal{X} \equiv \overline{\mathcal{K}}^g$ and $\mathcal{Y} \equiv \overline{\mathcal{K}}^m$ are idempotent vector spaces or semimodules, we may define a Galois connection for every $R \in K^{g \times m}$ by means of a scalar product $\langle \cdot | R | \cdot \rangle : X \times Y \rightarrow K$ and a scalar $\varphi \in \overline{\mathcal{K}}$ whose polar functions

$$x \uparrow_{R, \varphi} = \vee \{y \in Y \mid \langle x | R | y \rangle \leq \varphi\} \quad y \downarrow_{R, \varphi} = \vee \{x \in X \mid \langle x | R | y \rangle \leq \varphi\} \quad (8)$$

form the Galois connection $(\cdot \uparrow_{R, \varphi}, \cdot \downarrow_{R, \varphi}) : \mathcal{X} \bowtie \mathcal{Y}$. \square

2.4. Complete congruences of idempotent semimodules

In FCA, the quotient sets of the polars is important because it determines a fundamental limitation of the formal context: where it is not capable of distinguishing between two sets of objects (or attributes). It is used, extensively for reducing the context and the concept lattice [1, notes to Ch.1]. In our approach, this treatment has to be extended to semimodules: the following is borrowed and adapted from [23, 26, 27].

Given a right \mathcal{K} -semimodule \mathcal{X} a subset $\mathcal{W} \subset \mathcal{X}^2$ is called a *pre-congruence (of semimodules)*, if it is a subsemimodule $\langle W, \oplus, \odot \rangle$ such that $(x, x) \in W, \forall x \in X$, and if $(x_1, x_2) \in W$ and $(x_2, x_3) \in W$, then $(x_1, x_3) \in W$. Furthermore, it is a *congruence (of semimodules)* whenever $(x_1, x_2) \in W$ implies $(x_2, x_1) \in W$. Therefore, in the present case, congruences are equivalence relations closed with respect to the semimodule operation and have, in turn, a semimodule structure when thought of as a subsemimodule of \mathcal{X}^2 [26, § 2].

A natural way to define a congruence is to consider a continuous, right linear functional $F : \mathcal{X} \rightarrow \mathcal{Y}$ to a complete right \mathcal{K} -semimodule \mathcal{Y} , and define its *image* $\text{IM}(F)$ and *bikernel* $\text{KER}(F)$ as follows.

$$\begin{aligned} \text{IM}(F) &= \{y \in Y \mid \exists x \in X, y = F(x)\} = \{F(x) \mid x \in X\} = F(X) \\ \text{KER}(F) &= \{(x_1, x_2) \in X^2 \mid F(x_1) = F(x_2)\}. \end{aligned}$$

Conversely, every complete congruence \mathcal{W} arises in this way. The bikernel is an analogue to the kernel of a linear function in the setting of idempotent semimodules⁴.

⁴Note that in this paper we use *kernel* to refer to monotone, contractive idempotent endomorphisms of semimodules in Subsection 2.3.

To investigate analogous relationships between images and bikernels in the standard case we define:

Definition 3. Let \mathcal{K} be an idempotent semifield and $\mathcal{X} \cong \mathcal{K}^n$ a semi-vector space over \mathcal{K} . Then,

- The *orthogonal of a semimodule* $\mathcal{V} \subseteq \mathcal{X}$ is the congruence

$$\mathcal{V}^\perp = \{(x_1, x_2) \in \mathcal{X}^2 \mid \forall x \in V, x_1^*x = x_2^*x\}$$

- The *orthogonal of a congruence (as a semimodule)* $\mathcal{W} \subseteq \mathcal{X}^2$ is the semimodule

$$\mathcal{W}^\top = \{x \in X \mid \forall (x_1, x_2) \in W, x_1^*x = x_2^*x\}$$

We have the following propositions.

Proposition 2.3. [27, Ch. IV,1.2.2, generalized] For all (convex) closed semimodules $\mathcal{V} \subseteq \mathcal{K}^n$, $(\mathcal{V}^\perp)^\top = \mathcal{V}$.

Proposition 2.4. [23, Th. 10, generalized] A semimodule $\mathcal{W} \subseteq \mathcal{X}^2$ is a (convex) closed congruence if and only if $\mathcal{W} = (\mathcal{W}^\top)^\perp$.

The generalization is that they apply to semimodules over idempotent semifields in general, not a particular one. Both hold, a fortiori, for finitely generated semimodules, which are (convex) closed (see Section 2.2).

Specially, when a homomorphism F is represented by its matrix A :

Proposition 2.5. ([26, Lemma 7]) For any matrix $A \in K^{m \times n}$ we have $(\text{IM}(A))^\perp = \text{KER}(A^\top)$ and $(\text{KER}(A))^\top = \text{IM}(A^\top)$.

This is clearly the way to define congruences for Galois adjunctions because F and F^\sharp are a residuated pair, that is, a Galois adjunction.

When two linear functions $F : Z \rightarrow X$ and $G : X \rightarrow Y$ compose, the property that $\text{KER}(G)$ intersects $\text{IM}(F)$ at a single point is often called a *transversality condition* that provides the basis for a number of results in max-min-plus Control and Decision Theory [22]. Previously it had been used as a reason to call the relation between bikernels and images an *orthogonality*, whence the notation on bikernels and their orthogonals in Definition 3 [28]. The work below presents this orthogonality in the light of Galois connections and FCA.

But for Galois connections as first described in idempotent semimodules by [21] a better way may be the following. Given a pre-dual pair \mathcal{X}, \mathcal{Y} and a dot product $\langle X \mid Y \rangle$, we define the following correspondences between semimodules of \mathcal{X}^2 and \mathcal{Y} :

$$\begin{aligned} \mathcal{W} \subset \mathcal{X}^2 &\mapsto \mathcal{W}^\top = \{y \in Y \mid \langle x_1 \mid y \rangle = \langle x_2 \mid y \rangle, \forall (x_1, x_2) \in W\} \\ \mathcal{V} \subset \mathcal{Y} &\mapsto \mathcal{V}^\perp = \{(x_1, x_2) \in \mathcal{X}^2 \mid \langle x_1 \mid y \rangle = \langle x_2 \mid y \rangle, \forall y \in V\} \end{aligned} \quad (9)$$

Note that \mathcal{V} is a complete subsemimodule of \mathcal{Y} and $\mathcal{V} = (\mathcal{V}^\perp)^\top$ [23], and

Proposition 2.6. *Let $(\mathcal{X}, \mathcal{Y})$ be a pre-dual pair satisfying the property that if $W \in \mathcal{X}^2$ is a complete congruence and $(s, t) \notin W$, then there exists a $y \in Y$ such that if $(x_1, x_2) \in W$ then $\langle x_1 | y \rangle = \langle x_2 | y \rangle$ and $\langle s | y \rangle \neq \langle t | y \rangle$. Then a subsemimodule $\mathcal{W} \subset \mathcal{X}^2$ is a complete congruence if and only if $\mathcal{W} = (\mathcal{W}^\top)^\perp$.*

Next we investigate the structure of the equivalence classes of congruences. On a complete (as a semimodule) pre-congruence $\mathcal{W} \subset \mathcal{X}^2$ for $x \in X$ define:

$$\hat{x} = \vee \{x' \in X \mid (x', x) \in \mathcal{W}\} \quad (10)$$

Recall that \hat{x} is just the supremum in the equivalence class of $x \in X$ and it is a closure operator: $x \leq \hat{x} = \hat{\hat{x}}$ whence $x_1 \leq x_2$ implies $\hat{x}_1 \leq \hat{x}_2$, and in particular $\hat{x}_1 = \hat{x}_2$ if $(x_1, x_2) \in \mathcal{W}$. This will be expanded below.

2.5. Basic Spectral Theory over Dioids

Let $\mathcal{M}_n(\mathcal{S})$ be the semiring of square matrices over a semiring \mathcal{S} with the usual operations. Given $A \in \mathcal{M}_n(\mathcal{S})$ the *right (left) eigenproblem* is the task of finding the *right eigenvectors* $v \in S^{n \times 1}$ and *right eigenvalues* $\rho \in S$ (respectively *left eigenvectors* $u \in S^{1 \times n}$ and *left eigenvalues* $\lambda \in S$) satisfying:

$$u \otimes A = \lambda \otimes u \quad A \otimes v = v \otimes \rho \quad (11)$$

The left and right eigenspaces and spectra are the sets of these solutions:

$$\begin{aligned} \Lambda(A) &= \{\lambda \in S \mid \mathcal{U}_\lambda(A) \neq \{\epsilon^n\}\} & \text{P}(A) &= \{\rho \in S \mid \mathcal{V}_\rho(A) \neq \{\epsilon^n\}\} \\ \mathcal{U}_\lambda(A) &= \{u \in S^{1 \times n} \mid u \otimes A = \lambda \otimes u\} & \mathcal{V}_\rho(A) &= \{v \in S^{n \times 1} \mid A \otimes v = v \otimes \rho\} \\ \mathcal{U}(A) &= \bigcup_{\lambda \in \Lambda(A)} \mathcal{U}_\lambda(A) & \mathcal{V}(A) &= \bigcup_{\rho \in \text{P}(A)} \mathcal{V}_\rho(A) \end{aligned} \quad (12)$$

With so little structure it might seem hard to solve (11), but a very generic solution based in the concept of transitive closure of a matrix $A^+ = \sum_{i=1}^{\infty} A^i$ and transitive-reflexive closure $A^* = \sum_{i=0}^{\infty} A^i$ is given by the following theorem:

Theorem 2.7. [29, Theorem 1] *Let $A \in S^{n \times n}$. If A^* exists, the following two conditions are equivalent:*

1. $A_{.i}^+ \otimes \mu = A_{.i}^* \otimes \mu$ for some $i \in \{1 \dots n\}$, and $\mu \in S$.
2. $A_{.i}^+ \otimes \mu$ (and $A_{.i}^* \otimes \mu$) is an eigenvector of A for e , $A_{.i}^+ \otimes \mu \in \mathcal{V}_e(A)$.

A more algorithmically oriented approach to solving the all-eigenvectors problem in idempotent semifields can be found in [15, 16]. Specifically, note that the eigenspace for a particular eigenvalue $\mathcal{V}_\rho(A)$ is generated by its *fundamental eigenvectors* $FEV(A)$ so that $\mathcal{V}_\rho(A) = \langle FEV(A) \rangle_{\mathcal{K}}$.

3. Theory and Calculations

3.1. Dual scalar product and scaling

To be consistent with the way FCA handles the data in the incidence relation the proper way to define the polars is through a scalar product in the dual semimodules [6, Lemmas 2.8, 2.11 and 2.12, dualized].

Lemma 3.1. *When $\overline{\mathcal{K}}$ is a completed idempotent semifield and $\mathcal{X} \equiv \overline{\mathcal{K}}^g$ and $\mathcal{Y} \equiv \overline{\mathcal{K}}^m$ are idempotent vectors spaces or semimodules, we may define a Galois connection for every $R \in K^{g \times m}$ by means of $\langle x \mid R \mid y \rangle = y^\top \dot{\otimes}^d R \dot{\otimes}^d x$.*

$$\begin{aligned} x_{R,\varphi}^\uparrow &= \vee^d \{y \in Y \mid \langle x \mid R \mid y \rangle \leq^d \varphi\} = R^\top \dot{\otimes} x^{-1} \dot{\otimes} \varphi^{-1} \\ y_{R,\varphi}^\downarrow &= \vee^d \{x \in X \mid \langle x \mid R \mid y \rangle \leq^d \varphi\} = R \dot{\otimes} y^{-1} \dot{\otimes} \varphi^{-1} \end{aligned} \quad (13)$$

Proof. The scalar product, written in standard operations is $\langle x \mid R \mid y \rangle = x \setminus R / y^\top = x^* \dot{\otimes} R \dot{\otimes} y^{-1}$. From (8), in the dual semimodules, we have to solve $x^* \dot{\otimes} R \dot{\otimes} y^{-1} \geq \varphi$ in both x and y . Solving for y^{-1} we have $y^{-1} = (x^* \dot{\otimes} R) \setminus \varphi = R^* \dot{\otimes} x \dot{\otimes} \varphi$, whence inverting $x_{R,\varphi}^\uparrow = R^\top \dot{\otimes} x^{-1} \dot{\otimes} \varphi^{-1}$. Since $x^* \dot{\otimes} R \dot{\otimes} y^{-1} \geq \varphi \iff y^* \dot{\otimes} R^\top \dot{\otimes} x^{-1} \geq \varphi$ by transposition, a (transpose) dual expression arises for $y_{R,\varphi}^\downarrow$, with $x \leftrightarrow y$ and $R \leftrightarrow R^\top$ on all semimodules. \square

Our main intent, in this paper, is to see how this construction develops in terms of linear algebra over idempotent semifields. For that purpose, we leave the corner cases when $\varphi \in \{\perp, \top\}$ for Section 3.7. Then we can reduce the studying of the connection with a generic φ to a simpler setting inspired by [30]: when $\{\varphi, \gamma, \mu\} \in \overline{\mathcal{K}} \setminus \{\perp, \top\}$ we consider $\varphi = \gamma \dot{\otimes} \mu$ so:

$$\begin{aligned} x^* \dot{\otimes} R \dot{\otimes} y^{-1} \geq \gamma \dot{\otimes} \mu &\iff \gamma \setminus (x^* \dot{\otimes} R \dot{\otimes} y^{-1}) / \mu \geq e \\ &\iff (\gamma^{-1} \dot{\otimes} x^* \dot{\otimes} R \dot{\otimes} y^{-1} \dot{\otimes} \mu^{-1}) \geq e \\ &\iff (x \dot{\otimes} \gamma)^* \dot{\otimes} R \dot{\otimes} (y \dot{\otimes} \mu)^{-1} \geq e, \end{aligned}$$

Then the products $x \dot{\otimes} \gamma = \tilde{x}^\gamma$ and $y \dot{\otimes} \mu = \tilde{y}^\mu$ have the interpretation of (finite) scalings in the original spaces

$$x^* \dot{\otimes} R \dot{\otimes} y^{-1} \geq \gamma \dot{\otimes} \mu \iff (\tilde{x}^\gamma)^* \dot{\otimes} R \dot{\otimes} (\tilde{y}^\mu)^{-1} \geq e$$

whence we need only consider the case where $\varphi = e$. The following development presupposes this setting and we treat $x \in \tilde{X}^\gamma$ and $y \in \tilde{Y}^\mu$ simply as placeholders. Note that elements in $(\tilde{X}^\gamma)^\top$ are treated as row (left) vectors, while those of \tilde{Y}^μ are treated as column (right) vectors, with those roles swapped by transposition. This is the origin of *row-column duality* often invoked in proofs.

From (13) with $\varphi = e$, we recall the following:

Lemma 3.2. $(\cdot_{R,\varphi}^\uparrow, \cdot_{R,\varphi}^\downarrow) : \tilde{\mathcal{X}}^\gamma \times \tilde{\mathcal{Y}}^\mu$ with

$$x_{R,\varphi}^\uparrow = R^\top \dot{\otimes} x^{-1} \qquad y_{R,\varphi}^\downarrow = R \dot{\otimes} y^{-1} \quad (14)$$

is a Galois connection between the semimodules $\tilde{\mathcal{X}}^\gamma \cong (\overline{\mathcal{K}}^g)^\gamma$ and $\tilde{\mathcal{Y}}^\mu \cong (\overline{\mathcal{K}}^m)^\mu$ if and only if for $x \in X$, $y \in Y$, we have $y \leq x_{R,\varphi}^\uparrow \iff x \leq y_{R,\varphi}^\downarrow$.

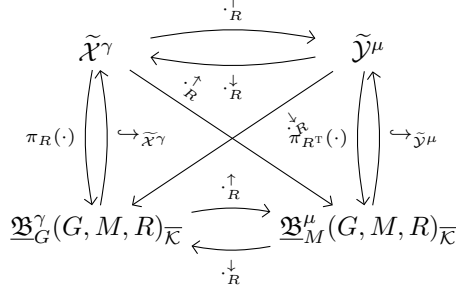


Figure 2: $(\cdot \uparrow_R, \cdot \downarrow_R) : \tilde{\mathcal{X}}^\gamma \times \tilde{\mathcal{Y}}^\mu$, the Galois connection between **scaled** spaces. Refer to the text for the notation.

Proof. We need only prove in one sense, since the other is similar. If $y \leq x \uparrow_R = R^T \dot{\otimes} x^{-1}$, then by inversion, $R^* \dot{\otimes} x \leq y^{-1}$ whence, by residuation $x \leq R^* \setminus y^{-1} = R \dot{\otimes} y^{-1} = y \downarrow_R$. \square

The diagram in Fig. 2 summarizes this Galois connection [6, 25]. Lemma 3.2 puts at our disposal a number of results that could be obtained as Corollaries from it and the theory of Galois Connections [25]. However, the purpose of this paper is to make explicit the advantages of idempotent semifields in developing a generalization of FCA. In this particular instance, we highlight the connection between their order and algebraic properties and for this reason we insist on algebraic manipulation in some proofs, like that of the following Proposition. :

Proposition 3.3. *Consider the Galois connection $(\cdot \uparrow_R, \cdot \downarrow_R) : \tilde{\mathcal{X}}^\gamma \times \tilde{\mathcal{Y}}^\mu$. Then:*

1. *The polars are antitone, join-inverting functions:*

$$(x_1 \dot{\oplus} x_2) \uparrow_R = x_1 \uparrow_R \dot{\oplus} x_2 \uparrow_R \quad (y_1 \dot{\oplus} y_2) \downarrow_R = y_1 \downarrow_R \dot{\oplus} y_2 \downarrow_R. \quad (15)$$

2. *The compositions of the polars: $\pi_R(\cdot) : \tilde{\mathcal{X}}^\gamma \rightarrow \tilde{\mathcal{X}}^\gamma$, $\pi_{R^T}(\cdot) : \tilde{\mathcal{Y}}^\mu \rightarrow \tilde{\mathcal{Y}}^\mu$ are closures, that is, extensive and idempotent operators,*

$$\begin{aligned} \pi_R(x) &\geq x & \pi_{R^T}(y) &\geq y \\ \pi_R(\pi_R(x)) &= \pi_R(x) & \pi_{R^T}(\pi_{R^T}(y)) &= \pi_{R^T}(y) \end{aligned}$$

with algebraically closed expressions

$$\pi_R(x) = (x \uparrow_R) \downarrow_R = R \dot{\otimes} (R^* \dot{\otimes} x) \quad \pi_{R^T}(y) = (y \downarrow_R) \uparrow_R = R^T \dot{\otimes} (R^{-1} \dot{\otimes} y). \quad (16)$$

3. *The polars are mutual pseudo-inverses:*

$$(\cdot) \uparrow_R \circ (\cdot) \downarrow_R \circ (\cdot) \uparrow_R = (\cdot) \uparrow_R \quad (\cdot) \downarrow_R \circ (\cdot) \uparrow_R \circ (\cdot) \downarrow_R = (\cdot) \downarrow_R$$

Proof. We only prove for left vectors, since the proofs for right vectors are row-column dual. Note that the techniques in this proof will be used in many places below. First:

$$(x_1 \oplus x_2)_R^\uparrow = R^\top \dot{\otimes} (x_1 \oplus x_2)^{-1} = R^\top \dot{\otimes} x_1^{-1} \dot{\oplus} R^\top \dot{\otimes} x_2^{-1} = x_{1R}^\uparrow \dot{\oplus} x_{2R}^\uparrow$$

From Proposition 2.2.4 we find that $\pi_R(x) = R \dot{\otimes} (R^* \otimes x) \geq x$, that is $\pi_R(\cdot)$ is extensive. From Proposition 2.2.3 we know that $R \dot{\otimes} (R^* \otimes (R \dot{\otimes} M)) = R \dot{\otimes} M$, whence

$$\pi_R(\pi_R(x)) = R \dot{\otimes} (R^* \otimes (R \dot{\otimes} (R^* \otimes x))) = R \dot{\otimes} (R^* \otimes x) = \pi_R(x).$$

Finally,

$$((y_R^\downarrow)_R^\uparrow)^\downarrow = R \dot{\otimes} (R^\top \dot{\otimes} (R \dot{\otimes} y^{-1})^{-1})^{-1} = R \dot{\otimes} (R^* \otimes [R \dot{\otimes} y^{-1}]) = R \dot{\otimes} y^{-1} = y_R^\downarrow$$

where the reduction step also comes from Proposition 2.2.3. \square

Consider the effect of applying the polars to the whole of the ambient spaces \tilde{X}^γ and \tilde{Y}^μ . In this way we define two sets, the *system of extents* $\mathfrak{B}_G^\gamma(G, M, R)$ and the *system of intents* $\mathfrak{B}_M^\mu(G, M, R)$ of the Galois connection.

$$\mathfrak{B}_G^\gamma(G, M, R) = (\tilde{Y}^\mu)_R^\downarrow \quad \mathfrak{B}_M^\mu(G, M, R) = (\tilde{X}^\gamma)_R^\uparrow \quad (17)$$

For this reason, we call $\cdot_R^\uparrow : \tilde{X}^\gamma \rightarrow \mathfrak{B}_M^\mu(G, M, R)$ the *polar of (or generating) intents* and $\cdot_R^\downarrow : \tilde{Y}^\mu \rightarrow \mathfrak{B}_G^\gamma(G, M, R)$ the *polar of (or generating) extents*. The rationale for these names will be made evident below.

One of the advantages of working in idempotent semimodules is that we can strengthen statement 1 in Proposition 3.3 to reveal that the polars are idempotent semimodule morphisms:

Proposition 3.4. *The polar of intents of the Galois connection transforms a $\bar{\mathcal{K}}$ -semimodule of object vectors into a $\bar{\mathcal{K}}^d$ -semimodule of intents, and dually for the polar of the extents that transforms a $\bar{\mathcal{K}}$ -semimodule of attribute vectors into a $\bar{\mathcal{K}}^d$ -semimodule of extents.*

Proof. For linearity, consider $x_{1R}^\uparrow = R^\top \dot{\otimes} x_1^{-1}$ and $x_{2R}^\uparrow = R^\top \dot{\otimes} x_2^{-1}$.

$$\begin{aligned} (\lambda_1 \otimes x_1 \oplus \lambda_2 \otimes x_2)_R^\uparrow &= R^\top \dot{\otimes} (\lambda_1 \otimes x_1 \oplus \lambda_2 \otimes x_2)^{-1} = \\ &= R^\top \dot{\otimes} (\lambda_1^{-1} \dot{\otimes} x_1^{-1} \dot{\oplus} \lambda_2^{-1} \dot{\otimes} x_2^{-1}) = \\ &= (\lambda_1^{-1} \dot{\otimes} R^\top \dot{\otimes} x_1^{-1}) \dot{\oplus} (\lambda_2^{-1} \dot{\otimes} R^\top \dot{\otimes} x_2^{-1}) = \\ &= (\lambda_1^{-1} \dot{\otimes} x_{1R}^\uparrow) \dot{\oplus} (\lambda_2^{-1} \dot{\otimes} x_{2R}^\uparrow) \end{aligned}$$

and recalling the definition of the operations of the dual, we may write:

$$(\lambda_1 \dot{\otimes} x_1 \dot{\oplus} \lambda_2 \dot{\otimes} x_2)_R^\dagger = \lambda_1 \dot{\otimes}^d x_1 \dot{\oplus}^d \lambda_2 \dot{\otimes}^d x_2.$$

For the polar of extents the proof is similar. \square

Note that this is the \mathcal{K} -FCA analogue of the fact that the polars are join-inverting. But the novelty is that the scalings for one semimodule and the other are inverted, hence they are not exactly homomorphisms of semimodules. This theme will recur in our results: how to enrich the Galois connection in the setting of idempotent semimodules. The following corollary is immediate.

Corollary 3.5. *The systems of extents and intents are $\overline{\mathcal{K}}^d$ -semimodules.*

3.2. The Semimodules of Closed Elements and Formal Concepts

We next explain the images of the closure operators. For that purpose, call the sets of fixpoints of the closure operators

$$\text{fix}(\pi_R(\cdot)) = \{a \in \tilde{X}^\gamma \mid \pi_R(a) = a\} \quad \text{fix}(\pi_{R^t}(\cdot)) = \{b \in \tilde{Y}^\mu \mid \pi_{R^t}(b) = b\}.$$

These are also called *closed elements* of the Galois connection that generates $\pi_R(\cdot)$ and $\pi_{R^t}(\cdot)$.

Again, the advantages of working in complete idempotent semifields make themselves evident in the following proposition relating the fixpoints with structures in the ambient spaces.

Proposition 3.6. *$\text{fix}(\pi_R(\cdot))$ and $\text{fix}(\pi_{R^t}(\cdot))$ are complete $\overline{\mathcal{K}}^d$ -subsemimodules of \tilde{X}^γ and \tilde{Y}^μ respectively.*

Proof. Consider a generic vector $z \in \tilde{Y}^\mu$, then by Lemma 2.2.2 we have

$$\text{fix}(\pi_R(R \dot{\otimes} z)) = R \dot{\otimes} (R^* \dot{\otimes} (R \dot{\otimes} z)) = R \dot{\otimes} z.$$

This means that any $\overline{\mathcal{K}}^d$ -combination of columns of R is a fixpoint of $\pi_R(\cdot)$, that is $\langle R \rangle_{\overline{\mathcal{K}}^d} \subseteq \text{fix}(\pi_R(\cdot))$. Now, consider $a \in \text{fix}(\pi_R(\cdot))$. Then $R \dot{\otimes} (R^* \dot{\otimes} a) = a$ whence $R \dot{\otimes} z = a$, so $\text{fix}(\pi_R(\cdot)) \subseteq \langle R \rangle_{\overline{\mathcal{K}}^d}$. Dually $\text{fix}(\pi_{R^t}(\cdot)) = \langle R^t \rangle_{\overline{\mathcal{K}}^d}$. \square

The next corollary is not difficult to prove algebraically, but follows if follows directly from Lemma 3.2. In fact, these are not new elements.

Corollary 3.7. *The sets of extents and intents are the sets of fixpoints of the closure operators of the Galois connection.*

$$\mathfrak{B}_G^\gamma(G, M, R) = \text{fix}(\pi_R(\cdot)) \quad \mathfrak{B}_M^\mu(G, M, R) = \text{fix}(\pi_{R^t}(\cdot)) .$$

Due to this and from now on, when we assert the use of the row-column duality and the duality of the polars, as above, we will invoke the *duality of the Galois connection (between row and column semimodules)* as "GC-duality".

The previous Lemma and Proposition 3.6 yield the following Corollary.

Corollary 3.8. *The system of extents and intents of the Galois connection are $\overline{\mathcal{K}}^d$ -subsemimodules of \tilde{X}^γ and \tilde{Y}^μ , generated by the columns of R and R^t , respectively.*

$$\mathfrak{B}_G^\gamma(G, M, R) = \langle R \rangle_{\overline{\mathcal{K}}^d} \quad \mathfrak{B}_M^\mu(G, M, R) = \langle R^t \rangle_{\overline{\mathcal{K}}^d}$$

Finally, remark how in the proof of Lemma 3.7 a is the extent due to the intent b and vice-versa in the previous demonstration, also written $b = a_R^\uparrow \iff a = b_R^\downarrow$. This is the lower part of the diagram in Figure 2, otherwise said:

Corollary 3.9. \cdot_R^\downarrow and \cdot_R^\uparrow are mutual inverses on $\mathfrak{B}_G^\gamma(G, M, R)$ and $\mathfrak{B}_M^\mu(G, M, R)$.

Perhaps the most important affordance of FCA is the following concept motivated by the previous remark⁵.

Definition 4. We call the pairs (a, b) such that $a_R^\uparrow = b$ and $b_R^\downarrow = a$ (γ, μ) -formal concepts (of (G, M, R)) and call their set $\mathfrak{B}^{\gamma \otimes \mu}(G, M, R)$.

For $(a, b) \in \mathfrak{B}^{\gamma \otimes \mu}(G, M, R)$ we say a is its *extent* and b its *intent* of (a, b) and $\mathfrak{B}_G^\gamma(G, M, R)$, respectively $\mathfrak{B}_M^\mu(G, M, R)$ are the *systems of extents and intents* of the Galois connection. \square

3.3. The Lattices of Extents and Intents

In the previous Section we evidenced a bijection between the systems of extents and intents, but in fact, the Galois Connection theorem implies that the sets of extents and intents are dually isomorphic as orders. How is this expressed in our framework?

Due to Lemma 3.2, we already know that they are complete lattices. But these lattices of extents $\mathfrak{B}_G^\gamma(G, M, R)$ and intents $\mathfrak{B}_M^\mu(G, M, R)$ are only complete meet-subsemilattices of their ambient spaces $\tilde{\mathcal{X}}^\gamma$ and $\tilde{\mathcal{Y}}^\mu$ but not their join-subsemilattices. Luckily, Proposition 3.11 will allow us to characterize their carrying sets of extents and intents as \mathcal{K} -semimodules, that is, as complete join-semilattices, too. First consider the following structures on the set of extents and intents:

$$\mathcal{B}_G = \langle \mathfrak{B}_G^\gamma(G, M, R), \tilde{\oplus}, \tilde{\odot}, \epsilon_G \rangle \quad \mathcal{B}_M = \langle \mathfrak{B}_M^\mu(G, M, R), \tilde{\oplus}, \tilde{\odot}, \epsilon_M \rangle$$

with the two additions:

$$a_1 \tilde{\oplus} a_2 = \pi_R(a_1 \oplus a_2) \quad b_1 \tilde{\oplus} b_2 = \pi_{R^t}(b_1 \oplus b_2) \quad (18)$$

⁵This use of the word “concept” is at the meta-level. The following definition of “(formal) concept” is at the object level. Confusing them leads to a belief that FCA is about representing (cognitive) concepts. This is not so. Formal concepts may be used to model cognitive concepts, but their application is much wider. Of course, the meta-concept of object-(formal) concept can be considered a cognitive concept in present day (cognitive) Concept Theory. So when using formal concepts for conceptual modeling, we suggest strictly using “concept” for the meta level and “formal concept” for the object level. And this is our criterion in this paper too!

two right traslations:

$$(a, \lambda) \mapsto \lambda \overset{\cdot}{\otimes} a = \pi_R(\lambda \otimes a) \quad (b, \mu) \mapsto \mu \overset{\cdot}{\otimes} b = \pi_{R^T}(\mu \otimes b) \quad (19)$$

and bottom elements:

$$\epsilon_G = R \overset{\cdot}{\otimes} \perp^M \quad \epsilon_M = R^T \overset{\cdot}{\otimes} \perp^G \quad (20)$$

We next prove that:

Proposition 3.10. *Let $\lambda, \mu \in K$, a_1 and a_2 be extents of $\mathfrak{B}_G^\gamma(G, M, R)$ and b_1 and b_2 be intents of $\mathfrak{B}_M^\mu(G, M, R)$. Then*

$$\lambda \overset{\cdot}{\otimes} a_1 \overset{\cdot}{\oplus} \mu \overset{\cdot}{\otimes} a_2 = \pi_R(\lambda \otimes a_1 \oplus \mu \otimes a_2) \quad \lambda \overset{\cdot}{\otimes} b_1 \overset{\cdot}{\oplus} \mu \overset{\cdot}{\otimes} b_2 = \pi_{R^T}(\lambda \otimes b_1 \oplus \mu \otimes b_2) \quad (21)$$

Proof. Call $a = \lambda \overset{\cdot}{\otimes} a_1 \overset{\cdot}{\oplus} \mu \overset{\cdot}{\otimes} a_2$, then:

$$\begin{aligned} a &= \left(R \overset{\cdot}{\otimes} (R^* \otimes (\lambda \otimes a_1)) \right) \overset{\cdot}{\oplus} \left(R \overset{\cdot}{\otimes} (R^* \otimes (\mu \otimes a_2)) \right) \\ &= R \overset{\cdot}{\otimes} (R^* \otimes [(R \overset{\cdot}{\otimes} (R^* \otimes (\lambda \otimes a_1))) \overset{\cdot}{\oplus} (R \overset{\cdot}{\otimes} (R^* \otimes (\mu \otimes a_2)))] \end{aligned}$$

Distributing R^* over $\overset{\cdot}{\oplus}$ and applying the matrix equalities:

$$\begin{aligned} &= R \overset{\cdot}{\otimes} [R^* \otimes (R \overset{\cdot}{\otimes} [R^* \otimes (\lambda \otimes a_1)]) \overset{\cdot}{\oplus} R^* \otimes (R \overset{\cdot}{\otimes} [R^* \otimes (\mu \otimes a_2)])] \\ &= R \overset{\cdot}{\otimes} [(R^* \otimes (\lambda \otimes a_1)) \overset{\cdot}{\oplus} (R^* \otimes (\mu \otimes a_2))] \end{aligned}$$

Using the distributivity and applying the definition of the closure we get:

$$= R \overset{\cdot}{\otimes} (R^* \otimes (\lambda \otimes a_1 \oplus \mu \otimes a_2)) = \pi_R(\lambda \otimes a_1 \oplus \mu \otimes a_2)$$

The proof for intents is GC-dual. \square

We are now ready to prove the following proposition:

Proposition 3.11. \mathcal{B}_G and \mathcal{B}_M are right complete idempotent \overline{K} -semimodules.

Proof. Consider first extents, and the neutral element as defined. Addition is clearly commutative and idempotent from the definition $a \overset{\cdot}{\oplus} a = \pi_R(a \oplus a) = \pi_R(a) = a$. Associativity follows a pattern exploited in the rest of the proofs:

$$\begin{aligned} a_1 \overset{\cdot}{\oplus} (a_2 \overset{\cdot}{\oplus} a_3) &= R \overset{\cdot}{\otimes} (R^* \otimes [a_1 \overset{\cdot}{\oplus} (R \overset{\cdot}{\otimes} [R^* \otimes (a_2 \overset{\cdot}{\oplus} a_3)])]) \\ &= R \overset{\cdot}{\otimes} (R^* \otimes a_1 \overset{\cdot}{\oplus} R^* \otimes [R \overset{\cdot}{\otimes} (R^* \otimes [a_2 \overset{\cdot}{\oplus} a_3])]) \\ &= R \overset{\cdot}{\otimes} (R^* \otimes a_1 \overset{\cdot}{\oplus} R^* \otimes [a_2 \overset{\cdot}{\oplus} a_3]) = R \overset{\cdot}{\otimes} (R^* \otimes [a_1 \overset{\cdot}{\oplus} a_2 \overset{\cdot}{\oplus} a_3]) \\ &= \pi_R(a_1 \overset{\cdot}{\oplus} a_2 \overset{\cdot}{\oplus} a_3) \end{aligned}$$

and the result follows by the associativity of \oplus , and the commutativity of $\tilde{\oplus}$ and \oplus . The additive identity, that is, the bottom element, is correctly-defined:

$$\begin{aligned}\epsilon_G \tilde{\oplus} a &= R \dot{\otimes} (R^* \dot{\otimes} [\epsilon_G \oplus a]) = R \dot{\otimes} (R^* \dot{\otimes} [R \dot{\otimes} \perp^M \oplus a]) \\ &= R \dot{\otimes} ([R^* \dot{\otimes} (R \dot{\otimes} \perp^M)] \oplus [R^* \dot{\otimes} a]) = R \dot{\otimes} (\perp^M \oplus [R^* \dot{\otimes} a]) \\ &= R \dot{\otimes} (R^* \dot{\otimes} a) = a\end{aligned}$$

Where we have used that $R^* \dot{\otimes} (R \dot{\otimes} x) \leq x$ is a kernel operator, that is a contractive, idempotent function, whence $R^* \dot{\otimes} (R \dot{\otimes} \perp^M) = \perp^M$.

Only the external laws are left to be proven: first $\perp_{\mathcal{K}} \tilde{\odot} a = \pi_R(\perp_{\mathcal{K}} \otimes a) = \pi_R(\perp^G) = R \dot{\otimes} (R^* \dot{\otimes} \perp^G) = R \dot{\otimes} \perp^M = \epsilon_G$ —indeed this might be taken for its definition. Next:

$$\lambda \tilde{\odot} \epsilon_G = R \dot{\otimes} (R^* \dot{\otimes} (\lambda \otimes R \dot{\otimes} \perp^M)) = R \dot{\otimes} (\lambda \otimes (R^* \dot{\otimes} (R \dot{\otimes} \perp^M))).$$

whence $\lambda \tilde{\odot} \epsilon_G = R \dot{\otimes} (\lambda \otimes \perp^M) = R \dot{\otimes} \perp^M = \epsilon_G$. From the definition of the scalar action $e_{\mathcal{K}} \tilde{\odot} a = \pi_R(e_{\mathcal{K}} \odot a) = \pi_R(a) = a$. From (21) the rest of the laws follow by simple instantiation. The proof is GC-dual for intents. \square

Thus each $\mathfrak{B}_G^\gamma(G, M, R)$ and $\mathfrak{B}_M^\mu(G, M, R)$ carries a double semimodule structure:

1. A \mathcal{K}^d -subsemimodule of their ambient spaces, as in Corollary 3.8, whereby we call them the lattices of extents $\underline{\mathfrak{B}}_G^\gamma(G, M, R)$ and intents $\underline{\mathfrak{B}}_M^\mu(G, M, R)$.
2. A \mathcal{K} -semimodule as in Proposition 3.11 baptized as \mathcal{B}_G and \mathcal{B}_M . The semimodule structure is idiosyncratic in that it is defined with the help of the closure operators of each particular formal context.

Although the polars are bijective in the sets of extents and intents we do not have yet a full characterization in terms of $\overline{\mathcal{K}}$ -semimodules. But:

Proposition 3.12. *The polars are dual isomorphisms from the \mathcal{K} - to \mathcal{K}^d -semimodules.*

$$(\mathcal{B}_G)_R^\uparrow = \underline{\mathfrak{B}}_M^\mu(G, M, R) \quad (\mathcal{B}_M)_R^\downarrow = \underline{\mathfrak{B}}_G^\gamma(G, M, R)$$

Proof. Recall that we know that the carrier sets of \mathcal{B}_G and \mathcal{B}_M , the systems of extents and intents are in a bijection through the polars, which act as mutual inverses. What we care now is for their semimodule structures.

Let $\lambda, \mu \in K$, a_1 and a_2 be extents with b_1 and b_2 their intents. Then:

$$\begin{aligned}(\lambda \tilde{\otimes} a_1 \tilde{\oplus} \mu \tilde{\otimes} a_2)_R^\uparrow &= (\lambda \otimes a_1 \oplus \mu \otimes a_2)_{RRR}^{\uparrow \downarrow \uparrow} = (\lambda \otimes a_1 \oplus \mu \otimes a_2)_R^\uparrow = \\ &= \lambda^{-1} \dot{\otimes} b_1 \dot{\oplus} \mu^{-1} \dot{\otimes} b_2 = \lambda \otimes^d b_1 \oplus^d \mu \otimes^d b_2\end{aligned}$$

The action of the polar of extents is:

$$\begin{aligned}
(\lambda \overset{\circ}{\otimes} b_1 \overset{\circ}{\oplus} \mu \overset{\circ}{\otimes} b_2) \overset{\circ}{\downarrow}_R &= (\lambda^{-1} \overset{\circ}{\otimes} b_1 \overset{\circ}{\oplus} \mu^{-1} \overset{\circ}{\otimes} b_2) \overset{\circ}{\downarrow}_R = R \overset{\circ}{\otimes} (\lambda \overset{\circ}{\otimes} b_1^{-1} \overset{\circ}{\oplus} \mu \overset{\circ}{\otimes} b_2^{-1}) \\
&= R \overset{\circ}{\otimes} (\lambda \overset{\circ}{\otimes} R^* \overset{\circ}{\otimes} a_1 \overset{\circ}{\oplus} \mu \overset{\circ}{\otimes} R^* \overset{\circ}{\otimes} a_2) \\
&= R \overset{\circ}{\otimes} (R^* \overset{\circ}{\otimes} [\lambda \overset{\circ}{\otimes} a_1 \overset{\circ}{\oplus} \mu \overset{\circ}{\otimes} a_2]) = \lambda \overset{\circ}{\otimes} a_1 \overset{\circ}{\oplus} \mu \overset{\circ}{\otimes} a_2
\end{aligned}$$

That is, the polars are dual order embeddings in each direction, and GC-dually for intents. Taken together, these results mean that both their semimodule structures are in dual isomorphisms. \square

Then we have proven the systems of extents and intents as sets carrying double \mathcal{K} - and \mathcal{K}^d -semimodule structures.

Corollary 3.13. *The system of extents and intents are dually isomorphic double complete semimodules,*

$$\mathcal{B}_G = (\mathcal{B}_M) \overset{\circ}{\downarrow}_R \qquad \mathcal{B}_M = (\mathcal{B}_G) \overset{\circ}{\uparrow}_R$$

and, a fortiori, dually isomorphic complete lattices following the algebra pattern $\langle L, \wedge, \vee, \perp, \top \rangle$.

$$\begin{aligned}
\mathfrak{B}_G^\gamma(G, M, R) &= \langle \mathfrak{B}_G^\gamma(G, M, R), \overset{\circ}{\oplus}, \overset{\circ}{\otimes}, R \overset{\circ}{\otimes} \perp^M, \top^G \rangle \\
\mathfrak{B}_M^\mu(G, M, R) &= \langle \mathfrak{B}_M^\mu(G, M, R), \overset{\circ}{\oplus}, \overset{\circ}{\otimes}, R^\top \overset{\circ}{\otimes} \perp^G, \top^M \rangle
\end{aligned}$$

3.4. The Semimodule of Formal Concepts

Proposition 3.6 says that the sets of extents and intents are already idempotent semimodules, that is the idempotent analogue of a vector space. Could we endow the set of concepts with a similar structure? The following definition and name make sense due to Corollary 3.5.

Definition 5 (*The double semimodule of formal concepts*). Define upper and lower addition of formal concepts:

$$(a_1, b_1) \overset{\circ}{\oplus} (a_2, b_2) = (a_1 \overset{\circ}{\oplus} a_2, b_1 \overset{\circ}{\oplus} b_2) \quad (a_1, b_1) \overset{\circ}{\otimes} (a_2, b_2) = (a_1 \overset{\circ}{\otimes} a_2, b_1 \overset{\circ}{\otimes} b_2) \quad (22)$$

which we call *generalization* and *specialization*. This translates into arbitrary joins in complete idempotent semifields, where instead of $\overset{\circ}{\oplus}$ we write $\overset{\circ}{\sum}$ and instead of $\overset{\circ}{\oplus}^d$ we write $\overset{\circ}{\sum}^d = \overset{\circ}{\sum}$.

Next, we consider endowing the set of concepts with a right scalar action and its “dual”:

$$\lambda \overset{\circ}{\otimes} (a, b) = (\lambda \overset{\circ}{\otimes} a, \lambda \overset{\circ}{\otimes} b) \quad \lambda \overset{\circ}{\oplus} (a, b) = (\lambda \overset{\circ}{\oplus} a, \lambda \overset{\circ}{\oplus} b) \quad (23)$$

When $\lambda > e$ we call \odot (*formal-concept*) *scalar abstraction* and $\dot{\odot}$ (*formal-concept*) *scalar instantiation*.

We call this structure the *double semimodule of formal concepts*

$$\langle \underline{\mathfrak{B}}^{\gamma \dot{\otimes} \mu}(G, M, R), \odot, \dot{\odot}, \oplus, \dot{\oplus} \rangle.$$

These definitions provide the basis for the following extended theorem of \mathcal{K} -FCA:

Theorem 3.14. *The double (γ, μ) -concept semimodule $\underline{\mathfrak{B}}^{\gamma \dot{\otimes} \mu}(G, M, R)$ is a dually isomorphic pair of double complete idempotent semimodules in which infimum and supremum combinations are given by:*

$$\sum_{i \in I} \lambda_i \otimes (a_i, b_i) = \left(\sum_{i \in I} \tilde{\lambda}_i \tilde{\otimes} a_i, \sum_{i \in I} \lambda_i \otimes^d b_i \right) = \left(\pi_R \left(\sum_{i \in I} \lambda_i \odot a_i \right), \sum_{i \in I} \lambda_i^{-1} \dot{\odot} b_i \right) \quad (24)$$

$$\sum_{i \in I} \lambda_i \dot{\otimes} (a_i, b_i) = \left(\sum_{i \in I} \lambda_i \otimes^d a_i, \sum_{i \in I} \tilde{\lambda}_i \tilde{\otimes} b_i \right) = \left(\sum_{i \in I} \lambda_i^{-1} \dot{\odot} a_i, \pi_{R^T} \left(\sum_{i \in I} \lambda_i \odot b_i \right) \right) \quad (25)$$

Proof. This is a corollary of Proposition 3.12 and Corollary 3.13 written in the language of the previous definitions. \square

Note that we do not use special notation for the meets $\dot{\oplus}$ and joins \oplus of concepts, as we did for the component lattices, and that the closure operations are hidden in the definitions of the new joins.

3.5. Join-dense and meet-dense vectors

Standard concept lattices have “natural” building algorithms in terms of the object-intents and attribute-extents. We have just seen that \mathcal{K} -concept lattices are generated in terms of the dual semifield very straightforwardly, and in terms of the original semifield in a more convoluted way. We next present a way to reconcile both views.

First, we define concept-building operators from sets of objects and attributes respectively:

$$\begin{aligned} \gamma : \tilde{X}^\gamma &\rightarrow \underline{\mathfrak{B}}^{\gamma \dot{\otimes} \mu}(G, M, R) & \mu : \tilde{Y}^\mu &\rightarrow \underline{\mathfrak{B}}^{\gamma \dot{\otimes} \mu}(G, M, R) \\ x \mapsto \gamma(x) &= (\pi_R(x), x_R^\uparrow) & y \mapsto \mu(y) &= (y_R^\downarrow, \pi_{R^T}(y)) \end{aligned}$$

Next, let I_g and I_m be the identity matrices of dimension $g \times g$ and $m \times m$ in $\overline{\mathcal{K}}$, whose columns are naturally conceived as the unitary vectors of objects and attributes, respectively. By simple application of the polars to the identities we have the following lemma.

Lemma 3.15. For the Galois connection $(\cdot \uparrow_R, \cdot \downarrow_R) : \tilde{\mathcal{X}}^\gamma \bowtie \tilde{\mathcal{Y}}^\mu$, the object- and attribute-concepts, taken as pairs of matrices of co-indexed vectors, are:

$$\gamma_R(I_g) = (R \dot{\otimes} R^*, R^\top) \quad \mu_R(I_m) = (R, R^\top \dot{\otimes} R^{-1})$$

Note how the matrix notation allows us to carry out multiple computations at the same time. We may now conclude the following:

Corollary 3.16. For the Galois connection $(\cdot \uparrow_R, \cdot \downarrow_R) : \tilde{\mathcal{X}}^\gamma \bowtie \tilde{\mathcal{Y}}^\mu$, its system of extents is $\bar{\mathcal{K}}^d$ -generated by the attribute-extents. Dually, its system of intents is $\bar{\mathcal{K}}^d$ -generated by the object-intents.

Proof. From Corollary 3.8 and Lemma 3.15. □

This is a result that has a nice analogue with standard FCA where the $\gamma_R(I_g)$ are join-dense and the $\mu_R(I_m)$ are meet-dense. Furthermore, since these semimodules are complete and finitely-generated we can always find a subset of these $\dot{\oplus}$ -dense sets that acts as a basis [4]. The schematic diagram of Fig. 3 makes these mechanisms evident.

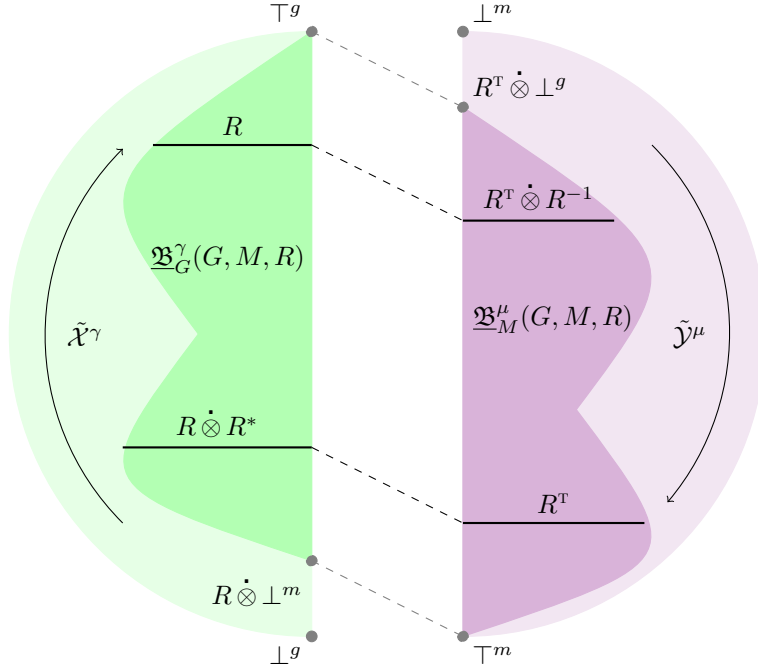


Figure 3: Extended schematics of the Galois connection between $\tilde{\mathcal{X}}^\gamma$ and $\tilde{\mathcal{Y}}^\mu$ (outer clouds). Extents $\mathfrak{B}_G^\gamma(G, M, R)$ (left inner cloud) and intents $\mathfrak{B}_M^\mu(G, M, R)$ (right inner cloud) are dually isomorphic \mathcal{K}^d -semimodules generated by R^\top and R respectively. Similarly, they are dually isomorphic \mathcal{K} -semimodules generated by closing $R \dot{\otimes} R^*$ and $R^\top \dot{\otimes} R^{-1}$ (see text).

Can we expect to find a similar mechanism for \oplus -dense sets, that is object-extents and attribute intents? The answer suggestively blends the spectral theory of matrices and \mathcal{K} -FCA. First, consider a property of the object-extents $P_G = R \dot{\otimes} R^*$ and attribute intents $P_M = R^T \dot{\otimes} R^{-1}$.

Proposition 3.17. *The object extents (respectively, attribute extents) are fundamental eigenvectors of P_G (respectively, P_M) for the eigenvalue e .*

Proof. Consider the power $(R \dot{\otimes} R^*) \otimes (R \dot{\otimes} R^*) = R \dot{\otimes} R^*$, where the equality comes from the matrix product laws. It is easy to see by induction that $(R \dot{\otimes} R^*)^n = R \dot{\otimes} R^*$. Furthermore, its diagonal only has the elements $\{e, \top\}$ wherefore $I_G \leq P_G$, so

$$P_G^{\oplus} = \sum_{n=0}^{\infty} P_G^n = I_G \oplus \sum_{n=1}^{\infty} P_G^n = P_M^{\oplus}$$

hence by Theorem 2.7 we know that the columns of P_G are all eigenvectors of P_G for e . And dually for P_M . \square

From this we obtain that:

Proposition 3.18. *The closures of the eigenspaces of P_G and P_M generate the system of extents and intents.*

$$\mathfrak{B}_G^\gamma(G, M, R) = \pi_R(\mathcal{V}_e(R \dot{\otimes} R^*)) \quad \mathfrak{B}_M^\mu(G, M, R) = \pi_{R^T}(\mathcal{V}_e(R^{-1} \dot{\otimes} R^T))$$

Proof. We prove it for extents: recall that the eigenspace of e generated by the columns of P_G is $\mathcal{V}_e(R \dot{\otimes} R^*) = \langle R \dot{\otimes} R^* \rangle_{\mathcal{K}}$. Furthermore, we know that the eigenspaces are \mathcal{K} -semimodules, that the polars transform \mathcal{K} -semimodules into \mathcal{K}^d -semimodules, and $(R \dot{\otimes} R^*)_R^\uparrow = R^T \dot{\otimes} (R^{-1} \dot{\otimes} R^T) = R^T$, hence

$$(\mathcal{V}_e(R \dot{\otimes} R^*))_R^\uparrow = (\langle R \dot{\otimes} R^* \rangle_{\mathcal{K}})_R^\uparrow = \langle (R \dot{\otimes} R^*)_R^\uparrow \rangle_{\mathcal{K}^d} = \langle R^T \rangle_{\mathcal{K}^d} = \mathfrak{B}_M^\mu(G, M, R).$$

We directly use the extent polar:

$$\mathfrak{B}_G^\gamma(G, M, R) = (\mathfrak{B}_M^\mu(G, M, R))_R^\downarrow = ((\mathcal{V}_e(R \dot{\otimes} R^*))_R^\uparrow)_R^\downarrow = \pi_R(\mathcal{V}_e(R \dot{\otimes} R^*)).$$

And GC-dually for extents. \square

In passing, we have proven the following corollary.

Corollary 3.19. *Both the set of extents and intents are generated from the object extents, and dually from the attribute intents.*

$$\begin{aligned} \mathfrak{B}_G^\gamma(G, M, R) &= \langle R \rangle_{\mathcal{K}^d} = \pi_R(\langle R \dot{\otimes} R^* \rangle_{\mathcal{K}}) \\ \mathfrak{B}_M^\mu(G, M, R) &= \langle R^T \rangle_{\mathcal{K}^d} = \pi_{R^T}(\langle R^T \dot{\otimes} R^{-1} \rangle_{\mathcal{K}}) \end{aligned}$$

3.6. Orthogonal congruences of concept lattices

In the setting of \mathcal{K} -FCA our first interest should be in the congruences induced by the polars, but we might also wonder about those induced by the closure operators.

Definition 6. Let $(\cdot \uparrow_R, \cdot \downarrow_R) : \mathcal{X} \bowtie \mathcal{Y}$ be a Galois connection. Then define the congruences induced by the polar of intents $\text{KER}(\cdot \uparrow_R)$ and extents $\text{KER}(\cdot \downarrow_R)$, respectively:

$$\begin{aligned} \text{KER}(\cdot \uparrow_R) &= \{(x_1, x_2) \in X^2 \mid x_1 \uparrow_R = x_2 \uparrow_R = b, \forall b \in \mathfrak{B}_M^\mu(G, M, R)\} \\ \text{KER}(\cdot \downarrow_R) &= \{(y_1, y_2) \in Y^2 \mid y_1 \downarrow_R = y_2 \downarrow_R = a, \forall a \in \mathfrak{B}_G^\gamma(G, M, R)\} \end{aligned}$$

and the congruence on \mathcal{X} induced by the closure of extents $\text{KER}(\pi_{R^\tau})$ and, dually, that induced by the closure of intents $\text{KER}(\pi_R)$:

$$\begin{aligned} \text{KER}(\pi_R(\cdot)) &= \{(x_1, x_2) \in X^2 \mid \pi_R(x_1) = \pi_R(x_2) = a, \forall a \in \mathfrak{B}_G^\gamma(G, M, R)\} \\ \text{KER}(\pi_{R^\tau}(\cdot)) &= \{(y_1, y_2) \in Y^2 \mid \pi_{R^\tau}(y_1) = \pi_{R^\tau}(y_2) = b, \forall b \in \mathfrak{B}_M^\mu(G, M, R)\} \quad \square \end{aligned}$$

But these lead to the same results, as the next result shows.

Lemma 3.20. *Let $(\cdot \uparrow_R, \cdot \downarrow_R) : \mathcal{X} \bowtie \mathcal{Y}$ be a Galois connection. Then the congruence induced by the polar of intents and the closure of extents are the same, and GC-dually for those of the polar of extent and closure of intents.*

$$\text{KER}(\cdot \uparrow_R) = \text{KER}(\pi_R(\cdot)) \quad \text{KER}(\cdot \downarrow_R) = \text{KER}(\pi_{R^\tau}(\cdot)) \quad (26)$$

Proof. If $(x_1, x_2) \in \text{KER}(\cdot \uparrow_R)$ then $x_1 \uparrow_R = x_2 \uparrow_R = b$ whence $(x_1 \uparrow_R) \downarrow_R = (x_2 \uparrow_R) \downarrow_R = a$, with $a = b \downarrow_R$ and $(x_1, x_2) \in \text{KER}(\pi_{R^\tau})$.

On the other hand, if $(x_1, x_2) \in \text{KER}(\pi_R(\cdot))$ then $(x_1 \uparrow_R) \downarrow_R = (x_2 \uparrow_R) \downarrow_R = a$ whence $(x_1 \uparrow_R) \downarrow_{RR} \uparrow = (x_2 \uparrow_R) \downarrow_{RR} \uparrow = a \uparrow_R$, and $x_1 \uparrow_R = x_2 \uparrow_R = b$, by the properties of the Galois connection and the fact that $a = b \downarrow_R \iff a \uparrow_R = b$. For intents the proof is GC-dual. \square

Note that the image of the polars (and the closures) have special status in our theory: they are the system of extents and intents (see § 3.2).

$$\text{IM}(\cdot \uparrow_R) = \mathfrak{B}_M^\mu(G, M, R) = \text{IM}(\pi_{R^\tau}(\cdot)) \quad \text{IM}(\cdot \downarrow_R) = \mathfrak{B}_G^\gamma(G, M, R) = \text{IM}(\pi_R(\cdot))$$

Of course, these are swapped with respect to the closures, as pointed out above. In keeping with standard practice in linear algebra, we will use the first pair of these definitions.

The adequate definition of orthogonals in our Galois connection setting is that of (9). We now turn to relating these two concepts:

Lemma 3.21. *Let $(\cdot \uparrow_R, \cdot \downarrow_R) : \mathcal{X} \bowtie \mathcal{Y}$ be a Galois connection. Then:*

$$\text{KER}(\cdot \uparrow_R)^\top = \mathfrak{B}_M^\mu(G, M, R) \quad \text{KER}(\cdot \downarrow_R)^\top = \mathfrak{B}_G^\gamma(G, M, R)$$

Proof. First, if $(x_1, x_2) \in \text{KER}(\cdot \uparrow_R)$, say $x_1 \uparrow_R = x_2 \uparrow_R = b$ for a certain $b \in \mathfrak{B}_M^\mu(G, M, R)$, therefore $\text{KER}(\cdot \uparrow_R)^\top \subseteq \mathfrak{B}_M^\mu(G, M, R)$.

Last, for $b \in \mathfrak{B}_M^\mu(G, M, R)$ then $\{x \in X \mid x \uparrow_R = b\}$ is clearly a class of $\text{KER}(\cdot \uparrow_R)$, whence $b \in \text{KER}(\cdot \uparrow_R)^\top$ by definition, therefore $\text{KER}(\cdot \uparrow_R)^\top \supseteq \mathfrak{B}_M^\mu(G, M, R)$. For the bikernel of extents the proof is GC-dual. \square

The natural definitions for the orthogonals of the systems of extents and intents are the following, adapted from (3).

$$\begin{aligned} \mathfrak{B}_M^\mu(G, M, R)^\perp &= \{(x_1, x_2) \in X^2 \mid x_1 \uparrow_R = x_2 \uparrow_R = b, \forall b \in \mathfrak{B}_M^\mu(G, M, R)\} \\ \mathfrak{B}_G^\gamma(G, M, R)^\perp &= \{(y_1, y_2) \in Y^2 \mid y_1 \downarrow_R = y_2 \downarrow_R = a, \forall a \in \mathfrak{B}_G^\gamma(G, M, R)\} \end{aligned}$$

Corollary 3.22. *Let $(\cdot \uparrow_R, \cdot \downarrow_R) : \mathcal{X} \times \mathcal{Y}$ be a Galois connection. Then:*

$$\mathfrak{B}_M^\mu(G, M, R)^\perp = \text{KER}(\cdot \uparrow_R) \quad \mathfrak{B}_G^\gamma(G, M, R)^\perp = \text{KER}(\cdot \downarrow_R)$$

Proof. By inspection of their definitions. \square

When the connection between \mathcal{X} and \mathcal{Y} is an adjunction, Cuninghame-Green has proven that the Chebychev distance between x and its closure $\pi_R(x)$ is minimal among all closures [2], and this is also the case for the kernel operator in the adjunction. For these reasons, Gaubert et al. have decided to call this projector the *orthogonal projection* with respect to the closure systems, and, as proven above, this is also the orthogonal with respect to the polars.

We now undertake to understand the structure of the classes in the bikernels.

Proposition 3.23. *Every class in the bikernel of intents $\text{KER}(\cdot \uparrow_R)$ intersects the system of extents at a single point, and GC-dually for the bikernel of extents and the system of intents.*

Proof. Let $x \uparrow_R = b$, whence $(x \uparrow_R)^\downarrow = b \downarrow_R = a$, where a is the extent univocally related to intent b by the concept lattice. Since $a \uparrow_R = b$, this means that $a \in [x]_{\text{KER}(\cdot \uparrow_R)}$ and since $a = b \downarrow_R$ then $a \in \mathfrak{B}_G^\gamma(G, M, R)$. However, if we suppose that there is another $a' \in [x]_{\text{KER}(\cdot \uparrow_R)} \cap \mathfrak{B}_G^\gamma(G, M, R)$, then we will have $a' \uparrow_R = b$ and therefore $(a' \uparrow_R)^\downarrow = b \downarrow_R = a$, so a' is not closed whence $a' \notin \mathfrak{B}_G^\gamma(G, M, R)$, a contradiction. Whence $a = b \downarrow_R$ is the only contact point of the class and the system of extents.

For the bikernel of extents and the system of intents the proof is GC-dual. \square

In fact for each class $C(b)$ mapping onto a $b \in \mathfrak{B}_M^\mu(G, M, R)$ as in Lemma 3.21, $a = b \downarrow_R$ as the closure of every element in the class has a special status:

Lemma 3.24. *For every class $C(b) \subseteq \text{KER}(\cdot \uparrow_R)$ mapping to $b \in \mathfrak{B}_M^\mu(G, M, R)$, the extent of b , $a = b \downarrow_R$ is the greatest element in the class,*

$$\pi_R(x) = \hat{x} = a, \forall x \in C(b)$$

and dually for the bikernel of extents and the classes mapping to a single extent $C(a) \subseteq \text{KER}(\cdot \downarrow_R)$ with $b = a \uparrow_R$

$$\pi_{R^v}(y) = \hat{y} = b, \forall y \in C(a)$$

Proof. This follows because $a \in C(b)$ as proven in the previous lemma, $a \geq x, \forall x \in C(b)$ given that a is the closure of any of the elements in $C(b)$ and hence $a = \vee C(b)$. The other assertion is GC-dual. \square

In fact, by means of the formal concept (a, b) such that $a = b \downarrow_R \iff b = a \uparrow_R$ both classes $C(b) \in \text{KER}(\cdot \uparrow_R)$ and $C(a) \in \text{KER}(\cdot \downarrow_R)$ are biunivocally related. The two previous results suggest, when possible, to name the classes of equivalence after their top-element, equivalently, their intersection with the systems of extents and intents, e.g. $C(b) = [a = b \downarrow_R]_{\text{KER}(\cdot \uparrow_R)}$. A final fact for the classes of equivalence follows.

Lemma 3.25. *Let $(\cdot \uparrow_R, \cdot \downarrow_R) : \mathcal{X} \times \mathcal{Y}$ be a Galois connection. Then the equivalence classes of $\text{KER}(\cdot \uparrow_R)$ and $\text{KER}(\cdot \downarrow_R)$ are join-subsemilattices of their ambient spaces.*

Proof. Let $x_2, x_1 \in [a]_{\text{KER}(\cdot \uparrow_R)}$ with $a \uparrow_R = b \iff b \downarrow_R = a$. Recall that \oplus is the join in \mathcal{X} and $\cdot \uparrow_R$ transforms joins into meets, whence: $(x_1 \vee x_2) \uparrow_R = (x_1 \oplus x_2) \uparrow_R = x_1 \uparrow_R \dot{\oplus} x_2 \uparrow_R = b \dot{\oplus} b = b$ so that $x_1 \vee x_2 \in [a]_{\text{KER}(\cdot \uparrow_R)}$. And GC-dually. \square

3.7. The case of non-finite φ

We now retrace our original decision in Section 3.1 to carry out the scaling when $\varphi \in \bar{K} \setminus \{\perp, \top\}$ and consider the corner cases $\varphi \in \{\perp, \top\}$.

First, notice that for $\varphi = \perp, \forall x \in X, \forall y \in Y$, the polars of (8) appear as:

$$x \uparrow_{R, \perp} = R^{\top} \dot{\otimes} x^{-1} \dot{\otimes} \top = \top^m \quad y \downarrow_{R, \perp} = R \dot{\otimes} y^{-1} \dot{\otimes} \top = \top^g$$

This situation is the simplest, since $(\top^g) \uparrow_{R, \perp} = \top^m \iff (\top^m) \downarrow_{R, \perp} = \top^g$.

Corollary 3.26. *For $R \in \mathcal{M}_{g \times m}(\bar{K})$, and the context (G, M, R) , the polars $(\cdot \uparrow_{R, \perp}, \cdot \downarrow_{R, \perp}) : \mathcal{X} \times \mathcal{Y}$ establish a Galois connection between*

$$\mathfrak{B}_G^{\perp}(G, M, R)_{\bar{K}} = \{\top^g\} \quad \text{and} \quad \mathfrak{B}_M^{\perp}(G, M, R)_{\bar{K}} = \{\top^m\}$$

so that

$$\underline{\mathfrak{B}}^{\perp}(G, M, R)_{\bar{K}} = \{(\top^g, \top^m)\} \cong \mathbb{1}.$$

and the whole spaces are the equivalence classes of the elements in the lattices:

$$\text{KER}(\cdot \uparrow_{R, \perp}) = \{\mathcal{X}\} \quad \text{KER}(\cdot \downarrow_{R, \perp}) = \{\mathcal{Y}\}$$

The conclusion is that this choice of $\varphi = \perp$ essentially conflates all the information in the context and is the least informative possible.

On the other hand, for $\varphi = \top$ the polars of (8) appear as:

$$x_{R,\top}^\uparrow = R^\top \dot{\otimes} x^{-1} \dot{\otimes} \perp \qquad y_{R,\top}^\downarrow = R \dot{\otimes} y^{-1} \dot{\otimes} \perp$$

Proposition 3.27. *For $R \neq \top^{g \times m} \in \mathcal{M}_{g \times m}(\overline{\mathcal{K}})$, and the context (G, M, R) , the polars $(\cdot_{R,\top}^\uparrow, \cdot_{R,\top}^\downarrow) : \mathcal{X} \overleftrightarrow{\times} \mathcal{Y}$ establish a Galois connection between $\mathfrak{B}_G^\top(G, M, R)_{\overline{\mathcal{K}}} = \{\top^g, R \dot{\otimes} \top^m\}$ and $\mathfrak{B}_M^\top(G, M, R)_{\overline{\mathcal{K}}} = \{\top^m, R^\top \dot{\otimes} \top^g\}$ so that*

$$\mathfrak{B}^\top(G, M, R)_{\overline{\mathcal{K}}} = \{(R^\top \dot{\otimes} \perp^m, \top^m), (\top^g, R \dot{\otimes} \perp^g)\} \quad (27)$$

and

$$\underline{\mathfrak{B}}^\top(G, M, R)_{\overline{\mathcal{K}}} \cong 2.$$

Proof. Notice that when $\varphi = \top$ then we have

$$x^* \dot{\otimes} R \dot{\otimes} y^{-1} \geq \top \iff x^* \dot{\otimes} R \dot{\otimes} y^{-1} = \top.$$

To use readily available results in [15, 16] we invert the equation to read: $x^\top \dot{\otimes} R^{-1} \dot{\otimes} y = \perp \iff y^\top \dot{\otimes} R^* \dot{\otimes} x = \perp$ and write both equations together as $z^\top = [x^\top \ y^\top]^\top$:

$$z^\top \dot{\otimes} A \dot{\otimes} z = \perp \qquad A = \begin{bmatrix} \perp_{g \times g} & R^{-1} \\ R^* & \perp_{m \times m} \end{bmatrix} \quad (28)$$

Since $A = A^\top$ and $\overline{\mathcal{K}}$ is zerosumfree, the equation holds for generic z if and only if $A \dot{\otimes} z = \perp^{(g+m)}$. But in that case $A \dot{\otimes} z = \perp^{(g+m)} \iff A \dot{\otimes} z = z \dot{\otimes} \perp$, that is if z is a right eigenvector of A for \perp , $z \in \mathcal{V}_\perp(A)$. Now, $I_{(g+m)}$ is the identity matrix in $\overline{\mathcal{K}}^{(g+m) \times (g+m)}$, but from [15, 16] we know that $z \in \mathcal{V}_\perp(A)$ if and only if it is a combination of the fundamental eigenvectors of A for \perp , that is those columns of $I_{(g+m)}$ indexed by the empty columns of A . Due to (28) we know that this can only happen:

- as combinations of empty columns of R^{-1} . Then for $y \in Y$ we have $R^{-1} \dot{\otimes} y = \perp^g$ which entails that y belongs to the right nullspace $\mathcal{N}_r(R^{-1})$. Notice that the empty columns of R^{-1} are the saturated columns of R .
- row-column dually, as combinations of empty columns of R^* . Then, for $x \in X$, we have $R^* \dot{\otimes} x = \perp^m$, that is, x belongs to the right nullspace $\mathcal{N}_r(R^*)$. Notice also that the empty columns of R^* are the saturated rows of R .

Consider $x \in \mathcal{N}_r(R^*)$, then $x_{R,\top}^\uparrow = (R^* \dot{\otimes} x)^{-1} \dot{\otimes} \perp = \top^m \dot{\otimes} \perp = \top^m$ whence

$$(x_{R,\top}^\uparrow)_{R,\top}^\downarrow = (\top^m)_{R,\top}^\downarrow = R \dot{\otimes} \perp^m \dot{\otimes} \perp = R \dot{\otimes} \perp^m$$

and $(R \dot{\otimes} \perp^m, \top^m)$ is its formal concept. GC-dually, if $y \in \mathcal{N}_r(R^{-1})$, then $y_{R, \top}^\downarrow = \top^g$ and $(y_{R, \top}^\downarrow)_{R, \top}^\uparrow = R \dot{\otimes} \perp^g$ leading to concept $(\top^g, R \dot{\otimes} \perp^g)$.

There are no other elements in the closures, since they would not fulfill the equation, and they are paired as (27) describes and the polars prove. The order between the concepts is induced from that of the ambient spaces. \square

We would also like to understand the structure of the congruences associated to these polars. For that purpose, first recall that the right nullspace of R^{-1} is not reduced to the zero vector if and only if R^{-1} has null columns, that is for $\forall j \in \{1 \dots g\}, J \subseteq \{1 \dots m\}, R_{i, J}^{-1} = \perp$. In such case, the right nullspace is generated by the unitary vectors e_j corresponding to each of the empty columns, and dually for the left null eigenspace and the null rows $\forall j \in \{1 \dots m\}, I \subseteq \{1 \dots g\}, R_{I, j} = \perp$.

$$\mathcal{N}_r(R^*) = \langle e_i \mid i \in I \rangle_{\overline{\mathcal{K}}} \subseteq \mathcal{X} \quad \mathcal{N}_r(R^{-1}) = \langle e_j \mid j \in J \rangle_{\overline{\mathcal{K}}} \subseteq \mathcal{Y}$$

Proposition 3.28. *For $R \in \mathcal{M}_{g \times m}(\overline{\mathcal{K}})$, and the context (G, M, R) ,*

- *The right nullspaces of R^{-1} and R^* are the equivalence classes of the bottom elements in the lattices:*

$$[R \dot{\otimes} \perp^m]_{\text{KER}(\cdot_{R, \top}^\uparrow)} = \mathcal{N}_r(R^*) \quad [R^\top \dot{\otimes} \perp^g]_{\text{KER}(\cdot_{R, \top}^\downarrow)} = \mathcal{N}_r(R^{-1})$$

- *The rest of the spaces are the equivalence classes of the top elements in the lattices:*

$$[\top^g]_{\text{KER}(\cdot_{R, \top}^\uparrow)} = \mathcal{X} \setminus \mathcal{N}_r(R^*) \quad [\top^m]_{\text{KER}(\cdot_{R, \top}^\downarrow)} = \mathcal{Y} \setminus \mathcal{N}_r(R^{-1})$$

Proof. For the first statement, the last part of the proof of the previous proposition proved that $[R \dot{\otimes} \perp^m]_{\text{KER}(\cdot_{R, \top}^\uparrow)} = [(\top^m)_{R, \top}^\downarrow]_{\text{KER}(\cdot_{R, \top}^\uparrow)} = \mathcal{N}_r(R^*)$, and GC-dually for intents. The second statement follows from the fact that $\text{KER}(\cdot_{R, \top}^\uparrow)$ and $\text{KER}(\cdot_{R, \top}^\downarrow)$ are equivalences on \mathcal{X} and \mathcal{Y} , respectively with only two classes: that related to the top and that related to the bottom. \square

We make several final remarks:

- Now we can see that if R has no saturated columns—even if we have an “empty” $R = \perp^{g \times m}$ —then $\mathcal{N}_r(R^{-1}) = \langle \emptyset \rangle_{\overline{\mathcal{K}}} = \{\perp^m\}$ and only the bottom of \mathcal{Y} maps to the top of \mathcal{X} . And GC-dually if R has no saturated rows.
- On the other hand, if R has all columns saturated $R = \top^{g \times m}$ (equivalently, all rows) then $\mathcal{N}_r(R^{-1}) = \mathcal{Y}$ and $\mathcal{N}_r(R^*) = \mathcal{X}$ with the closure systems collapsed onto $\mathfrak{B}^\top(G, M, \top^{g \times m}) = \{(\top^g, \top^m)\} \cong \mathbb{1}$.

As in the FCA case, when $\varphi = \top$ a “full” contexts has a concept lattice that is isomorphic to the unitary lattice. But in all other cases—including when R is completely empty—the concept lattice is isomorphic to $\mathbb{2}$. Although this agrees with FCA when $R = \perp^{g \times m}$ it does not when that is not the case.

This supports the intuition that $\varphi = \top$ is too coarse a grain to “observe” the context. In the next section we show how we can recover the standard behavior of FCA on empty and full contexts when $\varphi = e$ besides carrying a more meaningful exploration of the information in (G, M, R) .

3.8. FCA as Linear Algebra

FCA can easily be carried out as processing vectors in linear algebra over a complete semifield $\bar{\mathcal{K}}$. For that purpose, we need a procedure to encode and decode FCA into \mathcal{K} -FCA and vice-versa.

1. The first step is to encode the binary context (G, M, I) into a \mathcal{K} -valued context (G, M, R_I) . When an object g_i is incident to an attribute m_j —that is, $g_i I m_j$ —we consider the degree of that attribute in that object is *saturated*—that is, its value is $(R_I)_{ij} = \top$ —otherwise the degree is *zero*—that is, the bottom in the semifield, $(R_I)_{ij} = \perp$.

Note that for this step there is no need to state what concrete algebra $\bar{\mathcal{K}}$ stands for, since these values are available for every completed semifield.

2. Similarly, the characteristic vector for a set of attributes— $y_j = e \iff m_j \in y$ and $y_j = \perp$ otherwise—only needs the neutral element and the bottom. Hence, to encode a binary content into a $\bar{\mathcal{K}}$ -context we need only consider its complete sub-semifield $\mathfrak{3}$.
3. The next step is to choose the φ parameter. As seen in the previous section, choosing $\varphi \in \{\perp, \top\}$ leads to special, non-informative lattices, hence we choose $\varphi \in \bar{\mathcal{K}} \setminus \{\perp, \top\}$. Since the case where $\varphi \neq e$ can be normalized using the technique in Section 3.1, we use $\varphi = e$.

4. To obtain the object intents and extents in bulk we may close the identity matrix in $K^{g \times g}$ that encodes the singleton object sets as columns $(I_g)_{R_I}^\uparrow = R_I^\top \dot{\otimes} I_g^{-1} = R_I^\top$ so $((I_g)_{R_I}^\uparrow)_{R_I}^\downarrow = R_I \dot{\otimes} R_I^*$. And GC-dually for the attribute extents $(I_m)_{R_I}^\downarrow = R_I$ and intents, $((I_m)_{R_I}^\downarrow)_{R_I}^\uparrow = R_I^\top \dot{\otimes} R_I^{-1}$. This parallelism is one more advantage of the algebraization of FCA.

To see the object concepts Γ and attribute concepts M we build two matrices: Γ from the object intents and extents, and M from the attribute intents and extents

$$\Gamma = \begin{bmatrix} (I_g)_{R_I}^\uparrow \\ ((I_g)_{R_I}^\uparrow)_{R_I}^\downarrow \end{bmatrix} \quad M = \begin{bmatrix} ((I_m)_{R_I}^\downarrow)_{R_I}^\uparrow \\ (I_m)_{R_I}^\downarrow \end{bmatrix}$$

and read the concepts off the columns. Note also that since the entries in R_I are in $\mathbb{2} = \{\perp, \top\}$, the object and attribute extents and intents will also be in this subset of K .

5. The final step is to generate the lattice, which is described in the proof of Proposition 3.29 .

Proposition 3.29. *Given a formal context (G, M, I) , with R_I encoded as above,*

$$\underline{\mathfrak{B}}(G, M, I) \cong \underline{\mathfrak{B}}^e(G, M, R_I)_2 .$$

Proof. We may generate the lattice in terms of $\overline{\mathcal{K}}^d$, but since the entries in the extents and intents so far are in $\{\perp, \top\}$ we can restrict ourselves to using only these scalars. To see this, in equation (24) describing concept joins, consider the meet of intents $\sum_{i \in I} \lambda_i^{-1} \dot{\circ} b_i$. Clearly, when $\lambda_i = \perp$ then the intent b_i in the meet of intents is multiplied by $\lambda_i^{-1} = \top$ and left out of the sum (the min $\dot{\circ}$ acting as meet); however, if $\lambda_i = \top$ (or $\lambda_i = e$), then intent b_i is taken into consideration as is. So this equation performs an intersection of intents in the encoding mentioned above. GC-dually happens with the equation describing the meet of concepts in (25) and their meet of extents.

To generate from the attribute intents, choose the subset of index J of the columns of attribute intents (rows I) which are different, M_{IJ} . By the encoding above, we know that these include an encoding of the meet-irreducible intents of $\underline{\mathfrak{B}}(G, M, I)$. If $|J| = n$, encode all the possible characteristic subsets of n elements into a matrix $C \in 2^{n \times 2^n}$. Then by the reasoning above, $B_M = M_{IJ} \dot{\otimes} C^{-1}$ implements all possible combinations of meet irreducible intents—possibly with repetitions—so B_M includes an encoding of all the system of intents of $\underline{\mathfrak{B}}(G, M, I)$.

To find an encoding of the system of extents, just apply the polar $B_{G_M} = R \dot{\otimes} B_G^{-1}$. So the concept lattice is encoded into the correlative columns of B_{G_M} and B_G .

Of course nothing precludes, GC-dually, calculating the concept lattice by means of the system of extents of the object concepts. \square

Note that from the previous procedure and Proposition 3.29 we have proven:

Proposition 3.30. *Given a binary context (G, M, I) , with R_I as above,*

$$\underline{\mathfrak{B}}^e(G, M, R_I)_2 = \underline{\mathfrak{B}}^e(G, M, R_I)_{\overline{\mathcal{K}}}$$

for every possible complete idempotent semifield $\overline{\mathcal{K}}$.

Proof. Recall that any complete idempotent semifield whose carrier set is reduced to $\{\perp, e, \top\}$ is isomorphic to $\mathfrak{3}$, whose multiplications and additions are internal laws. It is easy to see that given that the object and attribute extents and intents of R_I only have non-finite $\{\perp, \top\}$ coordinates, the multiplication by any finite scalar produces the same vector, hence their linear combinations are actually closed, a sub-semimodule of $\mathfrak{3}$. Since also $\varphi = e$, reproducing FCA with \mathcal{K} -FCA can be done generically on any complete idempotent semifield with the procedure above, hence Proposition 3.29 provides the result. \square

3.9. Calculation: binary concept lattices using \mathcal{K} -Formal Concept Analysis

Consider the context of [1, Fig. 1.5] reproduced in Figure 4.a and associated concept lattice in Figure 4.b. The binary relation can be encoded as

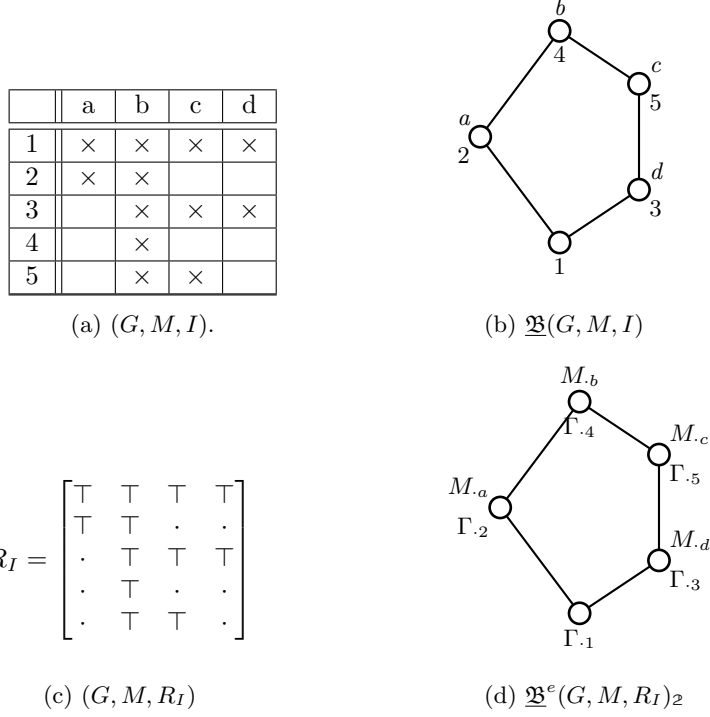


Figure 4: The context and its concept lattice from [1, Figs. §1.5 and 1.6.]. An encoding R_I of the incidence I into a complete semifield $\bar{\mathcal{K}}$, and its e -lattice annotated in the object concepts (see text).

shown in Figure 4.c where we have written \cdot instead of \perp to lessen the visual cluttering. In semifield 2, encoding the unitary set $\{m_1\}$ results in the vector $m_1 = [\top \cdot \cdot \cdot]^\top$.

Given that $\varphi = e$ is fixed, the result of step 4 in the above procedure is— where we have placed intents “above” extents, and used E to index the rows for

the extents and I for the intents of the object Γ and attribute concepts M :

$$\Gamma_I = \begin{bmatrix} \top & \top & \cdot & \cdot & \cdot \\ \top & \top & \top & \top & \top \\ \top & \cdot & \top & \cdot & \top \\ \top & \cdot & \top & \cdot & \cdot \end{bmatrix} \quad M_I = \begin{bmatrix} \top & \cdot & \cdot & \cdot \\ \top & \top & \top & \top \\ \cdot & \cdot & \top & \top \\ \cdot & \cdot & \cdot & \top \end{bmatrix}$$

$$\Gamma_E = \begin{bmatrix} \top & \top & \top & \top & \top \\ \cdot & \top & \cdot & \top & \cdot \\ \cdot & \cdot & \top & \top & \top \\ \cdot & \cdot & \cdot & \top & \cdot \\ \cdot & \cdot & \cdot & \top & \top \end{bmatrix} \quad M_E = \begin{bmatrix} \top & \top & \top & \top \\ \top & \top & \cdot & \cdot \\ \cdot & \top & \top & \top \\ \cdot & \top & \cdot & \cdot \\ \cdot & \top & \top & \cdot \end{bmatrix}$$

1 2 3 4 5
 a b c d

So, for instance,

$$\Gamma_{\cdot 4} = [\cdot \quad \top \quad \cdot \quad \cdot \quad \parallel \quad \top \quad \top \quad \top \quad \top \quad \top]^T \cong \bar{\gamma}(4) = \bar{\mu}(b) \cong M_b.$$

We can also see that attribute intents are all different $n = m = 4$, whence $C \in 2^{4 \times 16}$ is relatively large. The more so since $|\mathfrak{B}(G, M, I)| = 5$ as per Figure 4.b rather than $n = 16$. We obviate the verbose matrix equations but have annotated in Figure 4.d where the object and attribute concepts appear which means that the reducing labeling of FCA is also available for \mathcal{K} -FCA.

Note that in the example above we did not have to choose whether $\bar{\mathcal{K}}$ was $\bar{\mathbb{R}}_{\max,+}$ or $\bar{\mathbb{R}}_{\max,\times}$. Indeed, due to the order- and GC-duality we have been tout-ing, we could have also decided to use $\bar{\mathbb{R}}_{\min,+}$, by inverting all the expressions.

4. Discussion of Results

4.1. Summary

Our main result is that \mathcal{K} -concept lattices are both \mathcal{K}^d -subsemimodules and \mathcal{K} -semimodules, as well as lattice-ordered, and that the polars are also dual isomorphisms of semimodules, on top of dual lattice isomorphisms.

These results stem from the fact that the polars in the Galois connection are actually linear in that dual idempotent semifield $\bar{\mathcal{K}}^d$, which can be expressed succinctly in the opposite semifield $\bar{\mathcal{K}}^{-1}$. This entails not only that major FCA-related concepts such as the systems of extents and intents or the concept lattice have a direct counterpart in linear algebra over the dual idempotent semifield, but also that the signature of their algebras is enriched, e.g. the action of a scalar on a formal concept, and that linear algebra concepts known to be important, e.g. bikernels, lend importance to heretofore overlooked concepts in FCA, e.g. the quotient sets of the polars.

The analysis in the presence of the extreme elements $\{\perp, \top\}$ is a novelty of this paper even in the work that originally defined the Galois connection for idempotent semimodules [21].

As a summary of results we present Table 1 where we enter an isomorphic lattice to $\underline{\mathfrak{B}}^\varphi(G, M, R)_3$ for each value of $\varphi \in \{\perp, e, \top\}$ and three different values

Table 1: Lattices isomorphic to $\mathfrak{B}^\varphi(G, M, R)_3$ when sweeping in φ and R for some specific values. Note that $\mathfrak{B}(G, M, \times) \cong \mathbf{1}$ and $\mathfrak{B}(G, M, \circ) \cong \mathbf{2}$ —where \times indicates a full relation and \circ an empty one—and also that $\perp^{g \times m} \leq R_I \leq \top^{g \times m}$.

$\varphi \backslash R$	$\perp^{g \times m}$	R_I	$\top^{g \times m}$
\perp	$\mathbf{1}(\cong \mathfrak{B}(G, M, \times))$	$\mathbf{1}(\cong \mathfrak{B}(G, M, \times))$	$\mathbf{1}(\cong \mathfrak{B}(G, M, \times))$
e	$\mathbf{2}(\cong \mathfrak{B}(G, M, \circ))$	$\mathfrak{B}^e(G, M, R_I)_2(\cong \mathfrak{B}(G, M, I))$	$\mathbf{1}(\cong \mathfrak{B}(G, M, \times))$
\top	$\mathbf{2}(\cong \mathfrak{B}(G, M, \circ))$	$\mathbf{2}(\cong \mathfrak{B}(G, M, \circ))$	$\mathbf{1}(\cong \mathfrak{B}(G, M, \times))$

of the multivalued incidence: when $R = \perp^{g \times m}$, when $R = \top^{g \times m}$ and when it comes from encoding a binary incidence $R = R_I$, as in Section 3.8. We also enclose in parentheses how to obtain such lattices in FCA, since we believe that it shows some insights into how FCA is encoded as \mathcal{K} -FCA.

4.2. Discussion

Other generalizations of FCA. A generalization of FCA similar to \mathcal{K} -FCA is *Fuzzy-FCA* [30, 31], that is, where relations have incidence degrees over fuzzy semirings. This is quite extensively developed but its algebraic connection is not as advanced as the one we present in this paper. We believe this to be due to the fact that fuzzy semirings, being inclines, lack a multiplicative inverse (see Sec. 2).

Another proper generalization of FCA, the *Logic of Information Systems* [32] is based on logical contexts where formulas are used to replace attributes, that is, an intension is the set of formulas that hold for the objects in an extent. Such framework has been mainly developed for information systems, especially for modeling records whose fields admit formulaic descriptions—dates, restricted vocabularies, etc.—and the mathematical framework is far from linear algebra.

In yet another extension, Pattern structures [33], the concept of a *formal concept* is specialized to model specific methodological entities in its domain of application: the descriptions of positive and negative examples in supervised classification [34].

Review of similar work. The standard reference for modern idempotent linear algebra is [4], but in his seminal work Cuninghame-Green [2, Ch.22] already developed a construction similar to that of Sec. 3 but describing an adjunction. The pairs of closed elements that constitute concepts were also not evident in his exposition: recognizing their importance is the merit of FCA.

The former is also the setting of most work in idempotent linear algebra where usually the connection between spaces is established as a (dual or not) residuated pair [22, 23, 26–28]. This leads to left or right Galois adjunctions and in terms of *extended* FCA to object or attribute neighborhood lattices [6]. See [25] for a revision of the genesis and importance of Galois connections and adjunctions, as well as a discussion of the different notation and nomenclatures for these concepts.

On the other hand, our scaling is reminiscent of that of Cuninghame-Green’s—technicalities aside—but inspired by [30], although the interpretation of scaling

as abstraction or concretion only makes sense in terms of similar concepts of FCA, like the join representing a generalization and the meet a specialization.

Congruences of linear functions were the inspiration for those of the polars. Since the former have a very specific meaning in the geometric approach to observability and controllability in idempotent Control Theory [26], we conjecture that the latter are also important in a different approach to Control Theory based on Galois connections, which by the present paper would involve a generalization of FCA. This is to be explored in future work.

Inverses in \mathcal{K} -concept lattices. Since \mathcal{K} is an idempotent *semifield* where an inversion is available, as it is in the ambient spaces, we might wonder whether the semimodules of extents and intents have a similar inversion available. In general this is not the case, since $a^{-1} = (a^*)^T$ we have for $a \in \underline{\mathfrak{B}}_G^\gamma(G, M, R)$ that $a^{-1} = R \otimes (R^* \dot{\otimes} a^{-1})$. That is, the inverses of extents are actually fixpoints of a kernel operator, and likewise for intents. Dually, if the inverse a^{-1} were an extent, then a would also have to be a fixpoint of that kernel. This would only happen on very particular extents, and in general φ -formal concepts have no inverses.

Notational matters. Finally, a clarification: previous uses of \mathcal{K} -FCA for data mining—notably [35]—have started from the scalar products $\langle x \mid R \mid y \rangle = x^T \otimes R \otimes y$ defined in the idempotent semifield $\mathbb{R}_{\max,+}$ or in $\mathbb{R}_{\min,+}$. Such scalar products produce Galois connections that can also be interpreted in the framework of FCA—and do not involve the concept of the dual semimodule—but they have the drawback that the isomorphism in Section 3.8— $\underline{\mathfrak{B}}(G, M, I) \cong \underline{\mathfrak{B}}^e(G, M, R_I)_3$ —is not so straightforward. For this reason, we have decided to start all our development where [6] left off, with the scalar product of Section 3.1 that uses duals. The most important results of this paper in the alternate definition of the scalar products above can be found in [36].

4.3. Conclusions

This work has proven that the relationship between FCA and Linear Algebra over idempotent semifields is tighter than suspected.

On the one hand, \mathcal{K} -FCA is enriched and set in the wider context of Linear Algebra over idempotent semifields: concept lattices are complete idempotent semimodules shaped by the polars which are linear functions in some algebra.

Reciprocally, the unique characteristics of Linear Algebra over idempotent semifields are better understood by relating to concepts and methods of FCA: antitone functions, Galois connections and lattices. We believe this should be a fruitful partnership for future developments in both fields.

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References

- [1] B. Ganter, R. Wille, Formal Concept Analysis: Mathematical Foundations, Springer, Berlin, Heidelberg, 1999.
- [2] R. Cuninghame-Green, Minimax Algebra, no. 166 in Lecture notes in Economics and Mathematical Systems, Springer, 1979.
- [3] F. Baccelli, G. Cohen, G. Olsder, J. Quadrat, Synchronization and Linearity, Wiley, 1992.
- [4] P. Butkovič, Max-linear Systems. Theory and Algorithms, Monographs in Mathematics, Springer, 2010.
- [5] M. Erné, J. Koslowski, A. Melton, G. Strecker, A primer on Galois connections, in: A. Todd (Ed.), Proceedings of the 1991 Summer Conference on General Topology and Applications in Honor of Mary Ellen Rudin and Her Work, Vol. 704 of Annals of the New York Academy of Sciences, New York Academy of Science, Madison, WI, USA, 1993, pp. 103–125.
- [6] F. J. Valverde-Albacete, C. Peláez-Moreno, Extending conceptualisation modes for generalised Formal Concept Analysis, Information Sciences 181 (2011) 1888–1909.
- [7] F. J. Valverde-Albacete, C. Peláez-Moreno, Towards a generalisation of Formal Concept Analysis for data mining purposes, in: Concept Lattices. Proceedings of the International Conference on Formal Concept Analysis, ICFCA 06, Vol. 3874 of LNAI, 2006, pp. 161–176.
- [8] F. J. Valverde-Albacete, C. Peláez-Moreno, Further Galois connections between semimodules over idempotent semirings, in: J. Diatta, P. Eklund (Eds.), Proceedings of the 4th Conference on Concept Lattices and Applications (CLA 07), Montpellier, France, 2007, pp. 199–212.
- [9] H. Nobuhara, D. B. K. Trieu, T. Maruyama, B. Bede, Max-plus algebra-based wavelet transforms and their FPGA implementation for image coding, Information Sciences 180 (17) (2010) 3232–3247.
- [10] W. Fang, X. Yin, Y. An, N. Xiong, Q. Guo, J. Li, Optimal scheduling for data transmission between mobile devices and cloud, Information Sciences 301 (C) (2015) 169–180.
- [11] J. R. Deller, J. G. Proakis, J. H. Hansen, Discrete-Time Processing of Speech Signals, Prentice-Hall, 1987.
- [12] G. Ritter, T. Beaver, Morphological perceptrons, Neural Networks.
- [13] G. Ritter, J. Diaz-de Leon, P. Sussner, Morphological bidirectional associative memories., Neural Networks.

- [14] F. J. Valverde-Albacete, C. Peláez-Moreno, Spectral lattices of $(R_{max,+})$ -formal contexts, in: Concept Lattices. Proceedings of the International Conference on Formal Concept Analysis, ICFCA 08, Vol. LCNS 4933, 2008, pp. 124–139.
- [15] F. J. Valverde-Albacete, C. Peláez-Moreno, The spectra of irreducible matrices over completed idempotent semifields, *Fuzzy Sets and Systems* 271 (2015) 46–69. doi:10.1016/j.fss.2014.09.022.
- [16] F. J. Valverde-Albacete, C. Peláez-Moreno, The spectra of reducible matrices over complete commutative idempotent semifields and their spectral lattices, *International Journal of General Systems* 45 (2) (2016) 86–115.
- [17] J. S. Golan, *Semirings and their Applications*, Kluwer Academic, 1999.
- [18] M. Gondran, M. Minoux, *Graphs, Dioids and Semirings. New Models and Algorithms.*, Operations Research Computer Science Interfaces series., Springer, 2008.
- [19] J. J. Moreau, Inf-convolution, sous-additivité, convexité des fonctions numériques, *J. Math. pures et appl.* 49 (1970) 109–154.
- [20] T. Blyth, M. Janowitz, *Residuation Theory*, Pergamon press, 1972.
- [21] G. Cohen, S. Gaubert, J.-P. Quadrat, Duality and separation theorems in idempotent semimodules, *Linear Algebra and Its Applications* 379 (2004) 395–422.
- [22] G. Cohen, S. Gaubert, J.-P. Quadrat, Projection and aggregation in maxplus algebra, in: *Current trends in nonlinear systems and control*, Birkhäuser Boston, Boston, MA, 2006, pp. 443–454.
- [23] S. Gaubert, R. D. Katz, The tropical analogue of polar cones, *Linear Algebra and its Applications* 431 (5-7) (2009) 608–625.
- [24] B. Davey, H. Priestley, *Introduction to lattices and order*, 2nd Edition, Cambridge University Press, Cambridge, UK, 2002.
- [25] M. Ern e, Adjunctions and Galois connections: Origins, history and development, in: Denecke et al. [37], pp. 1–138.
- [26] M. Di Loreto, S. Gaubert, R. D. Katz, J.-J. Loiseau, Duality Between Invariant Spaces for Max-Plus Linear Discrete Event Systems, *SIAM Journal on Control and Optimization* 48 (8) (2010) 5606–5628.
- [27] S. Gaubert, *Th orie des syst emes lin aires dans les dio ides*, Th ese,  cole des Mines de Paris (July 1992).
- [28] G. Cohen, S. Gaubert, J. Quadrat, Kernels, images and projections in dioids, *Proceedings of the Workshop on Discrete Event Systems (WODES)* (1996) 1–8.

- [29] M. Gondran, M. Minoux, Valeurs propres et vecteurs propres dans les dioïdes et leur interprétation en théorie des graphes, EDF, Bulletin de la Direction des Etudes et Recherches, Serie C, Mathématiques Informatique 2 (1977) 25–41.
- [30] R. Bělohlávek, Fuzzy Relational Systems. Foundations and Principles, Vol. 20 of IFSR International Series on Systems Science and Engineering, Kluwer Academic, 2002.
- [31] A. Burusco, R. Fuentes-González, The study of the L-fuzzy Concept Lattice, Mathware and Soft Computing 1 (3) (1994) 209–218.
- [32] S. Ferré, O. Ridoux, Introduction to logical information systems, Information Processing and Management 40 (2004) 383–419.
- [33] B. Ganter, S. Kuznetsov, Pattern structures and their projections, in: Conceptual Structures: Broadening the Base, Vol. 2120 of LNCS, Springer, 2001, pp. 129–142.
- [34] M. Kaytoue, S. O. Kuznetsov, A. Napoli, S. Duplessis, Mining gene expression data with pattern structures in Formal Concept Analysis, Information Sciences 181 (10) (2011) 1989 – 2001, special Issue on Information Engineering Applications Based on Lattices. doi:10.1016/j.ins.2010.07.007.
- [35] F. J. Valverde-Albacete, J. M. González-Calabozo, A. Peñas, C. Peláez-Moreno, Supporting scientific knowledge discovery with extended, generalized formal concept analysis, Expert Systems with Applications 44 (2016) 198 – 216. doi:http://dx.doi.org/10.1016/j.eswa.2015.09.022.
- [36] F. J. Valverde-Albacete, C. Peláez-Moreno, The Linear Algebra in Formal Concept Analysis over Idempotent Semifields, in: Formal Concept Analysis, no. 9113 in LNAI, Springer Berlin Heidelberg, Heidelberg, 2015, pp. 97–113.
- [37] K. Denecke, M. Ern e, S. Wismath (Eds.), Galois Connections and Applications, no. 565 in Mathematics and Its Applications, Kluwer Academic, Dordrecht, Boston and London, 2004.