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Differentiability of the value function and Euler equation in non–concave discrete time stochastic dynamic programming*

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Abstract

We consider a stochastic, non–concave dynamic programming problem admitting interior solutions and prove, under mild conditions, that the expected value function is differentiable along optimal paths. This property allows us to obtain rigorously the Euler equation as a necessary condition of optimality for this class of problems.

Keywords— Dynamic programming, Euler equation, Envelope Theorem

1 Introduction

The Euler equation is a useful tool to analyze discrete time dynamic programming problems with interior solutions. A way to obtain the Euler equation is from the Envelope Theorem developed by Mirman and Zilcha (1975) and Benveniste and Scheinkman (1979). This result asserts that interiority of solutions plus concavity imply differentiability of the value function, providing an expression for the derivative. Hence, under this approach, the validity of the Euler equation as a necessary condition of optimality for interior solutions depends on the differentiability of the value function. From a large body of literature that uses the Euler equation to analyze stochastic models, it is worth mentioning Brock and Mirman (1972), Donaldson and Mehra (1983), Majumdar et al (1989), Coleman (1991), Nishimura and Stachurski (2005), Nishimura et al (2012) or Cai et al (2014), to cite only a few of them. The aim of this paper is to show that the Euler Equation holds in stochastic, non–concave models where the optimal correspondence admits a selection that is interior to the technological constraint. Variants of this result have previously been obtained in the *deterministic case* by Dechert and Nishimura (1983), Amir (1996), Askri and Le Van (1998), Cotter and Park (2006) and Morand et al (2018), among others, using different techniques and assumptions. Clausen and Strub (2016) provide envelope theorems based on the so called 'Sandwich Differentiable Lemma' and apply them to a great variety of models, including *stochastic models*, but it seems that there is no specific claim in their paper that may apply directly to our framework without further

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work[1]. Dechert and Nishimura focus on the optimal growth model; Amir (1996) uses increasing differences in the return function to show the differentiability of the value function; Askri and Le Van (1998) impose Lipschitzianity of the value function and differentiability of the instantaneous utility function, working with the generalized gradient of Clarke; while Morand el al (2018) study Lipschitz dynamic programs without smoothness and without concavity, and find lower and upper bounds in the directional derivatives of the value function, even when the optimal policy is non–interior. Cotter and Park (2006) base their approach on the results of Milgrom and Segal (2002), supposing that the utility function is equi-differentiable and that the feasible set correspondence is independent of the state variables[2].

In this paper, we deal with the stochastic case with an underlying Markov chain, and impose only the differentiability of the return function and some natural and indispensable integrability conditions. The contribution of our paper is best summarized in the following table, while also indicating the state of the art concerning the differentiability of the value function when the optimal policy function is interior. The assertions in the cells of the table concern the value function at interior (endogenous) states and interior optimal policies, provided that the return function is smooth. The contribution of this paper is to fill in the bottom, righthand side cell and thus to obtain the Euler equation as a necessary condition of optimality.

	concave	non–concave
deterministic	the value function is differentiable	the value function is differentiable at the optimal policy
stochastic	the value function is differentiable	the *expected* value function is differentiable at the optimal policy

Table 1: Differentiability properties of the value function in concave/non–concave and deterministic/stochastic models.

The paper is organized as follows. Section 2 describes the dynamic programming problem and establishes the basic hypotheses. Section 3 contains the main result, consisting in the differentiability of the expected value function at the optimal policy and showing the validity of the Euler equation. Section 4 closes the paper with some conclusions.

2 Dynamic programming

Consider a dynamic programming model $(X, Z, \Gamma, Q, F, \beta)$, where

[1]Notice that what we prove in this paper is the differentiability of the expected value function along the optimal path; the differentiability of the plain value function along the optimal path cannot be inferred without further assumptions.

[2]The consideration of a choice set correspondence depending on the endogenous variable requires some additional regularity, even if it is possible to select an interior policy function, as is imposed later in this paper. See Rincón–Zapatero and Santos (2009, 2102) for an Envelope Theorem in a concave dynamic problem with constraints, and Rincón–Zapatero and Zhao (2018) for the deterministic recursive utility case.

1. $X \times Z$ is the set of possible states of the system. X and Z are non–empty Borel sets in $\mathbb{R}^l$ and $\mathbb{R}^m$, respectively.

2. Γ is a correspondence that assigns each state (x, z) a nonempty set $\Gamma(x, z)$ of feasible actions at (x, z). We let $Y = \bigcup_{(x,z)\in X\times Z} \Gamma(x, z)$ and $\Omega = \{(x, y, z) : (x, y) \in X \times Z, y \in \Gamma(x, z)\}$ be the graph of Γ.

3. Q is the transition function, which associates a conditional probability distribution $Q(\cdot|z)$ on Z to each $z \in Z$. Hence, the law of motion is assumed to be a first-order Markov process, which could be degenerated, giving rise to a deterministic model.

4. F is the one–period return function, defined on Ω.

5. $\beta \in (0, 1)$ is a discount factor.

Starting at some state (x_0, z_0), the agent chooses an action $x_1 \in \Gamma(x_0, z_0)$, obtaining a return of $F(x_0, x_1, z_0)$ and the system moves to the next state (x_1, z_1), where z_1 is drawn according to the probability distribution $Q(\cdot|z_0)$. Iteration of this process yields a random sequence $(x_0, z_0, x_1, z_1, \ldots)$ and a total discounted return $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}, z_t)$. A history of length t is $z^t = (z_0, z_1, \ldots, z_t)$. Let Z^t be the set of all histories of length t. A (feasible) plan π is a constant value $\pi_0 \in X$ and a sequence of measurable functions $\pi_t : Z^t \longrightarrow X$, such that $\pi_t(z^t) \in \Gamma(\pi_{t-1}(z^{t-1}), z_t)$, for all $t = 1, 2, \ldots$. Denote by $\Pi(x_0, z_0)$ the set of all feasible plans starting at the state (x_0, z_0). Any feasible plan $\pi \in \Pi(x_0, z_0)$, along with the transition function Q, defines a distribution $\mathbf{P}^{\pi,(x_0,z_0)}$ on all possible futures of the system $\{(x_t, z_t)\}_{t=1}^{\infty}$, as well as the expected total discounted utility

$$u(\pi, x_0, z_0) = \mathbf{E}^{\pi,(x_0,z_0)} \left(\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}, z_t) \right).$$

The expectation $\mathbf{E}^{\pi,(x_0,z_0)}$ is taken with respect to the distribution $\mathbf{P}^{\pi,(x_0,z_0)}$. The problem is then to find a plan $\pi \in \Pi(x_0, z_0)$ such that $u(\pi, (x_0, z_0)) \geq u(\widehat{\pi}, (x_0, z_0))$ for all $\widehat{\pi} \in \Pi(x_0, z_0)$, for all $(x_0, z_0) \in X \times Z$. The value function of the problem is $v(x_0, z_0) = \sup_{\pi\in\Pi(x_0,z_0)} u(\pi, (x_0, z_0))$.

The following functional equation plays a key role in the solution of the problem

$$v(x, z) = \sup_{y\in\Gamma(x,z)} \left(F(x, y, z) + \beta \int_Z v(y, z')Q(dz'|z) \right), \quad (x, z) \in X \times Z. \tag{2.1}$$

This equation is also referred as the "optimality equation" or "Bellman equation". If there exists a function v satisfying (2.1), then the associated policy correspondence G is

$$G(x, z) = \left\{ y \in \Gamma(x, z) : v(x, z) = F(x, y, z) + \beta \int_Z v(y, z')Q(dz'|z) \right\}.$$

If G is non–empty and admits a measurable selection, then we say that the plan π is generated by G when there is a measurable selections g from G, such that $\pi_0 = g(x_0, z_0)$ and $\pi_t(z^t) = g(\pi_{t-1}(z^{t-1}), z_t)$, all $z^t \in Z^t$, $t = 1, 2, \ldots$.

There is a close connection between solutions of the functional equation and the value function. Under suitable hypotheses, the value function solves the equation. However, a famous example due to Blackwell (1965), shows that the value function may not be Borel measurable, even when all the primitives of the problem are Borel measurable, and hence the Bellman equation cannot characterize the value function in this case. On the other hand, there are sufficient conditions that, applied to solutions of (2.1), allow us to identify the value function, see e.g., Theorem 9.2 in Stokey and Lucas with Prescott (1989).

We define the Markov operator

$$Mf(x,z) = \int_Z f(y,z')Q(dz'|z), \tag{2.2}$$

for integrable functions f.

The Euler equation is obtained for plans that are at interior states and choose interior actions. We say that the state (x,z) is interior, if x is in the interior of the set X; and we say that the action y at (x,z) is interior, if y is in the interior of the set $\Gamma(x,z)$. A plan π is interior at (x,z), if (x,z) is interior and, with probability one under $\mathbf{P}^{\pi,(x,z)}$, only interior states are reached and only interior actions are taken.

We shall use the notation $D_iF(x,y,z)$ for the partial derivatives of F at (x,y,z) with respect to the ith coordinate, for $i = 1,\ldots,2l$. We let the vectors $D_xF = (D_1F,\ldots,D_lF)$ and $D_yF = (D_{l+1}F,\ldots,D_{2l}F)$ and use a similar notation for the partial derivatives for other functions. No role is played by the derivatives of F with respect to z. Also, given $x \in X$, N_x denotes a neighborhood of x in X, that is, the intersection of a neighborhood of x in $\mathbb{R}^l$ with X.

We impose the following assumptions.

B1: The correspondence Γ is non–empty, compact-valued, and continuous. The space of shocks Z is compact.

B2: The correspondence G is nonempty and permits a measurable selection $g(x,z)$ at any $(x,z) \in \text{int}(X) \times Z$, that generates an optimal and interior plan, π. Moreover, for all $x \in \text{int}(X)$, the mapping $z \to g(x,z)$ is bounded.

B3: The utility function F is continuous, and for each $z \in Z$, the functions $(x,y) \to D_iF(x,y,z)$ are continuous on the interior of $\Omega_z = \{(x,y) : (x,y,z) \in \Omega\}$, for all $i = 1,\ldots,2l$.

B4: The value function v satisfies the Bellman equation and is continuous on $\text{int}(X) \times Z$.

B5: For every interior state (x_0,z_0) and interior action $y_0 \in \Gamma(x_0,z_0)$, there exists a neighborhood $N_{x_0} \ni x_0$ *independent* of z_0, such that $y_0 \in \Gamma(x,z_0)$, for all $x \in N_{x_0}$.

Hypothesis B1 is standard. The compactness of Z is usually assumed in applications. In our context, it is important for dealing with integrability issues. To weaken this assumption would require additional assumptions on F and on the sequences of shocks, complicating the obtention of the differentiability results. Le Van and Stachurski (2007) study the dependence of the stationary distribution when the space of shocks is not compact. Hypothesis B2 establishes the boundedness of g with respect to z, for any fixed x. This is imposed for technical reasons. Regarding B3, it is more stringent than the smoothness condition imposed on F, when F is concave in (x,y). Under concavity, only the differentiability of $x \to D_iF(x,y,z)$ is required. Lack of concavity has to be compensated with the differentiability of F, not only with respect to the state, but also with respect to the decision variable. If we consider uncertainty in addition to non–concavity, as it is the case here, the hypothesis has to be strengthened to get B3. Hypothesis B4 holds under the standard hypotheses we have imposed if F is bounded, as well as in the unbounded case under further conditions that link the discount factor and the growth of F on the feasible correspondence. See Boyd III (1990) or Rincón–Zapatero and Rodríguez–Palmero (2003, 2007) to cite only some works that deal with this problem. B4 is taken for granted here, since we are interested in the differentiability properties of the value function. Continuity of v in (x, z) is imposed for the purpose of clarity in the exposition. Since v satisfies the Bellman equation by hypothesis, the proof of Theorem 3.1 below is still valid by merely supposing that $v(\cdot, z)$ is continuous in the interior of X for each $z \in Z$. B5 is a mild interiority assumption that plays an important role in the construction of differentiable upper and lower envelopes of the value function[3].

3 Differentiability of value function and Euler equation

In the next theorem, we use the notation $x \cdot y$ for the inner product of vectors in $\mathbb{R}^l$ and $\|x\|$ for the Euclidean norm. The next theorem is the main result of the paper. Part (a) provides an envelope theorem for non–concave models with interior solutions. It has two distinctive features: (i) it is the expected value function which is differentiable, not the value function itself; and (ii) it is differentiable at the optimal path. Of course, (ii) has been well known in the deterministic case since the work of Dechert and Nishimura (1983). Regarding (i), this is new, to our knowledge. Part (b) asserts that the Euler equation holds. Note that, in the proof, we start the reasoning at (π_0, z_1) and not at (x_0, z_0), that is, a period ahead of the initial state. This is because, to follow the method of Mirman–Zilcha or Benveniste–Scheinkman to get differentiability in the non–concave case, we cannot use the well known fact that a concave function which is the upper envelope of a smooth function is also smooth. In our case, we need to complete the information with the fact that the continuation value function will be a lower envelope of a smooth function. These two properties hold at the same time for *tomorrow's states* and not at current ones. Recall the expression of the Markov operator M given in (2.2).

Theorem 3.1. *Let the problem $(X, Z, \Gamma, Q, F, \beta)$ satisfy Assumptions B1–B5. Then, for any interior state $(x_0, z_0) \in X \times Z$, the expected value function $Mv(\cdot, z_0)$ is continuously differentiable at π_0 and the following holds with probability one under $\mathbf{P}^{\pi,(x_0,z_0)}$:*
(a) the envelope equation

$$D_x Mv(\pi_t(z^t), z_t) = \int_Z D_x F(\pi_t(z^t), \pi_{t+1}(z^{t+1}), z_{t+1}) Q(dz_{t+1}|z_t), \quad t = 0, 1, \ldots;$$

(b) the stochastic Euler equation

$$D_y F(\pi_t(z^t), \pi_{t+1}(z^{t+1}), z_{t+1}) + \beta \int_Z D_x F(\pi_{t+1}(z^{t+1}), \pi_{t+2}(z^{t+2}), z_{t+2}) Q(dz_{t+2}|z_{t+1}) = 0, \quad t = 0, 1, \ldots.$$

Proof. Let the state be (x_0, z_0) and let $\pi_0 \in G(x_0, z_0)$. The optimal plan π is interior, thus by B5 , there exists N_{π_0} such that for all $z_1 \in Z$, $\pi_1(z^1) \in \Gamma(y, z_1)$ for all $y \in N_{\pi_0}$. We take N_{π_0} to be a compact neighborhood of π_0. The function $\phi : N_{\pi_0} \times Z \longrightarrow \mathbb{R}$ given by

$$\phi(y, z_1) := F(y, \pi_1(z^1), z_1) + \beta Mv(\pi_1(z^1), z_1) \qquad (3.1)$$

[3]A previous version of the paper claimed that B1 and B2 imply B5. That this is wrong was pointed out by one referee, who provided the following counterexample: let a deterministic model with $X = [-1, 1]$ and $\Gamma(x) := \{y \in [-1, 1] : |y| \geq |x|\}$, for $x \in X$. The correspondence obeys B1 and B2, $y_0 = 0$ belongs to the interior of $\Gamma(0) = [-1, 1]$, but $0 \notin \Gamma(x)$ for $x \neq 0$.

is well defined. Clearly, it is of class C^1 with respect to y. Let us show that it is measurable with respect to z_1 and bounded in $N_{\pi_0} \times Z$. We will use repetitively the following property that can be found in Bergé (1963): since Γ is continuous with compact values and $N_{\pi_0} \times Z$ is compact, the set $\Gamma(N_{\pi_0} \times Z)$ is compact. The first summand of the right hand side of (3.1), $F(y, \pi_1(z^1), z_1)$, is measurable, since $\pi_1(z^1)$ is measurable and F is continuous. The continuity of F guarantees that $F(y, \pi_1(z^1), z_1)$ is bounded in $N_{\pi_0} \times Z$. The second summand in (3.1) is $Mv(\pi_1(z^1), z_1)$. Since v is a solution of the Bellman equation, it holds

$$Mv(\pi_1(z^1), z_1) = \frac{1}{\beta} v(\pi_0, z_1) - \frac{1}{\beta} F(\pi_0, \pi_1(z^1), z_1).$$

Clearly, by continuity, both terms on the right hand side of the equality are measurable and bounded in z_1. Thus, the function $z_1 \to \phi(y, z_1)$ is integrable with respect to z_1 for $y \in N_{\pi_0}$. As a consequence, the function Φ, given by

$$\Phi(y, z_0) = \int_Z \phi(y, z_1) Q(dz_1|z_0), \tag{3.2}$$

is well defined for all $y \in N_{\pi_0}$. Moreover, since ϕ is continuous in y and bounded in $N_{\pi_0} \times Z$, by Theorem 20.3 in Aliprantis and Burkinshaw (1990), function Φ is continuous in y. By (3.1), $\phi(\pi_0, z_1) = v(\pi_0, z_1)$ and $v(y, z_1) \geq \phi(y, z_1)$ in N_{π_0}. Hence, integrating, we have

$$\Phi(\pi_0, z_0) = \int_Z \phi(\pi_0, z_1) Q(dz_1|z_0) = \int_Z v(\pi_0, z_1) Q(dz_1|z_0) = Mv(\pi_0, z_0). \tag{3.3}$$

and

$$\Phi(y, z_0) = \int_Z \phi(y, z_1) Q(dz_1|z_0) \leq \int_Z v(y, z_1) Q(dz_1|z_0) = Mv(y, z_0), \tag{3.4}$$

all $y \in N_{\pi_0}$. Now we show that the function Φ is differentiable with respect to x. To prove this, note that the partial derivatives $D_i \phi(y, z_1) = D_i F(y, \pi_1(z^1), z_1)$, $i = 1, \ldots, l$, are measurable. Also, $|D_i \phi(y, z_1)| \leq C$ in $\Gamma(N_{\pi_0} \times Z)$ for some constant C. This is because $\Gamma(N_{\pi_0} \times Z)$ is a compact subset of the interior of X and then, by continuity, $|D_i F|$ is bounded. By Theorem 20.4 in Aliprantis and Burkinshaw (1990), $\Phi(\cdot, z_0)$ is differentiable at π_0 and differentiating under the integral is allowed, to obtain

$$D_x \Phi(\pi_0, z_0) = \int_Z D_x \phi(\pi_0, z_1) Q(dz_1|z_0) = \int_Z D_x F(\pi_0, \pi_1(z^1), z_1) Q(dz_1|z_0). \tag{3.5}$$

Observe that, by (3.3) and (3.4)

$$\begin{aligned} &\liminf_{y \to \pi_0} \frac{Mv(y, z_0) - Mv(\pi_0, z_0) - D_x \Phi(\pi_0, z_0) \cdot (y - \pi_0)}{\|y - \pi_0\|} \\ &\qquad \geq \liminf_{y \to \pi_0} \frac{\Phi(y, z_0) - \Phi(\pi_0, z_0) - D_x \Phi(\pi_0, z_0) \cdot (y - \pi_0)}{\|y - \pi_0\|} = 0, \end{aligned} \tag{3.6}$$

where, the equality holds since $\Phi(\cdot, z_0)$ is differentiable.

On the other hand, since π_0 belongs to the interior of $\Gamma(x_0, z_0)$, there is a neighborhood N'_{π_0} of π_0 which is in the interior of $\Gamma(x_0, z_0)$. For any $y \in N'_{\pi_0}$, let us define the function

$$\psi(y, z_0) = \frac{1}{\beta} \left(v(x_0, z_0) - F(x_0, y, z_0) \right).$$

Note that ψ is continuously differentiable. Moreover,

$$\psi(\pi_0, z_0) = \frac{1}{\beta}\left(v(x_0, z_0) - F(x_0, \pi_0, z_0)\right) = Mv(\pi_0, z_0)$$

and, for $y \in N'_{\pi_0}$, $F(x_0, y, z_0) + \beta Mv(y, z_0) \leq v(x_0, z_0)$, hence $\psi(y, z_0) \geq Mv(y, z_0)$. Then we can compute

$$\begin{aligned}&\limsup_{y\to\pi_0} \frac{Mv(y, z_0) - Mv(\pi_0, z_0) - D_x\psi(\pi_0, z_0)\cdot(y-\pi_0)}{\|y-\pi_0\|}\\ &\qquad\leq \limsup_{y\to\pi_0} \frac{\psi(y, z_0) - \psi(\pi_0, z_0) - D_x\psi(\pi_0, z_0)\cdot(y-\pi_0)}{\|y-\pi_0\|} = 0,\end{aligned} \tag{3.7}$$

since $\psi(\cdot, z_0)$ is differentiable. Now, a well known method of reasoning will prove that $D_x\Phi(\pi_0, z_0) = D_x\psi(\pi_0, z_0)$. Note that

$$\begin{aligned}&\frac{Mv(y, z_0) - Mv(\pi_0, z_0) - D_x\psi(\pi_0, z_0)\cdot(y-\pi_0)}{\|y-\pi_0\|}\\ &\quad - \frac{Mv(y, z_0) - Mv(\pi_0, z_0) - D_x\Phi(\pi_0, z_0)\cdot(y-\pi_0)}{\|y-\pi_0\|}\\ &\quad = \left(D_x\Phi(\pi_0, z_0) - D_x\psi(\pi_0, z_0)\right)\cdot\frac{(y-\pi_0)}{\|y-\pi_0\|}.\end{aligned}$$

If we let $y \to \pi_0$ along the direction $D_x\Phi(\pi_0, z_0) - D_x\psi(\pi_0, z_0)$, that is, if we take $y = \pi_0 + \lambda(D_x\Phi(\pi_0, z_0) - D_x\psi(\pi_0, z_0))$, then

$$\begin{aligned}&\frac{Mv(y, z_0) - Mv(\pi_0, z_0) - D_x\psi(\pi_0, z_0)\cdot(y-\pi_0)}{\|y-\pi_0\|}\\ &\quad - \frac{Mv(y, z_0) - Mv(\pi_0, z_0) - D_x\Phi(\pi_0, z_0)\cdot(y-\pi_0)}{\|y-\pi_0\|}\\ &\quad = \|D_x\Phi(\pi_0, z_0) - D_x\psi(\pi_0, z_0)\|.\end{aligned}$$

Letting $\lambda \to 0$, by the definition of lim sup and lim inf, we have

$$\begin{aligned}&\|D_x\Phi(\pi_0, z_0) - D_x\psi(\pi_0, z_0)\|\\ &\quad\leq \limsup_{y\to\pi_0} \frac{Mv(y, z_0) - Mv(\pi_0, z_0) - D_x\psi(\pi_0, z_0)\cdot(y-\pi_0)}{\|y-\pi_0\|}\\ &\qquad - \liminf_{y\to\pi_0} \frac{Mv(y, z_0) - Mv(\pi_0, z_0) - D_x\Phi(\pi_0, z_0)\cdot(y-\pi_0)}{\|y-\pi_0\|}.\end{aligned}$$

Hence, by (3.6) and (3.7), $\|D_x\Phi(\pi_0, z_0) - D_x\psi(\pi_0, z_0)\| \leq 0$ and thus $D_x\Phi(\pi_0, z_0) = D_x\psi(\pi_0, z_0)$ as claimed above. Consequently, $Mv(\cdot, z_0)$ is differentiable at π_0 and

$$D_xMv(\pi_0, z_0) = D_x\Phi(\pi_0, z_0) = D_x\psi(\pi_0, z_0).$$

Now, by (3.5) and since $D_x\psi(y, \pi_0) = -\frac{1}{\beta}D_yF(x_0, y, z_0)$, we have

$$D_xMv(\pi_0, z_0) = \int_Z D_xF(\pi_0, \pi_1(z^1), z_1)Q(dz_1|z_0) = -\frac{1}{\beta}D_yF(x_0, \pi_0, z_0),$$

or

$$D_yF(x_0,\pi_0,z_0)+\beta\int_Z D_xF(\pi_0,\pi_1(z^1),z_1)Q(dz_1|z_0)=0.$$

By induction, this process can be iterated, starting now at $(\pi_1(z^1),z_1)$, since $(\pi_1(z^1),z_1)$ is an interior state by assumption. In this way, we get the Envelope Equation (a) and the Euler Equation (b) of the theorem at any $t\geq 1$. □

3.1 Deterministic problems

If the problem is deterministic, that is, if Q is degenerated, then the theorem holds with weaker assumptions, as the integrability is not a issue. We replace B3 by the following assumption and state the corresponding theorem below.

B3': The function $F:\Omega\longrightarrow\mathbb{R}$ is continuous and differentiable in the interior of Ω.

In the theorem below, the assumptions have to be interpreted as independent of z. The theorem has already been established and proved by Dechert and Nishimura (1983) in the particular case of a nonconvex optimal growth model.

Theorem 3.2. *Let the problem (X,Γ,F,β) satisfy Assumptions B1, B2, B3', B4 and B5. Then, for any interior state $x_0\in X$, the function v is differentiable at $g(x_0)$ and the following hold:*
(a) the Envelope equation

$$D_xv(x_t)=D_xF(x_t,x_{t+1}),\quad t=1,2,\ldots;$$

(b) the Euler equation

$$D_yF(x_t,x_{t+1})+\beta D_xF(x_{t+1},x_{t+2})=0,\quad t=1,2,\ldots,$$

where $x_{t+1}=g(x_t)$, $t=1,2,\ldots$.

4 Conclusions

We have obtained the Euler equation as a necessary condition of optimality for interior solutions in non–concave problems. To do it, we have proved that the expected value function is differentiable at tomorrow's optimal states. This property suffices to construct the Euler equation. Future work should be directed to exploring the case where the endogenous state variable obeys a law of motion that depends on the previous state, actions, and stochastic shocks. This seems to be a challenging problem as, to our knowledge, there are no general results, even in the concave case; one exception being Blume et al (1982) in their study of the higher order differentiability of the value and the policy function in concave problems.

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References

[1] Aliprantis, C., and O. Burkinshaw (1990): *Principles of Real Analysis.* London: Academic Press.

[2] Amir, R. (1996): "Sensitivity Analysis in Multisector Optimal Economic Dynamics," *Journal of Mathematical Economics,* 25, 123-141.

[3] Askri, K., and C. Le Van (1998): "Differentiability of the Value Function of Nonclassical Optimal Growth Models," *Journal of Optimization Theory and Applications*, 97, 591–604.

[4] Benveniste, L.M., and J.A. Scheinkman (1979): "On the Differentiability of the Value Function in Dynamic Models of Economics," *Econometrica,* 47, 727-732.

[5] Bergé, C. (1963): *Topological Spaces.* New York: Macmillan.

[6] Blackwell, D. (1965): "Discounted Dynamic Programming,". *Annals of Applied Statistics,* 36, 226–235.

[7] Blume, L., D. Easly, and M. O'Hara (1982): "Characterization of Optimal Plans for Stochastic Dynamic Programs," *Journal of Economic Theory,* 28, 221–234.

[8] Boyd III, J. H. (1990): "Recursive Utility and the Ramsey Problem," *Journal of Economic Theory,* 50, 326–345.

[9] Brock, W. A., and L. J. Mirman (1972): "Optimal Economic Growth and Uncertainty: The Discounted Case," *Journal of Economic Theory,* 4, 479–513.

[10] Cai, Y., T. Kamihigashi, and J. Stachurski (2014): "Stochastic Optimal Growth with Risky Labor Supply," *Journal of Mathematical Economics,* 50, 167–176.

[11] Clausen, A. and C. Strub (2016): "A General and Intuitive Envelope Theorem," *Edinburgh School of Economics Discussion Paper Series #274.*

[12] Coleman, W. J. (1991): "Equilibrium in a Production Economy with an Income Tax," *Econometrica,* 59, 1091–1104.

[13] Cotter, K. D., and J.–H. Park (2006): "Non–Concave Dynamic Programming," *Economic Letters,* 90, 141–146.

[14] Donaldson, J. B., and R. Mehra (1983): "Stochastic Growth with Correlated Production Shocks," *Journal of Economic Theory,* 29, 282–312.

[15] Dechert, W.D., and K. Nishimura (1983): "A complete characterization of optimal growth paths in an aggregated model with a non–concave production function," *Journal of Economic Theory,* 31, 332–354.

[16] Le Van, C., and J. Stachurski (2007): "Parametric Continuity of Stationary Distributions," *Economic Theory,* 33, 333–348.

[17] Majumdar, M., T. Mitra, and Y. Nyarko (1989): “Dynamic Optimization Under Uncertainty: Non-convex Feasible Set.” In: Feiwel G.R. (eds) *Joan Robinson and Modern Economic Theory,* 545–590. Palgrave Macmillan, London

[18] Milgrom, P., and I. Segal (2002): “Envelope Theorems for Arbitrary Choice Sets,” *Econometrica,* 70, 583–601.

[19] Mirman, L. J., and I. Zilcha (1975): “On Optimal Growth under Uncertainty,” *Journal of Economic Theory,* 11, 329-339.

[20] Morand, O., K. Reffett, and S. Tarafdar (2018): “Generalized Envelope Theorems: Applications to Dynamic Programming,” *Journal of Optimization Theory and Applications,* 176, 650–687.

[21] Nishimura, K., R. Rudnicki, and J. Stachurski (2012): “Stochastic Optimal Growth with Nonvonvexities.” In: Stachurski, J., A. Venditti, and M. Yano (eds) *Nonlinear Dynamics in Equilibrium Models,* 261–288. Springer, Berlin, Heidelberg.

[22] Nishimura, K., and J. Stachurski (2005): “Stability of Stochastic Optimal Growth Models: A New Approach,” *Journal of Economic Theory,* 122, 100–118.

[23] Rincón–Zapatero, J.P., and C. Rodríguez–Palmero (2003): “Existence and Uniqueness of Solutions of the Bellman Equation in the Unbounded Case,” *Econometrica,* 71, 1519-1555.

[24] Rincón–Zapatero, J.P., and C. Rodríguez–Palmero (2007): “Recursive Utility with Unbounded Aggregators,” *Economic Theory,* 33, 381-391.

[25] Rincón–Zapatero, J.P., and M.S. Santos (2009): “Differentiability of the Value Function without Interiority Assumptions,” *Journal of Economic Theory,* 144: 1948-1964.

[26] Rincón–Zapatero, J.P., and Santos, M. (2012): “Differentiability of the Value Function in Continuous–Time Economic Models,” *Journal of Mathematical Analysis and Applications,* 394, 305–323.

[27] Rincón–Zapatero, J.P., and Zhao, Y. (2018): “Envelope Theorem in Dynamic Economic Models with Recursive Utility,” *Economics Letters,* 163: 10–12.

[28] Stokey, N., and R. Lucas, with E. Prescott (1989): *Recursive Methods in Economic Dynamics*. Harvard University Press