

CHARACTERIZATION OF BANACH SPACES OF CONTINUOUS VECTOR VALUED FUNCTIONS WITH THE WEAK BANACH-SAKS PROPERTY

BY

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Introduction

A Banach space E is said to have the Banach-Saks property (resp. weak Banach-Saks property) if for every bounded sequence (resp. weakly convergent sequence) (x_n) in E , you can choose a subsequence (x'_n) of (x_n) such that the sequence

$$(y_n) = \left(\frac{x'_1 + \cdots + x'_n}{n} \right)$$

converges in the E -norm.

We shall refer to these properties as the B.S.P. and the W.B.S.P.

It is known that a Banach space E with the B.S.P. is reflexive. So, it is clear that a $C(K)$ space (being $C(K)$, the Banach space of the continuous functions from K to \mathbb{R} , and being K , a compact Hausdorff space) has the B.S.P. iff K is finite.

Much more interesting in this context of $C(K)$ spaces is the W.B.S.P. The following characterization of $C(K)$ spaces with the W.B.S.P. is due essentially to N. Farnum (see [2]).

THEOREM 1. *Let K be a compact Hausdorff space. Then $C(K)$ possesses the W.B.S.P. if and only if*

$$K^{(\omega)} = \bigcap_{n=1}^{\infty} K^{(n)} = \emptyset$$

where $K^{(0)} = K$ and $K^{(n)}$ is the set of all accumulation points of $K^{(n-1)}$ for $n \in \mathbb{N}$.

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The target of this note is to characterize when $C(K, E)$, the Banach space of all continuous functions defined on a compact Hausdorff space K with values in a Banach space E , endowed with the supremum norm, has the W.B.S.P. Later, in Section 2, we'll show a Banach space E and a compact K such that

- (a) $C(K)$ and E have the W.B.S.P.
- (b) $C(K, E)$ has not the W.B.S.P.

Finally, in Section 3, we'll talk a little about two other properties that a $C(K)$ space may enjoy or not: the hereditary Dunford-Pettis and the alternate Banach-Saks properties.

The notations and terminology used and not explained here can be found in [2]. We only want to recall the definition of the spaces $E = (\Sigma \oplus E_n)_p$. If $(E_n, \|\cdot\|_n)$ is a Banach space, and $p, 1 \leq p < \infty$ (resp. $p = 0$) we define E as the Banach space of all sequences (x_n) , with $x_n \in E_n, (\|x_n\|_n) \rightarrow 0$ and such that

$$\left(\sum_n \|x_n\|_n^p \right)^{1/p} < \infty \quad (\text{resp. } \sup\{\|x_n\|_n: n = 1, \dots\} < \infty)$$

being these expressions the norm of E , for $p, 1 \leq p < \infty$ and $p = 0$ respectively.

1. When a $C(K, E)$ space has the weak Banach-Saks property

If K is a finite compact Hausdorff space, then it is immediate that $C(K, E)$ possesses the W.B.S.P. if and only if E does it. If K is infinite, we have the following result. First of all, we recall that $c_0(E)$ is the Banach space of all null sequences in E , endowed with the supremum norm.

THEOREM 2. *Let K be an infinite compact Hausdorff space. Then $C(K, E)$ has the W.B.S.P. if and only if $C(K)$ and $c_0(E)$ have the W.B.S.P.*

Proof. It is identical to the proof of Theorem 3 of [3], so we omit it.

Now the question is: when $c_0(E)$ has the W.B.S.P.? The following theorem gives us the answer.

THEOREM 3. *$c_0(E)$ has the W.B.S.P. if and only if E has the uniform W.B.S.P. That is to say, there exists a sequence $(a(n))$ of positive real numbers converging to 0 such that, for every sequence*

$$(x_n) \subset B(E), \quad (x_n) \xrightarrow{\omega} 0,$$

and for every $m \in \mathbf{N}$, we can choose $n(1) < \dots < n(m)$, these numbers depending on m , satisfying

$$\left\| \frac{x_{n(1)} + \dots + x_{n(m)}}{m} \right\| < a(m).$$

Proof. Suppose E has not the uniform W.B.S.P. Then, there exist a strictly increasing sequence of integers $(i(m))$, an $\varepsilon > 0$, and sequences

$$(x_n^{(1)}), \dots, (x_n^{(m)}), \dots$$

weakly convergent to 0, $(x_n^{(m)}) \subset B(E)$, such that

$$\left\| \frac{x_{n(1)}^{(m)} + \dots + x_{n(i(m))}^{(m)}}{i(m)} \right\| > \varepsilon$$

for every $n(1) < \dots < n(i(m))$. We can suppose $i(m) \geq 2^m$ without problem.

Let $(f_n) \subset B(c_0(E))$ be the sequence defined as follows:

$$f_1 = (x_1^{(1)}, 0, \dots),$$

$$f_n = (x_n^{(1)}, x_{n-1}^{(2)}, \dots, x_1^{(n)}, 0, \dots).$$

That is, $f_n(m) = 0$ if $n < m$; $f_n(m) = x_{n-m+1}^{(m)}$ if $n \geq m$. It is clear that, for every m fixed

$$(f_n(m): n = 1, \dots) \xrightarrow{\omega} 0$$

and we can deduce that

$$(f_n) \xrightarrow{\omega} 0$$

(for instance, see [5]).

Let (f'_n) be any subsequence of (f_n) . It is clear that the sequence

$$(g_n) = ((f'_1 + \dots + f'_n)/n)$$

does not converge in the $c_0(E)$ -norm. In fact, if (g_n) converges to anything, it must be to 0. But

$$\begin{aligned} \|g_{i(m)+m}\|_\infty &= \left\| \frac{f'_1 + \cdots + f'_{i(m)+m}}{i(m) + m} \right\|_\infty \\ &\geq \left\| \frac{f'_1(m) + \cdots + f'_{i(m)+m}(m)}{i(m) + m} \right\|_E \\ &= \left\| \frac{x_{n(1)}^{(m)} + \cdots + x_{n(j(m))}^{(m)}}{i(m) + m} \right\|_E \end{aligned}$$

It is easy to see that $i(m) \leq j(m) \leq i(m) + m$. And now, the inequality continues with

$$\begin{aligned} &\geq \left\| \frac{x_{n(1)}^{(m)} + \cdots + x_{n(i(m))}^{(m)}}{i(m) + m} \right\|_E - m/(i(m) + m) \\ &> \frac{\varepsilon i(m)}{i(m) + m} - \frac{m}{i(m) + m} \\ &> \frac{\varepsilon}{2} - \frac{m}{2^m} \end{aligned}$$

So we deduce that $c_0(E)$ does not have the W.B.S.P.

Let's suppose now that E has the uniform W.B.S.P. and let's see that $c_0(E)$ has the W.B.S.P. First of all, we need the following technical result.

LEMMA 4. *Let E be a Banach space with the uniform weak Banach-Saks property. Then, there exists a sequence (δ_m) of positive real numbers such that $(\delta_m) \rightarrow 0$ for which given any sequence (x_n) in $B(E)$ with $(x_n) \xrightarrow{\omega} 0$, there is a subsequence (y_n) of (x_n) such that for every subsequence (y'_n) of (y_n) , we have*

$$\left\| \sum_{j \leq m} y'_j/m \right\| \leq \delta_m$$

We leave the proof of this lemma to the end, and first finish the proof of our theorem. Let

$$(f_n) \xrightarrow{\omega} 0, \quad (f_n) \subset B(c_0(E)).$$

We need a subsequence (f'_n) of (f_n) such that

$$(*) \quad ((f'_1 + \cdots + f'_n)/n) \rightarrow 0$$

in the $c_0(E)$ -norm. We suppose first that

$$(f_n) \subset c_{00}(E) = \{f \in c_0(E) : \max\{k : f(k) \neq 0\} = N_f \in \mathbf{N}\}.$$

Since $c_{00}(E)$ is dense on $c_0(E)$, if we prove $(*)$ for a sequence $(f_n) \subset c_{00}(E)$, we also have $(*)$ for any sequence $(g_n) \subset c_0(E)$, $(g_n) \xrightarrow{\omega} 0$.

So let's suppose $(f_n) \subset B(c_0(E))$, $(f_n) \subset c_{00}(E)$, $(f_n) \xrightarrow{\omega} 0$. For every $n \in \mathbf{N}$, we define

$$M(n) = \max\{k : f_n(k) \neq 0\}$$

If the sequence $(M(n))$ is bounded (for instance, by M), we can apply Lemma 4 to the sequences

$$(f_n(1)), \dots, (f_n(M)) \subset E$$

and we can choose a subsequence (f'_n) of (f_n) such that

$$\left\| \sum_{j \leq m} f'_j(k)/m \right\| \leq \delta_m$$

for $m \in \mathbf{N}$, and $1 \leq k \leq M$. So, due to the fact that $f_n(k) = 0$ if $k > M$, we have $\|\sum_{j \leq m} f'_j/m\|_\infty \leq \delta_m$ and we are finished.

If the sequence $(M(n))$ is not bounded, then we can assume (by passing to a subsequence if necessary) that $(M(n))$ is strictly increasing.

Now, we build (f'_n) subsequence of (f_n) by induction.

Case $n = 1$. Let $(f_n^{(1)}) = (f_n)$, $f'_1 = f_1^{(1)}$. We also define $N(0) = 0$ and

$$N(1) = \max\{n : f'_1(n) \neq 0\}$$

Case $n = 2$. We consider the sequences

$$(f_n^{(1)}(1) : n = 1, \dots), \dots, (f_n^{(1)}(N(1)) : n = 1, \dots)$$

If we apply Lemma 4 to these sequences, we have a subsequence $(f_n^{(2)})$ of $(f_n^{(1)})$ such that for every increasing sequence of integers $(n(j))$,

$$\left\| \sum_{j \leq m} f_{n(j)}^{(2)}(k)/m \right\| \leq \delta_m$$

for $m \in \mathbf{N}$ and $1 \leq k \leq N(1)$.

We choose $f_2' = f_2^{(2)}$ and define

$$N(2) = \max\{n: f_2'(n) \neq 0\}.$$

Case $n = r + 1$. Let's suppose that we have chosen the sequences $(f_n^{(t)}: n = 1, \dots)$, $t = 1, \dots, r$, the functions f_1', \dots, f_r' and the numbers $N(0) < \dots < N(r)$ satisfying:

- (1) If $t > 1$, $(f_n^{(t)})$ is a subsequence of $(f_n^{(t-1)})$.
- (2) $f_i' = f_i^{(t)}$, $1 \leq i \leq r$.
- (3) $N(i) = \max\{n: f_i'(n) \neq 0\}$, $1 \leq i \leq r$.
- (4) For every $t = 2, \dots, r$ and for every increasing sequence of integers $(n(j))$, we have

$$\left\| \sum_{j \leq m} f_{n(j)}^{(t)}(k)/m \right\| \leq \delta_m$$

for $m \in \mathbf{N}$ and $1 \leq k \leq N(t-1)$.

Now we consider the sequences

$$(f_n^{(r)}(N(r-1)+1): n = 1, \dots), \dots, (f_n^{(r)}(N(r)): n = 1, \dots)$$

If we apply Lemma 4 to these sequences, we have a subsequence $(f_n^{(r+1)})$ of $(f_n^{(r)})$ such that for every increasing sequence of integers $(n(j))$, we have

$$\left\| \sum_{j \leq m} f_{n(j)}^{(r+1)}(k)/m \right\| \leq \delta_m$$

for $m \in \mathbf{N}$ and $N(r-1) < k \leq N(r)$. Note that the previous inequality is also true for every k , $1 \leq k \leq N(r-1)$, since $(f_n^{(r+1)})$ is a subsequence of $(f_n^{(r)})$. So, it is true for $1 \leq k \leq N(r)$. Now, we define $f_{r+1}' = f_{r+1}^{(r+1)}$, $N(r+1) = \max\{n: f_{r+1}'(n) \neq 0\}$ and the induction is finished.

Now, let's prove (*). For every $m \in \mathbf{N}$, if $k \leq N(1)$ then

$$\begin{aligned} \|(f_1' + \dots + f_m')(k)/m\| &= \|(f_1' + f_n^{(2)} + \dots + f_{n(m-1)}^{(2)})(k)/m\| \\ &\leq 1/m + ((m-1)/m) \cdot \delta_{m-1} \end{aligned}$$

where $n(1) = 2 < \dots < n(m-1)$ are suitable numbers.

If k is such that $N(1) < k \leq N(2)$ then

$$\begin{aligned} \|(f_1' + f_m')(k)/m\| &= \|(f_2' + f_{n(1)}^{(3)} + \dots + f_{n(m-2)}^{(3)})(k)/m\| \\ &\leq 1/m + ((m-2)/m) \cdot \delta_{m-2} \end{aligned}$$

where $n(1) = 3 < \dots < n(m-2)$ are suitable numbers.

Continuing in this way, it is clear that

$$\|(f_1' + \dots + f_m')(k)/m\| < 1/m + ((m-t)/m)\delta_{m-t}$$

if $N(t-1) < k \leq N(t)$ (with the convention that δ_0 is any real number), $1 \leq t \leq m$, and it is immediate that

$$\|(f_1' + \dots + f_m')(k)/m\| = 0 \quad \text{if } N(m) < k.$$

So it only remains to prove that if $(\delta_m) \rightarrow 0$, then

$$(\eta_m) = (\sup\{(m-t)/m \cdot \delta_{m-t}: 1 \leq t \leq m-1\}) \rightarrow 0$$

We leave it as an easy exercise to the reader.

Proof of Lemma 4. Let $(x_n) \xrightarrow{\omega} 0$ and $(x_n) \subset B(E)$. If (x_n) has a subsequence (x_n') such that $\|x_n'\| \rightarrow 0$, we have finished. If not, then (x_n) has a subsequence (x_n') with the following good property: For every b_1, \dots, b_r , if $n(1) < \dots < n(r)$ then the limit

$$\lim_{n(1) \rightarrow \infty} \left\| \sum_{j \leq r} b_j x_{n(j)}' \right\|$$

exists. We call that limit $L(\sum_{j \leq r} b_j e_j)$ for convenience. See [1], Chapter 1, for a proof.

Since E has the uniform W.B.S.P. (see the definition at the beginning of Theorem 3) we can deduce that if we take $s(r) \leq r$, $s(r) \in \mathbf{N}$ and $b_1 = \dots = b_{s(r)} = 0$, and $b_{s(r+1)} = \dots = b_r = 1/r$, we have

$$(+)$$

$$\left| L\left(\sum_{j \leq r} b_j e_j\right) \right| \leq a(r-s(r)) \cdot (r-s(r))/r$$

This holds because

$$\begin{aligned} & \lim_{n(1) \rightarrow \infty} \left\| \sum_{j \leq r} b_j x'_{n(j)} \right\| \\ &= \frac{r - s(r)}{r} \lim_{n(1) \rightarrow \infty} \left\| \sum_{j=s(r)+1}^r \frac{x'_{n(j)}}{r - s(r)} \right\| \\ &\leq \frac{r - s(r)}{r} a(r - s(r)). \end{aligned}$$

The last inequality is due to the fact that that limit exists and the definition of $a(m)$, for every $m \in \mathbf{N}$.

Let $s(m) = [\sqrt{m}]$ (where $[\cdot]$ is the greatest integer function). Now, we consider the finite set

$$A(i) = \{m \in \mathbf{N} : s(m) = i\}$$

It is clear that for every i we have an integer $N(i)$ such that if $n(i) \geq N(i)$ then

$$(+ +) \quad \left| L \left(\sum_{j=i+1}^m \frac{e_j}{m} \right) - \left\| \sum_{j=i+1}^m \frac{x'_{n(j)}}{m} \right\| \right| < \frac{1}{m}$$

for every m , $s(m) = i$. Then, if we define $M(0) = 1$ and

$$M(i) = \max(N(i), M(i-1) + 1)$$

the sequence $(x'_{M(i)})$ satisfies Lemma 4. In fact, if $(x'_{n(j)})$ is a subsequence of $(x'_{M(j)})$, we have

$$\left\| \sum_{j=s(m)+1}^m \frac{x'_{n(j)}}{m} \right\| \leq \left| L \left(\sum_{j=s(m)+1}^m \frac{e_j}{m} \right) \right| + \frac{1}{m}$$

This inequality follows from $(+ +)$ $n(i) \geq M(i) \geq N(i)$. Now, by $(+)$, we continue the inequality with

$$\leq a(m - s(m)) \frac{m - s(m)}{m} + \frac{1}{m}.$$

And finally, it is clear that

$$\begin{aligned} \left\| \sum_{j \leq m} \frac{x'_{n(j)}}{m} \right\| &\leq \left\| \sum_{j \leq s(m)} \frac{x'_{n(j)}}{m} \right\| + \left\| \sum_{j=s(m)+1}^m \frac{x'_{n(j)}}{m} \right\| \\ &\leq \frac{s(m)}{m} + a(m - s(m)) \frac{m - s(m)}{m} + \frac{1}{m} \\ &= \delta_m \end{aligned}$$

It is obvious that $(\delta_m) \rightarrow 0$. So the sequence $(y_i) = (x'_{M(i)})$ satisfies our lemma.

2. A Banach space E with the weak Banach-Saks property but not in the uniform sense

We begin this section with the following question: If a $C(K)$ space has the W.B.S.P., does $C(K)$ possess the uniform W.B.S.P.? The answer is yes, and we deduce it in this way:

(a) As we saw in Theorem 1, a $C(K)$ space has the W.B.S.P. if and only if

$$(+) \quad K^{(\omega)} = \bigcap_n K^{(n)} = \emptyset.$$

(b) $C(K)$ has the uniform W.B.S.P. if and only if $c_0(C(K))$ has the W.B.S.P. (by Theorem 3).

(c) As $c_0(C(K))$ is isomorphic to $C(\mathbf{N}^* \times K)$, where \mathbf{N}^* is the Alexandroff compactification of \mathbf{N} , $c_0(C(K))$ has the W.B.S.P. if and only if

$$(+ +) \quad (\mathbf{N}^* \times K)^{(\omega)} = \bigcap_n (\mathbf{N}^* \times K)^{(n)} = \emptyset.$$

(d) Proposition 10 of [3] proves that $(+) \Rightarrow (+ +)$, so we have finished.

The problem that we want to solve now is the following: we have seen that for a $C(K)$ space, the properties uniform W.B.S. and W.B.S. are equivalent, but is that true for any Banach space E ? The answer is no. First of all, we need the following result. Remember that a Banach space E has the W.B.S.P. if and only if for every sequence $(x_n) \xrightarrow{\omega} 0$ there exists a subsequence (x'_n) of (x_n) such that for every subsequence (x''_n) of (x'_n) , we have

$$\left\| \sum_{j \leq m} x''_j / m \right\| \rightarrow 0 \quad \text{as } m \rightarrow \infty$$

See [1], Chapter 2, for a proof.

THEOREM 5. *If each Banach space E_n has the W.B.S. property then so does $(\Sigma \oplus E_n)_1$.*

Proof. Let $(x^m) \xrightarrow{\omega} 0$ in $(\Sigma \oplus E_n)_1$. Then, it is known that if

$$x^m = (x_1^m, \dots, x_n^m, \dots)$$

where $x_n^m \in E_n$ and $\|\cdot\|_n$ is the norm of the Banach space E_n we have

$$(+) \quad \limsup_{k \rightarrow 0} \left\{ \sum_{j=k}^{\infty} \|x_j^m\|_j : m = 1, \dots \right\} = 0$$

Consider the sequence $(x_1^m) \xrightarrow{\omega} 0$ in E_1 . E_1 has the W.B.S. property, so we can choose a subsequence $(^{(1)}x_1^m)$ of (x_1^m) so that

$$\|^{(1)}x_1^{m(1)} + \dots + ^{(1)}x_1^{m(r)}\|_1 / r \rightarrow 0 \quad \text{in } E_1 \text{ as } r \rightarrow \infty,$$

for every increasing sequence of natural numbers $(m(j))$. Now let's consider the sequence

$$(^{(1)}x_2^m) \xrightarrow{\omega} 0 \quad \text{in } E_2.$$

As E_2 has the W.B.S. property, there exists $(^{(2)}x_2^m)$ a subsequence of $(^{(1)}x_2^m)$, such that

$$\|^{(2)}x_2^{m(1)} + \dots + ^{(2)}x_2^{m(r)}\|_2 / r \rightarrow 0 \quad \text{in } E_2 \text{ as } r \rightarrow \infty,$$

for every increasing sequence of natural numbers $(m(j))$.

In the same way, for every k there exists $(^{(k)}x^m : m = 1, \dots)$, a subsequence of $(^{(k-1)}x^m : m = 1, \dots)$, such that

$$\|^{(k)}x_k^{m(1)} + \dots + ^{(k)}x_k^{m(r)}\|_k / r \rightarrow 0 \quad \text{in } E_k \text{ as } r \rightarrow \infty,$$

for every increasing sequence of natural numbers $(m(j))$.

Define the subsequence (y^m) of (x^m) by

$$y^m = ^{(m)}x^m$$

We will show that $\|y^1 + \dots + y^m\| / m \rightarrow 0$. Let $\varepsilon > 0$. Then, by (+), there exists $k(\varepsilon)$ such that

$$\sum_{j=k(\varepsilon)+1}^{\infty} \|y_j^m\|_j < \varepsilon/2 \quad \text{for every } m.$$

For each $j = 1, \dots, k(\varepsilon)$, we have

$$\{y_j^m : m = j, \dots\} = \{^{(j)}x_j^{k(m)} : m = j, \dots\}$$

where $k(m)$ is an increasing sequence of integers with $k(j) = j$. So, it is clear that, for every $j = 1, \dots, k(\varepsilon)$, there exist i_j such that for every $i > i_j$ we have

$$\|y_j^1 + \dots + y_j^i\|_j / i < \varepsilon/2k(\varepsilon)$$

Now, taking $i_0 = \max\{i_1, \dots, i_{k(\varepsilon)}\}$, if $i > i_0$ we obviously have

$$\begin{aligned} \left\| \sum_{s=1}^i y^s \right\| / i &= \sum_{j=1}^{\infty} \left\| \sum_{s=1}^i y_j^s \right\|_j / i \\ &= \sum_{j=1}^{k(\varepsilon)} \left\| \sum_{s=1}^i y_j^s \right\|_j / i + \sum_{j=k(\varepsilon)+1}^{\infty} \left\| \sum_{s=1}^i y_j^s \right\|_j / i \\ &< \sum_{j=1}^{k(\varepsilon)} \varepsilon/2k(\varepsilon) + \sum_{s=1}^i \left(\sum_{j=k(\varepsilon)+1}^{\infty} \|y_j^s\|_j \right) / i \\ &< \varepsilon \end{aligned}$$

So the Banach space $(\Sigma \oplus E_n)_1$ has the W.B.S. property.

Now we can establish the main result of this section.

COROLLARY 6. *Let \mathbf{N}^* be the Alexandroff compactification of \mathbf{N} . There exists a Banach space E with the weak Banach-Saks property such that $c(\mathbf{N}^*, E)$ does not have the weak Banach-Saks property despite the fact that $c(\mathbf{N}^*)$ has the W.B.S.P.*

Proof. Take $E_n = c_0(\omega^n)$. It is known that E_n is isomorphic to c_0 (see [6]), so every E_n has the W.B.S. property. Define $E = (\Sigma \oplus E_n)_1$. As we have seen before, E has the W.B.S. property. But $c_0(E)$ does not (and so neither does $c(\mathbf{N}^*, E)$). To prove my point, we consider the subspace of $c_0(E)$,

$$Z = \{f: \mathbf{N} \rightarrow E, f \in c_0(E) / f(n) \in E_n\}.$$

It is immediately seen that this subspace is isometric to

$$(\Sigma \oplus E_n)_0 = (\Sigma \oplus c_0(\omega^n))_0$$

which, in fact, is isometric to $c_0(\omega^\omega)$, a very well known example of a Banach space which does not have the W.B.S. property (see [2] for a proof). So $c_0(E)$ has not this property, either.

The previous corollary is remarkable because it shows (using Theorem 3) that the properties W.B.S. and uniform W.B.S. are not equivalent. In fact we have a better result.

THEOREM 7. *There is a Banach space E with the Banach-Saks property such that E does not have the uniform weak Banach-Saks property.*

Proof. Using Lemma 5.2 of [1], it is very easy to prove that if each Banach space E_n has the Banach-Saks property, so does $E = (\Sigma \oplus E_n)_2$.

For any n , one can take

$$E_n = \{x: \mathbf{N} \rightarrow \mathbf{R} \text{ such that } (*) < +\infty\}$$

where

$$(*) = \sup \left\{ \left(\sum_m \left(\sum_{k \in A_m} |x(k)| \right)^2 \right)^{1/2} : \text{Card}(A_m) = n, \bigcup_m A_m = \mathbf{N} \text{ and } A_m \cap A_{m'} = \emptyset \text{ if } m \neq m' \right\}$$

We take $\|\cdot\|_n = (*)$. It is clear that $(E_n, \|\cdot\|_n)$ is isomorphic to l^2 , so it has the Banach-Saks property. By Lemma 5.2 of [1], E has the Banach-Saks property. But, for every $n \in \mathbf{N}$, if we take

$$x_k^{(n)} = (0, \dots, {}^n e_k, 0, \dots)$$

it is clear that $(x_k^{(n)}) \xrightarrow{\omega} 0$ and, for every $m(1) < \dots < m(n)$, we have

$$\left\| \left(x_{m(1)}^{(n)} + \dots + x_{m(n)}^{(n)} \right) / n \right\| = 1$$

So E does not have the uniform weak Banach-Saks property.

3. Alternate Banach-Saks and the hereditary Dunford-Pettis properties of $C(K, E)$ spaces

DEFINITION 8. (a) A Banach space E is said to have the alternate Banach-Saks property (A.B.S.P.) if for every bounded sequence (x_n) in E , we can choose a subsequence (x'_n) of (x_n) such that the sequence

$$(y_n) = \left(\frac{(-1)x'_1 + \dots + (-1)^n x'_n}{n} \right)$$

converges in the E -norm.

(b) A Banach space E is said to have the hereditary Dunford-Pettis property (H.D.P.P.) if for every sequence weakly convergent to 0, not convergent in norm, there is a subsequence (x'_n) of (x_n) which is equivalent to the unit vector basis of c_0 (see [3] for this definition).

Everything we have done in Section 1 with the W.B.S.P. we can do it with the A.B.S.P. For instance:

THEOREM 9. (a) $C(K, E)$ has the A.B.S.P. if and only if $C(K)$ and $c_0(E)$ have the A.B.S.P.

(b) $C(K)$ has the A.B.S.P. if and only if $K^{(\omega)} = \bigcap_n K^{(n)} = \emptyset$

(c) $c_0(E)$ has the A.B.S.P. if and only if E has the uniform A.B.S.P. That is there exists a sequence $(a(n))$ of positive real numbers converging to 0 such that, for every sequence $(x_n) \subset B(E)$, and for every $m \in \mathbf{N}$, we can choose $n(1) < \dots < n(m)$, these numbers depending on m , such that

$$\left\| \left((-1)x_{n(1)} + \dots + (-1)^m x_{n(m)} \right) / m \right\| < a(m)$$

(d) The space E of Theorem 7 has the A.B.S.P. but not the uniform A.B.S.P. Then the space $C(\mathbf{N}', E)$ does not have the A.B.S.P. although

(i) $c(\mathbf{N}')$ has the A.B.S.P. where \mathbf{N}' is the Alexandroff compactification of \mathbf{N} , and

(ii) E has the A.B.S.P.

The hereditary Dunford-Pettis property on $C(K, E)$ spaces was intensely studied in [3]. The uniform H.D.P.P. was defined there as follows:

(*) There exists $M > 0$ such that every normalized weakly null sequence $(x_n) \subset E$, has a subsequence (x'_n) that is equivalent to the unit vector basis of c_0 and satisfies

$$\left\| \sum_n a_n y_n \right\| \leq M \sup |a_n| \quad \text{for all } (a_n) \in c_0$$

and the problem "does every Banach space with the H.D.P.P. satisfy (*)" is still open.

We do not have the answer to this difficult question. Someone suggested that the space $E = (\Sigma \oplus E_n)_1$ with $E_n = c_0(\omega^n)$ could be the answer, and we are going to prove that it is not the case. Of course, we saw in Corollary 6 that $c_0(E)$ has a subspace isometric to $c_0(\omega^\omega)$, a well known example of a Banach space without the H.D.P.P. (see [2], for a proof), and so $c_0(E)$ does not have the H.D.P.P. The problem is that neither does E have this property. Let's prove this.

THEOREM 10. *The space $E = (\Sigma \oplus c_0(\omega^n))_1$ is not hereditarily Dunford-Pettis (although E has the weak Banach-Saks and the Dunford-Pettis properties).*

Proof. Remember that a Banach space E has the Dunford-Pettis property if for every $(x_n) \xrightarrow{\omega} 0$, $(x_n) \subset E$ and for every $(x'_n) \xrightarrow{\omega} 0$, $(x'_n) \subset E'$, the sequence of real numbers $(\langle x_n, x'_n \rangle) \rightarrow 0$.

To prove that E has the Dunford-Pettis property is very easy with the ideas of Theorem 5.

To prove that E does not have the H.D.P.P., we begin with the fact that the space $(\Sigma \oplus c_0(\omega^n))_0$ isometric to $c_0(\omega^\omega)$ does not have the H.D.P.P. Then, using the same technique as Cembranos in [3], there is no $M > 0$ such that (*) is satisfied for every $c_0(\omega^n)$ with the same M . In other words, for every $M_k = 3^k$, there is an $n(k)$ (we take $n(k) > n(k-1)$) such that in the space $E_{n(k)} = c_0(\omega^{n(k)})$ there is a sequence

$$(x_j^{(k)}: j = 1, \dots)$$

with the following properties (where $\|\cdot\|_k$ is the norm of $E_{n(k)}$):

- (a) $(x_j^{(k)}) \xrightarrow{\omega} 0$ in $E_{n(k)}$, $\|x_j^{(k)}\|_k = 1$.
 (b) For every subsequence $(x_{m(j)}^{(k)}: j = 1, \dots)$ of $(x_j^{(k)}: j = 1, \dots)$ there is a sequence $a = (a_1, \dots, a_n, \dots) \in c_0$ (depending on the subsequence) such that

$$3^k \sup |a_j| \leq \|\sum a_j x_{m(j)}^{(k)}\|_k$$

Consider the following sequence in $(\Sigma \oplus E_{n(k)})_1$, a subspace of $(\Sigma \oplus c_0(\omega^n))_1$:

$$z_1 = (x_1^{(1)}/2, \dots, x_1^{(k)}/2^k, \dots),$$

$$z_m = (x_m^{(1)}/2, \dots, x_m^{(k)}/2^k, \dots)$$

It is very easy to prove that

- (1) $z_m \in (\Sigma \oplus E_{n(k)})_1$,
 (2) $(z_m) \xrightarrow{\omega} 0$ in $(\Sigma \oplus E_{n(k)})_1$.

But it is not difficult to see that, for every subsequence (z'_n) of (z_n) , and every $k \in \mathbf{N}$, there is a sequence depending on (z'_m) and k such that

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} a_j z'_j \right\| &= \sum_{k=1}^{\infty} \left\| \sum_{j=1}^{\infty} a_j z'_j(k) \right\|_k \\ &= \sum_{k=1}^{\infty} \left\| \sum_{j=1}^{\infty} \frac{a_j x_{m(j)}^{(k)}}{2^k} \right\|_k \\ &\geq \left\| \sum_{j=1}^{\infty} \frac{a_j x_{m(j)}^{(k)}}{2^k} \right\|_k \\ &\geq (3^k/2^k) \sup |a_j| \quad \text{due to the choice of } a = (a_1, \dots) \end{aligned}$$

So (z'_m) can not be equivalent to the canonic base of c_0 , and so $(\Sigma \oplus c_0(\omega^n))_1$ does not have the H.D.P.P.

REFERENCES

1. B. BEAUZAMY and J.T. LAPRESTE, *Modèles étalés des espaces de Banach*, Hermann, Paris, 1984.
2. J. BOURGAIN, *The Szlenk index and operators on $C(K)$ -spaces*, Bull. Soc. Math. Belg., vol. 30 (1978), pp. 83-87.
3. P. CEMBRANOS, *The hereditary Dunford-Pettis property on $C(K, E)$* , Illinois J. Math., to appear.
4. J. DIESTEL, *Geometry of Banach spaces-selected topics*, Lecture Notes in Mathematics, vol. 485, Springer-Verlag, New York, 1975.
5. D.R. LEWIS, *Conditional weak compactness in certain injective tensor products*, Math. Ann., vol. 201 (1973) pp. 201-209.
6. Z. SEMADENI, *Banach spaces of continuous functions*, PWN, Warsaw, 1971.

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