Palatini gravity with nonmetricity, torsion, and boundaries in metric and connection variables

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We prove the equivalence in the covariant phase space of the metric and connection formulations for Palatini gravity, with nonmetricity and torsion, on a spacetime manifold with boundary. To this end, we will rely on the cohomological approach provided by the relative bicomplex framework. Finally, we discuss some of the physical implications derived from this equivalence in the context of singularity identification through curvature invariants.

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I. INTRODUCTION

The study of general relativity on manifolds with boundary is of great importance as these can be used to account for the asymptotics of the gravitational field and the presence of horizons [1–3]. The original formulation of general relativity relied on a Lorentzian metric as the fundamental field. However, it is also possible to use tetrads for the same purpose. This is important from a physical point of view for several reasons. First, tetrads provide a natural way to incorporate fermion fields into the theory. Second, the tetrad formalism leads naturally to the Ashtekar formulation, one of the main avenues for the quantization of gravity [4,5]. Finally, local Lorentz gauge invariance plays a crucial role in this formalism and it may introduce significant differences in the treatment of the theory compared with the metric formulation. Within both frameworks, there are several choices for the action that can be classified according to their independent basic field variables. The most well known is standard GR, where the action is written in terms of a metric (metric-GR) or a tetrad (tetrad-GR). Another approach is to consider Palatini theories, where the action is written in terms of a metric and an independent connection (metric-Palatini) or a frame and an independent spin connection (tetrad-Palatini) [6–9].

Metric-Palatini (actually discovered as we understand it now by Einstein in 1925 [10]) has been studied for a long time as an alternative to metric-GR. From a mathematical point of view, its natural framework is the space of metrics times the space of "unrestricted" connections i.e., not forcing neither the torsion nor the nonmetricity to be zero. This general approach has also been proved physically relevant [8,11,12]. For instance, nonzero torsion has been used in the cosmological context as a mechanism that can prevent the formation of singularities [13]. There are yet even more general theories that might be studied with the tools provided in this paper such as $f(R)$-Palatini theories, which avoid certain dynamical instabilities present in metric-GR [14] and have the advantage of naturally generating an effective cosmological constant [8]. In addition, torsion and nonmetricity have a direct link with the microscopic properties of matter [15] and, as pointed out in [16], they can only be detected by test particles with microstructure.

The present paper attempts to unify and expand certain aspects of previous works [17–27] dealing with metric-Palatini and tetrad-Palatini, focusing on the comparison of both formulations and the treatment of boundaries. In this regard, we would like to highlight the pioneering work by Obukhov [28], where he introduced the appropriate surface terms for Palatini gravity. Here we generalize those results and improve on the variational treatment of the problem by relying on the recently proposed CPS algorithm [29], which provides a clean, consistent, and ambiguity-free procedure to obtain the solution spaces, the presymplectic forms canonically associated with the actions, and some relevant charges.

The equivalence of the metric-Palatini and tetrad-Palatini formalisms will be proven in two steps. First, we will obtain a precise description of the solution spaces for both theories which will allow us to map them appropriately. Second, we will show that the presymplectic forms given by the CPS algorithm are equivalent.

In the following, we consider a 4-dimensional spacetime $M \cong \Sigma \times \mathbb{R}$, where Σ is a 3-dimensional manifold with boundary $\partial\Sigma$ (possibly empty). $\partial_L M \cong \partial\Sigma \times \mathbb{R}$ denotes the *lateral boundary* of M and we restrict ourselves to the open set of metrics making $\partial_L M$ timelike. Greek letters will denote abstract indices for tensors in M and barred Greek indices for tensors on $\partial_L M$ (quite often the object itself will also carry an overbar). The inclusion map will be denoted as $\jmath : \partial_L M \hookrightarrow M$ and its tangent map as $\jmath^\alpha_{\bar\alpha}$.

II. METRIC-PALATINI

Given a connection $\tilde\nabla$, we define its torsion, Riemann, and Ricci tensors as

$$\begin{aligned}\widetilde{\mathrm{Tor}}^\alpha{}_{\mu\nu}(\mathrm{d}\phi)_\alpha &= -[\tilde\nabla_\mu, \tilde\nabla_\nu]\phi,\\ \widetilde{\mathrm{Riem}}^\alpha{}_{\beta\mu\nu}Z^\beta &= ([\tilde\nabla_\mu, \tilde\nabla_\nu] + \widetilde{\mathrm{Tor}}^\beta{}_{\mu\nu}\tilde\nabla_\beta)Z^\alpha,\\ \widetilde{\mathrm{Ric}}_{\beta\nu} &\coloneqq \widetilde{\mathrm{Riem}}^\mu{}_{\beta\mu\nu}.\end{aligned}$$

If we endow M with a connection $\tilde\nabla$ and a metric g, we have the nonmetricity tensor, the $(g, \tilde\nabla)$-scalar-curvature, the $(g, \tilde\nabla)$-extrinsic-curvature of $\partial_L M$, and its trace

$$\tilde M_{\alpha\beta\gamma} \coloneqq \tilde\nabla_\alpha g_{\beta\gamma}, \qquad \tilde R \coloneqq g^{\alpha\beta}\widetilde{\mathrm{Ric}}_{\alpha\beta},$$
$$\tilde K_{\bar\alpha\bar\beta} \coloneqq \frac{1}{2}\jmath^\alpha_{\bar\alpha}\jmath^\beta_{\bar\beta}(\tilde\nabla_\alpha \nu_\beta + g_{\alpha\gamma}\tilde\nabla_\beta \nu^\gamma), \qquad \tilde K \coloneqq \bar g^{\bar\alpha\bar\beta}\tilde K_{\bar\alpha\bar\beta},$$

where $\bar g \coloneqq \jmath^* g$ is the induced metric, ν^α the outward unit vector field, and $\nu_\beta = g_{\beta\gamma}\nu^\gamma$. Notice that $\tilde R$ and $\tilde K_{\bar\alpha\bar\beta}$ are (nonstandard) generalizations of the g-scalar $\overset{\circ}{\nabla}$ and g-extrinsic curvature $\overset{\circ}{K}$ defined by the g-Levi-Civita connection $\overset{\circ}{\nabla}$. Moreover, in general, $\tilde K_{\bar\alpha\bar\beta}$ is not symmetric if the nonmetricity is different from zero.

Given two connections ∇ and $\tilde\nabla$, their difference is a (2,1)-tensor $Q \equiv \tilde\nabla - \nabla$. For a (1,1)-tensor $S^\beta{}_\gamma$ we have

$$(\tilde\nabla_\alpha - \nabla_\alpha)S^\beta{}_\gamma = Q^\beta{}_{\alpha\mu}S^\mu{}_\gamma - Q^\mu{}_{\alpha\gamma}S^\beta{}_\mu$$

and analogously for higher order objects. Observe that if we choose a fiducial connection, usually the g-Levi-Civita one, there is a bijection between connections $\tilde\nabla$ and (2,1)-tensors Q. Working with tensors is usually easier, as they form a vector space while connections form an affine space. Thus, in the following, we will use the variables (g, Q) instead of the equivalent ones $(g, \tilde\nabla)$.

We will now use the CPS algorithm [29], which essentially consists in introducing a pair of bulk and boundary Lagrangians, compute the variations, extract the equations of motion and symplectic potentials, and get the presymplectic form on the space of solutions. The power of this method lies in its cohomological nature which renders it ambiguity-free: we can pick any representative Lagrangians and symplectic potentials to describe the solution spaces and compute the presymplectic form.

A. The action

We consider actions of the form

$$\mathbb{S} = \int_M L - \int_{\partial M} \bar\ell,$$

for some bulk Lagrangian L and some boundary Lagrangian $\bar\ell$. The metric-GR and the metric-Palatini actions are given, respectively, by the Lagrangian pairs

$$L^{(m)}_{\mathrm{EH}}(g) \coloneqq (\overset{\circ}{R} - 2\Lambda)\mathrm{vol}_g, \qquad \bar\ell^{(m)}_{\mathrm{GHY}}(g) \coloneqq -2\overset{\circ}{K}\mathrm{vol}_{\bar g},$$
$$L^{(m)}_{\mathrm{PT}}(g, Q) \coloneqq (\tilde R - 2\Lambda)\mathrm{vol}_g, \qquad \bar\ell^{(m)}_{\mathrm{PT}}(g, Q) \coloneqq -2\tilde K\mathrm{vol}_{\bar g}.$$

It is straightforward to rewrite the Palatini Lagrangians as those of standard GR (Einstein-Hilbert on the bulk and Gibbons-Hawking-York on the boundary) plus a coupling term

$$L^{(m)}_{\mathrm{PT}}(g, Q) = L^{(m)}_{\mathrm{EH}}(g) + L^{(m)}_{\mathrm{CP}}(g, Q),$$
$$\bar\ell^{(m)}_{\mathrm{PT}}(g, Q) = \bar\ell^{(m)}_{\mathrm{GHY}}(g) + \bar\ell^{(m)}_{\mathrm{CP}}(g, Q),$$

where

$$L^{(m)}_{\rm CP}(g,Q) \coloneqq (C_\lambda A^\lambda - Q^{\alpha\beta}{}_\lambda Q^\lambda{}_{\alpha\beta})\mathrm{vol}_g + \mathrm{d}(\iota_{\vec{A}-\vec{C}}\mathrm{vol}_g),$$
$$\bar{\ell}^{(m)}_{\rm CP}(g,Q) \coloneqq \jmath^*(\iota_{\vec{A}-\vec{C}}\mathrm{vol}_g),$$
$$A^\alpha \coloneqq g^{\beta\gamma}Q^\alpha{}_{\beta\gamma}, \qquad B_\beta \coloneqq Q^\mu{}_{\beta\mu}, \qquad C_\gamma \coloneqq Q^\mu{}_{\mu\gamma}.$$

As mentioned before, the results given by the CPS algorithm do not depend on the choice of Lagrangians as long as they define the same action (the pairs are equal up to a *relative* exact form, see [29] for more details). In the present case it is easy to see that

$$(L^{(m)}_{\rm PT}, \bar{\ell}^{(m)}_{\rm PT}) = (L^{(m)}_{\rm EH}, \bar{\ell}^{(m)}_{\rm GHY}) + (\hat{L}^{(m)}_{\rm CP}, 0) + \underline{\mathrm{d}}(\iota_{\vec{A}-\vec{C}}\mathrm{vol}_g, 0),$$
$$\hat{L}^{(m)}_{\rm CP} \coloneqq L^{(m)}_{\rm CP} - \mathrm{d}(\iota_{\vec{A}-\vec{C}}\mathrm{vol}_g).$$

Considering $(\hat{L}^{(m)}_{\rm CP}, 0)$ actually makes the computations a little shorter, but we will stick to $(L^{(m)}_{\rm CP}, \bar{\ell}^{(m)}_{\rm CP})$ as this will facilitate the comparison of our results with the existing literature.

B. Variations

The variations of the $L^{(m)}_{\rm EH}$ and $\bar{\ell}^{(m)}_{\rm GHY}$ terms are standard and they may be found, for instance, in [30]. The remaining variations are those of the coupling Lagrangians, which can be written as

$$\mathbb{d}L^{(m)}_{\rm CP} = (\mathfrak{E}^{\rm CP}_{(m)})^{\alpha\beta}\mathbb{d}g_{\alpha\beta} + (\mathcal{E}^{\rm CP}_{(m)})_\gamma{}^{\alpha\sigma}\mathbb{d}Q^\gamma{}_{\alpha\sigma} + \mathrm{d}\Theta^{(m)}_{\rm CP},$$
$$\mathbb{d}\bar{\ell}^{(m)}_{\rm CP} - \jmath^*\Theta^{(m)}_{\rm CP} = 0,$$

where

$$(\mathfrak{E}^{\rm CP}_{(m)})^{\alpha\beta}(g,Q) \coloneqq \bigg(Q^{\gamma\alpha}{}_\sigma Q^\sigma{}_\gamma{}^\beta - C_\sigma Q^{\sigma\alpha\beta} + \frac{1}{2}g^{\alpha\beta}(C_\sigma A^\sigma - Q^{\gamma\tau}{}_\sigma Q^\sigma{}_{\gamma\tau})\bigg)\mathrm{vol}_g,$$
$$(\mathcal{E}^{\rm CP}_{(m)})_\gamma{}^{\alpha\sigma}(g,Q) \coloneqq (\delta^\alpha_\gamma A^\sigma + g^{\alpha\sigma}C_\gamma - Q^\sigma{}_\gamma{}^\alpha - Q^{\alpha\sigma}{}_\gamma)\mathrm{vol}_g.$$

Notice that the latter is an equation of motion by itself, but the former is not. To obtain one we have to add $\mathfrak{E}^{\rm CP}_{(m)}$ to the Einstein equations coming from the variation of $L^{(m)}_{\rm EH}$ (which only depends on g).

In view of the previous result, we can choose the following representatives as the contributions of the CP terms to the symplectic potentials

$$\Theta^{(m)}_{\rm CP}(g,Q) \coloneqq \mathbb{d}(\iota_{\vec{A}-\vec{C}}\mathrm{vol}_{\bar{g}}), \qquad \bar{\theta}^{(m)}_{\rm CP}(g,Q) \coloneqq 0.$$

C. Space of solutions

The algebraic equation of motion $\mathcal{E}^{\rm CP}_{(m)}(g,Q) = 0$ can be solved for Q. Its general solution is $Q_0^{\alpha\beta\gamma} = g^{\alpha\gamma}U^\beta$ with arbitrary U^β (we sketch the proof of a similar fact in Sec. III C). This solution, moreover, satisfies $\mathfrak{E}^{\rm CP}_{(m)}(g,Q_0) = 0$ for any g. Hence, (g,Q) is a solution for Palatini if and only if $Q = Q_0$ and g satisfies the Einstein equations:

$$\mathrm{Sol}^{(m)}_{\rm PT} = \{(g_{\alpha\beta}, \delta^\alpha_\gamma U_\beta)/g \in \mathrm{Sol}^{(m)}_{\rm GR}, U_\beta \text{ arbitrary}\}.$$

This, in turn, proves that the metric sector of metric-Palatini is equivalent to metric-GR. Notice that the boundary only plays a role in the metric sector $\mathrm{Sol}^{(m)}_{\rm GR}$ of the solution space (which is discussed in detail in [30], where both Dirichlet and Neumann boundary conditions are considered). We have now the following ("on shell") identities over the space of solutions

$$\tilde{\mathrm{R}}\mathrm{iem}^\alpha{}_{\beta\mu\nu} = \mathring{\mathrm{R}}\mathrm{iem}^\alpha{}_{\beta\mu\nu} + g^\alpha_\beta(\mathrm{d}U)_{\mu\nu},$$
$$\tilde{\mathrm{R}}\mathrm{ic}_{\beta\nu} = \mathring{\mathrm{R}}\mathrm{ic}_{\beta\nu} + (\mathrm{d}U)_{\beta\nu}, \qquad \tilde{R} = \mathring{R},$$
$$\tilde{K}_{\bar{\alpha}\bar{\beta}} = \mathring{K}_{\bar{\alpha}\bar{\beta}} - \frac{1}{2}(U\wedge\nu)_{\bar{\alpha}\bar{\beta}}, \qquad \tilde{K} = \mathring{K},$$
$$\tilde{M}_{\alpha\beta\gamma} = -2g_{\beta\gamma}U_\alpha, \qquad \tilde{T}^\gamma{}_{\alpha\beta} = \delta^\gamma_\beta U_\alpha - \delta^\gamma_\alpha U_\beta. \quad (1)$$

The last two equations imply that, on solutions, $\tilde{M} = 0$ if and only if $\tilde{T} = 0$. Thus, we have either the Levi-Civita connection or one with nonmetricity and torsion.

D. Presymplectic form

The metric-Palatini presymplectic form on the space of solutions $\Omega^{\rm PT}_{(m)}$ has the same functional form as the metric-GR presymplectic form $\Omega^{\rm GR}_{(m)}$ since $\mathbb{d}\Theta^{(m)}_{\rm CP} = 0$. However, the solution spaces are different. In order to compare them, we need the projection $\pi_{(m)}(g,Q) = g$. In this way we get

$$\Omega^{\rm PT}_{(m)} = \pi^*_{(m)}\Omega^{\rm GR}_{(m)} \quad (2)$$

Clearly, all vectors of the form $(0,\mathbb{V}) \in T\mathrm{Sol}^{(m)}_{\rm PT}$, i.e., in the directions of the connection sector, define gauge directions since they belong to the kernel of $(\pi_{(m)})_*$ and hence annihilate $\Omega^{\rm PT}_{(m)}$.

III. TETRAD-PALATINI

Given a tetrad e^I_α, we have its η_{IJ}-dual E^α_I and the spacetime metric $g = \eta_{IJ}e^I e^J \eqqcolon \Phi_{\rm GR}(e)$ as explained in more detail in [30]. We can consider the Levi-Civita connection $\mathring{\nabla}$ of g which, in turn, allows us to build the 1-form connection

$$\mathring{\omega}_\mu{}^K{}_I \coloneqq e^K_\alpha \mathring{\nabla}_\mu E^\alpha_I. \quad (3)$$

If we now consider a generic 1-form connection $\tilde{\omega}_\mu{}^{IJ}$ (not necessarily antisymmetric in the internal indices), we can define the covariant derivative

$$\tilde{\nabla}_\mu Y^\beta = (\mathrm{d}Y^J)_\mu E_J^\beta + E_K^\beta \tilde{\omega}_\mu{}^K{}_J Y^J, \qquad Y^J \coloneqq e_\gamma^J Y^\gamma,$$

and extend it in the usual way to objects with more spacetime and internal indices. In particular, applying this definition to E_I^β leads to

$$\tilde{\omega}_\mu{}^K{}_I = e_\alpha^K \tilde{\nabla}_\mu E_I^\alpha, \tag{4}$$

which is analogous to (3). We also have the $\tilde{\omega}$-curvature and the $\tilde{\omega}$-covariant derivative over forms given by

$$\tilde{F}_{IJ} = \mathrm{d}\tilde{\omega}_{IJ} + \tilde{\omega}_{IK} \wedge \tilde{\omega}^K{}_J,$$
$$\tilde{\mathcal{D}}\alpha_I = \mathrm{d}\alpha_I + \tilde{\omega}_I{}^J \wedge \alpha_J,$$

where the latter is extended, as usual, to objects with more antisymmetric internal indices.

Given two 1-form connections ω and $\tilde{\omega}$, we have two covariant derivatives ∇ and $\tilde{\nabla}$ which are related by a (2,1)-tensor Q. In particular, if we take $\nabla = \mathring{\nabla}$, then

$$Q^\beta{}_{\mu\alpha} = E_K^\beta e_\alpha^J (\tilde{\omega}_\mu{}^K{}_J - \mathring{\omega}_\mu{}^K{}_J) \eqqcolon \varphi(e,\tilde{\omega})^\beta{}_{\mu\alpha}.$$

This allows us to define the map

$$\Phi_{\mathrm{PT}}(e,\tilde{\omega}) \coloneqq (\Phi_{\mathrm{GR}}(e), \varphi(e,\tilde{\omega}))$$

which is surjective but not injective. In fact,

$$\Phi_{\mathrm{PT}}(e,\tilde{\omega}) = \Phi_{\mathrm{PT}}(e',\tilde{\omega}')$$

if and only if there exists some $\Psi \in SO(1,3)$ such that $e'_I = \Psi_I{}^J e_J$ and $\tilde{\omega}'^{J}_I = \Psi_I{}^K \mathrm{d}\Psi^J{}_K + \Psi_I{}^K \Psi^J{}_L \tilde{\omega}_K{}^L$.

A. The action

In order to introduce the tetrad formalism, we perform the "change of variables" given by Φ_{PT}

$$L^{(t)}_{\mathrm{PT}}(e,\tilde{\omega}) \coloneqq L^{(m)}_{\mathrm{PT}} \circ \Phi_{\mathrm{PT}}(e,\tilde{\omega}),$$
$$\bar{\ell}^{(t)}_{\mathrm{PT}}(e,\tilde{\omega}) \coloneqq \bar{\ell}^{(m)}_{\mathrm{PT}} \circ \Phi_{\mathrm{PT}}(e,\tilde{\omega}).$$

We could write again these Lagrangians as the GR part plus a coupling term, but in this case it is more convenient to split the 1-form connection in its antisymmetric and symmetric parts (in its internal indices)

$$\tilde{\omega}_{IJ} = \hat{\omega}_{IJ} + S_{IJ},$$

and consider the equivalent variables $(e,\hat{\omega},S)$ instead. Following the ideas of [30], one obtains the following expressions for these Lagrangians

$$L^{(t)}_{\mathrm{PT}}(e,\hat{\omega},S) = \frac{1}{2}\epsilon_{IJKL}\left(\hat{F}^{IJ} - \frac{\Lambda}{6} e^I \wedge e^J + S^I{}_M \wedge S^{MJ}\right) \wedge e^K \wedge e^L,$$

$$\bar{\ell}^{(t)}_{\mathrm{PT}}(e,\hat{\omega}) = -\frac{1}{2}\epsilon_{IJKL}(2N^I \mathrm{d}N^J - \bar{\hat{\omega}}^{IJ}) \wedge \bar{e}^K \wedge \bar{e}^L,$$

where $\bar{e}^I \coloneqq \jmath^* e^I$, $\bar{\hat{\omega}}^{IJ} \coloneqq \jmath^* \hat{\omega}^{IJ}$ and $N^I = \nu^\alpha e_\alpha^I$. Notice that $\bar{\ell}^{(t)}_{\mathrm{PT}}$ does not depend on S.

B. Variations

Computing the variations, one easily obtains

$$\mathbb{d}L^{(t)}_{\mathrm{PT}} = \mathfrak{E}^{(t)}_L \wedge \mathbb{d}e^L + \mathcal{E}^{(t)}_{KL} \wedge \mathbb{d}\hat{\omega}^{KL} + \mathscr{E}^{(t)}_{JM} \wedge \mathbb{d}S^{JM} + \mathrm{d}\Theta^{(t)}_{\mathrm{PT}},$$
$$\mathbb{d}\bar{\ell}^{(t)}_{\mathrm{PT}} - \jmath^*\Theta^{(t)}_{\mathrm{PT}} = \bar{b}^{(t)}_I \wedge \mathbb{d}\bar{e}^I - \mathrm{d}\bar{\theta}^{(t)}_{\mathrm{PT}},$$

where the Euler-Lagrange equations are

$$\mathfrak{E}^{(t)}_L \coloneqq \epsilon_{IJKL}\left(\hat{F}^{IJ} + S^I{}_M \wedge S^{MJ} - \frac{\Lambda}{3} e^I \wedge e^J\right) \wedge e^K,$$
$$\mathcal{E}^{(t)}_{KL} \coloneqq \frac{1}{2}\hat{D}(\epsilon_{IJKL} e^I \wedge e^J),$$
$$\mathscr{E}^{(t)}_{JM} \coloneqq \frac{1}{2}(\epsilon_{IKLJ}\delta^R_M + \epsilon_{IKLM}\delta^R_J) S_R{}^I \wedge e^K \wedge e^L,$$
$$\bar{b}^{(t)}_I \coloneqq \epsilon_{IJKL}(2N^K \mathrm{d}N^L - \bar{\hat{\omega}}^{KL}) \wedge \bar{e}^J + 2\epsilon_{MJKL} N^L (\iota_{\bar{E}^J} \mathrm{d}\bar{e}^K) \wedge \bar{e}^M N_I,$$

and we take the symplectic potentials

$$\Theta^{(t)}_{\mathrm{PT}} \coloneqq \frac{1}{2}\epsilon_{IJKL} e^I \wedge e^J \wedge \mathbb{d}\hat{\omega}^{KL}, \tag{5}$$

$$\bar{\theta}^{(t)}_{\mathrm{PT}} \coloneqq \epsilon_{IJKL} \bar{e}^I \wedge \bar{e}^J \wedge N^K \mathbb{d}N^L. \tag{6}$$

Notice that the boundary term $\bar{b}_I$ does not appear in the Dirichlet case as in the variational principle appears multiplied by $\mathbb{d}\bar{e}^I = 0$, see [29] for a careful discussion.

C. Space of solutions

As in the metric case, we can exactly solve some of the equations. We begin by expanding $S_{IJ} = S_{IJK} e^K$ and using the unique decomposition $S_{IJK} = \not{S}_{IJK} + \eta_{IJ} U_K$ with $\not{S}^I{}_{IK} = 0$. Plugging this into $\mathscr{E}^{(t)} = 0$ we get

$$\not{S}_{RLI} + \not{S}_{ILR} - \not{S}_{RJ}{}^J \eta_{LI} - \not{S}_{IJ}{}^J \eta_{LR} = 0,$$

and taking its trace shows that $\not{S}_{IJ}{}^J = 0$ which, in turn, implies $\not{S}_{IJK} = 0$. The general solution to $\mathscr{E}^{(t)} = 0$ is then $S_{IJK} = \eta_{IJ} U_K$, with arbitrary U_K. Similarly, from $\mathcal{E}^{(t)} = 0$ we obtain the solution $\hat{\omega}_{IJ} = \mathring{\omega}_{IJ}$. Thus

$$\mathrm{Sol}_{\mathrm{PT}}^{(t)} = \{(e^I_\alpha, \overset{\circ}{\omega}{}^{IJ}_\mu + \eta^{IJ} U_\mu)/e^I_\alpha \in \mathrm{Sol}_{\mathrm{GR}}^{(t)}, U_\mu \text{ arbitrary}\}$$

We see, again, that the tetrad sector of Palatini is equivalent to tetrad-GR and that the boundary plays no role in the connection part. As in the metric case, the Dirichlet or Neumann boundary conditions for the tetrads are incorporated in $\mathrm{Sol}_{\mathrm{GR}}^{(t)}$, which is studied in detail in [30].

D. Presymplectic form

It is easy to see that, over the space of solutions, both symplectic potentials on (5) are given by the same expressions as the symplectic potentials obtained in [30] for tetrad-GR (but each living in different spaces). In fact, using the projection $\pi_{(t)}(e, \tilde{\omega}) = e$, we have

$$\Omega^{\mathrm{PT}}_{(t)} = \pi^*_{(t)} \Omega^{\mathrm{GR}}_{(t)} \tag{7}$$

in analogy with Eq. (2). We conclude as well that the vectors of the form $(0, \mathbb{W}) \in T\mathrm{Sol}_{\mathrm{PT}}^{(t)}$ correspond to degenerate directions of $\Omega^{\mathrm{PT}}_{(t)}$.

IV. CONCLUSIONS AND COMMENTS

In this paper, we have studied Palatini gravity, both in the metric and tetrad formulations, in a manifold with boundary, and including nonmetricity and torsion. As can be seen in (2) and (7), the presymplectic structures defined on the space of solutions of metric-Palatini and tetrad-Palatini are related to those of metric-GR and tetrad-GR by the projection on the first factor. Also, the solution spaces of metric-Palatini and tetrad-Palatini are related to each other by Φ_{PT}. Finally, since we have from [30] that

$$\Omega^{\mathrm{GR}}_{(t)} = \Phi^*_{\mathrm{GR}} \Omega^{\mathrm{GR}}_{(m)} \tag{8}$$

it is immediate to obtain, using $\pi_{(m)} \circ \Phi_{\mathrm{PT}} = \Phi_{\mathrm{GR}} \circ \pi_{(t)}$, the following important relation

$$\Omega^{\mathrm{PT}}_{(t)} = \Phi^*_{\mathrm{PT}} \Omega^{\mathrm{PT}}_{(m)} \tag{9}$$

This is one of the main results of the paper: the equivalence, up to the internal gauge transformations given by the kernel of $(\Phi_{\mathrm{PT}})_*$, of metric-Palatini and tetrad-Palatini. Furthermore, Eqs. (2), (7)–(9) show the equivalence (up to the gauge transformations given by the kernel of the corresponding push-forwards) of all these four formulations of gravity.

$$\begin{array}{ccc} \left(\Omega^{\mathrm{PT}}_{(t)}, \mathrm{Sol}^{\mathrm{PT}}_{(t)}\right) & \xrightarrow{\Phi_{\mathrm{PT}}} & \left(\Omega^{\mathrm{PT}}_{(m)}, \mathrm{Sol}^{\mathrm{PT}}_{(m)}\right) \\ \pi_{(t)} \downarrow & & \downarrow \pi_{(m)} \\ \left(\Omega^{\mathrm{GR}}_{(t)}, \mathrm{Sol}^{\mathrm{GR}}_{(t)}\right) & \xrightarrow[\Phi_{\mathrm{GR}}]{} & \left(\Omega^{\mathrm{GR}}_{(m)}, \mathrm{Sol}^{\mathrm{GR}}_{(m)}\right) \end{array} \tag{10}$$

This has been proven for both Dirichlet and homogeneous Neumann boundary conditions, although the same result can be obtained in other instances as long as all four formulations are related in the way shown in the diagram (hence with the appropriate boundary Lagrangians, [30]). Notice that this may also be seen by realizing that the projection of the spaces of solutions over the metric/tetrad sector is well defined (in the sense that it is not necessary to know the second factor to get the first). On the other hand, the main result regarding the connection sector is that boundaries have no influence whatsoever (once the correct boundary Lagrangian is considered). We have also found the well-known projective symmetry in both metric and tetrad formulations, which can be expressed, respectively, in the form

$$Q^\alpha{}_{\beta\gamma} = \delta^\alpha_\gamma U_\beta, \qquad S^{IJ}_\beta = \eta^{IJ} U_\beta \tag{11}$$

for arbitrary U_β. This projective symmetry plays an important role to define observables as they must be U-independent. For instance, the g-Kretschmann curvature $\overset{\circ}{\mathrm{K}}\mathrm{res} \coloneqq \overset{\circ}{\mathrm{R}}\mathrm{iem}_{\alpha\beta\gamma\delta} \overset{\circ}{\mathrm{R}}\mathrm{iem}^{\alpha\beta\gamma\delta}$ leads to an observable for every (g, Q) as, in fact, it is independent of Q. However, the (g, Q)-Kretschmann $\tilde{\mathrm{K}}\mathrm{res} \coloneqq \tilde{\mathrm{R}}\mathrm{iem}_{\alpha\beta\gamma\delta} \tilde{\mathrm{R}}\mathrm{iem}^{\alpha\beta\gamma\delta}$ cannot provide an observable. Indeed, using (1), it is easy to show that on the space of solutions we have

$$\tilde{\mathrm{K}}\mathrm{res} = \overset{\circ}{\mathrm{K}}\mathrm{res} + (\mathrm{d}U)_{\alpha\beta} (\mathrm{d}U)^{\alpha\beta}$$

which depends on U. This is important if one plans to use these geometric objects as indicators of the presence of singularities [31].

Finally, it is relevant to mention that although we have proven the equivalence of the four theories in vacuum, the equivalence between Palatini and GR may be broken by matter fields. For instance, pregeodesics computed by using the connection are insensitive to the U-arbitrariness [32]. However, if matter is introduced, this may no longer be the case. Since we consider Palatini theories where both nonmetricity and torsion are allowed to be nonzero, matter fields can be coupled in ways that are not possible in standard GR. This may produce interesting changes in the dynamics of the gravity-matter system.

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