Working paper

2021-07

Statistics and Econometrics
ISSN 2387-0303

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Abstract

Markov bridges may be useful models in finance to describe situations in which information on the underlying processes is known in advance. However, within the framework of optimal stopping problems, Markov bridges are inherently challenging processes as they are time-inhomogeneous and account for explosive drifts. Consequently, few results are known in the literature of optimal stopping theory related to Markov bridges, all of them confined to the simplistic Brownian bridge.

In this paper we make a rigorous analysis of the existence and characterization of the free boundary related to the optimal stopping problem that maximizes the mean of an Ornstein–Uhlenbeck bridge. The result includes the Brownian bridge problem as a limit case. The methodology hereby presented relies on a time-space transformation that casts the original problem into a more tractable one with infinite horizon and a Brownian motion underneath. We conclude by commenting on two different numerical algorithms to compute the free-boundary equation and discuss illustrative cases that shed light on the boundary's shape. In particular, the free boundary does not generally share the monotonicity of the Brownian bridge case.

Keywords: Free-boundary problem; Optimal stopping; Ornstein–Uhlenbeck bridge; Time-inhomogeneity.

1 Introduction

Since their first appearance in the seminal monograph of Wald (1947), Optimal Stopping Problems (OSP) have become ubiquitous tools in mathematical finance, stochastic analysis, and mathematical statistics, among many other fields. Particularly, OSPs which are non-homogeneous in time are known to be mathematically challenging and, compared to the time-homogeneous counterpart, the literature addressing this topic is scarce, non-comprehensive, and often heavy on smoothing conditions. Markov bridges are not only time-inhomogeneous processes, but they also fail to meet the common assumption of Lipschitz continuity of the underlying drift (see, e.g., Krylov and Aries (1980, Chapter 3), or Jacka and Lynn (1992)), as their drifts explode when time approaches the horizon, thus inherently adding an extra layer of complexity.

The first result in OSPs with Markov bridges was given by Shepp (1969), who circumvented the complexity of dealing with a Brownian Bridge (BB) by using a time-space transformation that allowed to reformulate the problem into a more tractable one with a Brownian motion underneath. Since then, more than fifty years ago, the use of Markov bridges in the context of OSPs has been narrowed to extending the result of Shepp (1969): Ekström and Wanntorp (2009) and Ernst and Shepp (2015) studied alternative methods of solutions; Ekström and Wanntorp (2009) and de Angelis and Milazzo (2020) looked at a broader class of gain functions, Glover (2020) randomized the horizon while Föllmer (1972), Leung et al. (2018), and Ekström and Vaicenavicius (2020) analyzed the randomization of the bridge's terminal point.

In finance, the use of a BB in OSPs has been motivated by several applications. Boyce (1970) applied it to the optimal selling of bonds; Baurdoux et al. (2015) suggested the use of a BB to model misspriced assets that could rapidly return to their fair price, or perishable commodities that become useless after a given deadline; and Ekström and Wanntorp (2009) used a BB to model the *stock-pinning* effect, that is, the phenomenon in which the price of a stock tends to be pulled towards

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the strike price of one of its underlying options with massive trading volumes at the expiration date. While these motivations encourage the investor to rely on a model with added information at the horizon, none of them are exclusive to a BB, its usage being rather driven by tractability issues. Thus, in those same scenarios, other bridge processes could be more appealing than the over-simplistic BB. In particular, we drive our attention to an Ornstein–Uhlenbeck Bridge (OUB) process, since its version without added information, the Ornstein–Uhlenbeck (OU) process, is often the reference model in many financial problems.

Indeed, OU processes are a go-to in finance when it comes to modeling assets with prices that fluctuate around a given level. This mean-reverting phenomenon has been systematically observed in a wide variety of markets. A good reference for either theory, applications, and empirical evidence of mean-reverting problems is Leung and Li (2016). An example is given by the pair trading strategy, which consists on holding a position in one asset as well the opposite position in another, both assets known to be correlated in a way that the spread between their prices shows mean reversion. Recently, many authors have tackled pair trading by using an OSP approach with an OU process. Ekström et al. (2011) found the best time to liquidate the spread in the presence of a stop-loss level; Leung and Li (2015) used a discounted double OSP to compute the optimal buy-low-sell-high strategy in a perpetual frame; and Kitapbayev and Leung (2017) extended that result to a finite horizon and took the viewpoint of investors entering the spread either buying or shorting.

In this paper we solve the finite-horizon OSP featuring the identity as the gain function and an OUB as the underlying process. The solution is provided in terms of an non-linear, Volterra-type integral equation. Similarly to Shepp (1969), our methodology relies on a time-space change that casts the original problem into an infinite horizon OSP with a Brownian motion as the underlying process. Due to the complexity of our resulting OSP, we use a direct approach to solve it rather than using the common candidate-verification scheme. We then show that one can either apply the inverse transformation to recover the solution of the original OSP or, equivalently, solve the Volterra integral equation reformulated back in terms of OUB. It is worthwhile to highlight that the BB framework is included in our analysis as a limit case.

The rest of the paper is structured as follows. Section 2 introduces the main problem and some useful notation. In Section 3 we derive the transformed OSP and establish its equivalence to the original one. The most technical part of the paper is relegated to Section 4, in which we derive the solution of the reformulated OSP. From it, we use the reversed transformation to get back the solution to the original OSP in Section 5, where we also remark that both a BB and an OUB with general pulling level and terminal time are immediate consequences of our results. An algorithm for numerical approximations of the solution is given in Section 6, along with a compendium of illustrative cases for different values of the OUB's parameters. Concluding remarks are relegated to Section 7.

2 Formulation of the problem

Let $X = \{X_t\}_{t\in[0,1]}$ be an OUB with terminal value $X_1 = z$, $z \in \mathbb{R}$, and defined in the filtered space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,1]})$. That is, for an OU process $\widetilde{X} = \{\widetilde{X}_t\}_{t\in[0,1]}$, take X such that $\mathrm{Law}(X, \mathbb{P}) = \mathrm{Law}(\widetilde{X}, \widetilde{\mathbb{P}}_z)$, where $\widetilde{\mathbb{P}}_z := \mathbb{P}\big(\cdot\,|\widetilde{X}_1 = z\big)$. It is well known (see, e.g., Barczy and Kern (2013)) that X solves the Stochastic Differential Equation (SDE)

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \gamma\mathrm{d}B_t, \quad 0 \le t \le 1, \tag{1}$$

with $\gamma > 0$ and

$$\mu(t, x) = \alpha\frac{z - \cosh(\alpha(1-t))x}{\sinh(\alpha(1-t))}, \quad \alpha \ne 0. \tag{2}$$

Note that we can take $\{\mathcal{F}_t\}_{t\in[0,1]}$ as the natural filtration of the underlying standard Brownian motion $\{B_s\}_{t\in[0,1]}$ in (1).

Consider the finite-horizon OSP

$$V(t,x) := \sup_{\tau\leq 1-t} \mathbb{E}_{t,x}\left[X_{t+\tau}\right], \tag{3}$$

where V is the value function and $\mathbb{E}_{t,x}$ represents the expectation under the probability measure $\mathbb{P}_{t,x}$ defined as $\mathbb{P}_{t,x}(\cdot) := \mathbb{P}(\cdot|X_t = x)$. The supremum above is taken under all random times τ in the underlying filtration, such that $t+\tau$ is a stopping time in $\{\mathcal{F}_t\}_{t\in[0,1]}$. Henceforth, we will call τ a stopping time while keeping in mind that $t+\tau$ is the actual stopping time.

3 Reformulation of the problem

Barczy and Kern (2013) provide the following space-time transformed representation for X:

$$X_t = a_1(t, X_0, z) + a_2(t)B_{\psi(t)},$$

where the functions a_1 and a_2 take the form

$$a_1(t,x,z) := x\frac{\sinh(\alpha(1-t))}{\sinh(\alpha)} + z\frac{\sinh(\alpha t)}{\sinh(\alpha)}, \quad a_2(t) := \gamma e^{\alpha t}\frac{\kappa(1)-\kappa(t)}{\kappa(1)},$$

and $\psi : [0,1) \to \mathbb{R}_+$ is the time transformation $\psi(t) := \kappa(t)\kappa(1)/(\kappa(1)-\kappa(t))$, with $\kappa(t) := (2\alpha)^{-1}(1-e^{-2\alpha t})$. Notice that $t = \kappa^{-1}\left(\psi(t)\kappa(1)/(\psi(t)+\kappa(1))\right)$, where $\kappa^{-1}(s) = -(2\alpha)^{-1}\ln(1-2\alpha s)$. The following identities can be easily checked:

$$a_1(t,x,z) = \left(x + z\frac{\psi(t)e^{-\alpha}}{\kappa(1)}\right)\frac{1}{f\left(\frac{\psi(t)e^{-\alpha}}{\kappa(1)}\right)}, \quad a_2(t) = \frac{\gamma}{f\left(\frac{\psi(t)e^{-\alpha}}{\kappa(1)}\right)},$$

with

$$f(s) := \sqrt{\left(e^{\alpha}+s\right)\left(e^{-\alpha}+s\right)}. \tag{4}$$

Therefore, if we set the time change $s = \psi(t)e^{-\alpha}/\kappa(1-u)$, we get the space change

$$X_t = \frac{X_0 + zs}{f(s)} + \frac{\gamma}{f(s)}B_{s\kappa(1-u)e^{\alpha}} = \frac{zs + \gamma\sqrt{\kappa(1-u)e^{\alpha}}}{f(s)}\left(B_s + \frac{X_0}{\gamma\sqrt{\kappa(1-u)e^{\alpha}}}\right). \tag{5}$$

Let $Y = \{Y_s\}_{s\geq 0}$ be a Brownian motion starting at $Y_0 = y$ under the probability measure $\mathbb{P}_y$, that is, $\mathbb{P}_y\left(Y_0 = y\right) = 1$. Consider the infinite-horizon OSP

$$W_c(s,y) := \sup_{\sigma} \mathbb{E}_y\left[G_c(s+\sigma, Y_\sigma)\right], \tag{6}$$

with gain function

$$G_c(s,y) := \frac{cs+y}{f(s)} \tag{7}$$

and $c \in \mathbb{R}$. The operator $\mathbb{E}_y$ emphasizes that we are taking the mean with respect to $\mathbb{P}_y$, and the supremum in (6) is taken over all the stopping times σ in the natural filtration of $\{Y_s\}_{s\geq 0}$.

Solving an OSP means to give a tractable expression for the value function and to find a stopping time in which the supremum is attained. Thereby, we show in the next proposition the equivalence between (3) and (6), by providing formulae that relate V to W, and switch from a stopping time that is optimal in the former problem (if it exists) to one optimal in the latter.

Proposition 1. *(Time-space equivalence)*
Consider the time change $\upsilon : [0, t] \to \mathbb{R}$ such that $\upsilon(t) = \psi(t)e^{-\alpha}/\kappa(1)$. Take $(t, x) \in [0, 1) \times \mathbb{R}$ and set $s = \upsilon(t)$, $c_z := z/(\gamma\sqrt{\kappa(1)e^{\alpha}})$, and $y = c_x$. Then:

(i) The following equation holds:

$$V(t, x) = \frac{z}{c_z} W_{c_z}(s, y) . \tag{8}$$

(ii) The stopping time $\sigma^*(s, y)$ is optimal in (6) under $\mathbb{P}_y$ for $c = c_z$ if and only if

$$\tau^*(t, x) := \upsilon^{-1}(\sigma^*(s, y)) \tag{9}$$

is optimal in (3) under $\mathbb{P}_{t,x}$.

Proof. (i) We have already proved this part of the proposition. Indeed, (8) follows trivially from (3) and (5)–(7).

(ii) Suppose that $\sigma^* = \sigma^*(s, y)$ is optimal in (6) under $\mathbb{P}_y$ for $c = c_z$. Assume that there exists a stopping time $\tau' = \tau'(t, x)$ that outperforms $\tau^* = \tau^*(t, x)$ defined in (9), and set $\sigma' = \sigma'(s, y) := \upsilon^{-1}(\tau')$. Then, by relying on (5), we get that

$$\mathbb{E}_y\left[G_{c_z}\left(s + \sigma', Y_{\sigma'}\right)\right] = \mathbb{E}_{t,x}\left[X_{t+\tau'}\right] > \mathbb{E}_{t,x}\left[X_{t+\tau'}\right] = \mathbb{E}_y\left[G_{c_z}\left(s + \sigma^*, Y_{\sigma^*}\right)\right],$$

which contradicts the fact that σ^* is optimal in (6). Then, we have proved the *only if* part of the statement. The *if* direction follows by similar arguments. □

4 Solution of the reformulated problem: a direct approach

In this section we will work out a solution for the OSP (6). For the sake of briefness and since there is no risk of confusion, throughout the section we will use the notations $W = W_c$ and $G = G_c$, so that (6) can be rewritten as

$$W(s, y) = \sup_{\sigma} \mathbb{E}_y\left[G(s + \sigma, Y_\sigma)\right] . \tag{10}$$

Notice that $0 \le s/f(s) \le 1$ and $f(s) \ge \sqrt{1 + s^2}$ for all $s \in \mathbb{R}_+$, $f(0) = 1$, and f is increasing. Hence, the following holds for $M := \mathbb{E}\left[\sup_{0 \le u \le 1} |B_u|\right]$ and all $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$:

$$\begin{aligned}
\mathbb{E}_y\left[\sup_{u \ge 0} |G(s + u, Y_u)|\right] &\le |c| + \mathbb{E}_y\left[\sup_{u \ge 0} \frac{|Y_u|}{f(u)}\right] \le |c| + |y| + \mathbb{E}\left[\sup_{u \ge 0} \frac{|B_u|}{\sqrt{1 + u^2}}\right] \\
&\le |c| + |y| + M + \mathbb{E}\left[\sup_{u \ge 1} \frac{|B_u|}{\sqrt{1 + u^2}}\right] \\
&= |c| + |y| + M + \mathbb{E}\left[\sup_{u \ge 1} \frac{u}{\sqrt{1 + u^2}} \left|B_{1/u}\right|\right] \\
&\le |c| + |y| + M + \mathbb{E}\left[\sup_{u \ge 1} \left|B_{1/u}\right|\right] = |c| + |y| + 2M,
\end{aligned} \tag{11}$$

where we used the time-inversion property of a Brownian motion in the first equality. Thereby, since $M < \infty$ and G is continuous, we get that (see, e.g., Corollary 2.9, Remark 2.10, and equation (2.2.80) in Peskir and Shiryaev, 2006) the first hitting time

$$\sigma^*(s, y) = \inf\left\{u \ge 0 : (s + u, Y_u) \in \mathcal{D}\right\} \tag{12}$$

into the stopping set $\mathcal{D} := \{W = G\}$ is optimal for (10). That is,

$$W(s,y) = \mathbb{E}_y\left[G\left(s + \sigma^*(s,y), Y_{\sigma^*(s,y)}\right)\right]. \tag{13}$$

After applying Itô's lemma to both (6) and (13) we get the following alternative representations of W:

$$W(s,y) - G(s,y) = \sup_\sigma \mathbb{E}_y\left[\int_0^\sigma \mathbb{L}G\left(s+u, Y_u\right)\,\mathrm{d}u\right] = \mathbb{E}_y\left[\int_0^{\sigma^*(s,y)} \mathbb{L}G\left(s+u, Y_u\right)\,\mathrm{d}u\right], \tag{14}$$

where $\mathbb{L} = \partial_t + \frac{1}{2}\partial_{xx}$ is the infinitesimal generator of $\{(s+u, Y_u)\}_{u\geq 0}$. Here and thereafter, ∂_t and ∂_x will stand, respectively, for the differential operator with respect to time and space, while ∂_{xx} is a shorthand for $\partial_x\partial_x$. Notice that $\mathbb{L}G = \partial_t G$. Since many of the proofs rely on the first order partial derivatives of the gain function, we display them next for a quick reference:

$$\partial_t G(s,y) = \frac{c\left(f(s) - sf'(s)\right) - f'(s)y}{f^2(s)}, \tag{15}$$

$$\partial_x G(s,y) = \frac{1}{f(s)}. \tag{16}$$

To keep track of the initial condition in a way that does not change the underlying probability measure, we introduce the process $Y^y = \{Y_s^y\}_{s\geq 0}$ such that

$$\mathrm{Law}\left(\{Y_s^y\}_{s\geq 0}, \mathbb{P}\right) = \mathrm{Law}\left(\{Y_s\}_{s\geq 0}, \mathbb{P}_y\right).$$

Notice that the characterization of the Optimal Stopping Time (OST) in (12) is too abstract to work with. In the next proposition we characterize $\sigma^*(s,y)$ by means of a function called the Optimal Stopping Boundary (OSB), which is the frontier between $\mathcal{D}$ and its complement $\mathcal{C} := \{W > G\}$. We also derive some properties about the shape of the OSB which shed light on the geometry of $\mathcal{D}$ and $\mathcal{C}$.

Proposition 2 (Existence and shape of the optimal stopping boundary)**.**
There exists a function $b : \mathbb{R}_+ \to \mathbb{R}$ such that $\mathcal{D} = \{(s,y) : y \geq b(s)\}$. Moreover, $c(f(s) - sf'(s))/f(s) < b(s) < \infty$ for all $s \in \mathbb{R}_+$.

Proof. The claimed shape for the stopping set, $\mathcal{D} = \{(s,y) : y \geq b(s)\}$, is a straightforward consequence of the fact that both $y \mapsto G(s,y)$ and $y \mapsto Y_s^y$ are increasing for all $s \in \mathbb{R}_+$.

We now see that $b(s) > c(f(s) - sf'(s))/f(s)$ for all $s > 0$. Fix a pair (s,y) such that $\partial_t G(s,y) > 0$. Then, the continuity of $\partial_t G$ allows to pick a ball $\mathcal{B}$ such that $(s,y) \in \mathcal{B}$ and $\partial_t G > 0$ in $\mathcal{B}$. After recalling (14) and setting $\sigma_{\mathcal{B}}$ as the first exit time of $\{(s+u, Y_u^y)\}_{u\geq 0}$ from $\mathcal{B}$, we get that

$$W(s,y) - G(s,y) \geq \mathbb{E}_y\left[\int_0^{\sigma_{\mathcal{B}}} \partial_t G\left(s+u, Y_u\right)\,\mathrm{d}u\right] > 0.$$

We conclude then that $(s,y) \in \mathcal{C}$. Finally, the claimed lower bound for b comes after using (15) to realize that $\partial_t G(s,y) > 0$ if and only if $y < c(f(s) - sf'(s))/f(s)$.

We now prove $b(s) < \infty$ for all $s > 0$. Let $\widetilde{X} = \{\widetilde{X}_t\}_{t\in[0,1]}$ be a BB with pinning point $\widetilde{X}_1 = z$. The drift of $\widetilde{X}$ has the form $\widetilde{\mu}(t,x) = (z-x)/(1-t)$. Define $m_z : [0,1) \to \mathbb{R}$ such that

$$m_z(t) = z\frac{\sinh(\alpha(1-t)) - \alpha(1-t)}{\sinh(\alpha(1-t)) - \alpha(1-t)\cosh(\alpha(1-t))},$$

take $\overline{M}_z := \sup_{t\in[0,1)} m_z(t) < \infty$ and notice the following relation:

$$X_t \leq m_z(t) + |X_t - m_z(t)| \leq m_z(t) + |\widetilde{X}_t - m_z(t)| \leq \overline{M}_z + |\widetilde{X}_t - \overline{M}_z|.$$

The second inequality holds since, due to the fact that $\mu(t,x) \leq \widetilde{\mu}(t,x)$ if and only if $x \geq m_z(t)$, the drift of the reflection of X with respect to m_z is lower than the drift of the reflection of $\widetilde{X}$ with respect to m_z, and therefore we can ensure that, pathwise, the first process is lower than the last one $\mathbb{P}$-a.s. (see Ikeda and Watanabe, 1977, Theorem 1.1). The third inequality is straightforward from the definition of $\overline{M}_z$. Therefore, if we consider the OSP

$$\widetilde{V}_{\overline{M}_z}(t,x) = \sup_{\tau \leq 1-t} \mathbb{E}_{t,x}\left[\overline{M}_z + |\widetilde{X}_{t+\tau} - \overline{M}_z|\right],$$

we are allowed to state that $V \leq \widetilde{V}_{\overline{M}_z}$. If we take a pair $(t,x) \in [0,1] \times [\overline{M}, \infty)$ within the stopping set related to $V_{\overline{M}_z}$, then $V(t,x) \leq V_{\underline{M}_z}(t,x) = x$, meaning that (t,x) lies in the stopping set of V. Since it is known that the OSB related to $V_{\underline{M}_z}$ is finite (actually, this is one of the few cases in which the explicit form of the OSP with finite horizon is available; see, e.g., Theorem 3.2 in Ekström and Wanntorp (2009)), so is the one related to V. Then, by means of (8), we conclude that b is bounded from above. □

We next show that W is Lipschitz continuous in sets of the type $\mathbb{R}_+ \times \mathcal{R}$, where $\mathcal{R}$ stands for a compact set in $\mathbb{R}$.

Proposition 3 (Lipschitz continuity of the value function)**.**
For any compact set $\mathcal{R} \subset \mathbb{R}$, there exists a constant $L_{\mathcal{R}} > 0$ such that

$$|W(s_1,y_1) - W(s_2,y_2)| \leq L_{\mathcal{R}}\left(|s_1 - s_2| + |y_1 + y_2|\right),$$

for all $(s_1,y_1),(s_2,y_2) \in \mathbb{R}_+ \times \mathcal{R}$.

Proof. Take $(s_1,y_1),(s_2,y_2) \in \mathbb{R}_+ \times \mathcal{R}$ and realize that

$$\begin{aligned} W(s_1,y_1) - W(s_2,y_2) = &\sup_\sigma \mathbb{E}_{y_1}\left[G(s_1+\sigma, Y_\sigma)\right] - \sup_\sigma \mathbb{E}_{y_2}\left[G(s_1+\sigma, Y_\sigma)\right] \\ &+ \sup_\sigma \mathbb{E}_{y_2}\left[G(s_1+\sigma, Y_\sigma)\right] - \sup_\sigma \mathbb{E}_{y_2}\left[G(s_2+\sigma, Y_\sigma)\right]. \end{aligned}$$

Notice from (15) that the following relation holds:

$$|\partial_t G(s,y)| \leq K\left(1 + \frac{|y|}{f(u)}\right).$$

Then, since $|\sup_\sigma a_\sigma - \sup_\sigma b_\sigma| \leq \sup_\sigma |a_\sigma - b_\sigma|$, alongside Jensen's inequality, and (15) and (16), we get that

$$\begin{aligned} &\left|\sup_\sigma \mathbb{E}_{y_1}\left[G(s_1+\sigma, Y_\sigma)\right] - \sup_\sigma \mathbb{E}_{y_2}\left[G(s_1+\sigma, Y_\sigma)\right]\right| \\ &\qquad \leq \sup_\sigma \mathbb{E}\left[\left|G(s_1+\sigma, Y_\sigma^{y_1}) - G(s_1+\sigma, Y_\sigma^{y_2})\right|\right] \\ &\qquad = \sup_\sigma \mathbb{E}\left[\frac{|Y_\sigma^{y_1} - Y_\sigma^{y_2}|}{f(s_1+\sigma)}\right] = \frac{|y_1 - y_2|}{f(s_1)} \leq |y_1 - y_2|, \end{aligned}$$

and

$$\begin{aligned} &\left|\sup_\sigma \mathbb{E}_{y_2}\left[G(s_1+\sigma, Y_\sigma)\right] - \sup_\sigma \mathbb{E}_{y_2}\left[G(s_2+\sigma, Y_\sigma)\right]\right| \\ &\qquad \leq \sup_\sigma \mathbb{E}\left[\left|G(s_1+\sigma, Y_\sigma^{y_2}) - G(s_2+\sigma, Y_\sigma^{y_2})\right|\right] \\ &\qquad = |s_1 - s_2| \sup_\sigma \mathbb{E}\left[\left|\partial_t G(\xi, Y_\sigma^{y_2})\right|\right] \\ &\qquad \leq |s_1 - s_2| K\left(1 + \mathbb{E}\left[\sup_{s \geq 0} \frac{|Y_s^{y_2}|}{f(s)}\right]\right), \end{aligned}$$

where $\xi \in (\min\{s_1,s_2\}, \max\{s_1,s_2\})$ follows from the mean value theorem. Since we already proved in (11) that $\mathbb{E}\left[\sup_{s\geq 0}|Y_s^{y_2}|/f(s)\right] < \infty$, the Lipschitz continuity of W in $\mathbb{R}_+ \times \mathcal{R}$ follows. □

Beyond Lipschitz continuity, it turns out that the value function attains a higher smoothness away from the boundary. While this assertion is trivial in the interior of the stopping region, where $W = G$, we prove in the next proposition that it also holds in the continuation set. In addition, we show that $\mathbb{L}W$ vanishes in $\mathcal{C}$, which establishes the equivalence between (10) and a free-boundary problem.

Proposition 4 (Higher smoothness of the value function and the free-boundary problem). *$W \in C^{1,2}(\mathcal{C})$ and $\mathbb{L}W = 0$ in $\mathcal{C}$.*

Proof. The fact that $\mathbb{L}W = 0$ in $\mathcal{C}$ comes right after the strong Markov property of $\{(s+u, Y_u)\}_{u\geq 0}$; see Peskir and Shiryaev (2006, Section 7.1) for more details.

Since W is continuous in $\mathcal{C}$ (see Proposition 3) and the coefficients in the parabolic operator $\mathbb{L}$ are smooth enough (it suffices to require local α-Hölder continuity), then standard theory from parabolic partial differential equations (Friedman, 1964, Section 3, Theorem 9) guarantees that, for an open rectangle $R \subset \mathcal{C}$, the first initial-boundary value problem

$$\mathbb{L}f = 0 \quad \text{in } R, \tag{17a}$$

$$f = V \quad \text{on } \partial R \tag{17b}$$

has a unique solution $f \in C^{1,2}(R)$. Therefore, we can use Itô's formula on $f(s+u, Y_u)$ at $u = \tau_{R^c}$, that is, the first time $(s+u, Y_u)$ exits R, and then take $\mathbb{P}_y$-expectation with $y \in R$, which guarantees the vanishing of the martingale term and yields, together with (17a) and (17b), the equality $\mathbb{E}_y[W(s+\tau_{R^c}, Y_{\tau_{R^c}})] = f(t, x)$. Finally, due to the strong Markov property, $\mathbb{E}_y[W(s+\tau_{R^c}, Y_{\tau_{R^c}})] = W(s, y)$. □

Not only the gain function has continuous partial derivatives away from the boundary, but we can provide relatively explicit forms for those derivatives, as shown in the next proposition.

Proposition 5 (Partial derivatives of the value function). *Let $\sigma^* = \sigma^*(s, y)$, for $(s, y) \in \mathcal{C}$, and $a := e^{-\alpha} + e^{\alpha}$. Then,*

$$\partial_t W(s, y) = \partial_t G(s, y) + \mathbb{E}\left[\int_s^{s+\sigma^*} \frac{1}{f^3(u)}\left(-c\,(a+3u) + \frac{3\,(a+2u)^2}{4f^2(u)} - Y_{u-s}^y\right) \mathrm{d}u\right] \tag{18}$$

and

$$\partial_x W(s, y) = \mathbb{E}\left[\frac{1}{f(s+\sigma^*)}\right]. \tag{19}$$

Proof. Take $(s, y) \in \mathcal{C}$ and $\varepsilon > 0$. Due to (10) and (13), one gets the following for $\sigma^* = \sigma^*(s, y)$:

$$\varepsilon^{-1}\left(W(s, y) - W(s-\varepsilon, y)\right) \leq \varepsilon^{-1}\mathbb{E}\left[G(s+\sigma^*, Y_{\sigma^*}^y) - G(s-\varepsilon+\sigma^*, Y_{\sigma^*}^y)\right].$$

Hence, by letting $\varepsilon \to 0$ and recalling that $W \in C^{1,2}(\mathcal{C})$ (see Proposition 4), we get that

$$\partial_t W(s, y) \leq \mathbb{E}\left[\partial_t G(s+\sigma^*, Y_{\sigma^*}^y)\right] = \partial_t G(s, y) + \mathbb{E}\left[\int_s^{s+\sigma^*} \mathbb{L}\partial_t G(u, Y_{s-u}^y)\,\mathrm{d}u\right]. \tag{20}$$

In the same fashion we obtain

$$\varepsilon^{-1}\left(W(s+\varepsilon, y) - W(s, y)\right) \geq \varepsilon^{-1}\mathbb{E}\left[G(s+\varepsilon+\sigma^*, Y_{\sigma^*}^y) - G(s+\sigma^*, Y_{\sigma^*}^y)\right].$$

Thus, by arguing as in (20) we get the reverse inequality, and therefore (18) gets proved after computing $\mathbb{L}\partial_t G(u, Y_{s-u}^y) = \partial_{tt} G(u, Y_{s-u}^y)$.

To get the analog result for the space coordinate, notice that

$$\varepsilon^{-1}\left(W(s, y) - W(s, y-\varepsilon)\right) \leq \varepsilon^{-1}\mathbb{E}\left[W(s+\sigma^*, Y_{\sigma^*}^y) - W(s+\sigma^*, Y_{\sigma^*}^{y-\varepsilon})\right]$$

$$\leq \varepsilon^{-1}\mathbb{E}\left[G(s+\sigma^*, Y^y_{\sigma^*}) - G(s+\sigma^*, Y^{y-\varepsilon}_{\sigma^*})\right]$$
$$= \mathbb{E}\left[\frac{1}{f(s+\sigma^*)}\right],$$

while the same reasoning yields the inequality $\varepsilon^{-1}(W(s,y+\varepsilon) - W(s,y)) \geq \mathbb{E}[1/f(s+\sigma^*)]$, and then, by letting $\varepsilon \to 0$, we get (19). □

So far we have proved that solving (10) is equivalent to solving the free-boundary problem

$$\mathbb{L}W(s,y) = 0 \qquad \text{for } y < b(t), \tag{21a}$$
$$W(s,y) > G(s,y) \qquad \text{for } y < b(t), \tag{21b}$$
$$W(s,y) = G(s,y) \qquad \text{for } y \geq b(t). \tag{21c}$$

However, an additional condition for the value function on the free boundary is required to guarantee a unique solution. Roughly speaking, that condition comes in the form of smoothly binding the value and the gain functions with respect to the space coordinate, provided that the optimal boundary is (probabilistically) regular for the underlying process, that is, if after starting at a point $(s,y) \in \partial\mathcal{C}$, the process enters D immediately $\mathbb{P}_y$-a.s. This type of regularity is proved to hold true for piecewise monotonic and continuous boundaries in Cox and Peskir (2015) whenever the underlying process is a recurrent diffusion. In the next proposition we show that the boundary is differentiable with bounded derivative on any real interval, which implies piecewise monotonicity. The proof is inspired on Theorem 4.3 from De Angelis and Stabile (2019), which states the boundary's Lipschitz continuity for time-homogeneous processes satisfying some regularity conditions.

Proposition 6 (Lipschitz continuity of the optimal stopping boundary)**.**
The function b is differentiable. Moreover, for any closed interval $I := [\underline{s}, \overline{s}] \subset \mathbb{R}_+$, there exists a constant $L_I > 0$ such that

$$|b'(s)| \leq L_I, \tag{22}$$

whenever $s \in I$.

Proof. Consider the function $H : I \times \mathbb{R} \to \mathbb{R}_+$, for a closed interval $I \subset \mathbb{R}_+$, defined as $H(s,y) = W(s,y) - G(s,y)$. Proposition 2 entails that b is bounded from below, and thus we can choose a constant $r \in \mathbb{R}$ such that $r < \inf\{b(s) : s \in I\}$. Since $I \times \{r\} \subset \mathcal{C}$, H is continuous (see Proposition 3) and $H|_{I\times\{r\}} > 0$. Then, there exists $a > 0$ such that $H(s,r) > a$ for all $s \in I$. Therefore, for all δ such that $0 < \delta \leq a$, the equation $H(s,y) = \delta$ has a solution in $\mathcal{C}$ for all $s \in I$. Moreover, this solution is unique for each s since $\partial_x H < 0$ in $\mathcal{C}$ (see Proposition 5), and we denote it by $b_\delta(s)$, where $b_\delta : I \to \mathbb{R}$. Away from the boundary, H is regular enough to apply the implicit function theorem that guarantees that b_δ is differentiable and

$$b'_\delta(s) = -\partial_t H(s, b_\delta(s))/\partial_x H(s, b_\delta(s)). \tag{23}$$

Notice that b_δ is decreasing in δ and therefore it converges pointwisely to some limit function b_0, which satisfies $b_0 \leq b$ in I as $b_\delta < b$ for all δ. Since $H(s, b_\delta(s)) = \delta$ and H is continuous, it follows that $H(s, b_0(s)) = 0$ after taking $\delta \to 0$, which means that $b_0 \geq b$ in I and hence $b_0 = b$ in I.

Take $(s,y) \in \mathcal{C}$ such that $y > r$. Set $\sigma^* = \sigma^*(s,y)$ and consider

$$\sigma_r = \sigma_r(s,y) := \inf\{u \geq 0 : (s+u, Y^y_u) \notin I \times (r,\infty)\}.$$

Recalling (18), it is easy to check that there exists a constant $K^{(1)}_I > 0$ such that

$$|\partial_t H(s,y)| \leq K^{(1)}_I m(s,y) \tag{24}$$

with

$$m(s,y) := \mathbb{E}_y\left[\int_0^{\sigma^*}\left(1+\frac{|Y_u|}{f^2(s+u)}\right)\,\mathrm{d}u\right].$$

Using the tower property of conditional expectation, alongside the strong Markov property, we get

$$\begin{aligned}
&m(s,y)\\
&\quad= \mathbb{E}_y\left[\int_0^{\sigma^*\wedge\sigma_r}\left(1+\frac{|Y_u|}{f^2(s+u)}\right)\,\mathrm{d}u + \mathbb{1}\left(\sigma_r\le\sigma^*\right)\int_{\sigma_r}^{\sigma^*}\left(1+\frac{|Y_u|}{f^2(s+u)}\right)\,\mathrm{d}u\right]\\
&\quad= \mathbb{E}_y\left[\int_0^{\sigma^*\wedge\sigma_r}\left(1+\frac{|Y_u|}{f^2(s+u)}\right)\,\mathrm{d}u + \mathbb{1}\left(\sigma_r\le\sigma^*\right)\mathbb{E}_y\left[\int_{\sigma_r}^{\sigma_r+\sigma^*(\sigma_r,Y_{\sigma_r})}\left(1+\frac{|Y_u|}{f^2(s+u)}\right)\,\mathrm{d}u\Big|\mathcal{F}_{\sigma_r}\right]\right]\\
&\quad= \mathbb{E}_y\left[\int_0^{\sigma^*\wedge\sigma_r}\left(1+\frac{|Y_u|}{f^2(s+u)}\right)\,\mathrm{d}u + \mathbb{1}\left(\sigma_r\le\sigma^*\right)\mathbb{E}_r\left[\int_0^{\sigma^*(\sigma_r,Y_{\sigma_r})}\left(1+\frac{|Y_u|}{f^2(s+\sigma_r+u)}\right)\,\mathrm{d}u\right]\right]\\
&\quad= \mathbb{E}_y\left[\int_0^{\sigma^*\wedge\sigma_r}\left(1+\frac{|Y_u|}{f^2(s+u)}\right)\,\mathrm{d}u + \mathbb{1}\left(\sigma_r\le\sigma^*\right)m(s+\sigma_r,Y_{\sigma_r})\right]. \qquad (25)
\end{aligned}$$

Notice that, for $c<r<y<b(s)$, $(s+\sigma_r,Y^y_{\sigma_r})\in\Gamma_s$ on the set $\{\sigma_r\le\sigma^*\}$, with $\Gamma_s := \{(s,\bar{s})\times\{r\}\}\cup\{\bar{s}\times[r,b(\bar{s}))\}$ and $\bar{s} := \sup\{s : s\in I\}$. Hence, the following holds true on the set $\{\sigma_r\le\sigma^*\}$:

$$\begin{aligned}
m\left(s+\sigma_r,Y^y_{\sigma_r}\right) &\le \sup_{(t,x)\in\Gamma_s} m(t,x)\\
&\le \sup_{(t,x)\in\Gamma_s}\mathbb{E}_x\left[\int_0^\infty\left(1+\frac{|Y_u|}{f^2(t+u)}\right)\,\mathrm{d}u\right]\\
&\le \sup_{(t,x)\in\Gamma_s}\int_0^\infty\left(1+\frac{|x|}{f^2(t+u)}\right)\,\mathrm{d}u + \int_0^\infty\frac{\mathbb{E}\left[|B_u|\right]}{f^2(t+u)}\,\mathrm{d}u\\
&\le \int_0^\infty\left(1+\frac{|b(\bar{s})|}{f^2(u)}\right)\,\mathrm{d}u + \int_0^\infty\sqrt{\frac{2}{\pi}}\frac{\sqrt{u}}{f^2(u)}\,\mathrm{d}u < \infty. \qquad (26)
\end{aligned}$$

By plugging (26) into (25), after observing that $\left(1+|Y_u|/f^2(s+u)\right)\le 1+\max\{|\sup_{s\in I}b(s)|,|r|\}$, and recalling (24), we obtain the following for some constant $K_I^{(2)}>0$:

$$|\partial_t H(s,y)| \le K_I^{(2)}\mathbb{E}_y\left[\sigma_\delta\wedge\sigma_r + \mathbb{1}\left(\sigma_r\le\sigma_\delta\right)\right]. \qquad (27)$$

Arguing as in (25) and recalling (16) along with (19), we get that

$$\begin{aligned}
|\partial_x H(s,y)| &= \mathbb{E}_y\left[\frac{1}{f(s)} - \frac{1}{f(s+\sigma^*)}\right] = \mathbb{E}_y\left[\int_0^{\sigma^*} -\partial_t(1/f)(s+u)\,\mathrm{d}u\right]\\
&= \mathbb{E}_y\left[\int_0^{\sigma^*\wedge\sigma_r} -\partial_t(1/f)(s+u)\,\mathrm{d}u + \mathbb{1}\left(\sigma_r\le\sigma^*\right)|\partial_x H(s+\sigma_r,Y_{\sigma_r})|\right]\\
&\ge \mathbb{E}_y\left[\int_0^{\sigma^*\wedge\sigma_r} -\partial_t(1/f)(s+u)\,\mathrm{d}u + \mathbb{1}\left(\sigma_r\le\sigma^*,\sigma_r<\bar{s}-s\right)|\partial_x H(s+\sigma_r,r)|\right]. \qquad (28)
\end{aligned}$$

Take $\varepsilon>0$ such that $I\times\{r+\varepsilon\}\subset\mathcal{C}$, and consider the stopping time $\sigma_\varepsilon = \inf\{u\ge 0 : Y^r_u > r+\varepsilon\}$. Observe that $\sigma^*(s,r)>\sigma_\varepsilon$ for all $s\in I$. Then,

$$|\partial_x H(s+\sigma_r,r)| \ge \inf_{s\in I}|\partial_x H(s,r)| = \inf_{s\in I}\mathbb{E}_r\left[\frac{1}{f(s)} - \frac{1}{f(s+\sigma^*(s,r))}\right]$$

$$
\begin{aligned}
&\geq \inf_{s\in I} \mathbb{E}_r \left[\frac{1}{f(s)} - \frac{1}{f(s+\sigma_\varepsilon)} \right] \\
&\geq \inf_{s\in I} \left(\frac{1}{f(s)} - \frac{1}{f(s+\varepsilon)} \right) \mathbb{P}_r \left(\sigma_\varepsilon > \varepsilon \right) \\
&= \left(\frac{1}{f(\overline{s})} - \frac{1}{f(\overline{s}+\varepsilon)} \right) \mathbb{P} \left(\sup_{u\leq\varepsilon} B_u < \varepsilon \right) > 0,
\end{aligned} \tag{29}
$$

where we used the fact that $s \mapsto 1/f(s) - 1/f(s+u)$ is decreasing for all $u \geq 0$. After noticing that $-\partial_t(1/f)$ is positive and decreasing, which means that $-\partial_t(1/f)(s+u) \geq -\partial_t(1/f)(\overline{s}) > 0$ for all $u \leq \sigma_r$, and by plugging (29) into (28), we obtain, for a constant $K_{I,\varepsilon}^{(3)} > 0$,

$$
|\partial_x H(s,y)| \geq K_I^{(3)} \mathbb{E}_y \left[\sigma^* \wedge \sigma_r + \mathbb{1} \left(\sigma_r \leq \sigma^*, \sigma_r < \overline{s} - s \right) \right]. \tag{30}
$$

Therefore, using (27) and (30) in (23) yields the following bound for some constant $K_I^{(4)} > 0$, $y_\delta = b_\delta(s)$, and $\sigma_\delta = \sigma^*(s, y_\delta)$:

$$
\begin{aligned}
\left| b'_\delta(s) \right| &\leq K_I^{(4)} \frac{\mathbb{E}_{y_\delta} \left[\sigma_\delta \wedge \sigma_r + \mathbb{1} \left(\sigma_r \leq \sigma_\delta \right) \right]}{\mathbb{E}_{y_\delta} \left[\sigma_\delta \wedge \sigma_r + \mathbb{1} \left(\sigma_r \leq \sigma_\delta, \sigma_r < \overline{s} - s \right) \right]} \\
&\leq K_I^{(4)} \left(1 + \frac{\mathbb{P}_{y_\delta} \left(\sigma_r \leq \sigma_\delta \right)}{\mathbb{E}_{y_\delta} \left[\sigma_\delta \wedge \sigma_r + \mathbb{1} \left(\sigma_r \leq \sigma_\delta, \sigma_r < \overline{s} - s \right) \right]} \right) \\
&\leq K_I^{(4)} \left(1 + \frac{\mathbb{P}_{y_\delta} \left(\sigma_r \leq \sigma_\delta, \sigma_r = \overline{s} - s \right)}{\mathbb{E}_{y_\delta} \left[\sigma_\delta \wedge \sigma_r \right]} + \frac{\mathbb{P}_{y_\delta} \left(\sigma_r \leq \sigma_\delta, \sigma_r < \overline{s} - s \right)}{\mathbb{E}_{y_\delta} \left[\mathbb{1} \left(\sigma_r \leq \sigma_\delta, \sigma_r < \overline{s} - s \right) \right]} \right) \\
&\leq K_I^{(4)} \left(2 + \frac{\mathbb{P}_{y_\delta} \left(\sigma_r \leq \sigma_\delta, \sigma_r = \overline{s} - s \right)}{\mathbb{E}_{y_\delta} \left[\mathbb{1} \left(\sigma_r \leq \sigma_\delta, \sigma_r = \overline{s} - s \right) \left(\sigma_\delta \wedge \sigma_r \right) \right]} \right) \\
&\leq K_I^{(4)} \left(2 + \frac{1}{\overline{s} - s} \right).
\end{aligned} \tag{31}
$$

If we set $I_\varepsilon = [\underline{s}, \overline{s} - \varepsilon]$ for $\varepsilon > 0$ small enough, then, by relying on (31), we obtain the existence of a constant $L_{I_\varepsilon} > 0$, independent from δ, such that $|b'_\delta(s)| < L_{I_\varepsilon}$ for all $s \in I_\varepsilon$ and $0 < \delta \leq a$. We are thus able to use the Arzelà–Ascoly's theorem to guarantee that b_δ converges to b uniformly with respect to δ in I_ε. Since $\varepsilon > 0$ and I were arbitrarily chosen, we then conclude that b is anywhere differentiable and (22) holds true. $\square$

Once we have the Lipschitz continuity of the boundary on real bounded sets, this implying piecewise monotonicity, we proceed to illustrate in the following proposition how to obtain the principle of smooth fit, which, as we highlighted before, is required to provide a unique solution to the associated free-boundary problem (21a)–(21c).

Proposition 7 (The smooth-fit condition)**.**
For all $s \geq 0$, $y \mapsto W(s,y)$ is differentiable at $y = b(s)$. Moreover, $\partial_x W(s, b(s)) = \partial_x G(s, b(s))$.

Proof. Recall that we have already obtained in (19) an explicit form for $\partial_x W$ away from the boundary, namely,

$$
\partial_x W(s,y) = \mathbb{E} \left[\frac{1}{f(s + \sigma^*(s,y))} \right], \quad (s,y) \in \mathcal{C}.
$$

The principle of smooth fit is just the validation of such a formula on the boundary points $y = b(s)$, $s \in \mathbb{R}_+$.

We have that $\partial_x W(s, b(s)^+) = \partial_x G(s, b(s)) = 1/f(s)$, as $\sigma^*(s,y) = 0$ for all $y \geq b(s)$. By relying on Cox and Peskir (2015, Corollary 8), alongside the fact that our OSB is piecewise monotonic and continuous, we get that $\sigma^*(s, b(s)^-) = \sigma^*(s, b(s)) = 0$ $\mathbb{P}$-a.s., and hence the Dominated Convergence Theorem (DCT) entails that $\partial_x W(s, b(s)^-) = 1/f(s) = \partial_x G(s, b(s))$, thus concluding that the smooth-fit condition holds. $\square$

We are now in the position of getting a tractable characterization of both the value function and the OSB. Propositions 2–7 allow us to use an extension of the Itô's lemma (D'Auria et al., 2020, Lemma A2) on the function $W(s+t, Y_t)$ for $t \geq 0$. By recalling that $\mathbb{L}W = 0$ on $\mathcal{C}$ and $W = G$ on $\mathcal{D}$, and after taking $\mathbb{P}_y$-expectation (which cancels the martingale term), we get

$$
\begin{aligned}
W(s,y) &= \mathbb{E}_y\left[W(s+t, Y_t)\right] - \mathbb{E}_y\left[\int_0^t \left(\mathbb{L}W\right)(s+u, Y_u)\, \mathrm{d}u\right] \\
&= \mathbb{E}_y\left[W(s+t, Y_t)\right] - \mathbb{E}_y\left[\int_0^t \partial_t G\left(s+u, Y_u\right) \mathbb{1}\left(Y_u \geq b(s+u)\right)\, \mathrm{d}u\right],
\end{aligned} \tag{32}
$$

where the local-time term does not appear due to the smooth-fit condition.

Lemma 1. *For all* $(s,y) \in \mathbb{R}_+ \times \mathbb{R}$*,*

$$
\lim_{u\to\infty} \mathbb{E}_y\left[W(s+u, Y_u)\right] = c.
$$

Proof. The Markov property of Y, together with the fact that both $s \mapsto s/f(s)$ and $s \mapsto f(s)$ are increasing and $s/f(s) \to 1$ as $s \to \infty$, implies that

$$
\begin{aligned}
\mathbb{E}_y\left[W(s+u, Y_u)\right] &= \mathbb{E}_y\left[\sup_\sigma \mathbb{E}_{Y_u}\left[G\left(s+u+\sigma, Y_\sigma\right)\right]\right] \leq \mathbb{E}_y\left[\mathbb{E}_{Y_u}\left[\sup_{r\geq 0} G\left(s+u+r, Y_r\right)\right]\right] \\
&= \mathbb{E}_y\left[\mathbb{E}_{Y_u}\left[\sup_{r\geq 0}\left\{c\frac{s+u+r}{f(s+u+r)} + \frac{Y_r}{f(s+u+r)}\right\}\right]\right] \\
&\leq c\left(\mathbb{1}(c>0) + \frac{s+u}{f(s+u)}\mathbb{1}(c\leq 0)\right) + \mathbb{E}_y\left[\sup_{r\geq 0}\frac{Y_{u+r}}{f(u+r)}\right],
\end{aligned} \tag{33}
$$

and

$$
\begin{aligned}
\mathbb{E}_y\left[W(s+u, Y_u)\right] &\geq \mathbb{E}_y\left[\mathbb{E}_{Y_u}\left[\inf_{r\geq 0} G\left(s+u+r, Y_r\right)\right]\right] \\
&\geq c\left(\mathbb{1}(c<0) + \frac{s+u}{f(s+u)}\mathbb{1}(c\geq 0)\right) + \mathbb{E}_y\left[\inf_{r\geq 0}\frac{Y_{u+r}}{f(s+u+r)}\right].
\end{aligned} \tag{34}
$$

Notice that

$$
\lim_{u\to\infty} \mathbb{E}_y\left[\sup_{r\geq 0}\frac{Y_{u+r}}{f(u+r)}\right] = \mathbb{E}_y\left[\lim_{u\to\infty}\sup_{r\geq u}\frac{Y_r}{f(r)}\right] = \mathbb{E}_y\left[\limsup_{u\to\infty}\frac{Y_u}{f(u)}\right] = 0,
$$

where in the first equality we applied the monotone convergence theorem and in the second one we used the law of the iterated logarithm as an estimate of the convergence of the process in the numerator. A similar argument yields

$$
\lim_{u\to\infty} \mathbb{E}_y\left[\inf_{r\geq 0}\frac{Y_{u+r}}{f(s+u+r)}\right] = 0.
$$

Thus, we can take $u \to \infty$ in both (33) and (34) to complete the proof. □

By taking $t \to \infty$ in (32) and relying on Proposition 1, we get the following pricing formula for the value function:

$$
\begin{aligned}
W(s,y) &= c - \mathbb{E}_y\left[\int_0^\infty \left(\mathbb{L}W\right)(s+u, Y_u)\, \mathrm{d}u\right] \\
&= c - \mathbb{E}_y\left[\int_0^\infty \partial_t G\left(s+u, Y_u\right) \mathbb{1}\left(Y_u \geq b(s+u)\right)\, \mathrm{d}u\right].
\end{aligned} \tag{35}
$$

We can obtain a more tractable version of (35) by exploiting the linearity of $y \mapsto \partial_t G(s,y)$ (see (15)) as well as the Gaussianity of Y_u. Specifically, since $Y_u \sim \mathcal{N}(y,u)$ under $\mathbb{P}_y$, then $\mathbb{E}_y\left[Y_u \mathbb{1}\left(Y_u \geq x\right)\right] = \bar{\Phi}((x-y)/\sqrt{u})y + \sqrt{u}\phi((x-y)/\sqrt{u})$, where $\bar{\Phi}$ and ϕ denote the survival and the density functions of a standard normal random variable, respectively. By shifting the integrating variable s units to the right, we get that

$$W(s,y) = c - \int_s^\infty \frac{1}{f(u)}\left(c\bar{\Phi}_{s,y,u,b(u)} - \frac{(a+2u)\left((y+cu)\bar{\Phi}_{s,y,u,b(u)} + \sqrt{u-s}\phi_{s,y,u,b(u)}\right)}{2f^2(u)}\right) \mathrm{d}u, \quad (36)$$

where $a = e^{-\alpha} + e^{\alpha}$ and

$$\bar{\Phi}_{s_1,y_1,s_2,y_2} := \bar{\Phi}\left(\frac{y_2 - y_1}{\sqrt{s_2 - s_1}}\right), \quad \phi_{s_1,y_1,s_2,y_2} := \phi\left(\frac{y_2 - y_1}{\sqrt{s_2 - s_1}}\right), \quad y_1, y_2 \in \mathbb{R}, s_2 \geq s_1 \geq 0.$$

Take now $y \downarrow b(s)$ in both (35) and (36) to derive the free-boundary equation

$$G(s,b(s)) = c - \mathbb{E}_{b(s)}\left[\int_0^\infty \partial_t G\left(s+u, Y_u\right)\mathbb{1}\left(Y_u \geq b(s+u)\right)\mathrm{d}u\right], \quad (37)$$

alongside its more explicit expression

$$\begin{aligned} &G(s,b(s)) \\ &= c - \int_s^\infty \frac{1}{f(u)}\left(c\bar{\Phi}_{s,b(s),u,b(u)} - \frac{(a+2u)\left((b(s)+cu)\bar{\Phi}_{s,b(s),u,b(u)} + \sqrt{u-s}\phi_{s,b(s),u,b(u)}\right)}{2f^2(u)}\right)\mathrm{d}u. \end{aligned}$$

It turns out that there exists a unique function b that solves (37), as we state in the next theorem. The proof for such an assertion follows from adapting the methodology used in Peskir (2005, Theorem 3.1), where it is addressed the uniqueness of the solution of the free-boundary equation for an American put option with a geometric Brownian motion.

Theorem 1. *The integral equation* (37) *admits a unique solution among the class of continuous functions $\beta : \mathbb{R}_+ \to \mathbb{R}$ of bounded variation and such that $\beta(s) > c$ for all $s \in \mathbb{R}_+$.*

Proof. Suppose there exists a function $\beta : \mathbb{R}_+ \to \mathbb{R}$ solving the integral equation (37), and define W^β as in (35), but with β instead of b. We can conclude from (35) that the integrand is twice continuously differentiable with respect to y and, therefore, we can obtain $\partial_x W^\beta$ and $\partial_{xx} W^\beta$ by differentiating inside the integral symbol and ensure they are continuous functions on $\mathbb{R}_+ \times \mathbb{R}$. Besides, the following expression for $\mathbb{L}W^\beta$ can be easily computed from (35):

$$\mathbb{L}W^\beta(s,y) = \partial_t G(t,y)\mathbb{1}(y \geq \beta(s)).$$

Define the sets

$$\mathcal{C}_\beta := \left\{(s,y) \in \mathbb{R}_+ \times \mathbb{R} : y < \beta(s)\right\}, \quad \mathcal{D}_\beta := \left\{(s,y) \in \mathbb{R}_+ \times \mathbb{R} : y \geq \beta(s)\right\}.$$

It turns out that, on both sets, W^β is regular enough to apply the extension of the Itô's formula given in Lemma A2 in D'Auria et al. (2020), which yields

$$W^\beta(s,y) = \mathbb{E}_y\left[W^\beta(s+t, Y_t)\right] - \mathbb{E}_y\left[\int_0^t \partial_t G\left(s+u, Y_u\right)\mathbb{1}\left(Y_u \geq \beta(s+u)\right)\mathrm{d}u\right], \quad (38)$$

where the martingale term is canceled after taking $\mathbb{P}_y$-expectation and the local time term is missing due to the continuity of $\partial_x W^\beta$ on $\partial\mathcal{C}_\beta$. In addition,

$$G(s,y) = \mathbb{E}_y\left[G(s+t, Y_t)\right] - \mathbb{E}_y\left[\int_0^t \partial_t G\left(s+u, Y_u\right)\mathrm{d}u\right]. \quad (39)$$

Consider the first hitting time $\sigma_{\mathcal{C}_\beta}$ into $\mathcal{C}_\beta$, fix $(s,y) \in \mathcal{D}_\beta$, and notice that $\mathbb{P}_y(Y_u \geq \beta(t+s)) = 1$ for all $0 \leq u \leq \rho_{\mathcal{C}_\beta}$. Recall that $W^\beta(s, \beta(s)) = G(s, \beta(s))$ for all $s \in \mathbb{R}_+$, as β solves (37). Due to the law of the iterated logarithm, the DCT, the fact that W^β satisfies (35) with β instead of b, and recalling (7), we get

$$\lim_{u\to\infty} W^\beta(s+u, Y_u) = \lim_{u\to\infty} G(s+u, Y_u) = c$$

$\mathbb{P}_y$-a.s. for all $y \in \mathbb{R}$. Hence, $W^\beta\big(s + \sigma_{\mathcal{C}_\beta}, Y_{\sigma_{\mathcal{C}_\beta}}\big) = G\big(s + \sigma_{\mathcal{C}_\beta}, Y_{\sigma_{\mathcal{C}_\beta}}\big)$. From (38) and (39) it follows that

$$\begin{aligned} W^\beta(s,y) &= \mathbb{E}_y\left[W^\beta\big(s + \sigma_{\mathcal{C}_\beta}, Y_{\sigma_{\mathcal{C}_\beta}}\big)\right] - \mathbb{E}_y\left[\int_0^{\sigma_{\mathcal{C}_\beta}} \partial_t G\left(s+u, Y_u\right) \mathrm{d}u\right] \\ &= \mathbb{E}_y\left[G^\beta\big(s + \sigma_{\mathcal{C}_\beta}, Y_{\sigma_{\mathcal{C}_\beta}}\big)\right] - \mathbb{E}_y\left[\int_0^{\sigma_{\mathcal{C}_\beta}} \partial_t G\left(s+u, Y_u\right) \mathrm{d}u\right] \\ &= G(s,y), \end{aligned}$$

which proves that $W^\beta = G$ on $\mathcal{D}_\beta$.

Define now the first hitting time $\sigma_{\mathcal{D}_\beta}$ into $\mathcal{C}_\beta$. Notice that either $\sigma_{\mathcal{D}_\beta} = 0$ for $(s,y) \in \mathcal{D}_\beta$, on which $W^\beta = G$, or $Y_u < \beta(s+u)$ for all $0 \leq u < \sigma_{\mathcal{D}_\beta}$. We derive from (38) that

$$W^\beta(s,y) = \mathbb{E}_y\left[W^\beta\left(s + \sigma_{\mathcal{D}_\beta}, Y_{\sigma_{\mathcal{D}_\beta}}\right)\right] = \mathbb{E}_y\left[G\left(s + \sigma_{\mathcal{D}_\beta}, Y_{\sigma_{\mathcal{D}_\beta}}\right)\right],$$

for all $(s,y) \in \mathbb{R}_+ \times \mathbb{R}$, which, after recalling the definition of W in (6), proves that $W^\beta \leq W$.

Take $(s,y) \in \mathcal{D}_\beta \cap \mathcal{D}$ and consider the first hitting time $\sigma_{\mathcal{C}}$ into the continuation set $\mathcal{C}$. Since $W = G$ on $\mathcal{D}$ and $W^\beta = G$ on $\mathcal{D}_\beta$, by relying on (35), (38), and the fact that $\mathbb{P}_y\left(Y_u \geq b(s+u)\right) = 1$ for all $0 \leq u < \sigma_{\mathcal{C}}$, we get

$$\begin{aligned} \mathbb{E}_y\left[W\left(s + \sigma_{\mathcal{C}}, Y_{\sigma_{\mathcal{C}}}\right)\right] &= G(s,y) + \mathbb{E}_y\left[\int_0^{\sigma_{\mathcal{C}}} \partial_t G\left(s+u, Y_u\right) \mathrm{d}u\right], \\ \mathbb{E}_y\left[W^\beta\left(s + \sigma_{\mathcal{C}}, Y_{\sigma_{\mathcal{C}}}\right)\right] &= G(s,y) + \mathbb{E}_y\left[\int_0^{\sigma_{\mathcal{C}}} \partial_t G\left(s+u, Y_u\right) \mathbb{1}\left(Y_u \geq \beta(s+u)\right) \mathrm{d}u\right]. \end{aligned}$$

After recalling that $W^\beta \leq W$, we can merge the two previous equalities into

$$\mathbb{E}_y\left[\int_0^{\sigma_{\mathcal{C}}} \partial_t G\left(s+u, Y_u\right) \mathbb{1}\left(Y_u \geq \beta(s+u)\right) \mathrm{d}u\right] \leq \mathbb{E}_y\left[\int_0^{\sigma_{\mathcal{C}}} \partial_t G\left(s+u, Y_u\right) \mathrm{d}u\right],$$

which, alongside the fact that $\partial_t G(s,y) < 0$ for all $(s,y) \in \mathcal{D}$ (otherwise we get from (32) that the first exit time from a ball around (s,y) small enough will yield a better strategy than stopping immediately) and the continuity of β, implies that $b \geq \beta$.

Suppose that there exists a point $s \in \mathbb{R}_+$ such that $b(s) > \beta(s)$ and fix $y \in (\beta(s), b(s))$. Consider the stopping time $\sigma^* = \sigma^*(s,y)$ and plug it into both (35) and (38) to obtain

$$\begin{aligned} \mathbb{E}_y\left[W^\beta\left(s + \sigma^*, Y_{\sigma^*}\right)\right] &= \mathbb{E}_y\left[G\left(s + \sigma^*, Y_{\sigma^*}\right)\right] \\ &= W^\beta(s,y) + \mathbb{E}_y\left[\int_0^{\sigma^*} \partial_t G\left(s+u, Y_u\right) \mathbb{1}\left(Y_u \geq \beta(s+u)\right) \mathrm{d}u\right] \end{aligned}$$

and

$$\mathbb{E}_y\left[W\left(s + \sigma^*, Y_{\sigma^*}\right)\right] = \mathbb{E}_y\left[G\left(s + \sigma^*, Y_{\sigma^*}\right)\right] = W(s,y).$$

Thus, since $W^\beta \leq W$, we get

$$\mathbb{E}_y\left[\int_0^{\sigma^*} \partial_t G\left(s+u, Y_u\right) \mathbb{1}\left(Y_u \geq \beta(s+u)\right) \mathrm{d}u\right] \geq 0.$$

By using the fact that $y < b(s)$, the continuity of b, and the time-continuity of the process Y, we can state that $\sigma^* > 0$ $\mathbb{P}_y$-a.s. Therefore, since $\partial_t G(s,y) < 0$ for all $(s,y) \in \mathcal{D}_\beta$ (the same arguments used to prove that $\partial_t G < 0$ in $\mathcal{D}$ lead to this conclusion) the previous inequality can only stand if $\mathbb{1}(Y_u \geq \beta(s+u)) = 0$ for all $0 \leq u \leq \sigma^*$, meaning that $b(s+u) \leq \beta(s+u)$ in the same interval, which contradicts the assumption $b(s) > \beta(s)$ due to the continuity of both b and β. □

5 Solution of the original problem and some extensions

Recall that the OSPs (6) and (3) are equivalent, meaning that the value functions and the OSTs of both problems are linked through a homeomorphic transformation. Details on how to actually translate one problem into the other were given in Proposition 1. It then follows that the stopping time $\tau^*(t,x)$ defined in (9) is optimal for (6) and it admits the following alternative representation under $\mathbb{P}_x$:

$$\tau^*(t,x) = \inf\{u \geq 0 : X_{t+u} \geq \beta(t+u)\}, \quad \beta(t) = \frac{z}{c_z} G_{c_z}(s, b(s)), \tag{40}$$

where β is the OSB associated to (3), and $s = \upsilon(t)$ and c_z are defined in Proposition 1. We can obtain both V and β without requiring the computation of W and b. Indeed, consider the infinitesimal generator of $\{(t,X_t)\}_{t\in[0,1]}$, $\mathbb{L}_X$, and set $y = c_x$, $s_\varepsilon = s + \varepsilon$, and $t_\varepsilon = \upsilon^{-1}(s_\varepsilon)$ for $\varepsilon \in \mathbb{R}$. By means of (8) and the chain rule, we get that

$$\begin{aligned}
\frac{z}{c_z}(\mathbb{L}W_{c_z})(s,y) &:= \lim_{\varepsilon\to 0} \varepsilon^{-1}\left(\mathbb{E}_y\left[\frac{z}{c_z}W_{c_z}(s_\varepsilon, Y_\varepsilon)\right] - \frac{z}{c_z}W_{c_z}(s,y)\right) \\
&= \lim_{\varepsilon\to 0} \varepsilon^{-1}\left(\mathbb{E}_{t,x}\left[V(t_\varepsilon, X_{t_\varepsilon})\right] - V(t,x)\right) \\
&= (\mathbb{L}_X V)(t,x)\left(\upsilon^{-1}\right)'(s).
\end{aligned}$$

Hence, after multiplying both sides of (32) by z/c_z, integrating with respect to $\upsilon^{-1}(u)$ instead of u, and recalling that $\mathbb{L}_X V(t,x) = 0$ for all $x \leq \beta(t)$ and $V(t,x) = x$ for all $x \geq \beta(t)$, we get the pricing formula

$$\begin{aligned}
V(t,x) &= z - \mathbb{E}_{t,x}\left[\int_0^{1-t} (\mathbb{L}_X V)(t+u, X_{t+u})\,\mathrm{d}u\right] \\
&= z - \mathbb{E}_{t,x}\left[\int_0^{1-t} \mu(t+u, X_{t+u})\mathbb{1}(X_{t+u} \geq \beta(t+u))\,\mathrm{d}u\right].
\end{aligned} \tag{41}$$

In the same fashion we obtained (36), we can take advantage of the linearity of $x \mapsto \mu(t,x)$ and the Gaussian marginal distributions of X to come up with the following refined version of (41):

$$W(t,x) = z - \int_t^1 K(t,x,u,\beta(u))\,\mathrm{d}u, \tag{42}$$

where, for $x_1, x_2 \in \mathbb{R}$ and $0 \leq t_1 \leq t_2 \leq 1$,

$$K(t_1,x_1,t_2,x_2) := \alpha\frac{z\widetilde{\Phi}_{t_1,x_1,t_2,x_2} - \cosh(\alpha(1-t_2))(m_{t_2}(t_1,x_1)\widetilde{\Phi}_{t_1,x_1,t_2,x_2} + \upsilon_{t_2}(t_1)\widetilde{\phi}_{t_1,x_1,t_2,x_2})}{\sinh(\alpha(1-t_2))}, \tag{43}$$

with

$$\widetilde{\Phi}_{t_1,x_1,t_2,x_2} := \bar{\Phi}\left(\frac{x_2 - m_{t_2}(t_1,x_1)}{\upsilon_{t_2}(t_1)}\right), \quad \widetilde{\phi}_{t_1,x_1,t_2,x_2} := \phi\left(\frac{x_2 - m_{t_2}(t_1,x_1)}{\upsilon_{t_2}(t_1)}\right)$$

and

$$m_{t_2}(t_1,x_1) := \mathbb{E}_{t_1,x_1}[X_{t_2}] = \frac{x_1\sinh(\alpha(1-t_2)) + z\sinh(\alpha(t_2-t_1))}{\sinh(\alpha(1-t_1))},$$

$$v_{t_2}(t_1) := \sqrt{\mathbb{V}\mathrm{ar}_{t_1}[X_{t_2}]} = \sqrt{\frac{\gamma^2}{\alpha}\frac{\sinh(\alpha(1-t_2))\sinh(\alpha(t_2-t_1))}{\sinh(\alpha(1-t_1))}}.$$

Consequently, by taking $x \downarrow \beta(t)$ in (41) (or by directly transforming (37) in the same way we obtained (41) from (35)), we get the free-boundary equation

$$\begin{aligned}\beta(t) &= z - \mathbb{E}_{t,\beta(t)}\left[\int_0^{1-t} (\mathbb{L}_X V)(t+u, X_{t+u})\,\mathrm{d}u\right] \\ &= z - \mathbb{E}_{t,\beta(t)}\left[\int_0^{1-t} \mu(t+u, X_{t+u})\mathbb{1}(X_{t+u} \geq \beta(t+u))\,\mathrm{d}u\right],\end{aligned}$$

which may also be expressed as

$$\beta(t) = z - \int_t^1 K(t, \beta(t), u, \beta(u))\,\mathrm{d}u. \tag{44}$$

The next three remarks broaden the scope of applicability of the OUB as the underlying model in (3). In particular, the two first reveal that setting the terminal time to 1 and the pulling level (coming from the asymptotic mean of the OU process underneath) to 0 does not take a toll on generality, while the last one shows that the OSP for the BB arises as a limit case when $\alpha \to 0$.

Remark 1 (OUB with a general pulling level)**.** *Let $\widetilde{X}^\theta = \{\widetilde{X}_t^\theta\}_{t\in[0,1]}$ be an OU process satisfying the SDE $\mathrm{d}\widetilde{X}_t^\theta = \alpha(\widetilde{X}_t^\theta - \theta)\,\mathrm{d}t + \gamma\,\mathrm{d}B_t$. That is, $X^{\theta,z}$ is pulled towards θ with a time-dependent strength dictated by α. Denote by $X^{\theta,z} = \{X_t^{\theta,z}\}_{t\in[0,1]}$ the OUB process build on top of $\widetilde{X}^\theta$ and such that $X_1^{\theta,z} = z$. It is easy to check that $X^{\theta,z} = X^{0,z-\theta} + \theta$, whenever $X_0^{0,z-\theta} = X_0^{\theta,z} - \theta$. Denote by $V^{\theta,z}$ and $\beta^{\theta,z}$ the value function and the OSB associated to the OSP (3) with X replaced by $X^{\theta,z}$. Then $V^{\theta,z}(t,x) = V^{0,z-\theta}(t, x-\theta) + \theta$ and $b^{\theta,z}(t) = b^{0,z-\theta}(t) + \theta$.*

Remark 2 (OUB with a general horizon)**.** *Denote by $X^{\alpha,\gamma,T} = \{X_t^{\alpha,\gamma,T}\}_{t\in[0,T]}$ an OUB with slope α, volatility γ, and horizon T. Likewise, let $V^{\alpha,\gamma,T}$ and $\beta^{\alpha,\gamma,T}$ be the corresponding value function and the OSB. By relying on the scaling property of a Brownian motion, one can easily verify that $X_t^{\alpha r,\gamma,T} = X_{rt}^{\alpha,\gamma r^{-1/2},rT}$ $\mathbb{P}_x$-a.s. for any $r > 0$. Consequently, $V^{\alpha r,\gamma,T}(t,x) = V^{\alpha,\gamma r^{-1/2},rT}(rt,x)$ and $\beta^{\alpha r,\gamma,T}(t) = \beta^{\alpha,\gamma r^{-1/2},rT}(rt)$. Thereby, by taking $r = 1/T$, one can derive $V^{\alpha,\gamma,T}$ and $\beta^{\alpha,\gamma,T}$ for any set of values α, γ, and T from the solution of the OSP in (3).*

Remark 3 (BB from an OUB)**.** *To emphasize the dependence on α, denote by $X(\alpha)$, V_α, and β_α, respectively, the OUB solving (1), the value function in (13), and the corresponding OSB. The process $X_t(\alpha)$ has the following integral representation under $\mathbb{P}_x$ (Barczy and Kern, 2013):*

$$X_t = x\frac{\sinh(\alpha(1-t))}{\sinh(\alpha)} + z\frac{\sinh(\alpha t)}{\sinh(\alpha)} + \sigma\int_0^t \frac{\sinh(\alpha(1-t))}{\sinh(\alpha(1-u))}\,\mathrm{d}B_u,$$

from where we can conclude, after taking $\alpha \to 0$ and using the DCT, that $X_t(\alpha) \to \widetilde{X}_t$ $\mathbb{P}_x$-a.s. for all $t \in [0,1)$, where $\widetilde{X}$ is a BB process with final value $\widetilde{X}_1 = z$. Then, by applying Theorem 5 from Coquet and Toldo (2007) we have that $V_\alpha \to \widetilde{V}$, and hence $\beta_\alpha \to \widetilde{\beta}$, as $\alpha \to 0$, where $\widetilde{V}$ and $\widetilde{\beta}$ are the value function and the OSB related to $\widetilde{X}$.

6 Numerical results

The free-boundary equation (44) does not admit a closed-form solution and thus numerical procedures come in handy to compute an approximate boundary. By exploiting the fact that the OSB at a given time t depends only on its shape from t up to the horizon, one can discretize the integral in (44) by means of a right Riemann sum and, since the terminal value $\beta(1)$ is known, the entire boundary can be computed in a backward form. This method of backward induction is detailed in Detemple (2005, Chapter 8) and examples of its implementation can be found, e.g., in Pedersen and Peskir (2002). Another approach to solve (44) is by using Picard iterations, that is, by treating (44) as a fixed-point problem in which the entire boundary is updated in each step. The works of Detemple and Kitapbayev (2020) and de Angelis and Milazzo (2020) use this approach to solve the associated Volterra-type integral equation characterizing the OSB. To the best of our knowledge, when it comes to non-linear integral equations arisen from OSPs, the convergence of both the Picard scheme and the backward induction technique are numerically checked rather than formally proved. Therefore, we chose to use the Picard scheme since empirical tests suggested a faster convergence rate while keeping a similar accuracy compared to the backward induction approach.

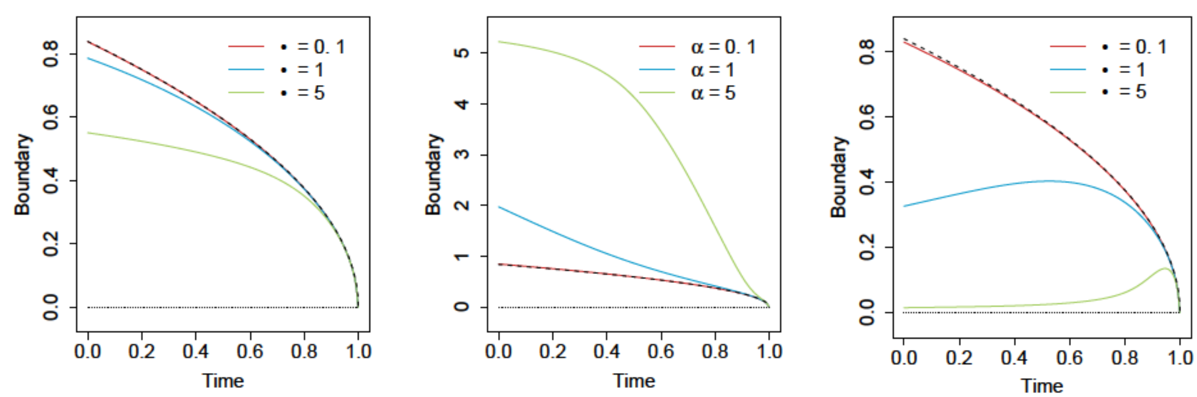
Define a partition of $[0,1]$, namely, $0 = t_0 < t_1 < \cdots < t_N = 1$ for $N \in \mathbb{N}$. Given that $\beta(1) = z$, we will initialize the Picard iterations by starting with the constant boundary $\beta^{(0)} : [0,1] \to \mathbb{R}$ with $\beta^{(0)} \equiv z$. The updating mechanism that generates subsequent boundaries is laid down in the following formula, which comes after discretizing the integral in (44) by using a right Riemann sum:

$$\beta_i^{(k)} = z - \sum_{j=i}^{N-2} K\left(t_i, \beta_i^{(k-1)}, t_{j+1}, \beta_{j+1}^{(k-1)}\right)(t_{j+1} - t_j), \quad k = 1, 2, \ldots$$

We neglect the $(N-1)$-addend and allow the sum to run only until $N-2$ since $K(t, x, 1, z)$ is not well defined, and therefore the last integral piece cannot be included in the right Riemann sum. As the overall integral is finite, the last piece vanishes as t_{N-1} gets closer to 1.

We chose to stop the fixed-point Picard algorithm after the m-th iteration if $m = \min\left\{k > 0 : \max_{i=1,\ldots,N} |\beta_i^{k-1} - \beta_i^k| < \varepsilon\right\}$ for $\varepsilon = 10^{-4}$. Empirical evidence suggested that the best performance of the algorithm was achieved when using a non-uniform mesh that lets the distances $t_i - t_{i-1}$ decrease smoothly as i increases. In our computations, we used the logarithmically-spaced partition $t_i = \ln(1 + i(e-1)/N)$, where $N = 500$ unless is otherwise specified.

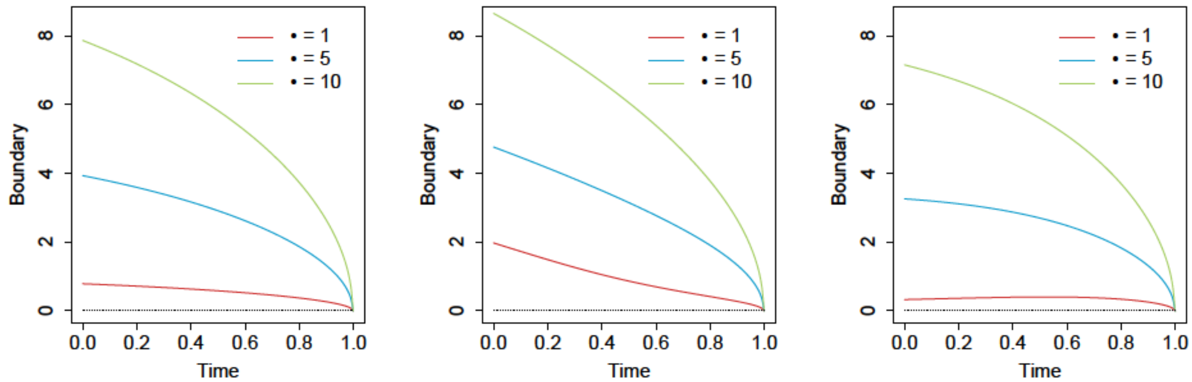
Figures 1, 2, and 3 reveal how the OSB's shape is affected by different set of values for the slope α, the volatility γ, and the anchor point z.



(a) $z = 0, \gamma = 1$ (b) $z = -5, \gamma = 1$ (c) $z = 5, \gamma = 1$

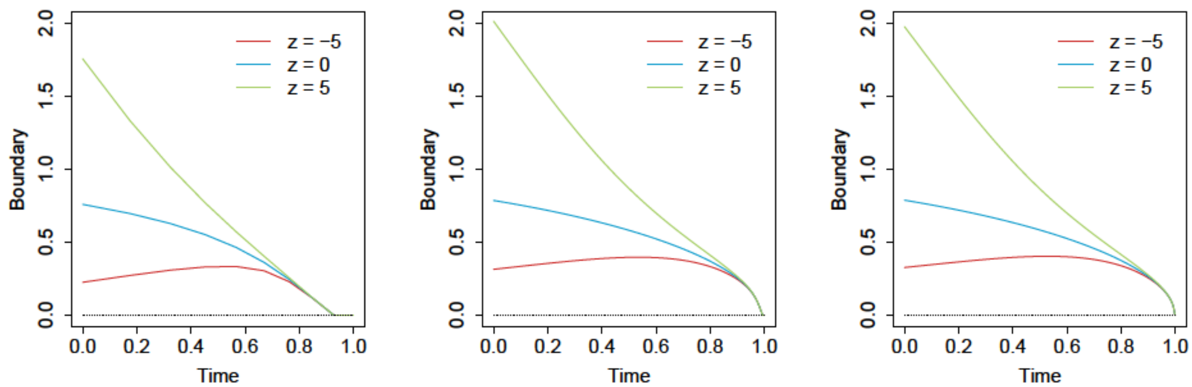
Figure 1: Optimal stopping boundary estimation for different values of α. The boundary is pulled towards 0 with a strength that increases as both $|\alpha|$ (values of α with equal absolute values yield the same boundary) and the residual time to the horizon $1-t$ get higher. As $\alpha \to 0$, the boundary estimation is shown to converge towards the OSB of a BB (dashed line), which is known to be $z + L\sqrt{1-t}$, for $L \approx 0.8399$.

The code implementing the boundary computation is available at D'Auria et al. (2021).



(a) $z = 0, \alpha = 1$ (b) $z = -5, \alpha = 1$ (c) $z = 5, \alpha = 1$

Figure 2: Optimal stopping boundary estimation for different values of γ. The boundary exhibits an increasing proportional relationship with respect to γ.



(a) $\alpha = \gamma = 1, N = 10$ (b) $\alpha = \gamma = 1, N = 100$ (c) $\alpha = \gamma = 1, N = 500$

Figure 3: Optimal stopping boundary estimation for different values of z and N. We display $t \mapsto \beta(t) - z$ to allow a clearer comparison across the different values of z. As N gets larger the boundary estimation is shown to converge.

7 Conclusions

In this paper we solved the finite-horizon OSP for an OUB process with the identity as the gain function. To the best of our knowledge, so far the only Markov bridge addressed by the optimal stopping literature has been the BB and some slight variations of it (see, e.g., Shepp (1969); Föllmer (1972); Ekström and Wanntorp (2009); Ernst and Shepp (2015); Leung et al. (2018); de Angelis and Milazzo (2020); Glover (2020); Ekström and Vaicenavicius (2020); D'Auria et al. (2020)). Markov bridges are potentially useful in mathematical finance as they allow to include additional information at some terminal time.

Arguing as Shepp (1969) for the BB, we worked out the OUB case by coming up with an equivalent OSP having a Brownian motion as the underlying process after time-space transforming the OUB. Contrary to Shepp (1969), the complexity of our problem did not allow to guess a candidate solution, and we directly characterized the value function and the OSB by means of the pricing formula and the free-boundary equation. However, the equivalence between both OSPs were used only to facilitate technicalities along the proofs, and it is not necessary to compute the solution, since both the pricing formula and the free-boundary equation are also provided in the original formulation. We discussed how to use a Picard iteration algorithm to numerically approximate the OSB and displayed some examples to illustrate how different sets of values for the OUB's parameters rule the shape of the OSB.

Acknowledgments

The first and third authors acknowledge the financial support by Spain's Ministry of Science and Innovation through the grant PID2020-116694GB-I00. The research of the first and second authors was also supported by the Community of Madrid through the framework of the multi-year agreement with Carlos III University of Madrid in its line of action "Excelencia para el Profesorado Universitario" (EPUC3M13). The second author acknowledges support by grant PGC2018-097284-B-100 by Spain's Ministry of Science, Innovation and Universities. The grant is co-funded with ERDF funds.

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