K.H. Kwon · D.W. Lee · F. Marcellán · S.B. Park

On Kernel polynomials and self-perturbation of orthogonal polynomials

Abstract. Given an orthogonal polynomial system $\{Q_n(x)\}_{n=0}^{\infty}$, def ne another polynomial system by

$$P_n(x) = Q_n(x) - \alpha_n Q_{n-t}(x), \quad n \geq 0,$$

where α_n are complex numbers and t is a positive integer. We fi d conditions for $\{P_n(x)\}_{n=0}^{\infty}$ to be an orthogonal polynomial system. When $t = 1$ and $\alpha_1 \neq 0$, it turns out that $\{Q_n(x)\}_{n=0}^{\infty}$ must be kernel polynomials for $\{P_n(x)\}_{n=0}^{\infty}$ for which we study, in detail, the location of zeros and semi-classical character.

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1. Introduction

For an orthogonal polynomial system $\{P_n(x)\}_{n=0}^{\infty}$ and a complex number λ with $P_n(\lambda) \neq 0$, $n \geq 1$, its kernel polynomial system $\{P_n^*(\lambda; x)\}_{n=0}^{\infty}$ can be introduced (cf. [3] and [3, p.35]) and $(x-\lambda)P_n^*(\lambda; x)$ can be expressed as a linear combination of $P_n(x)$ and $P_{n-1}(x)$. Conversely, $P_n(x)$ can be expressed as a linear combination of $P_n^*(\lambda; x)$ and $P_{n-1}^*(\lambda; x)$ (cf. (2.5)). In fact, this last property characterizes kernel polynomial systems: an OPS $\{Q_n(x)\}_{n=0}^{\infty}$ is a kernel polynomial system for some other orthogonal polynomial system if and only if

$$Q_n(x) - \alpha_n Q_{n-1}(x), \quad n \geq 0 \tag{1.1}$$

becomes an orthogonal polynomial system for some complex numbers α_n, with $\alpha_1 \neq 0$ (cf. Theorem 3.2). We may view (1.1) as a self-perturbation of orthogonal

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polynomials $\{Q_n(x)\}_{n=0}^{\infty}$. More generally, we may ask: for a fi ed integer $t \geq 1$, given an orthogonal polynomial system $\{Q_n(x)\}_{n=0}^{\infty}$, when is the polynomial system

$$Q_n(x) - \alpha_n Q_{n-t}(x), \quad (n \geq 0) \tag{1.2}$$

also an orthogonal polynomial system? Here, α_n are complex numbers. Geronimus [5] posed and solves completely the case $t = 1$. Later on Marcellán and Petronilho [12] stated the connection with the concept of coherent pairs of orthogonal polynomials. See also [2] for a similar problem.

In this work, we f rst study kernel polynomials (i.e., the case when $t = 1$) in some more detail, including their zero distribution and semi-classical character and then f nd necessary and suff cient conditions for polynomials def ned by (1.2) for $t \geq 2$ to be orthogonal polynomials.

2. Preliminaries

Let $\mathcal{P}$ be the space of all polynomials in one variable with complex coeff cients and $\deg(\phi)$ the degree of $\phi(x)$ in $\mathcal{P}$ with the convention that $\deg(0) = -1$. By a polynomial system (PS), we mean a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with $\deg(P_n) = n$, $n \geq 0$.

We call any linear functional σ on $\mathcal{P}$ a moment functional and denote its action on a polynomial $\phi(x)$ by $\langle \sigma, \phi \rangle$. We say that a moment functional σ is quasi-defi ite (respectively, positive-defi ite) if its moments $\sigma_n := \langle \sigma, x^n \rangle$ satisfy the Hamburger condition

$$\Delta_n(\sigma) := \det[\sigma_{i+j}]_{i,j=0}^{n} \neq 0$$
$$\text{(respectively, all } \sigma_n \text{ are real and } \Delta_n(\sigma) > 0), \quad n = 0, 1, \ldots .$$

Definition 2.1. *A PS $\{P_n(x)\}_{n=0}^{\infty}$ is called an orthogonal polynomial system (OPS) (respectively, a positive-definite OPS) if there is a moment functional σ such that*

$$\langle \sigma, P_m P_n \rangle = K_n \delta_{mn}, \qquad m, n = 0, 1, \ldots ,$$

where $K_n \neq 0$ (respectively, $K_n > 0$), $n \geq 0$. In this case, we call $\{P_n(x)\}_{n=0}^{\infty}$ an OPS (respectively, a positive-definite OPS) relative to σ.

It is well known (see Theorem 3.1, Chapter 1 in [4]) that a moment functional σ is quasi-defi ite if and only if there is an OPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ. Moreover, in this case, each $\{P_n(x)\}_{n=0}^{\infty}$ is uniquely determined by σ up to a non-zero constant factor. In particular, if each $P_n(x)$ is monic, we call $\{P_n(x)\}_{n=0}^{\infty}$ a monic OPS (MOPS) relative to σ. It's also well known that if $\{P_n(x)\}_{n=0}^{\infty}$ is an MOPS relative to a positive-defi ite moment functional σ, then each $P_n(x)$, $n \geq 1$, has n real simple zeros, which interlace with zeros of $P_{n+1}(x)$. We then let $[\xi, \eta]$ $(-\infty \leq \xi < \eta \leq \infty)$ be the true interval of orthogonality for $\{P_n(x)\}_{n=0}^{\infty}$ or σ, which is the smallest closed interval containing all zeros of $P_n(x)$, $n \geq 1$.

Due to Favard's theorem, a monic PS $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS (respectively, a positive-defi ite OPS) if and only if $\{P_n(x)\}_{n=0}^{\infty}$ satisfie a three-term recurrence relation:

$$P_{n+1}(x) = (x - b_n)P_n(x) - c_n P_{n-1}(x), \quad n \geq 0, \quad (P_{-1} = 0), \tag{2.1}$$

where $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are complex numbers with $c_n \neq 0$, $n \geq 1$ (respectively, b_n, $n \geq 0$, are real and $c_n > 0$, $n \geq 1$). For an MOPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ satisfying (2.1), we let $K_n(x, y)$, $\{P_n^{(1)}(x)\}_{n=0}^{\infty}$, and $\{P_n(c; x)\}_{n=0}^{\infty}$ be the n-th kernel polynomial, the associated MOPS of the f rst kind, and the monic co-recursive OPS, respectively define by

$$K_n(x, y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\langle \sigma, P_k^2 \rangle}, \quad n \geq 0;$$

$$P_{n+1}^{(1)}(x) = (x - b_{n+1})P_n^{(1)}(x) - c_{n+1} P_{n-1}^{(1)}(x), \quad n \geq 0 \quad (P_{-1}^{(1)}(x) = 0);$$

and

$$P_{n+1}(c; x) = (x - b_n)P_n(c; x) - c_n P_{n-1}(c; x), \quad n \geq 1,$$

where $P_0(c; x) = 0$, $P_1(c; x) = P_1(x) - c$, and c is a complex number.

For a moment functional σ, a polynomial $\pi(x)$, and a complex number λ, we let σ', $\pi\sigma$, and $(x - \lambda)^{-1}\sigma$ be the moment functionals def ned by

$$\langle \sigma', P \rangle = -\langle \sigma, P' \rangle, \quad \langle \pi\sigma, P \rangle = \langle \sigma, \pi P \rangle, \quad P \in \mathcal{P},$$

and

$$\langle (x - \lambda)^{-1}\sigma, P \rangle = \langle \sigma, \frac{P(x) - P(\lambda)}{x - \lambda} \rangle, \quad P \in \mathcal{P}.$$

Definition 2.2 (Maroni [13]). *A moment functional σ is called semi-classical if σ is quasi-definite and satisfies a Pearson-type functional equation*

$$(\alpha(x)\sigma)' - \beta(x)\sigma = 0 \tag{2.2}$$

for some polynomials $\alpha(x)$ and $\beta(x)$ with $\deg(\alpha) \geq 0$ *and* $\deg(\beta) \geq 1$.

For a semi-classical moment functional σ, we call

$$s := \min \max(\deg(\alpha) - 2, \deg(\beta) - 1)$$

the class number of σ, where the minimum is taken over all pairs $(\alpha, \beta) \neq (0, 0)$ of polynomials satisfying (2.2). We call an OPS relative to a semi-classical moment functional σ of class s a semi-classical OPS (SCOPS) of class s.

Next, we state two lemmas, which will be needed later.

Lemma 2.3. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS relative to a quasi-definite moment functional σ. Then for any moment functional τ, $\langle \tau, P_n \rangle = 0$, $n \geq k + 1$ for some integer $k \geq 0$ if and only if $\tau = \phi\sigma$ for some polynomial $\phi(x)$ of degree $\leq k$.*

Proof. See Lemma 2.2 in [9]. □

Lemma 2.4. *Let σ and τ be moment functionals and λ a complex number. Then $(x-\lambda)\sigma = \tau$ if and only if*

$$\sigma = (x-\lambda)^{-1}\tau + \sigma_0\delta(x-\lambda).$$

Proof. It is straightforward since the action of the moment functionals of both sides on $(x-\lambda)^n$, $n \geq 0$, coincides. □

We now recall a few well-known facts on kernel polynomials (cf. [4]).

Proposition 2.5. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an MOPS relative to σ. Then for a complex number λ, $(x-\lambda)\sigma$ is also quasi-definite if and only if $P_n(\lambda) \neq 0$, $n \geq 0$. In this case, the MOPS relative to $(x-\lambda)\sigma$ is given by*

$$P_n^*(\lambda; x) = \frac{1}{P_n(\lambda)} \cdot \frac{P_{n+1}(x)P_n(\lambda) - P_{n+1}(\lambda)P_n(x)}{x-\lambda} = \frac{\langle \sigma, P_n^2 \rangle}{P_n(\lambda)} K_n(x,\lambda), \quad n \geq 0, \tag{2.3}$$

where

$$K_n(x;\lambda) := \sum_{k=0}^{n} \frac{P_k(x)P_k(\lambda)}{\langle \sigma, P_k^2 \rangle}$$

is the n-th kernel polynomial for $\{P_n(x)\}_{n=0}^{\infty}$ and

$$\langle (x-\lambda)\sigma, P_n^*(\lambda; x)^2 \rangle = -\frac{P_{n+1}(\lambda)}{P_n(\lambda)} \langle \sigma, P_n^2 \rangle, \quad n \geq 0. \tag{2.4}$$

Moreover, if σ is positive-definite and $[\xi, \eta]$ is the true interval of orthogonality for σ, then the following are all equivalent:

(i) $(x-\lambda)\sigma$ *is positive-definite;*
(ii) $\operatorname{sgn} P_n(\lambda) = (-1)^n$, $n \geq 1$*;*
(iii) $\lambda \leq \xi$ *(in particular, $-\infty < \xi$ and λ must be real);*
(iv) $(x-\lambda)\sigma$ *is positive-definite on* $[\xi, \eta]$.

Proof. See Theorem 7.1 in [4, Chapter 1]. □

Due to (2.3), we call $\{P_n^*(\lambda; x)\}_{n=0}^{\infty}$ the monic kernel polynomial system (MKPS) for $\{P_n(x)\}_{n=0}^{\infty}$ (or for σ) with K-parameter λ. Conversely, we may express $P_n(x)$ in terms of $\{P_n^*(\lambda; x)\}_{n=0}^{\infty}$. By (2.3), we have

$$\frac{P_n(x)P_n(\lambda)}{<\sigma, P_n^2>} = K_n(x,\lambda) - K_{n-1}(x,\lambda) = \frac{P_n(\lambda)}{<\sigma, P_n^2>} P_n^*(\lambda; x) - \frac{P_{n-1}(\lambda)}{<\sigma, P_{n-1}^2>} P_{n-1}^*(\lambda; x)$$

so that

$$(2.5) \qquad P_n(x) = P_n^*(\lambda; x) - \frac{P_{n-1}(\lambda)}{P_n(\lambda)} c_n P_{n-1}^*(\lambda; x), \quad n \geq 0 \ \ (P_{-1}^*(\lambda; x) = 0),$$

where c_n are the coefficient of the three-term recurrence relation (2.1) for $\{P_n(x)\}_{n=0}^\infty$. Hence $P_n(x)$ is quasi-orthogonal (see Definitio 5.1 in [4, Chapter 2]) of order 1 relative to $(x-\lambda)\sigma$. In fact, relation (2.5) characterizes kernel polynomials completely as we shall see later (see Theorem 3.2).

3. Kernel polynomials

By Proposition 2.5, we see that for any MOPS $\{P_n(x)\}_{n=0}^\infty$ relative to σ and any K-parameter λ with $P_n(\lambda) \neq 0$, $n \geq 1$, its MKPS $\{P_n^*(\lambda; x)\}_{n=0}^\infty$ is uniquely determined by (2.3). We now ask: when is a given MOPS $\{Q_n(x)\}_{n=0}^\infty$ also an MKPS for some other MOPSs?

Lemma 3.1. *Let σ be a quasi-definite moment functional and $u = \sigma + c\delta(x-\lambda)$ $(c, \lambda \in \mathbb{C})$. Then, u is also quasi-definite if and only if*

$$d_n := 1 + cK_n(\lambda, \lambda) \neq 0, \quad n \geq 0,$$

where $K_n(x,y)$ is the n-th kernel polynomial for σ. In this case, the monic OPS $\{R_n(x)\}_{n=0}^\infty$ relative to u is given by

$$R_n(x) = P_n(x) - c\frac{P_n(\lambda)}{d_{n-1}} K_{n-1}(x, \lambda), \quad n \geq 0 \ (d_{-1} = 1, \ K_{-1}(x, y) = 0).$$

Proof. See Corollary 3.2 in [10] (see also [11]). □

Theorem 3.2. *For an MOPS $\{Q_n(x)\}_{n=0}^\infty$, the following are all equivalent:*

(i) $\{Q_n(x)\}_{n=0}^\infty$ *is an MKPS for some other OPS;*

(ii) *there are complex numbers λ, $a_n \neq 0$ $(n \geq 0)$, and an MOPS $\{P_n(x)\}_{n=0}^\infty$ such that*

$$(3.1) \qquad (x-\lambda)Q_n(x) = P_{n+1}(x) - a_n P_n(x), \quad n \geq 0;$$

(iii) *there are complex numbers α_n $(n \geq 1)$ such that $\alpha_1 \neq 0$ and*

$$Q_n(x) - \alpha_n Q_{n-1}(x), \quad n \geq 0 \quad (Q_{-1}(x) = 0),$$

form an MOPS.

In this case, $\{Q_n(x)\}_{n=0}^\infty$ is an MKPS for a quasi-definite moment functional τ with K-parameter λ if and only if

$$(3.2) \qquad \tau = a\sigma + b\delta(x-\lambda),$$

where σ is an orthogonalizing moment functional for $\{P_n(x)\}_{n=0}^\infty$ in (ii) and a $(\neq 0)$ and b are complex numbers satisfying

$$(3.3) \qquad a + bK_n(\lambda, \lambda) \neq 0, \quad n \geq 0,$$

where $K_n(x,y)$ is the n-th kernel polynomial for σ.

Proof. (i)⇒(ii): Assume $\{Q_n(x)\}_{n=0}^{\infty} = \{P_n^*(\lambda; x)\}_{n=0}^{\infty}$. Then we have (3.1) with $a_n = \frac{P_{n+1}(\lambda)}{P_n(\lambda)}$ by Proposition 2.5.

(ii)⇒(i): Assume that (ii) holds. Then

$$\begin{aligned}\langle (x-\lambda)\sigma, x^k Q_n(x)\rangle &= \langle \sigma, x^k(P_{n+1}(x) - a_n P_n(x))\rangle \\ &= -a_n\langle \sigma, P_n^2\rangle \delta_{kn}, \quad 0 \le k \le n,\end{aligned}$$

so that $\{Q_n(x)\}_{n=0}^{\infty}$ is an MOPS relative to $(x-\lambda)\sigma$. Hence, $\{Q_n(x)\}_{n=0}^{\infty} = \{P_n^*(\lambda; x)\}_{n=0}^{\infty}$ by Proposition 2.5.

(i)⇒(iii): This follows immediately from (2.5).

(iii)⇒(i): See Theorem 4.2.

Now, assume $\{Q_n(x)\}_{n=0}^{\infty} = \{P_n^*(\lambda; x)\}_{n=0}^{\infty}$. If $\{Q_n(x)\}_{n=0}^{\infty}$ is also an MKPS for τ with K-parameter λ, then $\{Q_n(x)\}_{n=0}^{\infty}$ is an MOPS relative to $(x-\lambda)\sigma$ and $(x-\lambda)\tau$. Hence $(x-\lambda)\tau = a(x-\lambda)\sigma$ for some $a \neq 0$ so that we have (3.2) with $b = \tau_0$. Since τ is quasi-definite, (3.3) follows from Lemma 3.1. Conversely, if τ is given by (3.2) and (3.3) holds, then τ is quasi-definite by Lemma 3.1 and $(x-\lambda)\tau = a(x-\lambda)\sigma$. Hence $\{Q_n(x)\}_{n=0}^{\infty}$ is an MOPS relative to $(x-\lambda)\tau$ (and $(x-\lambda)\sigma$) so that $\{Q_n(x)\}_{n=0}^{\infty}$ must be an MKPS for τ with K-parameter λ. □

Theorem 3.2 characterizes MKPSs and shows that an MOPS can be an MKPS for infinitely many distinct MOPSs with the same K-parameter. In fact, by Lemma 3.1, if $\{Q_n(x)\}_{n=0}^{\infty}$ is an MKPS for $\{P_n(x)\}_{n=0}^{\infty}$ with K-parameter λ, then $\{Q_n(x)\}_{n=0}^{\infty}$ is also an MKPS for $\{P_n(x) - \frac{b}{a}\cdot\frac{P_n(\lambda)}{d_{n-1}}K_{n-1}(x,\lambda)\}_{n=0}^{\infty}$ with K-parameter λ for any complex numbers $a \neq 0$ and b satisfying (3.3).

Quasi-orthogonality relations like (2.5) and (3.1) imply that there are close relations for relative locations of zeros of $\{P_n(x)\}_{n=0}^{\infty}$ and $\{P_n^*(\lambda; x)\}_{n=0}^{\infty}$ when they are real polynomials and one of them is positive-definite.

Theorem 3.3. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an MOPS relative to a positive-definite moment functional σ and $[\xi, \eta]$ the true interval of orthogonality of σ. If λ is any real number such that $P_n(\lambda) \neq 0$, $n \ge 1$, then $P_n^*(\lambda; x)$ has n real simple zeros, which interlace with zeros of $P_n(x)$ and $P_{n+1}(x)$. More precisely, let $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ and $x_{n,1}^* < x_{n,2}^* < \ldots < x_{n,n}^*$ be zeros of $P_n(x)$ and $P_n^*(\lambda; x)$ respectively and $x_{n,0} = \xi$, $x_{n,n+1} = \eta$.*

(i) *If $\lambda \le \xi$, then $P_n^*(\lambda; x)$ has one zero in each $(x_{n,k}, x_{n+1,k+1})$, $k = 1, 2, \ldots, n$.*
(ii) *If $\lambda \in (x_{n,k-1}, x_{n+1,k})$ for some $k = 1, 2, \ldots, n+1$, then $P_n^*(\lambda; x)$ has one zero in each one of $(-\infty, x_{n+1,1})$ and $(x_{n,j-1}, x_{n+1,j})$, $j = 2, 3, \ldots, k-1, k+1, \ldots, n+1$.*
(iii) *If $\lambda \in (x_{n+1,k}, x_{n,k})$ for some $k = 1, 2, \ldots, n+1$, then $P_n^*(\lambda; x)$ has one zero in each one of $(x_{n+1,j}, x_{n,j})$, $j = 1, 2, \ldots, k-1, k+1, \ldots, n$ and $(x_{n+1,n+1}, \infty)$.*
(iv) *If $\lambda \ge \eta$, then $P_n^*(\lambda; x)$ has one zero in each $(x_{n+1,k}, x_{n,k})$, $k = 1, 2, \ldots, n$.*

Proof. We shall prove only (ii); the proofs for (i),(iii), and (iv) are similar. Assume $\lambda \in (x_{n,k-1}, x_{n+1,k})$ for some $1 \le k \le n+1$. Using

$$\begin{aligned}&\operatorname{sgn} P_n(-\infty) = (-1)^n, \quad \operatorname{sgn} P_n(x_{n+1,j}) = \operatorname{sgn} P_{n+1}(x_{n,j}) = (-1)^{n+1-j}, \\ &\quad 1 \le j \le n+1, \quad \operatorname{sgn} P_n(\infty) = +1,\end{aligned} \tag{3.4}$$

we can easily see that $\operatorname{sgn} P_n(\lambda)P_{n+1}(\lambda) = -1$ for $\lambda \in (x_{n,k-1}, x_{n+1,k})$. By (2.3),

$$P_n^*(\lambda; x_{n,j-1}) = (x_{n,j-1} - \lambda)^{-1} P_{n+1}(x_{n,j-1}), \quad 2 \le j \le n+1,$$

so that

$$\operatorname{sgn} P_n^*(\lambda; x_{n,j-1}) = \begin{cases} (-1)^{n+1-j}, & 2 \le j \le k \\ (-1)^{n-j}, & k+1 \le j \le n+1, \end{cases}$$

and

$$P_n^*(\lambda; x_{n+1,j}) = -(x_{n+1,j} - \lambda)^{-1} \frac{P_{n+1}(\lambda)}{P_n(\lambda)} P_n(x_{n+1,j}), \quad 1 \le j \le n+1,$$

so that

$$\operatorname{sgn} P_n^*(\lambda; x_{n+1,j}) = \begin{cases} (-1)^{n-j}, & 1 \le j \le k-1 \\ (-1)^{n+1-j}, & k \le j \le n+1. \end{cases}$$

Hence, $P_n^*(\lambda; x_{n,j-1})P_n^*(\lambda; x_{n+1,j}) < 0$ for $j = 2, 3, \ldots, k-1, k+1, \ldots, n+1$. Finally, $\operatorname{sgn} P_n^*(\lambda; -\infty) = (-1)^n$ so that $P_n^*(\lambda; -\infty)P_n^*(x_{n+1,1}) < 0$. Hence, the conclusion follows. □

In the case $\lambda \le \xi$ (and $\lambda \ge \eta$), $(x-\lambda)\sigma$ (respectively $(\lambda - x)\sigma$) is positive-def nite and Theorem 3.3 (i) was proved in [4]. Note that when $\xi < \lambda < \eta$, $\{P_n^*(\lambda; x)\}_{n=0}^\infty$ is never positive-defi ite. However, $\{P_n^*(\lambda; x)\}_{n=0}^\infty$ can be positive-defi ite even if $\{P_n(x)\}_{n=0}^\infty$ is not positive-def nite (cf. Theorem 4.4).

Theorem 3.4. *Let $\{P_n^*(\lambda; x)\}_{n=0}^\infty$ be an MKPS for a real MOPS $\{P_n(x)\}_{n=0}^\infty$ with real K-parameter λ. If $\{P_n^*(\lambda; x)\}_{n=0}^\infty$ is positive-definite, then $P_n(x)$ has n real simple zeros, which interlace with zeros of $P_{n-1}^*(\lambda; x)$ and $P_n^*(\lambda; x)$. To be more precise, we have*

$$x_{n-1,k-1}^* < x_{n,k} < x_{n,k}^*, \quad 1 \le k \le n \quad (x_{n-1,0}^* = -\infty),$$

if $c_n P_n(\lambda)P_{n-1}(\lambda) < 0$ and

$$x_{n,k}^* < x_{n,k} < x_{n-1,k}^*, \quad 1 \le k \le n \quad (x_{n-1,n} = \infty),$$

if $c_n P_{n-1}(\lambda)P_n(\lambda) > 0$.

Proof. It's a straightforward consequence of the relation (2.5) and (3.4). □

Finally in this section, let's consider the semi-classical character of MKPSs.

Lemma 3.5. *Let σ be a semi-classical moment functional satisfying the equation (2.2) and*

$$s := \max(\deg(\alpha) - 2, \deg(\beta) - 1). \tag{3.5}$$

Then, σ is of class s if and only if for any zero c of $\alpha(x)$

$$|r_c| + |\langle \sigma, q_c(x) \rangle| \neq 0,$$

where $\alpha(x) = (x-c)\alpha_c(x)$ and $\alpha_c(x) - \beta(x) = (x-c)q_c(x) + r_c$.

Proof. See Proposition 3.5 in [13]. □

Theorem 3.6. *If $\{P_n(x)\}_{n=0}^{\infty}$ is a SCOPS of class s relative to σ satisfying (2.2) and (3.5), then $\{P_n^*(\lambda; x)\}_{n=0}^{\infty}$ is a SCOPS of class $s-1$ when $\alpha(\lambda) = \beta(\lambda) = 0$, s when $\alpha(\lambda) = 0$ and $\beta(\lambda) \neq 0$, and $s+1$ when $\alpha(\lambda) \neq 0$. Conversely, if $\{P_n^*(\lambda; x)\}_{n=0}^{\infty}$ is a SCOPS of class s, then $\{P_n(x)\}_{n=0}^{\infty}$ is a SCOPS of class $s-1$ or s or $s+1$.*

Proof. Let $\tau = (x-\lambda)\sigma$ and $\alpha(x) = (x-c)\alpha_c(x)$, $\alpha_c(x)-\beta(x) = (x-c)q_c(x)+r_c$ for any zero c of $\alpha(x)$.

(a) Assume $\alpha(\lambda) = \beta(\lambda) = 0$. Then $(\alpha_\lambda(x)\tau)' = \beta_\lambda(x)\tau$, where $\beta(x) = (x-\lambda)\beta_\lambda(x)$ so that τ is of class $\leq s-1$. Let $\alpha_\lambda(x) = (x-c)\alpha_{\lambda,c}(x)$ and $\alpha_{\lambda,c}(x) - \beta_\lambda(x) = (x-c)q_{\lambda,c}(x) + r_{\lambda,c}$, for any zero c of $\alpha_\lambda(x)$. If $\alpha_\lambda(\lambda) = 0$, then $(\alpha_\lambda\sigma)' = (\beta_\lambda - \alpha_{\lambda,\lambda})\sigma$ so that σ is of class $\leq s-1$, which is a contradiction. Hence $\alpha_\lambda(\lambda) \neq 0$, that is, $c \neq \lambda$. Now, we have

$$r_c = \alpha_c(c) - \beta(c) = (c-\lambda)r_{\lambda,c}.$$

Hence, if $r_{\lambda,c} = 0$, then $r_c = 0$, $\langle\sigma, q_c\rangle \neq 0$, and

$$q_{\lambda,c}(x) = \frac{\alpha_{\lambda,c}(x) - \beta_\lambda(x)}{x-c} = \frac{1}{x-\lambda}\left[\frac{\alpha(x)}{(x-c)^2} - \frac{\beta(x)}{x-c}\right] = \frac{q_c(x)}{x-\lambda},$$

so that $\langle\tau, q_{\lambda,c}\rangle = \langle\sigma, (x-\lambda)q_{\lambda,c}\rangle = \langle\sigma, q_c\rangle \neq 0$. Therefore, τ is of class $s-1$ by Lemma 3.5.

(b) Assume $\alpha(\lambda) = 0$ and $\beta(\lambda) \neq 0$. Then $(\alpha\tau)' = (\alpha_\lambda + \beta)\tau$ so that τ is of class $\leq s$. Let $\alpha_c(x) - \alpha_\lambda(x) - \beta(x) = (x-c)\tilde{q}_c(x) + \tilde{r}_c$ for any zero c of $\alpha(x)$. First assume $c \neq \lambda$. Then $\alpha_\lambda(c) = 0$ so that $\tilde{r}_c = \alpha_c(c) - \beta(c) = r_c$. Hence if $\tilde{r}_c = 0$, then $r_c = 0$, $\langle\sigma, q_c\rangle \neq 0$, and

$$\tilde{q}_c(x) = q_c(x) - \frac{\alpha_\lambda(x)}{x-c}.$$

Hence

$$\begin{aligned}\langle\tau, \tilde{q}_c\rangle &= \langle\sigma, (x-\lambda)(q_c(x) - \tfrac{\alpha_\lambda(x)}{x-c})\rangle = \langle\sigma, (x-\lambda)q_c(x) - \alpha_c(x)\rangle \\ &= \langle\sigma, (x-c+c-\lambda)q_c(x) - \alpha_c(x)\rangle = (c-\lambda)\langle\sigma, q_c\rangle + \langle\sigma, \beta\rangle \\ &= (c-\lambda)\langle\sigma, q_c\rangle \neq 0\end{aligned}$$

since $\langle\sigma, \beta\rangle = \langle\beta\sigma, 1\rangle = \langle(\alpha\sigma)', 1\rangle = 0$ and $(x-c)q_c(x) = \alpha_c(x) - \beta(x)$. Now if $c = \lambda$, then $\tilde{r}_c = -\beta(\lambda) \neq 0$. Therefore, τ is of class s by Lemma 3.5.

(c) Assume $\alpha(\lambda) \neq 0$. Then $(\tilde{\alpha}\tau)' = \tilde{\beta}\tau$, where $\tilde{\alpha}(x) = (x-c)\alpha(x)$ and $\tilde{\beta}(x) = 2\alpha(x) + (x-\lambda)\beta(x)$, so that τ is of class $\leq s+1$. As before, let $\tilde{\alpha}(x) = (x-c)\tilde{\alpha}_c(x)$ and $\tilde{\alpha}_c(x) - \tilde{\beta}(x) = (x-c)\tilde{q}_c(x) + \tilde{r}_c$ for any zero c of $\alpha(x)$. If $c = \lambda$, then $\tilde{\alpha}_\lambda(x) = \alpha(x)$ so that $\tilde{r}_\lambda = -\alpha(\lambda) \neq 0$. Now assume $c \neq \lambda$. Then $\tilde{\alpha}_c(x) = (x-\lambda)\alpha_c(x)$ so that $\tilde{r}_c = \tilde{\alpha}_c(c) - \tilde{\beta}(c) = (c-\lambda)r_c$. Hence, if $\tilde{r}_c = 0$, then $r_c = 0$, $\langle\sigma, q_c\rangle \neq 0$, and

$$\tilde{q}_c(x) = \frac{\tilde{\alpha}_c(x) - \tilde{\beta}(x)}{x-c} = (x-\lambda)q_c(x) - 2\alpha_c(x) = (c-\lambda)q_c(x) - \beta(x) - \alpha_c(x),$$

so that

$$\langle \tau, \tilde{q}_c \rangle = \langle \sigma, (x-\lambda)\tilde{q}_c \rangle = \langle \sigma, (x-c+c-\lambda)\{(c-\lambda)q_c(x) - \beta(x) - \alpha_c(x)\} \rangle$$
$$= \langle \sigma, (c-\lambda)^2 q_c \rangle - \langle \sigma, \alpha \rangle - \langle \sigma, (x+c-2\lambda)\beta \rangle = (c-\lambda)^2 \langle \sigma, q_c \rangle \neq 0$$

since $(x-c)q_c(x) = \alpha_c(x) - \beta(x)$, $\langle \sigma, \beta \rangle = 0$, and $\langle \sigma, x\beta \rangle = -\langle \sigma, \alpha \rangle$. Therefore, τ is of class $s+1$ by Lemma 3.5.

Now, the converse is trivial. □

The last case when $\alpha(\lambda) \neq 0$ was proved by Belmehdi (see Theorem 3.1 in [1]), where the structure relation for $\{P_n^*(\lambda; , x)\}_{n=0}^{\infty}$ is also given.

As a consequence of Theorem 3.6, we have: if $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS satisfying a second-order linear differential equation

$$\alpha(x)P_n''(x) + \beta(x)P_n'(x) = \lambda_n P_n(x), \quad n \geq 0,$$

where $\alpha(x)$ and $\beta(x)$ are polynomials with $\deg(\alpha) \leq 2$ and $\deg(\beta) = 1$, then $\alpha^2(x) + \beta^2(x) \neq 0$ and for any λ with $P_n(\lambda) \neq 0$, $n \geq 1$, $\{P_n^*(\lambda; x)\}_{n=0}^{\infty}$ is a SCOPS of class 0 if $\alpha(\lambda) = 0$ or of class 1 if $\alpha(\lambda) \neq 0$. Moreover, $\{P_n^*(\lambda; x)\}_{n=0}^{\infty}$ is also a classical OPS only when $\{P_n(x)\}_{n=0}^{\infty}$ is the Jacobi polynomial sequence with $\lambda = \pm 1$ or Bessel or Laguerre polynomial sequences with $\lambda = 0$. For example, the MKPS for Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ with K-parameter 0 is $\{L_n^{(\alpha+1)}(x)\}_{n=0}^{\infty}$. Note also that $\{P_n^*(\lambda; x)\}_{n=0}^{\infty}$ may be a classical OPS even if $\{P_n(x)\}_{n=0}^{\infty}$ is not a classical OPS but a SCOPS of class 1. For example, let $\{P_n(x)\}_{n=0}^{\infty}$ be one of the three classical-type OPSs, which are eigenfunctions of the fourth-order linear differential equations

$$\sum_{i=1}^{4} \ell_i(x)y^{(i)}(x) = \lambda_n y(x),$$

where $\ell_i(x)$ is a polynomial of degree $\leq i$ and $\ell_4(x) \neq 0$. Then we know (see [7,8]) that $\{P_n(x)\}_{n=0}^{\infty}$ is orthogonal with respect to $\tau = \sigma + a\delta(x-\lambda)$, where σ is a classical moment functional satisfying $(\alpha(x)\sigma)' = \beta(x)\sigma$ and $\alpha(\lambda) = 0$. Hence τ satisfie

$$[(x-\lambda)\alpha(x)\tau]' = [(x-\lambda)\beta(x) - \alpha(x)]\tau$$

so that τ is a semi-classical moment functional of class 1 (cf. Theorem 5.2 in [10]). Therefore, by Theorem 3.6, $\{P_n^*(\lambda; x)\}_{n=0}^{\infty}$ must be a classical OPS relative to $(x-\lambda)\tau = (x-\lambda)\sigma$. For example, the differential equation

$$x^2y^{(4)} + (4x - 2x^2)y^{(3)} + [x^2 - (2R+6)x]y'' + [(2R+2)x - 2R]y'$$
$$= [(2R+2)n + n(n-1)]y$$

has an OPS $\{L_n(R; x)\}_{n=0}^{\infty}$, called the Laguerre-type OPS, as solutions for $R \neq 0, -1, -2, \ldots$. Since $\{L_n(R; x)\}_{n=0}^{\infty}$ is an OPS relative to $(e^{-x} + \frac{1}{R}\delta(x))dx$ on $[0, \infty)$, its MKPS with K- parameter 0 are the Laguerre polynomials $\{L_n^{(1)}(x)\}_{n=0}^{\infty}$.

4. Self-perturbation of orthogonal polynomials

Relation (2.5) for an MOPS and its MKPS leads to a question: given an MOPS $\{Q_n(x)\}_{n=0}^{\infty}$ and a sequence of complex numbers $\{\alpha_n\}_{n=0}^{\infty}$, def ne another monic PS by

$$P_n(x) = Q_n(x) - \alpha_n Q_{n-t}(x), \quad n \geq 0, \tag{4.1}$$

where t is a positive integer and $Q_{-t}(x) = Q_{1-t}(x) = \ldots = Q_{-1}(x) = 0$ (so that the choices of $\alpha_0, \alpha_1, \ldots, \alpha_{t-1}$ are redundant). When is $\{P_n(x)\}_{n=0}^{\infty}$ also an MOPS? In the following, we always let

$$P_{n+1}(x) = (x - b_n)P_n(x) - c_n P_{n-1}(x), \quad n \geq 0 \quad (c_n \neq 0,\ n \geq 1) \tag{4.2}$$

$$Q_{n+1}(x) = (x - \tilde{b}_n)Q_n(x) - \tilde{c}_n Q_{n-1}(x), \quad n \geq 0 \quad (\tilde{c}_n \neq 0,\ n \geq 1) \tag{4.3}$$

be the three-term recurrence relations for $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$, respectively, when they are MOPSs.

By (4.1) and (4.3), (4.2) gives

$$\begin{aligned}&(\tilde{b}_n - b_n)Q_n(x) + (\tilde{c}_n - c_n)Q_{n-1}(x) + (\alpha_{n+1} - \alpha_n)Q_{n-t+1}(x)\\ &+ (\alpha_n b_n - \alpha_n \tilde{b}_{n-t})Q_{n-t}(x) + (\alpha_{n-1}c_n - \alpha_n \tilde{c}_{n-t})Q_{n-t-1}(x) = 0\end{aligned} \tag{4.4}$$

for $n \geq t+1$. On the other hand, if $\{P_n(x)\}_{n=0}^{\infty}$ is an MOPS relative to σ and $\{Q_n(x)\}_{n=0}^{\infty}$ is an MOPS relative to τ, then $\langle \tau, P_n\rangle = \langle \tau, Q_n - \alpha_n Q_{n-t}\rangle = 0$, $n \geq t+1$ so that by Lemma 2.3

$$\tau = \phi_t(x)\sigma, \tag{4.5}$$

where $\phi_t(x)$ is a polynomial of degree $\leq t$. If we set

$$\phi_t(x) = \sum_{j=0}^{t} a_j P_j(x),$$

then

$$a_j = \frac{\langle \tau, P_j\rangle}{\langle \sigma, P_j^2\rangle} = \begin{cases} \frac{\tau_0}{\sigma_0} \neq 0, & j = 0\\ 0, & 1 \leq j \leq t-1\\ -\frac{\alpha_t \tau_0}{\langle \sigma, P_t^2\rangle}, & j = t.\end{cases}$$

Hence

$$\phi_t(x) = -\frac{\alpha_t \tau_0}{\langle \sigma, P_t^2\rangle} P_t(x) + \frac{\tau_0}{\sigma_0} \quad (\sigma_0 =< \sigma, 1 >, \tau_0 =< \tau, 1 >), \tag{4.6}$$

so that $\deg(\phi_t) = t$ if and only if $\alpha_t \neq 0$.

Assume that $\alpha_t = 0$ and $\{P_n(x)\}_{n=0}^{\infty}$ is an MOPS. Then by (4.5) and (4.6), $\tau = c\sigma$ for some non-zero constant c. Hence, $\{P_n(x)\}_{n=0}^{\infty}$ must be an MOPS relative to σ and τ so that $\{P_n(x)\}_{n=0}^{\infty} = \{Q_n(x)\}_{n=0}^{\infty}$, that is, $\alpha_n = 0, n \geq t$. Hence, from now on, we always assume $\alpha_t \neq 0$ so that $\deg \phi_t = t$.

Now, by the relation (4.4), we must consider the three cases $t = 1$, $t = 2$, and $t \geq 3$ separately.

Proposition 4.1. *If $t \geq 3$, then $\{P_n(x)\}_{n=0}^{\infty}$ defined by (4.1) cannot be an MOPS.*

Proof. Assume $t \geq 3$ and $\{P_n(x)\}_{n=0}^{\infty}$ is an MOPS. Then (4.2) for $n = t-1$ becomes, via (4.1) and (4.3),

$$Q_t(x) - \alpha_t = Q_t(x) + (\tilde{b}_{t-1} - b_{t-1})Q_{t-1}(x) + (\tilde{c}_{t-1} - c_{t-1})Q_{t-2}(x),$$

so that $\alpha_t = 0$, which is a contradiction. □

Hence, we only need to consider the cases $t = 1$ and $t = 2$.

Theorem 4.2. *Let $\{Q_n(x)\}_{n=0}^{\infty}$ be an MOPS relative to τ and define another monic PS $\{P_n(x)\}_{n=0}^{\infty}$ by*

$$P_n(x) = Q_n(x) - \alpha_n Q_{n-1}(x), \quad n \geq 0 \quad (P_{-1}(x) \equiv 0), \tag{4.7}$$

where α_n are complex numbers with $\alpha_1 \neq 0$. Then, the following are all equivalent:

(i) *$\{P_n(x)\}_{n=0}^{\infty}$ is an MOPS (respectively, a positive-definite MOPS);*
(ii) *$\alpha_n \neq 0$, $n \geq 1$ (respectively, $\alpha_n\alpha_{n+1}\tilde{c}_n > 0$, $n \geq 1$) and*

$$\frac{\tilde{c}_n}{\alpha_n} + \alpha_{n+1} + \tilde{b}_n = \lambda \text{ (constant)}, \quad n \geq 1,$$
$$c := \lambda - \alpha_1 - \tilde{b}_0 \neq 0 \text{ (respectively, } c\alpha_1 > 0);$$

(iii) *there are complex numbers λ and $c \neq 0$ such that*

$$Q_n(c;\lambda) \neq 0, \quad n \geq 0$$
$$\text{(respectively, } cQ_1(c;\lambda) > 0 \text{ and } Q_{n-1}(c;\lambda)Q_{n+1}(c;\lambda)\tilde{c}_n > 0,\ n \geq 1)$$

and

$$\alpha_n = \frac{Q_n(\lambda) - cQ_{n-1}^{(1)}(\lambda)}{Q_{n-1}(\lambda) - cQ_{n-2}^{(1)}(\lambda)} = \frac{Q_n(c;\lambda)}{Q_{n-1}(c;\lambda)}, \quad n \geq 1 \quad (Q_{-1}^{(1)}(x) = 0). \tag{4.8}$$

In this case, $\{P_n(x)\}_{n=0}^{\infty}$ is the MOPS relative to

$$\sigma = (x-\lambda)^{-1}\tau - \frac{\tau_0}{c}\delta(x-\lambda),$$

which is quasi-definite (respectively, positive-definite or negative-definite) and $\{Q_n(x)\}_{n=0}^{\infty} = \{P_n^*(\lambda;x)\}_{n=0}^{\infty}$ is the MKPS for $\{P_n(x)\}_{n=0}^{\infty}$ with K-parameter λ. Here, we call a moment functional σ negative-definite if $-\sigma$ is positive-definite.

Theorem 4.2 was proved by Marcellán and Petronilho when $\{P_n(x)\}_{n=0}^{\infty}$ is a quasi-def nite OPS (cf. Theorem 2 in [12]). But, the equivalent conditions for positive-def niteness of $\{P_n(x)\}_{n=0}^{\infty}$ follow immediately from (4.8) and the following relations between coefficient of three-term recurrence relations for $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$:

$$b_n = \tilde{b}_n + \alpha_{n+1} - \alpha_n, \quad n \geq 0; \tag{4.9}$$

$$c_n = \tilde{c}_n + \alpha_n(\alpha_{n+1} + \tilde{b}_n - \alpha_n - \tilde{b}_{n-1}), \quad n \geq 1; \tag{4.10}$$

$$\alpha_{n-1}\, c_n = \alpha_n \tilde{c}_{n-1}, \quad n \geq 0. \tag{4.11}$$

Note that Theorem 4.2 gives, in particular, a proof of (iii)⇒(i) in Theorem 3.2.

Maroni [14] considered a problem that is closely related to Theorem 4.2: given a quasi-def nite moment functional τ and two complex numbers a and λ, when is the moment functional σ given by

$$\sigma = (x - \lambda)^{-1}\tau + a\delta(x - \lambda)$$

also quasi-def nite? Equivalently, when does the division problem $(x - \lambda)\sigma = \tau$ have a quasi-def nite moment functional solution σ? It's easy to see (cf. Theorem 4.2 (iii) and [14, Theorem 1.1]) that σ is quasi-defi ite if and only if

$$a\, Q_n(\lambda) + \tau_0 Q_{n-1}^{(1)}(\lambda) \neq 0, \quad n \geq 0. \tag{4.12}$$

Maroni stated condition (4.12) as

$$a \neq -\frac{\tau_0 Q_{n-1}^{(1)}(\lambda)}{Q_n(\lambda)}, \quad n \geq 0,$$

which is not true in general since $Q_n(\lambda)$ may be 0 for some $n \geq 1$ unless σ is positive-def nite. First note that when σ is also quasi-definit and $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ are MOPSs relative to σ and τ respectively, $\{Q_n(x)\}_{n=0}^{\infty} = \{P_n^*(\lambda; x)\}_{n=0}^{\infty}$. Now, construct an MOPS $\{P_n(x)\}_{n=0}^{\infty}$ as $P_0(x) = 1$, $P_1(x) = x$, and

$$P_{n+1}(x) = (x - b_n)P_n(x) - c_n P_{n-1}(x), \quad n \geq 1,$$

where $c_1 = -1$ and c_n, $n \geq 2$, are arbitrary non-zero constants. We may choose b_n, $n \geq 1$, so that $P_n(1) \neq 0$, $n \geq 1$. Assume that $\{P_n(x)\}_{n=0}^{\infty}$ is an MOPS relative to σ and $\sigma_0 = 1$. Then $\tau = (x - 1)\sigma$ is also quasi-definit and the MOPS $\{Q_n(x)\}_{n=0}^{\infty}$ relative to τ is just $\{P_n^*(1; x)\}_{n=0}^{\infty}$. Then $Q_1(1) = \langle \sigma, P_1^2 \rangle K_1(1, 1) = 0$ (cf. (2.5)) since $\langle \sigma, P_1^2 \rangle = c_1 = -1$.

Proposition 2.5 gives conditions for an MKPS $\{P_n^*(\lambda; x)\}_{n=0}^{\infty}$ to be positive-defi ite when $\{P_n(x)\}_{n=0}^{\infty}$ is positive-defi ite. Conversely, we now have:

Corollary 4.3. *In Theorem 4.2, assume that τ is positive-definite, α_n's are real, and $\{P_n(x)\}_{n=0}^{\infty}$ is an MOPS. Then, the following are all equivalent:*

(i) $\{P_n(x)\}_{n=0}^{\infty}$ *is a positive-definite OPS;*
(ii) *either $c > 0$ and $\alpha_n < 0$, $n \geq 1$ or $c < 0$ and $\alpha_n > 0$, $n \geq 0$;*

(iii) *either* $c < 0$ *and* $\operatorname{sgn} Q_n(c;\lambda) = (-1)^n$, $n \geq 0$ *or* $c > 0$ *and* $Q_n(c;\lambda) > 0$, $n \geq 0$;
(iv) *either* σ *or* $-\sigma$ *is positive-definite.*

In this case, τ *is positive-definite on the true interval of orthogonality* $[\xi, \eta]$ *of* $\{P_n(x)\}_{n=0}^{\infty}$ *so that either* $\lambda \leq \xi$ *(if* $c < 0$*) or* $\lambda \geq \eta$ *(if* $c > 0$*). Hence, in particular, either* $-\infty < \xi$ *or* $\eta < \infty$.

Proof. The equivalences of (i)–(iv) follow immediately from Theorem 4.2. The last conclusion follows from Proposition 2.5 since $\tau = (x - \lambda)\sigma$ is positive-defi ite and either σ or $-\sigma$ is positive-defi ite. □

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a monic PS def ned by (4.7) and set $\tilde{P}_n(x) = \frac{1}{n+1} P'_{n+1}(x)$. Then, (4.7) yields

$$\tilde{P}_n(x) = \frac{1}{n+1} Q'_{n+1}(x) - \frac{1}{n+1} \alpha_n Q'_n(x), \quad n \geq 0.$$

Hence, if $\{\tilde{P}_n(x)\}_{n=0}^{\infty}$ is also an MOPS, then $\{\tilde{P}_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ are the so-called coherent pairs (cf. [6,12]).

With respect to this, Marcellán and Petronilho [12] raised a question: given an MOPS $\{P_n(x)\}_{n=0}^{\infty}$, def ne another PS $\{Q_n(x)\}_{n=0}^{\infty}$ recursively by

$$P_n(x) = Q_n(x) - \alpha_n Q_{n-1}(x) - \beta_n, \quad n \geq 0, \tag{4.13}$$

where α_n and β_n are complex numbers with $\alpha_0 = \beta_0 = \beta_1 = 0$. When is $\{Q_n(x)\}_{n=0}^{\infty}$ also an MOPS? In the following, we also assume $\alpha_1 \neq 0$ as before:

Theorem 4.4. *Let* $\{P_n(x)\}_{n=0}^{\infty}$ *be an MOPS satisfying (4.2) and define* $\{Q_n(x)\}_{n=0}^{\infty}$ *by (4.13). Then, the following are all equivalent:*

(i) $\{Q_n(x)\}_{n=0}^{\infty}$ *is an MOPS (respectively, a positive-definite MOPS);*
(ii) $\beta_n = 0$, $n \geq 0$ *and there is a complex number* λ *such that* $P_n(\lambda) \neq 0$, $n \geq 1$ *(respectively,* $P_{n-1}(\lambda)P_{n+1}(\lambda)c_n c_{n+1} > 0$, $n \geq 1$*) such that*

$$\alpha_n = \frac{P_{n-1}(\lambda)}{P_n(\lambda)} c_n, \quad n \geq 1;$$

(iii) $\beta_n = 0$, $n \geq 0$, $\alpha_n \neq 0$, $n \geq 1$, *(respectively,* $\alpha_n \alpha_{n+1} c_{n+1} > 0$, $n \geq 1$*), and*

$$\frac{c_n}{\alpha_n} + \alpha_{n-1} + b_{n-1} = \lambda \text{ (constant)}, \quad n \geq 1 \ (a_0 = 0).$$

In this case, $\{Q_n(x)\}_{n=0}^{\infty}$ *is the MKPS for* $\{P_n(x)\}_{n=0}^{\infty}$ *with* K*-parameter* λ*, which is orthogonal with respect to* $(x-\lambda)\sigma$ *and satisfy the three-term recurrence relation (4.3) with* $\tilde{b}_n$ *and* $\tilde{c}_n$ *determined by (4.9) and (4.10).*

Proof. See Theorem 1 in [12]. □

Theorem 4.4, in particular, shows that $\{P_n^*(\lambda; x)\}_{n=0}^{\infty}$ may be positive-defi ite even if $\{P_n(x)\}_{n=0}^{\infty}$ is not positive-def nite. For example, if $c_1 < 0$, $c_{n+1} > 0, n \geq 1$ and $\alpha_n \alpha_{n+1} > 0$, $n \geq 0$, then $\{P_n(x)\}_{n=0}^{\infty}$ is not positive-defi ite but $\{Q_n(x)\}_{n=0}^{\infty}$ is positive-defi ite. As in Corollary 4.3, we also have:

Corollary 4.5. *In Theorem 4.4, assume that σ is positive-definite and $\{Q_n(x)\}_{n=0}^{\infty}$ is an MOPS. Then, the following are all equivalent:*

(i) $\{Q_n(x)\}_{n=0}^{\infty}$ *is a positive-definite OPS;*
(ii) *either $\alpha_n > 0$, $n \geq 1$ or $\alpha_n < 0$, $n \geq 1$;*
(iii) *either $\tau = (x - \lambda)\sigma$ or $-\tau$ is positive-definite on $[\xi, \eta]$;*
(iv) *either $\lambda \leq \xi$ or $\lambda \geq \xi$,*

where $[\xi, \eta]$ is the true interval of orthogonality for σ.

Remark 4.6. Let $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ be two real PSs satisfying the relation (4.7). If $\{Q_n(x)\}_{n=0}^{\infty}$ is a positive-definit OPS, then each $P_n(x)$, $n \geq 1$, has n real simple zeros, which interlace with the zeros of $Q_n(x)$ and with the zeros of $Q_{n-1}(x)$. Here, $\{P_n(x)\}_{n=0}^{\infty}$ need not be an MOPS (cf. Theorem 3.4).

If $\{P_n(x)\}_{n=0}^{\infty}$ is a positive-defi ite OPS and $\{Q_n(x)\}_{n=0}^{\infty}$ is an MOPS, then each $Q_n(x)$, $n \geq 1$, has n real simple zeros (cf. Theorem 3.3).

If $\{P_n(x)\}_{n=0}^{\infty}$ is a positive-defi ite OPS but $\{Q_n(x)\}_{n=0}^{\infty}$ is not an OPS, then $Q_n(x)$, $n \geq 1$, need not have real or simple zeros. For example, by Wendroff's theorem [15], there is a positive-defi ite OPS $\{P_n(x)\}_{n=0}^{\infty}$ with $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = x^2 - 1$. Def ne a PS $\{Q_n(x)\}_{n=0}^{\infty}$ by (4.7), where α_n, $n \geq 1$, are real constants. Then $Q_2(x) = x^2 + \alpha_2 x + \alpha_1\alpha_2 - 1$ so that $Q_2(x)$ may have two distinct real zeros or one double zero or two complex conjugate zeros.

We now consider the case $t = 2$.

Theorem 4.7. *Let $\{Q_n(x)\}_{n=0}^{\infty}$ be an MOPS relative to τ and define another monic PS $\{P_n(x)\}_{n=0}^{\infty}$ by*

$$P_n(x) = Q_n(x) - \alpha_n Q_{n-2}(x), \quad n \geq 0, \tag{4.14}$$

where α_n are complex numbers with $\alpha_2 \neq 0$. Then, the following are all equivalent:

(i) $\{P_n(x)\}_{n=0}^{\infty}$ *is an MOPS (respectively, a positive-definite MOPS);*
(ii) $\alpha_n \neq 0$, $n \geq 2$ *and*

$$\tilde{c}_1 + \alpha_2 \neq 0 \quad (\text{respectively, } \tilde{c}_1 + \alpha_2 > 0), \tag{4.15}$$

$$\tilde{c}_2 + \alpha_3 - \alpha_2 \neq 0 \quad (\text{respectively, } \tilde{c}_2 + \alpha_3 - \alpha_2 > 0), \tag{4.16}$$

$$\tilde{c}_n + \alpha_{n+1} - \alpha_n = \frac{\alpha_n}{\alpha_{n-1}}\tilde{c}_{n-2}, \quad n \geq 3 \tag{4.17}$$

$$(\text{respectively, } \tilde{c}_n + \alpha_{n+1} - \alpha_n = \frac{\alpha_n}{\alpha_{n-1}}\tilde{c}_{n-2} > 0, \quad n \geq 3),$$

$$\tilde{b}_n = \tilde{b}_{n-2}, \quad n \geq 2. \tag{4.18}$$

In this case, $\{P_n(x)\}_{n=0}^{\infty}$ satisfy the three-term recurrence relation (4.2), where

$$b_n = \tilde{b}_n, \quad n \geq 0 \quad \text{and} \quad c_1 = \tilde{c}_1 + \alpha_2, \quad c_n = \tilde{c}_n, \ n \geq 2, \tag{4.19}$$

and is orthogonal with respect to σ satisfying

$$[x^2 - (\tilde{b}_0 + \tilde{b}_1)x + \tilde{b}_1\tilde{b}_2 - \mu]\sigma = \tau, \tag{4.20}$$

where $\mu = \alpha_n(1 + \frac{\tilde{c}_{n-1}}{\alpha_n})(1 + \frac{\tilde{c}_{n-2}}{\alpha_{n-1}}) \neq 0$, $n \geq 3$.

Proof. Assume $\{P_n(x)\}_{n=0}^{\infty}$ is an MOPS satisfying the three-term recurrence relation (4.2). For $n = 2$ and $n = 1$, we have, from (4.2) via (4.14) and (4.3):

$$(\tilde{b}_2 - b_2)Q_2(x) + (\tilde{c}_2 - c_2 + \alpha_3 - \alpha_2)Q_1(x) + (\alpha_2 b_2 - \alpha_2 \tilde{b}_0 + b_0 c_2 - \tilde{b}_0 c_2) = 0;$$
$$(\tilde{b}_1 - b_1 + \tilde{b}_0 - b_0)Q_1(x) + \tilde{c}_1 - c_1 + \alpha_2 + \tilde{b}_0^2 - b_0\tilde{b}_0 - \tilde{b}_0 b_1 + b_0 b_1 = 0.$$

On the other hand, for $t = 2$ (4.4) becomes

$$\begin{aligned}&(\tilde{b}_n - b_n)Q_n(x) + (\tilde{c}_n - c_n + \alpha_{n+1} - \alpha_n)Q_{n-1}(x)\\ &+ (\alpha_n b_n - \alpha_n \tilde{b}_{n-2})Q_{n-2}(x) + (\alpha_{n-1} c_n - \alpha_n \tilde{c}_{n-2})Q_{n-3}(x) = 0, \quad n \geq 3.\end{aligned}$$

Hence,

$$b_n = \tilde{b}_n, \quad n \geq 2; \tag{4.21}$$
$$c_n = \tilde{c}_n + \alpha_{n+1} - \alpha_n, \quad n \geq 2; \tag{4.22}$$
$$\alpha_n(b_n - \tilde{b}_{n-2}) = 0, \quad n \geq 3; \tag{4.23}$$
$$\alpha_{n-1}c_n - \alpha_n \tilde{c}_{n-2} = 0, \quad n \geq 3; \tag{4.24}$$
$$\alpha_2 b_2 - \alpha_2 \tilde{b}_0 + b_0 c_2 - \tilde{b}_0 c_2 = 0; \tag{4.25}$$
$$b_0 + b_1 = \tilde{b}_0 + \tilde{b}_1; \tag{4.26}$$
$$c_1 = \tilde{c}_1 + \alpha_2 - \tilde{b}_0\tilde{b}_1 + b_0 b_1. \tag{4.27}$$

Then, $\alpha_n \neq 0, n \geq 2$ by (4.24) and so $\tilde{b}_n = \tilde{b}_{n-2}, n \geq 3$ by (4.21) and (4.23). (4.16) and (4.17) follow from (4.22) and (4.24). Since $P_1(x) = Q_1(x)$, i.e., $b_0 = \tilde{b}_0$, we have $b_1 = \tilde{b}_1$, $b_2 = \tilde{b}_0$, and $c_1 = \tilde{c}_1 + \alpha_2$ by (4.25), (4.26), and (4.27) from which we have $\tilde{b}_2 = \tilde{b}_0$ and $\tilde{c}_1 + \alpha_2 \neq 0$. We also have, from (4.22) and (4.24),

$$\alpha_n\left(1 + \frac{\tilde{c}_{n-2}}{\alpha_{n-1}}\right) = \alpha_{n+1}\left(1 + \frac{\tilde{c}_n}{\alpha_{n+1}}\right), \quad n \geq 3.$$

Thus

$$\alpha_n\left(1 + \frac{\tilde{c}_{n-1}}{\alpha_n}\right)\left(1 + \frac{\tilde{c}_{n-2}}{\alpha_{n-1}}\right) = \mu\ (= \text{constant}), \quad n \geq 3.$$

Then, $\mu = 0$ if and only if $\tilde{c}_n + \alpha_{n+1} = 0$ for some $n \geq 1$. On the other hand, if $\tilde{c}_n + \alpha_{n+1} = 0$ for some $n \geq 3$, then by (4.17), $\tilde{c}_{n-2} + \alpha_{n-1} = 0$ so that either $\tilde{c}_1 + \alpha_2 = 0$ or $\tilde{c}_2 + \alpha_3 = 0$, both of which are impossible by (4.15) and (4.16). Hence, $\mu \neq 0$.

Conversely, assume that (ii) holds. Defi e $b_n, n \geq 0$ and $c_n, n \geq 1$ by (4.19). Then, (4.24) holds so that $c_n \neq 0, n \geq 1$. Now, it's easy to show that the three-term recurrence relation (4.2) holds. Hence, $\{P_n(x)\}_{n=0}^{\infty}$ is an MOPS. Finally assume that $\{P_n(x)\}_{n=0}^{\infty}$ is an MOPS relative to σ. Then we may assume (cf. (4.6)) that $\tau = \phi(x)\sigma$, $\phi(x) = x^2 + \alpha x + \beta$. Then, it's easy to see that

$$\phi(x)Q_n(x) = P_{n+2}(x) + A_n P_n(x), \quad n \geq 1, \tag{4.28}$$

where $A_n = \frac{\langle \tau, Q_n^2 \rangle}{\langle \sigma, P_n^2 \rangle} \neq 0$. For $n \geq 2$, (4.28) gives, by (4.14),

$$(\tilde{b}_n + \tilde{b}_{n+1} + \alpha) Q_{n+1}(x) + (\tilde{c}_n + \tilde{c}_{n+1} + \tilde{b}_n^2 + \alpha \tilde{b}_n + \beta - A_n + \alpha_{n+2}) Q_n(x)$$
$$+ (\tilde{b}_{n-1} + \tilde{b}_n + \alpha) \tilde{c}_n Q_{n-1}(x) + (\tilde{c}_{n-1} \tilde{c}_n + A_n \alpha_n) Q_{n-2}(x) = 0$$

so that

$$\tilde{b}_n + \tilde{b}_{n+1} + \alpha = \tilde{b}_{n-1} + \tilde{b}_n + \alpha = 0, \quad n \geq 2;$$
$$\tilde{c}_n + \tilde{c}_{n+1} + \tilde{b}_n^2 + \alpha \tilde{b}_n + \beta - A_n + \alpha_{n+2} = 0, \quad n \geq 2;$$
$$\tilde{c}_{n-1} \tilde{c}_n + A_n \alpha_n = 0, \quad n \geq 2.$$

Hence,

$$\begin{aligned}
\alpha &= -\tilde{b}_n - \tilde{b}_{n+1}, \quad n \geq 0 \text{ (cf. (4.18))}; \\
A_n &= -\frac{\tilde{c}_{n-1} \tilde{c}_n}{\alpha_n}, \quad n \geq 2; \\
\beta &= A_n - \alpha_{n+2} - \tilde{c}_n - \tilde{c}_{n+1} - \tilde{b}_n^2 + (\tilde{b}_n + \tilde{b}_{n+1}) \tilde{b}_n \\
&= \tilde{b}_n \tilde{b}_{n+1} - \frac{c_{n-1} \tilde{c}_n}{\alpha_n} - \alpha_{n+2} - \tilde{c}_n - \tilde{c}_{n+1} \\
&= \tilde{b}_n \tilde{b}_{n+1} - \mu, \quad n \geq 2,
\end{aligned}$$

which gives (4.20). Finally, $\{P_n(x)\}_{n=0}^{\infty}$ is positive-defi ite if and only if $c_n > 0$, $n \geq 1$ so that the conclusion concerning the positive-def niteness of $\{P_n(x)\}_{n=0}^{\infty}$ follows from (4.17) and (4.19). □

Corollary 4.8. *In Theorem 4.7, assume that τ is positive-definite and $\{P_n(x)\}_{n=0}^{\infty}$ is an MOPS. Then, $\{P_n(x)\}_{n=0}^{\infty}$ is a positive-definite MOPS if and only if*

$$\tilde{c}_1 + \alpha_2 > 0, \quad \tilde{c}_2 + \alpha_3 - \alpha_2 > 0, \quad \text{and} \quad \alpha_{n-1} \alpha_n > 0, \quad n \geq 3.$$

Corollary 4.9. *Let $\{Q_n(x)\}_{n=0}^{\infty}$ be an MOPS relative to τ with real numbers $\tilde{c}_n$, $n \geq 1$, and define $\{P_n(x)\}_{n=0}^{\infty}$ by* (4.14).

(i) *If $\{Q_n(x)\}_{n=0}^{\infty}$ is a positive-definite MOPS and α_n, $n \geq 2$, are real, then $P_n(x)$, $n \geq 3$, has at least $n - 2$ nodal zeros (i.e., zeros of odd multiplicity) so that is at least $n - 3$ simple zeros in $(\tilde{\xi}, \tilde{\eta})$.*
(ii) *If $\{P_n(x)\}_{n=0}^{\infty}$ is a positive-definite MOPS, then $Q_n(x)$ has at least $n - 2$ nodal zeros so that is at least $n - 3$ simple zeros in (ξ, η).*

Here, $[\xi, \eta]$ and $[\tilde{\xi}, \tilde{\eta}]$ are the true intervals of orthogonality for positive-definite MOPSs $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$, respectively.

Proof. (i) When $\{Q_n(x)\}_{n=0}^{\infty}$ is a positive-defi ite MOPS and α_n, $n \geq 2$, are real, $\{P_n(x)\}_{n=0}^{\infty}$ is a real PS. Let $x_1 < x_2 < \ldots < x_k$ be the nodal zeros of $P_n(x)$ in $(\tilde{\xi}, \tilde{\eta})$ and $\pi(x) = \prod_{j=1}^{k}(x - x_j)$. Then $\pi(x) P_n(x) \neq 0$ in $(\tilde{\xi}, \tilde{\eta})$ so that $\langle \tau, \pi(x) P_n(x) \rangle = \langle \tau, \pi(x)(Q_n(x) - Q_{n-2}(x)) \rangle \neq 0$. Hence, $k \geq n - 2$.

(ii) Assume that $\{P_n(x)\}_{n=0}^{\infty}$ is a positive-def nite MOPS. Then, by Theorem 4.7, α_n, $n \geq 2$, are real since $\tilde{c}_n$, $n \geq 1$, are real. Hence, $A_n = -\frac{\tilde{c}_{n-1}\tilde{c}_n}{\alpha_n}$, $n \geq 2$, are also real so that $\phi(x)Q_n(x) = P_{n+2}(x) + A_n P_n(x)$ (cf. (4.28)) has at least n nodal zeros in (ξ, η). Hence, $Q_n(x)$ has at least $n-2$ nodal zeros in (ξ, η). □

As a converse to Theorem 4.7, we also have:

Theorem 4.10. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an MOPS relative to σ and define another monic PS $\{Q_n(x)\}_{n=0}^{\infty}$ recursively by (4.14). Then, the following are all equivalent:*

(i) *$\{Q_n(x)\}_{n=0}^{\infty}$ is also an MOPS (respectively, a positive-definite MOPS);*
(ii) *$\alpha_n \neq 0$, $n \geq 2$; $b_n = b_{n+2}$, $n \geq 0$; $\frac{\alpha_{n+1}}{\alpha_{n+2}}c_{n+2} = c_n - \alpha_{n+1} + \alpha_n$, $n \geq 1$ ($a_1 = 0$) (respectively, $\frac{\alpha_{n+1}}{\alpha_{n+2}}c_{n+2} = c_n - \alpha_{n+1} + \alpha_n > 0$, $n \geq 1$).*

In this case, $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy a three-term recurrence relation (4.3), where

$$\tilde{b}_n = b_n, \quad n \geq 0 \quad \text{and} \quad \tilde{c}_n = \frac{\alpha_{n+1}}{\alpha_{n+2}}c_{n+2}, \quad n \geq 1,$$

and is orthogonal with respect to $\tau = [x^2 - (b_0 + b_1)x + b_0 b_1 - \mu]\sigma$, where

$$\mu = \alpha_n\left(1 + \frac{c_n}{\alpha_n}\right)\left(1 + \frac{c_{n+1}}{\alpha_{n+1}}\right) \neq 0, \quad n \geq 2.$$

Finally, we consider the following modifie problem: given an MOPS $\{Q_n(x)\}_{n=0}^{\infty}$, def ne a sequence of monic polynomials $P_n(x)$, $n \geq t$, by

$$P_n(x) = Q_n(x) - \alpha_n Q_{n-t}(x), \quad n \geq t, \tag{4.29}$$

and ask: When can we complete $\{P_n(x)\}_{n=t}^{\infty}$ to an MOPS $\{P_n(x)\}_{n=0}^{\infty}$ by def ning $P_0(x) = 1, P_1(x), \ldots, P_{t-1}(x)$. Here t must be an integer ≥ 2. First, we note that if there is an MOPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ, then we still have (4.4), (4.5), and (4.6). We consider firs the case where $\alpha_t = 0$.

Proposition 4.11. *Let $\{Q_n(x)\}_{n=0}^{\infty}$ be an MOPS relative to τ and define $P_n(x)$, $n \geq t$, by (4.29), where $\alpha_t = 0$. Then, $\{P_n(x)\}_{n=t}^{\infty}$ can be completed to an MOPS $\{P_n(x)\}_{n=0}^{\infty}$ if and only if $\alpha_n = 0$, $n \geq t$ and $P_n(x) = Q_n(x)$, $0 \leq n \leq t-1$, that is, $\{P_n(x)\}_{n=0}^{\infty} = \{Q_n(x)\}_{n=0}^{\infty}$.*

Proof. Assume that there is an MOPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ. Then $\tau = \phi_t(x)\sigma$, $\deg(\phi_t) \leq t-1$ (cf. (4.5) and (4.6)). If $\phi_t(x) \equiv c$, a non-zero constant, then $\{Q_n(x)\}_{n=0}^{\infty}$ must be an MOPS relative to τ and σ so that $\{P_n(x)\}_{n=0}^{\infty} = \{Q_n(x)\}_{n=0}^{\infty}$. If $\deg \phi_t = 1$, then we may assume $\tau = (x - \mu)\sigma$ for some constant μ so that $\{Q_n(x)\}_{n=0}^{\infty} = \{P_n^*(\mu; x)\}_{n=0}^{\infty}$, which is impossible by (2.5) and (4.29). Hence, $t \geq 3$. Then, (4.4) implies $\alpha_{n-1}c_n - \alpha_n\tilde{c}_{n-t} = 0$, $n \geq t+1$ so that $\alpha_n = 0$, $n \geq t$, that is, $P_n(x) = Q_n(x)$, $n \geq t$. On the other hand, if $P_n(x) = Q_n(x)$ and $P_{n+1}(x) = Q_{n+1}(x)$, then

$$(x - b_n)P_n(x) - c_n P_{n-1}(x) - P_{n+1}(x) = (x - b_n)Q_n(x) - c_n P_{n-1}(x) - Q_{n+1}(x)$$
$$= (\tilde{b}_n - b_n)Q_n(x) + \tilde{c}_n Q_{n-1}(x) - c_n P_{n-1}(x) = 0$$

so that $P_{n-1}(x) = Q_{n-1}(x)$. Hence, we must have $P_n(x) = Q_n(x)$, $n \geq 0$. The converse is trivial. □

Turning next to the case $\alpha_t \neq 0$, we consider the special case $t = 2$. Then, as in Theorem 4.7, we can easily obtain:

Theorem 4.12. *Let $\{Q_n(x)\}_{n=0}^{\infty}$ be an MOPS relative to τ and define $P_n(x)$, $n \geq 2$ by (4.29) with $t = 2$ and $\alpha_2 \neq 0$. Then, $\{P_n(x)\}_{n=2}^{\infty}$ can be completed to an MOPS $\{P_n(x)\}_{n=0}^{\infty}$ if and only if $\alpha_n \neq 0$, $n \geq 2$ and*

$$\begin{aligned}
\tilde{b}_n &= \tilde{b}_{n-2}, \quad n \geq 3;\\
\tilde{c}_n + \alpha_{n+1} - \alpha_n &= \frac{\alpha_n}{\alpha_{n-1}}\tilde{c}_{n-2}, \quad n \geq 3;\\
\tilde{c}_2 + \alpha_3 - \alpha_2 &\neq 0;\\
(\tilde{c}_2 + \alpha_3 - \alpha_2)^2(\tilde{c}_1 + \alpha_2 - \tilde{b}_0\tilde{b}_1)&\\
+ (\tilde{b}_0\tilde{c}_2 + \tilde{b}_0\alpha_3 - \alpha_2\tilde{b}_2)\{\tilde{b}_1(\tilde{c}_2 + \alpha_3 - \alpha_2) &+ \alpha_2\tilde{b}_2 - \tilde{b}_0\alpha_2\} \neq 0.
\end{aligned}$$

In this case, $\{P_n(x)\}_{n=0}^{\infty}$ satisfy the three-term recurrence relation (4.2), where

$$b_n = \begin{cases} \frac{1}{c_2}(\tilde{b}_0\tilde{c}_2 + \tilde{b}_0\alpha_3 - \alpha_2\tilde{b}_2), & n = 0\\ \frac{1}{c_2}(\tilde{b}_1 c_2 + \alpha_2\tilde{b}_2 - \alpha_2\tilde{b}_0), & n = 1\\ \tilde{b}_n, & n \geq 2, \end{cases}$$

and

$$c_n = \begin{cases} \tilde{c}_1 + \alpha_2 - \tilde{b}_0\tilde{b}_1 + \frac{1}{c_2^2}(\tilde{b}_0\tilde{c}_2 + \tilde{b}_0\alpha_3 - \alpha_2\tilde{b}_2)(\tilde{b}_1 c_2 + \alpha_2\tilde{b}_2 - \tilde{b}_0\alpha_2), & n = 1\\ \tilde{c}_n + \alpha_{n+1} - \alpha_n, \quad n \geq 2, \end{cases}$$

and is orthogonal relative to σ satisfying (4.20).

For $t > 2$, we have:

Theorem 4.13. *Let $\{Q_n(x)\}_{n=0}^{\infty}$ be an MOPS and define $P_n(x)$, $n \geq t$, by (4.29) with $t \geq 3$ and $\alpha_t \neq 0$. If $\{P_n(x)\}_{n=t}^{\infty}$ can be completed to an MOPS $\{P_n(x)\}_{n=0}^{\infty}$, then:*

$$\alpha_n \neq 0, \quad n \geq t \quad \text{and} \quad \alpha_n = \alpha, \quad n \geq t+1; \tag{4.30}$$

$$\tilde{b}_n = \tilde{b}_{n-t}, \quad n \geq t+1; \tag{4.31}$$

$$\tilde{c}_n = \tilde{c}_{n-t}, \quad n \geq t+2; \tag{4.32}$$

$$\tilde{c}_{t+1} = \frac{\alpha}{\alpha_{t+1}}\tilde{c}_1; \tag{4.33}$$

and $P_t(x)$ and $Q_{t-1}(x) + \frac{1}{\tilde{c}_t}(\alpha Q_t(x) - \alpha_t x + \alpha_t b_t)$ have no common zero. Conversely, if (4.30)–(4.33) hold and $Q_t(x) - \alpha_t$ and $Q_{t-1}(x) - \frac{1}{\tilde{c}_t}[\alpha Q_1(x) - \alpha_t x + \alpha_t b_t]$ have simple real interlacing zeros, then $\{P_n(x)\}_{n=t}^{\infty}$ can be completed to an MOPS $\{P_n(x)\}_{n=0}^{\infty}$.

Proof. First assume that $\{P_n(x)\}_{n=t}^{\infty}$ can be completed to an MOPS $\{P_n(x)\}_{n=0}^{\infty}$. Then we have, from (4.2), for $n = t$,

$$(\tilde{b}_t - b_t)Q_t(x) + \tilde{c}_t Q_{t-1}(x) - c_t P_{t-1}(x) + \alpha_{t+1} Q_1(x) - \alpha_t x + \alpha_t b_t = 0.$$

Hence, together with (4.4), we have

$$\tilde{b}_n - b_n = \tilde{c}_n - c_n, \quad n \geq t; \tag{4.34}$$

$$\alpha_{n+1} - \alpha_n = \alpha_n b_n - \alpha_n \tilde{b}_{n-t} = 0, \quad n \geq t+1; \tag{4.35}$$

$$\alpha_{n-1} c_n - \alpha_n \tilde{c}_{n-t} = 0, \quad n \geq t+1. \tag{4.36}$$

Since $\alpha_t \neq 0$, $\alpha_n \neq 0$ for $n \geq t$, $\alpha_{n+1} = \alpha_n$ for $n \geq t+1$ and so $b_n = \tilde{b}_{n-t}$ for $n \geq t+1$ by (4.36) and (4.35). Hence, (4.30) and (4.31) hold. Then, (4.32) follows from (4.34) and (4.36) and (4.33) follows from (4.36) for $n = t+1$. Hence, we have

$$P_{t-1}(x) = Q_{t-1}(x) + \frac{1}{\tilde{c}_t}(\alpha Q_1(x) - \alpha_t x + \alpha_t b_t)$$

so that $P_t(x)$ and $P_{t-1}(x)$ cannot have a common zero. Conversely, assume that (4.30)–(4.33) hold and $P_t(x)$ and $Q_{t-1}(x) + \frac{1}{\tilde{c}_t}(\alpha Q_1(x) - \alpha_t x + \alpha_t b_t)$ have only simple real interlacing zeros. Defin $P_{t-1}(x) = Q_{t-1}(x) + \frac{1}{\tilde{c}_t}(\alpha Q_1(x) - \alpha_t x + \alpha_t b_t)$ and $b_n = \tilde{b}_n$, $c_n = \tilde{c}_n$ for $n \geq t$. Then, $c_n \neq 0$, $n \geq t$ and it's easy to show

$$P_{n+1}(x) = (x - b_n)P_n(x) - c_n P_{n-1}(x), \quad n \geq t.$$

Moreover, $P_t(x)$ and $P_{t-1}(x)$ must be real polynomials so that, by Wendroff's theorem [15], there are monic polynomials $\{P_n(x)\}_{n=0}^{t-2}$ such that

$$P_{n+1}(x) = (x - b_n)P_n(x) - c_n P_{n-1}(x), \quad 0 \leq n \leq t-1, \quad \text{and } c_n \neq 0, \quad n \geq 1.$$

Hence, by Favard's theorem, $\{P_n(x)\}_{n=0}^{\infty}$ is an MOPS. □

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