UNIVERSIDAD CARLOS III DE MADRID working papers

Working Paper
Economic Series 10-18
September 2010

Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 916249875

Non-revelation Mechanisms in Many-to-One Markets.

Antonio Romero-Medina*
Matteo Triossi†

Abstract

This paper presents a sequential admission mechanism where students are allowed to send multiple applications to colleges and colleges sequentially decide the applicants to enroll. The irreversibility of agents decisions and the sequential structure of the enrollments make truthful behavior a dominant strategy for colleges. Due to these features, the mechanism implements the set of stable matchings in Subgame Perfect Nash equilibrium. We extend the analysis to a mechanism where colleges make proposals to potential students and students decide sequentially. We show that this mechanism implements the stable set as well.

JEL-Classification: C78, D78.
Key-words: stable matching, subgame perfect Nash equilibrium.

* Departmento de Economía, Universidad Carlos III, Calle Madrid 126, 28903 Getafe-Madrid, Spain. E-mail: aromero@eco.uc3m.es.

† Corresponding author. Centro de Economía Aplicada, Departmento de Ingeniería Industrial , Universidad de Chile. Avenida Republica 701, Santiago, Chile. E-mail: mtriossi@dii.uchile, tel: +56 (562) 978 4072, fax: + 56 (562) 978 4011

1 Introduction

An important number of real-life admission and hiring procedures are decentralized. Many of them are non-revelation mechanisms: agents are not required to submit their preferences. Also, in many admission (resp. hiring) mechanisms, students (resp. workers) apply to colleges (resp. firms). It is common that participants are allowed to send multiple applications. However, these features have been rarely analyzed together in a many-to-one matching market. The purpose of this paper is the analysis of decentralized non-revelation mechanisms with multiple applications or offers. Our objective is the implementation of efficient allocations in a many-to-one matching market through sequential mechanisms admitting multiple applications.

The first mechanism we present, the "students-apply-college-sequentially-choose" mechanism (CSM from now on) consists of two main stages. First, a group of students simultaneously send applications to colleges. Then, colleges choose sequentially among their applicants, with the restriction that no college can enroll an applicant who has been previously accepted by another college. The mechanism extends to the many-to-one matching framework the sequential procedure introduced by Sotomayor (2003) for one-to-one matching markets. We prove that it implements the stable set in Subgame Perfect Nash Equilibrium (SPE from now on). There are two features that explain this result. First, students are committed to their strategy: a student has to join the first college (among the ones she applied to) which accepts her. This means that choices are not reversible. Second, colleges decide sequentially. These features imply that every college has a dominant strategy: to enroll its favorite group of applicants among the ones that have not been accepted by a different college in a previous stage. So the mechanism is an extension of the serial dictatorship. In order to weight the importance of the non-reversibility assumption we consider an alternative mechanism, which is similar to the CSM, but where students are not committed by their first stage applications. Here, no college has a dominant strategy at its turn. More important, unstable allocations can arise at equilibrium (see also Haeringer and Wooders 2010 and Triossi 2009).

Finally, we consider a second sequential mechanism, the "colleges-propose-students-sequentially-choose" mechanism (SSM from now on). Here, colleges have to make simultaneous proposals to students. Then, each student sequentially decides which college to join among the ones that accepted her. This procedure resembles some real-world college admission mechanisms where, before entering the process students are ranked according to the result of a standardized test which determines the order in which they can select the proposals from colleges. For instance, the admission to traditional Universities in Chile works in a similar fashion. We show that this mechanism implements the stable set in SPE as well.

Other scholars have addressed the design of mechanisms able to implement the set of stable matchings. In particular, Kara and Sönmez (1997) show that the stable correspondence is implementable in Nash Equilibrium. Alcalde (1996) approaches the design of "natural mechanisms" able to implement the stable set of one-to-one matching markets. To the best of our knowledge, the procedure introduced in Sotomayor (2003) is the first non-revelation mechanism presented in the literature allowing for multiple applications in matching markets. Like ours, the mechanism is an extension of the serial-dictatorship, but it does not consider many-to-one matching markets as we do. The CSM can also be seen as a generalization of the "students-propose-and-colleges-choose mechanism" introduced in Alcalde and Romero-Medina (2000) for a many-to-one market. There students are allowed to send at most one application. A "natural" admission mechanism with multiple applications in a many-to-one matching market is analyzed in Triossi (2009). However, that mechanism might implement unstable allocations if applications are costless as in our paper. Finally, different non-revelation mechanisms are presented in Diamantoudi et al (2007) and Haeringer and

Wooders (2010).[1] Both papers consider one-to-one matching markets.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 contains the main results and Section 4 concludes.

2 The model

A college admission problem (see Gale and Shapley 1962) is a triple $(C, S, >)$ where $C = \{c_1, ..., c_k\}$ is the set of colleges, $S = \{s_1, ..., s_t\}$ is the set of students and $>= (>_C, >_S)$ is the vector of agents' preferences. More precisely, $>_C= (>_{c_1}, ..., >_{c_k})$ is the list of colleges' preferences over subsets of students and $>_S= (>_{s_1}, ... >_{s_t})$ is the list of students' preferences over colleges. For every $c \in C$, $>_c$ is a strict order defined on 2^S, the set of all subsets of S. Let $S' \subseteq S$ be a set of students. The favorite group of students for college c among the ones belonging to S' is called the **choice set from** S'. It is denoted by $Ch_c(S', >_c)$ or by $Ch_c(S')$, when no ambiguity is possible. Formally, $Ch_c(S', >_c) = \arg\max_{>_c} \{S'' : S'' \subseteq S'\}$. We will say that student s is **acceptable** to college c if $\{s\} >_c \emptyset$. Otherwise, we will say that student s is **unacceptable** to c. We will denote as $A(c)$ the set of acceptable students for college c. For every agent $x \in C \cup S$, $\geq_x$ denotes x's weak preference relation. The maximum numbers of students college c is willing to enroll is c's **quota** and it is denoted by q_c, formally $q_c = \max \{n \in \mathbf{N} : |S'| > n \Rightarrow \varnothing P_f S'\}$.[2] The preferences of the colleges on individual students (which is on subsets of students of cardinality one) will be represented by ordered lists of acceptable students: $>_c$: $s_{i_1}...s_{i_r}$ where where $\{s_{i_j}\} >_c \{s_{i_l}\}$ for $j < l \leq r$ and $s_{i_r} \in A(c)$. For every $s \in S$ $>_s$ is a strict order on $C \cup \{s\}$. We will say that college c is **acceptable** to student s if $c >_s s$. Otherwise, we will say that college c is **unacceptable** to s. We will denote as $A(s)$ the set of acceptable colleges for student s. The preferences of the students will be rep-

[1] See also Alcalde and Romero Medina (2005).

[2] For every finite set X, $|X|$ denotes the number of elements of X.

resented by ordered lists of acceptable colleges: $>_s$: $c_{i_1}...c_{i_r}$ where $c_{i_j} >_s c_{i_l}$ for $j < l \leq r \leq t$ and $c_{i_r} \in A(s)$.

A matching is an assignment of students to colleges. Formally, a **matching** on (C, S) is a function $\mu : C \cup S \rightarrow 2^S \cup C$ such that, for every $(c, s) \in C \times S$: (i) $\mu(c) \in 2^S$, (ii) $\mu(s) \in C \cup \{s\}$ and (iii) $\mu(s) = c \Leftrightarrow c \in \mu(s)$.[3] We denote by $\mathcal{M}$ the set of matchings on (C, S). A matching μ is **individually rational** if (i) $Ch_c(\mu(c)) = \mu(c)$ for all $c \in C$ and (ii) $\mu(s) \geq_s s$ for all $s \in S$. A matching μ is **blocked by the pair** $(c, s) \in C \times S$ if (i) $c >_s \mu(s)$ and (ii) $s \in Ch_c(\mu(c) \cup \{s\})$. A matching μ is **stable in** $(C, S, >)$ if it is individually rational and no pair blocks it. Otherwise, μ is **unstable**. $\Gamma(C, S, >)$ denotes the **stable set**, the set of matchings that are stable in market $(C, S, >)$. The stable set may be empty. To overcome this problem the literature has focused on preference restrictions where students are not seen as complements. A college c has substitutable preferences if it wants to hire a student even when other students become unavailable. Formally, $>_c$ are **substitutable** if, for every $S' \in 2^S$ and for all $s, s' \in W$, $s \neq s'$: $s \in Ch_c(S') \Rightarrow s \in Ch_c(S' \setminus \{s'\})$. Under this restriction, the deferred acceptance algorithm (Gale and Shapley, 1962) produces either the college-optimal or the student-optimal stable matching depending on whether the colleges or the students make the offers (see Roth and Sotomayor, 1990). Throughout the paper we assume that the preferences of the colleges are substitutable. A stronger assumption is responsiveness. A college c has responsive preferences if, for any two assignments that differ in one student only, it prefers the assignment containing the most preferred student. Formally, P_c are **responsive** if, for all $S' \subset S$ such that $|S'| \leq q_c - 1$ and for all s, $s' \in S$: (i) $S' \cup \{s\} >_c S' \cup \{s'\} \Leftrightarrow \{s\} >_c \{s'\}$ and (ii) $S' \cup \{s\} >_c S' \Leftrightarrow \{s\} >_c \emptyset$.
An **extensive form matching mechanism** is an array $G = (C \cup S, H, M, g)$. $C \cup S$ is the set of players, H is the set of histories and M is the strategy

[3] Property (iii) implies that a matching μ is uniquely defined by by $\mu_{|C}$ or $\mu_{|S}$, where $\mu_{|X}$ denotes the restriction to the set X of the function μ.

space, $M = \prod_{x \in S \cup C} M_x$, where $M_x = \prod_{h \in H} M_x^h$ for every $x \in C \cup S$. Set $M^h = \prod_{x \in F \cup W} M_x^h$. Histories and strategies are linked by the following property $M^h = \left\{m^h \mid \left(h, m^h\right) \in H\right\}$. There is an initial history $h^0 \in H$ and every history $h \in H$ is represented by a finite sequence $(h^0, m^1, ..., m^{r-1}) = h^r$. If $h^{r+1} = (h^r, m^r)$ then history h^{r+1} proceeds history h^r. The set $Z_T = \{z \in H \mid \text{there is no } h \in H \text{ proceeding z}\}$ is the set of terminal histories. Given the initial history, every strategy profile $m \in M$ defines a unique terminal history z_m. The outcome function $g : Z \to \mathcal{M}$ specifies an outcome matching for each terminal history, and hence for each strategy profile m. With abuse of notation, we use $g(m)$ to denote $g(z_m)$, where z_m is the terminal history corresponding to m. Given a preference profile $>$, $(G, >)$ constitutes an extensive form game. Every $h \in H \setminus Z$ identifies a subgame $G(h) = (C \cup S, H(h), M(h), g_h, P)$, where h is the initial history, $H(h) = \{h' \in H \mid h' \text{ proceeds } h\}$ and $M(h) = \prod_{h' \in H(h)} M^{h'}$. Let $m \in M(h)$. With abuse of language, we will identify the game $G(h)$ with the corresponding node. Given the initial history h, a profile of strategy m specifies a unique terminal history, z_m. The outcome function is defined by $g_h(m) = g(z_m)$. Given $m \in M$ and $h \in H$, let $m(h) \in M(h)$ be the strategy prescribed by m once h is reached. Formally, if $m = \left(m^h\right)_{h \in H}$, then $m(h) = \left(m^h\right)_{h \in H(h)}$.

A **Subgame Perfect Equilibrium** is a strategy profile that induces a Nash Equilibrium in every subgame. Formally, m^* is a SPE if for all $h \in H$ and for all $x \in C \cup S$: $g_h(m^*(h)) R_x g_h\left(m'_x, m^*_{-x}(h)\right)$ for every $m'_x \in M_x(h)$. The matching $g(m^*)$ is called SPE outcome of $(G, >)$ and the set of SPE outcomes of $(G, >)$ is denoted by $SPE(G, P)$. Let $\mathcal{S}$ be a set of matching markets and let $\Phi : \mathcal{S} \twoheadrightarrow \mathcal{M}$ be a correspondence. An extensive form matching mechanism G **implements** Φ in SPE if , for every $(C, S, >) \in \mathcal{S}$, $SPE(G, >) = \Phi(C, S, >)$, which is if every SPE outcome of $(\Gamma, >)$ belongs to $\Phi(F, W, P)$ and for every matching $\mu \in \Phi(C, S, >)$ there exists a SPE of $(G, >)$ yielding μ as outcome. Throughout the paper, only equilibria in pure strategies are considered.

3 The admission mechanisms

In this section we analyze two sequential mechanisms. Both are extensions to many-to-one environments of the serial dictator mechanism. If we restrict the attention to one-to-one matching markets they both coincide with the mechanism studied in Sotomayor (2003). For every natural number $n \geq 1$ let $\sum_n$ be the set of permutations of n objects, which is one-to-one functions $\sigma : \{1, ..., n\} \to \{1, ..., n\}$.

3.1 The students-apply-colleges-sequentially-choose mechanism

We present our first mechanism. Let $\sigma \in \sum_k$. The CSM is described by the following procedure:

0. **Application.** Each student s applies to a set of colleges. Let $C(s) \subseteq 2^C$ be the set of colleges student s applies to. For every college c let $S(c) = \bigcup_{c \in C(s)} \{s\}$ be the set of students applying to college c. Set $\mu_0 \left(c_{\sigma(0)}\right) = \emptyset$.

For $1 \leq r \leq k$:

r. **Enrollment.** College $c_{\sigma(r)}$ decides the students to accept among the ones in $S_r^\sigma = S\left(c_{\sigma(r)}\right) \cap \left\{s : s \notin \mu_{r'}\left(c_{\sigma(r')}\right) \text{for all } r' < r\right\}$. Denote by $\mu_r\left(c_{\sigma(r)}\right) \subseteq S_r^\sigma$ the decision of college $c_{\sigma(r)}$.

The outcome matching is defined by $\mu\left(c_{\sigma(r)}\right) = \mu_r\left(c_{\sigma(r)}\right)$ for all r, $1 \leq r \leq k$. If $s \notin \bigcup_r \mu_r\left(c_{\sigma(r)}\right)$then $\mu(s) = s$. This matching is well defined because, by definition, $\mu_r\left(c_{\sigma(r)}\right) \cap \mu_{r'}\left(c_{\sigma(r')}\right) = \emptyset$ if $r \neq r'$.

At stage 0, every student applies to one or more colleges. At stage r, college $c_{\sigma(r)}$ has to select a subset of students among those who applied to

$c_{\sigma(r)}$ and have not been previously enrolled by a college $c_{\sigma(r')}$ with $r' < r$. Thus, permutation σ represents the order in which colleges move.

The strategy space of each student is 2^C. For every r, $1 \leq r \leq k$, let Z_r^σ be the set of subgames at the enrollment stage when college $c_{\sigma(r)}$ has to move. Every $z_r \in Z_r^\sigma$ is characterized by the applications of the students and by the decisions made by colleges along the path leading to z_r. Formally, $z_1 \in Z_1^\sigma$ can be identified with $\{C_{z_1}(s)\}_{s\in S,}$, where $C_{z_1}(s)$ is the set of colleges student s applied to. For $r \geq 2$, z_r can be identified with $\left\{C_{z_1}(s), \mu_{z_{r'}}\left(c_{\sigma(r')}\right)\right\}_{s\in S}^{1\leq r'<r}$, where $z_{r'} = z_{r'}(z_r)$ is the unique stage r' node preceding z_r and $\mu_{z_{r'}}\left(c_{\sigma(r')}\right)$ is college $c_{\sigma(r')}$ decision at node $z_{r'}$ for every r', $1 \leq r' < r$. Let $S_{z_r}\left(c_{\sigma(r)}\right) = \bigcup_{c\in C_{z_1}(s)}\{s\}$, be the set of students that applied to $c_{\sigma(r)}$ along the history leading to z_r where $z_1 = z_1(z_r)$. The strategy space of college $c_{\sigma(r)}$ in the subgame z_r is $S_{z_r} = S_{z_r}\left(c_{\sigma(r)}\right) \cap \left\{s : s \notin \mu_{r'z_{r'}}\left(c_{\sigma(r')}\right) \text{ for all } r' < r\right\} \cup \{\emptyset\}$. It consists of the set of available applicants of $c_{\sigma(r)}$ and the empty set, should $c_{\sigma(r)}$ decide not to enroll any student.

The CSM generalizes the mechanism introduced by Sotomayor (2003) to many-to-one matching markets. Furthermore, it modifies the students-apply-colleges-choose mechanism studied in Alcalde and Romero-Medina (2000) by allowing for multiple applications and making colleges choose sequentially.

Consider the CSM. Let $\mu_{rz_r} = Ch_{c_{\sigma(r)}}(S_{z_r})$ be college $c_{\sigma(r)}$ choice set from S_{z_r}. College $c_{\sigma(r)}$ can enroll any student in S_{z_r} so μ_{rz_r} is the best response of college $c_{\sigma(r)}$ in subgame z_r independently on the strategy of any college $c_{\sigma(r')}$, for every $r' > r$. Strategy μ_{rz_r} is strictly dominant in subgame z_r whenever $S_{z_r} \neq \emptyset$. The unique strategy available to $c_{\sigma(r)}$ if $S_{z_r} = \emptyset$ is the strategy $\mu_{z_r} = \emptyset$. We thus have the following result.

Lemma 1 *In every SPE of the game induced by the CSM, college $c_{\sigma(r)}$ uses strategy μ_{rz_r} at subgame z_r.*

This result simplifies the analysis of the game. It implies that the equilibrium behavior of each college c depends only on the set of available applicants when c has to make its enrollment decision.

Before stating our main result we prove an ancillary Lemma.

Lemma 2 *Let $>_c$ be a profile of substitutable preferences for $c \in C$, let $s \in S$ and let $A \subset S$. Set $T = Ch_c(A)$. If $s \in Ch_c(T \cup \{s\})$ then $s \in Ch_c(A \cup \{s\})$.*

Proof. By contradiction, assume that $s \notin Ch_c(A \cup \{s\})$. Preferences $>_c$ are substitutable so $Ch_c(A \cup \{s\}) \backslash \{s\} \subseteq Ch_c(A) = T$. Thus, $Ch_c(A \cup \{s\}) = T$. From $s \in Ch_{c_r}(T \cup \{s\})$. It follows that $Ch_{c_r}(T \cup \{s\}) >_{c_r} Ch_{c_r}(A \cup \{s\})$, which yields a contradiction because $T \cup \{s\} \subseteq A \cup \{s\}$. ■

Assume that, at its turn, college c can select among the students belonging to $A \subseteq S$ and it enrolls the students in $T \subseteq S$. Furthermore, assume that, if student s was available jointly with the students in T, college c would enroll s as well. Lemma 2 states that student s would be enrolled even if it was available jointly with the students in A. The substitutability assumption implies that student s would be enrolled even if it was available jointly with the students in any subset $A' \subseteq A$. In the proof of next result, this fact will be used to determine the response of college c to unilateral deviations from equilibrium strategy of student s.

Proposition 1 *The CSM implements the stable set in SPE for every permutation σ.*

Proof. Without loss of generality assume that σ is the identity map on $\{1, ..., k\}$.[4] We first show that any stable matching is a SPE outcome. Let $\mu \in \Gamma(C, S, >)$. Consider the following strategy profile. Every student s applies to $\mu(s)$. Every college c_r plays according to Lemma 1. In order to prove that this profile of strategy is a SPE it suffices to prove that no student has a profitable deviation, but this follows directly from the stability of μ.

[4]If σ is not the identity map it suffices to relabel colleges and set $\tilde{c}_r = c_{\sigma(r)}$.

Next, we show show that every SPE outcome is stable. The proof is by contradiction. Let μ be a SPE outcome, by contradiction assume that there are r and $s \in S$ such that (c_r, s) blocks μ. Let $\left\{C(s), \left(\mu_{z_{r'}}(c_{r'})\right)_{z_{r'} \in Z_{r'}}\right\}_{s \in S}^{1 \leq r' \leq k}$ be a SPE yielding μ as outcome. Let $(z_r)_{1 \leq r \leq k}$ be the nodes on the equilibrium path. There are two possibilities.

(i) $\mu(s) = s$, or $\mu(s) = c_{\bar{r}}$ for some $\bar{r} > r$. Let $C(s)$ be student s equilibrium strategy. Consider the following deviation for s: apply to $C'(s) = C(s) \cup \{c_r\}$. Let $(z'_r)_{1 \leq r \leq k}$ the set of nodes on the path induced by this deviation. From Lemma 1, we have $S_{z'_{r'}} = S_{z_{r'}}$ for all $r' < r$ and $S_{z'_r} = S_{z_r} \cup \{s\}$. From Lemma 2 it follows that $s \in Ch_{c_r}(S_{z_r} \cup \{s\})$ so the deviation is profitable to s, yielding a contradiction.

(ii) $\mu(s) = c_{\bar{r}}$ for some $\bar{r} < r$. Consider the following deviation for s: $C'(s) = \{c_r\}$. Let $(z'_r)_{1 \leq r \leq k}$ the set of nodes on the path induced by this deviation. Let ν be be the outcome matching product of this deviation. From substitutability and Lemma 1 we have $S_{z'_{r'}} = S_{z_{r'}} \setminus \{s\}$ for all $r' \leq \bar{r}$. For the same reasons, $\nu(c_{r'}) = \mu(c_{r'})$ for $r' < \bar{r}$ and $\mu(c_{\bar{r}}) \setminus \{s\} \subseteq \nu(c_{\bar{r}})$. Next, we prove that the following two properties hold: a) $S_{z'_{r'}} \subseteq S_{z_{r'}}$ and b) $\mu(c_{r'}) \cap S_{z'_{r'}} \subseteq \nu(c_{r'})$ for all r', $\bar{r} \leq r' < r$. Notice that if a) holds then b) holds as well because preferences are substitutable. Thus, it suffices to show that a) holds. We prove the the claim by contradiction. We have already shown that properties a) and b) hold for $r' = \bar{r}$. By contradiction assume that there exits $r' > \bar{r}$ such that a) does not hold. Let $\hat{r} > \bar{r}$ be the minimum of such r'. Let $s' \in S_{z'_{\hat{r}}} \setminus S_{z_{\hat{r}}}$. It follows that $s' \in \mu(c_{r'}) \cap S_{z'_{r'}}$ for some r', $\bar{r} < r' < \hat{r}$. However, this is impossible because $s' \in \mu(c_{r'}) \cap S_{z'_{r'}} \subseteq \nu(c_{r'})$ by the minimality of $\hat{r}$. As every $r' < \hat{r}$ satisfies properties a) and b), we have that $S_{z'_r} \setminus \{s\} \subseteq S_{z_r}$ and $s \in S_{z'_r}$. From Lemma 2 it follows that $s \in Ch_{c_r}(S_{z_r} \cup \{s\})$. Preferences are substitutable so $s \in Ch_{c_r}\left(S_{z'_r}\right)$. Thus, the deviation is profitable to s, which yields a contradiction. ■

The proof of the sufficient part of Proposition 1 shows that, given a stable matching μ, the strategy profile where every student s applies to $\mu(s)$ and each college uses its dominant strategy is a SPE yielding μ as outcome.
The proof of the necessary part is in two steps. The first one shows that no pair blocking a SPE outcome matching μ can involve a college c and a student s such that student s joins a colleges which decides after college c. The results follows directly from Lemma 1: if the claim was false student s could profitably deviate by including college c in her application. Indeed, in the subgames induced by this deviation the set of the available applicants for the colleges preceding c would not vary with respect to the ones on the equilibrium path. So college c, at its turn, would admit student s. The second step shows that no pair blocking a SPE outcome matching μ can involve a college c and a student s such that student s joins a college which decides before college c. In the proof we show that if the claim was false student s could profitably deviate by applying to college c only. The result follows from the fact that, in the subgame induced by this deviation, every college moving before c must admit the same students that admitted on the equilibrium path and are still available. This implies that the set of available students shrinks (with respect to the equilibrium path) for every college which has to move before c. So college c would have at most the same available applicants as on the equilibrium path plus s. From Lemma 2 it follows that college c would admit student s, then the deviation would be profitable for s, which leads to a contradiction

There are two features of the mechanism that determine Proposition 1: colleges decision are sequential and no student is allowed to move twice during the procedure which means that decisions are irreversible. It follows that each college, at its turn, has a straightforward dominant strategy: to enroll its favorite available applicants. In this sense, the mechanism is an extension of a serial dictatorship. The fact that students are committed to the outcome of their first stage application key ingredient of the result. To understand this last point, consider a similar mechanism where colleges decide sequentially and students decisions are reversible. More precisely, consider a mechanism where, at the first stage students simultaneously apply to some colleges. Each college can select any among its applicants. Finally, every student decides which college to join among the one who admitted her. The "alternative sequential mechanism" (ASM from now on) is defined by the following procedure.

0. **Application.** Each student s applies to a set of colleges. Let $C(s) \subseteq 2^C$ be the set of colleges student s applies to. For every college c let $S(c) = \bigcup_{c \in C(s)} \{s\}$ be the set of students applying to college c.

For $1 \leq r \leq k$:

r. **Selection.** College $c_{\sigma(r)}$ decides the students to accept among the ones in $S\left(c_{\sigma(r)}\right)$. Denote by $S_1\left(c_{\sigma(r)}\right) \subseteq S\left(c_{\sigma(r)}\right)$ the decision of college $c_{\sigma(r)}$. Let $C_1(s) = \bigcup_{s \in S_1(c)} \{c\}$ be the set of colleges that have accepted student s.

$k+\mathbf{1}$. **Enrollment.** Each student s decide which college to join, among the ones belonging to $C_1(s)$. Formally, student s has to choose a mate $\mu(s) \in C_1(s) \cup \{s\}$.

The outcome of the game is the matching μ defined at stage $k+1$.

In this mechanism, students have to move twice: first when they apply to colleges at stage 0, then when they decide which offer to accept, at stage $k+1$. Colleges no longer have a dominant strategy: the selection of a student is not a definitive acceptance because it must be ratified by the student at the last stage of the game. This feature implies that a student who applies to more than one college generates a problem of coordination. The following example compares the performance of the two mechanisms in subgames where colleges move.

Example 1 *There are two colleges and four students:* $C = \{c_1, c_2\}$ *and* $S = \{s_1, s_2, s_3, s_4\}$*. The preferences of the colleges are responsive and* $>_{c_1}: s_3 s_2 s_1$, $>_{c_2}: s_1 s_4 s_3$*. Assume that quotas are* $q_{c_1} = 1$ *and* $q_{c_2} = 2$*. The preferences of the students are as follows:* $>_{s_1}: c_1 c_2$, $>_{s_2}: c_1 c_2$, $>_{s_3}: c_2, c_1$ *and* $>_{s_4}: c_2$*. There is a unique stable matching* μ*, where* $\mu(c_1) = \{s_3\}$ *and* $\mu(c_2) = \{s_1, s_4\}$*. Assume that college* c_1 *is the first to move.*
Let us start by analyzing the ASM. Consider the subgame where student s_1 *and student* s_3 *have applied to both colleges, student* s_2 *has not applied to any college and student* s_4 *has applied only to college* c_2*. Consider the following strategy profile for colleges. College* c_1 *accepts students* s_1 *only. College* c_2 *accepts* $\{s_1, s_4\}$ *in the nodes where she college* c_1 *has not accepted student* s_1*, she accepts* $\{s_3, s_4\}$ *in the nodes where she college* c_1 *has accepted student* s_1*. The outcome matching is defined by* $\nu(c_1) = \{s_1\}$ *and* $\nu(c_2) = \{s_3, s_4\}$*. The matching* ν *is blocked by* (c_1, s_2) *and is a SPE of the subgame. First, notice that college* c_2 *has no profitable deviations. In the nodes where* c_1 *has accepted* s_1 *she cannot improve because* c_1 *has already accepted* s_1 *and* s_1 *prefers* c_1 *to* c_2*. In the nodes where* c_1 *has not accepted* s_1 *college* c_2 *enrolls its favorite subset of students. Students play their dominant strategies at this point: to accept the best offer they hold. Student* s_1 *and* s_3 *benefit from the implementation of matching* ν*, because they end up joining their favorite college.*
Notice that the strategies of the colleges are not strongly nor weakly dominated. However, the strategy of college c_1 *is strongly dominated in the corresponding subgame when the CSM is used. Indeed, accepting student* s_3 *alone is a strictly dominant strategy for college* c_1*, independently on the moves of college* c_2*. It is easy to check that the unique SPE outcome of the subgame considered is the stable matching* μ*.*

Lemma 1 does not apply to the alternative sequential mechanism, so one can't use the same argument when dealing with this mechanism. More important, if the ASM is employed, unstable matchings can emerge as SPE outcomes.

Example 2 *Consider the ASM and the matching market of Example 1. Let* $z = \{C(s)\}_{s\in S}$ *be an application profile for the students. Define the profile of preferences* $>^z_S$ *for the students as follows: for every* $s \in S$ $c >^z_s s$ *if and only if* $c \in C(s) \cap A(s)$, $c >^z_s c'$ *if and only of* $c >_s c'$ *for all* $c, c' \in C(s) \cap A(s)$. *The preference profile* $>^z_S$ *coincides with* $>_s$ *on the set of acceptable colleges student* s *applied to and ranks as not acceptable all other colleges. For every possible profile of applications let* μ_z *be the students-optimal stable matching of the market where agents' preferences are* $>^z = (>_C, >^z_S)$. *Consider the following application strategy for students:* s_1 *and* s_3 *apply to both colleges, student* s_2 *does not apply to any college and student* s_4 *applies to college* c_2 *only. At the last stage of the game, each student accept the best offer she holds. Every college* c *accepts the students in* $\mu_z(c)$. *The profile of strategy yields the unstable matching* ν *of Example 1 as outcome.*
We now prove that this profile of strategy is a SPE. We first show that the profile of strategy is a SPE in the subgame where students have applied according to z, *for every* z. *By contradiction, assume that college* c *has a profitable deviation. Then, there exists* s *such that* s *applied to* c *and such that* $c >_s \mu_z(s)$ *and* $\{s\} >_c \{s'\}$ *for some* $s' \in \mu_z(c)$ *thus* (c, s) *blocks* μ_z *according to preferences* $>^z$, *which yields a contradiction. In order to complete the proof of the claim it suffices to prove that* s_2 *has no profitable deviations where she applies to* c_1, *because each of the other students joins her favorite college. If student* s_2 *applies to* $\{c_1\}$ *or to* $\{c_1, c_2\}$ *the outcome matching of the subgame induced by this deviation is* μ *and* $\mu(s_2) = \nu(s_2) = s_2$, *so she has no profitable deviation.*
Notice that the matching ν *is no longer a SPE outcome if the CSM is employed. The reader can easily check that in every strategy profile leading to matching* ν, *student* s_2 *has a profitable deviation: to apply to college* c_1.

Triossi (2009) introduces application costs in a mechanism with multiple applications, where colleges decide simultaneously and students select the best offer they receive. Applications costs pin down the number of equilibrium applications of each student to one, mimicking the result of Alcalde and Romero-Medina (2000), where each student cannot send more than one application by design. The CSM works without the introduction application costs and the number of SPE applications is not necessarily one. For instance, applying to all colleges is part of a SPE if the preferences are the ones used for Example 1.

From Proposition 1 it follows that the set of SPE outcomes does not depend on the order in which colleges decide. However equilibrium application strategies differ depending on σ.

Example 3 *Let $C = \{c_1, c_2\}$ and let $S = \{s_1, s_2, s_3\}$. Assume that the preference of the colleges are responsive. Let $>_{c_1}$: $s_3 s_1 s_2$ and $>_{c_2}$: $s_2 s_3 s_1$. Let $q_{c_1} = 1$ and $q_{c_2} = 2$. Let $>_{s_1}$: $c_2 c_1$, let $>_{s_2}$: $c_{1 s_2} c_2$ and let $>_{s_3}$: $c_2 c_1$. There are two stable matchings μ and ν, where, $\mu(c_2) = \{s_2, s_3\}$, $\nu(c_1) = \{s_2\}$, $\nu(c_2) = \{s_1, s_3\}$. Assume that c_1 moves first. Consider the following profile of strategy: $C(s_2) = C(s_3) = \{c_1, c_2\}$ and $C(s_1) = \{c_1\}$. Colleges play their equilibrium strategies (see Lemma 1) at each node of the enrollment stages. It is easy to check that this profile of strategies is a SPE yielding μ as outcome. Next, consider the case where c_2 is the first to choose. In this case, the application strategy $C(s_2) = C(s_3) = \{c_1, c_2\}$ and $C(s_1) = \{c_1\}$ is not part of any SPE. Indeed, if these strategies were part of a SPE the outcome matching would be the matching λ defined by$\lambda(c_1) = \{s_3\}$, $\lambda(c_2) = \{s_1, s_2\}$, which is blocked by (c_2, s_3). But this contradicts Proposition 1.*

3.2 The colleges-propose-students-sequentially-choose mechanism

Let $\sigma \in \Sigma_t$. The SSM is described by the following procedure:

0. **Proposals.** Each college c propose to a set of students. Let $S(c)$, be the set of students receiving an offer from c. For every student s let $C(s) = \bigcup_{s \in S(c)} \{s\}$ be the set of colleges that make an offer to s.

For $1 \leq r \leq t$

r. **Enrollment.** Student $s_{\sigma(r)}$ decides whether to join a college in $C_r(s_{\sigma(r)}) = C\left(s_{\sigma(r)}\right)$ or staying out of the system. Let $\mu_r\left(s_{\sigma(r)}\right)$ be student $s_{\sigma(r)}$ choice.

The outcome matching is defined by $\mu\left(s_{\sigma(r)}\right) = \mu_r\left(s_{\sigma(r)}\right)$ for every r, $1 \leq r \leq t$.

At the first stage of the game, the strategy set of each college is 2^S. Let Z_r^σ, $1 \leq r \leq l$ be the set of subgames at the enrollment stage when student $s_{\sigma(r)}$ has to move. Every $z_r \in Z_r^\sigma$ is completely characterized by the family of sets of students each college proposed to and by the decision made by the students who had to choose before student $s_{\sigma(r)}$, if any. Formally, $z_1 \in Z_1^\sigma$ is characterized by $\{S_{z_1}(c)\}_{c\in C}$, where $S_{z_1}(c)$ is the set of students college c proposed to. For $r \geq 2$, z_r is completely characterized by $\left\{S_{z_1}(c), \mu_{z_{r'}}\left(s_{\sigma(r')}\right)\right\}_{c\in C}^{1\leq r'<r}$, where $z_{r'} = z_{r'}(z_r) \in Z_1$ is the unique stage r' node preceding z_r and $\mu_{z_{r'}}\left(s_{\sigma(r')}\right)$ is the choice of student $s_{\sigma(r')}$ at z_r', for $1 \leq r' < r$. The strategy space of student $s_{\sigma(r)} \in S$ at z_r is $C_{z_1}(s) = \bigcup_{s_{\sigma(r)}\in S_{z_1}(c)} \{c\} \cup \{s\}$, the set of college who has offered a position to student $s_{\sigma(r)}$, where $z_1 = z_1(z_r)$.

Notice that the strategy space of of student s is depends only on the proposals she receives. At her turn, every student has a strictly dominant strategy, to accept the best offer she receives. For every $z_1 \in Z_1^\sigma$ set $\tau_{z_1}(s) = \arg\max_{P_s} C_{z_1}(s)$. We have:

Lemma 3 *At any SPE of the SSM, student $s_{\sigma(r)}$ plays strategy $\tau_{z_0}\left(s_{\sigma(r)}\right)$, where $z_0 = z_0(r)$.*

With this result in mind we can show that Theorems 1 and 2 in Sotomayor (2003) extend to the many-to-one case if colleges propose to students.

Proposition 2 *The SSM implement the stable set in SPE for every permutation σ.*

Proof. Without loss of generality assume that σ is the identity map. First, we prove that any stable matching is a SPE outcome. Let μ be stable in $(C, S, >)$. Consider the following strategy profile. Every college c proposes only the students in $\mu(c)$. At her turn, every student plays according to Lemma 3. The strategies are optimal at every subgame z_r. The stability of μ implies that no college can profitably deviate.
Next, we show that any SPE yields a stable matching. Let μ be a SPE outcome. We prove that if μ is not stable in $\Gamma(C, S, >)$, some agent has a profitable deviation. First, μ is individually rational for colleges and for students. If μ was not individually rational for student $s_{\sigma(r)}$, she could profitably deviate by dropping all proposals she holds. If μ was not individually rational for college c, proposing only to the students in $Ch_c(\mu(c))$ would be a profitable deviation for c. Indeed, in the subgame induced by this deviation, c would be the best offer received by the students in $Ch_c(\mu(c))$. Let (c, s) be a college-student pair. If (c, s) blocks μ, consider the following deviation for c: propose only to the students in $Ch_c(\mu(c) \cup \{s\})$. From Lemma 3 it follows that the deviation would be profitable to c. ■

4 Conclusions

This paper studies two sequential admission mechanisms in a many-to-one matching framework. Both mechanisms exhibit characteristics common to some real-world procedures and implement the stable set in SPE. When restricted to the one-to-one case they coincide with the admission mechanism presented by Sotomayor (2003). The procedures generalize a serial dictatorship and extend the model by Alcalde and Romero-Medina (2000). Also, the Students-Propose-Colleges-Sequentially-Choose mechanism solves the coordination problem which was observed in Triossi (2009).

There are straightforward extensions that would help to explain some features of real market procedures. In particular, they would shed some light on the anomalies related with the timing of the market (see, for instance, Niederle and Roth 2003). A first extension could consider a model where the order in which colleges or students act is endogenous. Indeed in many decentralized admission mechanism colleges endogenously determine admission dates. Another extension would consider a model where colleges act sequentially and make exploding proposals to students (see also Haeringer and Wooders 2010).

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