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ENTROPY-BASED SEGREGATION INDICES

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Abstract

Recent research has shown that two entropy-based segregation indices possess an appealing mixture of basic and subsidiary but useful properties. It would appear that the only fundamental difference between the mutual information, or *M* index, and the Entropy, Information or *H* index, is that the second is a normalized version of the first. This paper introduces another normalized index in that family, the H^* index that, contrary to what is often asserted in the literature, is the normalized entropy index that captures the notion of segregation as departures from evenness. More importantly, the paper shows that applied researchers may do better using the *M* index than using either *H* or H^* in two circumstances: (i) if they are interested in the decomposability of segregation measures for any partition of organizational units into larger clusters and of demographic groups into supergroups, and (ii) if they are interested in the invariance properties of segregation measures to changes in the marginal distributions by demographic groups and by organizational units.

Keywords: Multigroup Segregation Measurement; Axiomatic Properties; Entropy Based Indicators; Econometric Models.

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I. INTRODUCTION

Segregation measures describe differences in the distribution of two or more demographic groups (genders, racial/ethnic groups) over a set of organizational units (occupations, neighborhoods, schools). As with the measurement of other complex, multifaceted phenomena in the social sciences –such as income inequality or economic poverty– it should come as no surprise that there exists a plethora of indicators capturing different aspects of the same phenomenon.[2] In some circumstances, this multiplicity of potential measures does not cause any practical problem. In most applications, however, different indices will lead to different conclusions, making it relevant to seek criteria to discriminate between the admissible alternatives.

Recent methodological papers have emphasized the conceptual and practical properties satisfied by two entropy-based indicators of multigroup segregation known as the Information, the Entropy or the H index (Reardon and Firebaugh, 2002), and the Mutual Information or M index (Frankel and Volij, 2009a, and Mora and Ruiz-Castillo, 2009a). It should be noted that the H index is a normalization of M. Taking as reference the school segregation problem in the multigroup case, this paper makes a number of contributions of different importance to this literature.

1. Among other alternatives, the M index can be motivated as the weighted sum of "local" school segregation indices, and of "local" ethnic group segregation indices, with weights equal to the demographic importance of each school or each ethnic group, respectively. Nevertheless, a remark warns the reader about an erroneous use of such local segregation indices.

[2] Surveys include James and Taeuber (1985), Massey and Denton (1988), and Flückiger and Silber (1999).

2. Our second contribution is to point out that there are two ways to normalize the *M* index. Contrary to what is believed since Massey and Denton (1988), the *H* index captures the isolation or representative aspect of segregation. The second normalization, leading to what we call the H^* index, is the one that captures the evenness aspect of segregation in the classical sense of James and Taeuber (1985). Interestingly enough, the *M* index simultaneously captures the evenness and representative aspects of segregation.

3. In many practical situations it is important to study segregation at several levels simultaneously. For that purpose, it is convenient to use additively decomposable segregation indices, such as the entropy-based indices, that for any partition of organizational units into *clusters* or demographic groups into *supergroups* allow us to express overall segregation as the sum of a between-groups term and a within-groups term.[3] Assume, for example, that we want to assess the degree to which overall school or residential segregation is due to racial differences across cities or states of different size, or how much is due to segregation within a large supergroup consisting of all minority races in the U.S.. As pointed out in the income inequality literature, these deceptively simple questions raise a number of conceptual and methodological problems (Shorrocks, 1988, p. 435). In our third contribution, it is shown that the empirical questions usually asked in decomposability analysis receive the more unambiguous answers that are possible in a segregation context under a particular strong notion of the additively decomposability properties. According to these properties, the within-groups term is the

[3] Examples of clusters in the school segregation context are the set of public or private schools in a country, or the sets of schools in major regions, states, cities, school districts or neighborhoods. In the occupational segregation context, we can have clusters of occupations in professional categories, economic activity sectors, or two- or three-digit occupations. Of course, supergroups can only be defined in a multigroup segregation context. Examples in a school or residential context are when precisely-defined ethnic categories, like Mexican or Puerto Rican, are aggregated into a major category such as Hispanic. In an occupational context, supergroups appear when different categories of women and male workers are aggregated into people of both genders of different age and/or educational attainment.

weighted average of segregation in each cluster or supergroup with weights equal to their demographic shares.

4. In empirical contexts where it is advisable to use entropy-based segregation indices, such as in the situations pointed out in the previous paragraph, which one should we use, the H, the H^* or the M index? It turns out that, except for Frankel and Volij (2009a) in school segregation and Mora and Ruiz-Castillo (2003, 2004), and Herranz *et al.* (2005) in occupational segregation, the authors that have used an entropy-based index have preferred the H index.[4] The major contribution of this paper is to show the practical and conceptual advantages of the M index in the following two circumstances.

Firstly, the M index satisfies the strong decomposability properties. This ensures that its answers to the empirical questions usually asked in decomposability analysis are as unambiguous as is possible in a segregation context. On the other hand, the H and the H^* indices only satisfy some weaker decomposability properties. The decomposition of the organizational units into clusters according to the H index, and the decomposition of demographic groups into supergroups according to the H^* index are free from ambiguities. Unfortunately, this is not the case for the decomposition into supergroups according to the H index, as well as the decomposition into clusters according to the H^* index. Moreover, the weights in all the decompositions for the H and the H^* indices are not invariant to changes in the within-group distributions, leading to additional problems of interpretation due to the

[4] Theil and Finizza (1971) introduced the H index for the study of school segregation in the two-group case. Reardon *et al.* (2000) distinguishes between the central city and the suburbs in a study of within-cities school segregation in the multigroup case, while Miller and Quigley (1990) and Fisher (2003) on one hand, and Iceland (2002) on the other study within-cities and within-regions residential segregation. Fisher *et al.* (2004), which is the only contribution on residential segregation that develops a full multilevel approach using the H index, only reports pair-wise comparisons of racial/ethnic groups.

nature of the weights. The shortcomings of the H and H^* indices relative to the M index are illustrated both by means of numerical examples, and with school segregation data by ethnic group in the U.S. public school system between 1989 and 2005.

Secondly, a majority of empirical studies by sociologists and economists have used segregation indices that do not change if the number of people in a given demographic group is multiplied by the same positive constant (i.e., if the number of women in an occupational segregation context, or the number of blacks in a residential or school segregation context, raises by 10% throughout all occupations, locations, or schools). Similarly, at least in the literature on occupational segregation by gender in the 1980s, many authors would argue that segregation should remain constant if the only change between two situations under comparison is in the population marginal distributions by organizational units. It turns out that the three entropy-based measures M, H, and H^* violate both properties, that is, they mix up segregation changes with changes in the marginal distributions in segregation comparisons over time or across space. However, the M index admits two decompositions that isolate one term that captures segregation changes net of the impact of pure demographic factors (Mora and Ruiz-Castillo, 2009a). This paper presents the first evidence showing the advantages of using the M index rather than the H and H^* indices to deal with these issues by means of numerical examples, and in the context of inter-temporal changes of school segregation in the U.S. public school sector between 1989 and 2005.

In this scenario, the only advantage the H and H^* measures can claim is normalization. However, this is a subsidiary property that other authors have shown not only that, like any other axiom, its intuitive desirability is arguable, but also that it leads to the violation of the extremely convenient strong decomposability properties already mentioned (Clotfelter, 1979, and Frankel and Volij, 2009b).

The rest of this paper is organized into six Sections. Section II introduces the notation, presents the three entropy-based indices, and motivates two initial remarks. Sections III establishes that the empirical questions usually asked in decomposition analysis are free of ambiguities under the strong decomposability properties that are only satisfied by the M index. Section IV disentangles the different problems of interpretation that plague the weak decomposability properties satisfied by the H and the H^* indices. Sections V and VI discuss the invariance properties, and the normalization issue, while Section VII concludes.

II. ENTROPY-BASED INDICES

II.1. Notation

It would be useful to refer to a specific segregation problem. The case discussed throughout the paper is the school segregation problem. Assume a city X consisting of N schools, indexed by $n = 1,\ldots, N$. Each student belongs to any of G racial groups, indexed by $g = 1,\ldots, G$. However, given the racial diversity existing in many countries, this paper studies the multigroup case where $G \geq 2$. The data available can be organized into the following G x N matrix:

$$X = t_{gn} = \begin{bmatrix} t_{11} & \cdots & t_{1N} \\ \vdots & \ddots & \vdots \\ t_{G1} & \cdots & t_{GN} \end{bmatrix}$$

where t_{gn} is the number of individuals of racial group g attending school n, so that $t = \sum_{n=1}^{N} \sum_{g=1}^{G} t_{gn}$ is the total student population.

The information contained in the joint absolute frequencies of racial groups and schools,

t_{gn}, is usually summarized by means of numerical indices of segregation. Let $\Xi(G, N)$ be the set of all cities with G groups and N schools. A segregation index S is a real valued function defined in $\Xi(G, N)$, where $S(X)$ provides the extent of school segregation for any city $X \in \Xi(G, N)$. Let $p_{gn} = t_{gn}/t$, and denote by $P_{gn} = \left(p_{gn}\right)_{g=1,n=1}^{G,N}$ the joint distribution of racial groups and neighborhoods in a city $X \in \Xi(G, N)$. In the following, the discussion will be restricted to indices that capture a *relative* view of segregation in which all that matters is the joint distribution, i.e. indices which admit a representation as a function of P_{gn}.[5]

This paper considers two notions of segregation. Under the first one, referred to as "evenness", segregation is viewed as the tendency of racial groups to have different distributions across schools.[6] In contrast, the notion of "representativeness" asks to what extent schools have different racial compositions from the population as a whole.[7] As can be seen in city X, where the rows are racial groups and the columns are schools, evenness and representativeness are dual concepts: deviations from evenness (representativeness) correspond to differences in the row (column) percentages. The following observation indicates how close these two views are to each other.

[5] This property, satisfied by most segregation indices, is referred to as *Size Invariance* in James and Taeuber (1985) and as *Weak Scale Invariance* in Frankel and Volij (2009a). For a study that focuses on translation invariant segregation indices that represent an absolute view of segregation, see Chakravarty and Silber (1992).

[6] It is generally agreed that residential and school segregation are multifaceted concepts whose measurement may require a battery of indices, one for each facet. In the context of residential segregation, Massey and Denton (1988) distinguish five notions or dimensions, of which evenness is the one that agrees with the classic definition of James and Taeuber (1985).

[7] Frankel and Volij (2009b) view representativeness as the multigroup generalization of the notion of "isolation", the second dimension proposed by Massey and Denton (1988) in the two-group case. Racially isolated schools are, by definition, not representative of the population. But unlike isolation, in the multigroup case representativeness is not based on the exposure of one specific group to another. The remaining three dimensions –concentration, centralization, and clustering– require detailed geographic information and would not affect the measurement of, say, occupational segregation by gender or school segregation by race . Like the most commonly used measures of segregation, the entropy-based indices studied in this paper are "aspatial" measures that do not adequately account for the spatial relationships among geographical locations. See Reardon and O'Sullivan (2004) for a discussion of these issues.

Remark 1. If a segregation index S that captures the notion of evenness when applied to city $X \in \Xi(G, N)$ is applied to the city $X' \in \Xi(N, G)$, where the role of schools and racial groups are reversed so that $t_{gn} = t'_{ng}$ for all g and n, then what will be called the *reciprocal* index S^* applied to X' captures equally well the notion of representativeness (and vice versa).

In general, $S(X)$ and $S^*(X')$ will provide a different segregation value for the same data. When this is not the case, that is, when $S(X)$ is equal to $S^*(X')$ the segregation index under consideration is said to be *transpose-invariant*.

Before we present the entropy-based indices of segregation, the concept of entropy of a distribution must be introduced. Consider a discrete random variable x that takes Q probability values, indexed by $q = 1,\ldots, Q$. Let p_q be the probability of the qth value with $p_q \geq 0$ and $\sum_{q=1}^{Q} p_q = 1$. For instance, if x is the ethnic group of a randomly selected student, then p_q is the proportion of students in the city who are in the qth group. The *entropy* of the Q values of variable x is the real value function defined as

$$E\ P\ = -\sum_{q=1}^{Q} p_q \log(p_q) = \sum_{q=1}^{Q} p_q \log\left(\frac{1}{p_q}\right)$$

with $0 \log(1/0) = 0$.[8] Heuristically, the information brought about by observing the actual value of x is the opposite of the logarithm of its likelihood, $-\log(p_q) = \log(1/p_q)$: the observation of an unlikely value brings about a large amount of information once observed. Therefore, the entropy can be considered a measure of the expected information for the value of variable x brought about by an observation. On the other hand, it is straightforward to show

[8] The base of the logarithm is irrelevant, providing essentially a unit of measure. In this paper the natural logarithm will be used.

that the entropy is bounded, so that it reaches its maximum value at the discrete uniform distribution U_Q, whereby all values are equally likely to be observed, and attains its minimum value in any of the Q degenerate distributions. Since the entropy captures the degree of uniformity in the probabilities of each possible event described by x, it can be also interpreted as a measure of uncertainty or diversity of random variable x.

II. 2. Segregation as Departures from Representativeness

The M index is defined as follows. Suppose that a student is drawn randomly from the city, so that the uncertainty about her race is measured by the entropy of the city's ethnic distribution, $E(P_g)$, where $P_g = (p_g)$ and $p_g = \sum_{n=1}^{N} p_{gn}$. Suppose that, in addition, we are informed about the school the student attends. The uncertainty about her race is now measured by the entropy of her school's ethnic distribution, $E(P_{g|n})$, where $P_{g|n} = (p_{g|n})$ and $p_{g|n} = p_{gn} / \sum_{g=1}^{G} p_{gn}$. If the schools in the city are all segregated, then the latter entropy will tend to be lower because the student's school conveys some information about her race. The M index equals this change in entropy, $E(P_{g|n}) - E(P_g)$, averaged over the students in the city:

$$M = \sum_{n=1}^{N} p_n \left[E(P_g) - E(P_{g|n}) \right]. \tag{1}$$

In information theory, expression (1) is the expected information of the message that transforms the marginal distribution of groups in the city, P_g, to the conditional distribution of racial groups in school n, $P_{g|n}$. Since $E(P_{g|n}) - E(P_g)$ measures the extent to which the racial composition in school n differs from the one for the city as a whole, it can be interpreted as a local measure of discrepancy in racial shares or a *local index of segregation* in school n when segregation focuses on representativeness.[9] However, the following point should be well understood.

Remark 2. Local indices of segregation $E(P_{g|n}) - E(P_g)$ are not independent from each other. First, an independent change in the racial mix in one school (through the addition or removal of one student) necessarily affects the racial composition in the city, and hence the local measure of discrepancy in racial shares in the remaining schools. Second, a change in the racial composition of a school maintaining the total number of students of each race in the city as a whole, necessarily affects the local measure of discrepancy in some other school.

Therefore, while equation (1) may seem to permit the decomposition of overall segregation at the city level in N components, it is meaningless to talk of a single school's contribution to overall city segregation: segregation as deviations from representativeness arises from the comparison of the racial composition in the N schools –not by the racial characteristics of a school in isolation.

It can be shown that $M \in [0, \log G]$. In particular, M takes its minimum value whenever the racial entropy in each school coincides with the racial entropy at the city level, $E(P_{g|n}) = E(P_g)$, $n = 1,\ldots,N$, while it reaches its maximum value when the racial distribution at city level is the discrete uniform distribution U_G and there is no ethnic mix within schools. In other words, the notion of complete segregation for this measure demands two conditions: there must be no racial mix within organizational units, and races must be uniformly

[9] Entropy-based and other local segregation indices are axiomatically characterized in Alonso-Villar and Del Río (2010).

distributed at city level. For any given racial marginal distribution P_g, M attains its maximum at the city's racial entropy, $E(P_g)$. This fact suggests normalizing M by $E(P_g)$:

$$H \equiv \frac{M}{E(P_g)} = \sum_{n=1}^{N} p_n \left(\frac{E(P_g) - E(P_{g|n})}{E(P_g)} \right). \qquad (2)$$

The H index, referred to as the Entropy or Information index, first appears in Theil and Finizza (1971) and Theil (1972) in the context of racial segregation in a set of schools belonging to a given school district. Intuitively, it captures the proportion of the racial mix uniformity in the city that is not due to racial mix uniformity at school level. Note that, in contrast to M, it can only take values within the unit interval (regardless of the logarithmic base). More importantly, it reaches the unit whenever there is no racial mix within schools. On the other hand, equation (2) implies that, contrary to some previous claims in the literature, the entropy index H is a segregation index that measures departures from representativeness.[10]

II. 3. Segregation as Departures from Evenness

Note that $p_{g|n} p_n = p_{n|g} p_g$ so that $\log(p_g) - \log(p_{g|n}) = \log(p_n) - \log(p_{n|g})$: the information obtained about race from learning about the school the student attends equals the information gained about the school the student attends when learning about her race. Hence, the M index also equals the reduction in uncertainty about a students' school that comes from learning her race:

$$M = \sum_{g=1}^{G} p_g \left[E(P_n) - E(P_{n|g}) \right]. \qquad (3)$$

[10] We believe that the misunderstanding starts with Massey and Denton (1988, p. 304). For the data they analyzed, they found that the Entropy index shared a common latent factor, which they interpreted as an evenness factor, with some evenness measures of segregation.

In information theory, expression (3) can be interpreted as the expected message that transforms the set of proportions P_n to the set of proportions $P_{n|g}$. Since the term $E(P_n)-E(P_{n|g})$ measures the extent to which the distribution of students in group *g* across schools differs from the school size distribution for the population as a whole, it can be interpreted as a *local index of segregation in ethnic group* g when segregation is taken to mean deviations from evenness. Of course, a remark similar to Remark 2 applies here as well. In the words of Reardon and Firebaugh (2002), "segregation is defined by the *relationships* among the groups' distributions across organizational units –not by the distribution across units of each group in isolation".

Equations (1) and (3) show that the *M* index is transpose invariant and captures the criteria of evenness and representativeness in a symmetric fashion. As a function of the school distributions by gender, *M* reaches its minimum value, 0, whenever the school entropy is the same for all racial groups, $E(P_{n|g}) = E(P_n)$, $g = 1,\ldots,G$, while it reaches its maximum value, log *N*, when the school distribution at the city level is the discrete uniform distribution U_N but each racial group attends a disjoint set of schools (so that there is no ethnic mix within schools). Thus, the notion of complete segregation as departure from evenness for *M* also demands two conditions: in addition to requiring no racial mix within organizational units, schools must be uniformly distributed at the city level. For any given school distribution P_n, M attains its maximum at the schools entropy at the city level, $E(P_n)$. This fact suggests normalizing *M* by $E(P_n)$:

$$H^* \equiv \frac{M}{E(P_n)} = \sum_{g=1}^{G} p_g \left(\frac{E(P_n) - E(P_{n|g})}{E(P_n)} \right). \tag{4}$$

The H^* index has not been defined previously. Intuitively, it captures the proportion of the school distribution uniformity in the city that is not due to school-share uniformity within racial groups. It can only take values within the unit interval, and it reaches the unity whenever there is no racial mix within schools. Equation (4) implies that H^* is a segregation index that measures departures from evenness.

In terms of the definition introduced in Remark 1, H^* is the reciprocal index of *H*. From equations (2) and (4) we obtain that:

$$H = \frac{E\ \ P_g}{E\ \ P_n} H^*.$$

Clearly, since the two indices are different whenever $E\ \ P_n \neq E\ \ P_g$, neither of them is transpose-invariant.

III. STRONG DECOMPOSABILITY PROPERTIES

III.1. Strong School Decomposability

In many research situations it is useful to partition organizational units into clusters of different size. Consider a partition of the *N* schools into $K < N$ school districts indexed by k = 1,..., *K*. Let X^k be the set of schools which belong to district *k*, and N_k be the number of schools in X^k with $\sum_{k=1}^{K} N^k = N$. The data available in X^k can be organized into the following *G* x N^k matrix:

$$X^k = t_{gn^k} = \begin{bmatrix} t_{11^k} & \cdots & t_{1N^k} \\ \vdots & \ddots & \vdots \\ t_{G1^k} & \cdots & t_{GN^k} \end{bmatrix}$$

where t_{gn^k} denotes the number of individuals of racial group *g* attending school n^k in district *k*.

School and race frequencies at city level simply result from horizontal grouping of the school and race frequencies from all K districts, $X = \left[X^1 ... X^k ... X^K\right]$. Assume now that all schools in district k have the same racial composition as the district as a whole, i.e. let $\tilde{X}^k$ refer to the district such that $p_{g|n^k} = p_{g|k}$ for all n^k and all g, or the district in which the N^k original schools have been combined into a single school with conditional racial distribution $P_{g|k} = p_{g|k}$.[11] Then $S(\tilde{X}^k) = 0$ for every $k = 1,..., K$, according to any sensible segregation index S. Would this mean that city segregation should be equal to zero? As long as the racial composition of at least two districts differ from each other, it is to be expected that overall city segregation should be positive and equal to "between-districts" segregation, that is $S(X) = S(\tilde{X}^1,...,\tilde{X}^K)$ where $S(\tilde{X}^1,...,\tilde{X}^K) \equiv S\left[\tilde{X}^1 \quad \ldots \quad \tilde{X}^K\right]$.

These considerations motivate a decomposability property for a segregation index according to which, for any partition of the N schools into $K < N$ clusters, overall segregation can be expressed as the sum of two terms, one that captures between-groups segregation, and one that captures within-groups segregation and is equal to the weighted average of segregation levels within each of the clusters, with weights independent of the level of segregation within them. Generally, it would be convenient to have the weights adding up to unity. Moreover, it is natural to require that the weights coincide with the demographic importance of each cluster. Thus, we have

Definition 1. A school segregation index S is said to be *strongly school decomposable*, ***D1***, if and only if for any partition of the set of N schools into $K < N$ clusters, so that

[11] Alternatively, suppose that all racial groups in a district are equally distributed over the district schools.

$X = \left[X^1 ... X^k ... X^K\right] \in \Xi(N,G)$, $X^k \in \Xi(G, N_k)$, $k = 1,..., K$, and $\sum_{k=1}^{K} N_k = N$, overall segregation, *S(X)*, can be written as

$$S(X) = S(\tilde{X}^1,...,\tilde{X}^K) + \sum_{k=1}^{K} p_k S(X^k), \tag{5}$$

where $\tilde{X}^k$ refers to the cluster in which $p_{g|n^k} = p_{g|k}$ for all n^k and *g*, and p_k is the proportion of students in cluster *k*, $p_k = (1/t) \sum_{n^k \in X^k} \sum_{g=1}^{G} t_{gn^k}$.

For any partition of schools into clusters we have to make sure that the following three magnitudes are well defined: (i) the contribution to overall segregation of any individual cluster; (ii) the part of overall segregation accounted for by segregation within all clusters, and (iii) how much segregation can be attributed to racial differences across clusters of different size.

In the first place, note that if we are merely interested in ranking clusters' segregation levels the decomposability requirement is quite inessential. However, if the analysis involves comparisons between cluster and overall levels, then decomposability can be very useful indeed. As pointed out in the field of income inequality, a problem arises in the different interpretations that can be placed in statements like "*x* percent of overall segregation is attributed to cluster *k*" (see, *inter alia*, Shorrocks 1980, 1984, 1988). Fortunately, definition ***D1*** implies a satisfactory way of assigning segregation contributions to the clusters. For, when equation (5) holds for any partition of *N* schools into *K* clusters, it seems natural to define the contribution to overall segregation of cluster *k* by:

$$C_k = p_k S(X^k). \tag{6}$$

It is easy to check that this definition of C_k is consistent with the other two obvious interpretations of the sentence "contribution to segregation of cluster *k*". First, consider the situation in which the original frequencies of students across races and schools in the city is replaced by one in which all schools in cluster *k* are incorporated into a single school. Since in this case $S(\tilde{X}^k) = 0$, then from equation (5) it is immediate to see that

$$C_k = S(X) - S(X^1,...,X^{k-1},\tilde{X}^k,X^{k+1},...,X^K),$$

i.e. the contribution C_k can also be interpreted as the amount by which overall segregation falls if the segregation within cluster *k* is eliminated. Second, consider the situation by which the original joint frequencies are replaced by one in which all clusters except *k* become single school clusters. Since in this situation $S(\tilde{X}^j) = 0$, for all $j \neq k$, it follows that

$$C_k = S(\tilde{X}^1,...,\tilde{X}^{k-1},X^k,\tilde{X}^{k+1},...,\tilde{X}^K) - S(\tilde{X}^1,...,\tilde{X}^K),$$

i.e. C_k can also be interpreted as the amount by which overall segregation increases if segregation within cluster *k* is introduced starting from the position of zero segregation within each cluster. Therefore, under ***D1*** it is possible to provide the same answer to different interpretations of what is meant by the contribution of each cluster to overall segregation. Consequently, the problem of unambiguously comparing individual clusters' contributions is solved. For example, the ratio $S(X^k)/S(X)$ is greater, equal or smaller than one whenever cluster *k*'s contribution to the overall segregation level, $C_k / S(X)$, is greater than, equal to, or smaller than its demographic importance given by p_k.

In the second place, we must examine the contribution made to overall segregation by all clusters taken together, *C*. This question admits two sensible interpretations. First, a natural response is to compute the reduction in overall segregation that would arise if the segregation within all clusters were eliminated. In the partition into K clusters C will be:

$$C = S(X) - S(\tilde{X}^1, ..., \tilde{X}^K).$$

A second interpretation would consist of the sum of the individual contributions defined in expression (6), that is,

$$\sum_{k=1}^{K} C_k \equiv \sum_{k=1}^{K} p_k S(X^k).$$

It is immediate to see that for any segregation measure S satisfying ***D1***, $C = \sum_{k=1}^{K} C_k$ so that both interpretations provide the same answer.

Finally, consider the possibility of partitioning the set of schools in a country into clusters of different size, say regions, cities, or school districts. The empirical question to be addressed is "How much segregation can be attributed to racial differences across regions as opposed to other geographical levels." This may be interpreted as meaning: (i) by how much segregation would fall if racial differences across clusters were the only source of school segregation, or (ii) by how much segregation would fall if racial differences at the cluster level were eliminated. Interpretation (i) suggests a comparison of overall segregation with the amount that would arise if segregation within each of K clusters were made equal to zero but racial differences across districts remained the same. As was seen before, for measures satisfying ***D1*** this would eliminate the total within-groups term and leave only the between-groups contribution so that $S(X) = S(\tilde{X}^1, ..., \tilde{X}^K)$. Interpretation (ii) suggests a comparison of overall segregation with the segregation level that would result if all clusters had the same racial composition, equal to the one for the nation as a whole, but the segregation within each cluster remained unchanged.

Unfortunately, in contrast to the situation for relative measures of income inequality, this conceptual experiment is not possible for measures of segregation, a difficulty that deserves an explanation.

For any partition of an income distribution, any decomposable inequality index allows expressing overall income inequality as the sum of a between- and a within-groups term, where the between-groups term is the inequality of the distribution where each individual is assigned the mean income of the subgroup to which she belongs. In this situation, starting from an income distribution x and a partition of the population into sub-groups, there is no difficulty in constructing a new income distribution y satisfying two conditions: (a) the mean income of any subgroup is equal to the mean income for the entire population, so that the between-groups inequality of distribution is equal to zero, and (b) income inequality within each subgroup is preserved. Then it is easy to see that the difference between income inequality in the initial situation, say $I(x) = B(x) + W(x)$, and income inequality in the second situation, $I(y) = B(y) + W(y) = 0 + W(x)$, is equal to the between-groups term:

$$I(x) - I(y) = B(x) + W(x) - W(x) = B(x).$$

That is, according to interpretation (ii), between-groups income inequality is the amount by which overall income inequality is reduced when the differences between subgroup income means are eliminated by making them equal to the population income mean.[12]

The corresponding conceptual exercise in the segregation case is logically impossible. Starting from $X = \left[X^1 ... X^k ... X^K\right]$, let us attempt to construct another city Y satisfying two

[12] As a matter of fact, the answers to interpretations (i) and (ii) coincide and are equal to the between-groups term only when the weights in the within-groups term do not depend on the subgroup means. This is only the case for one of the members of the entropy family of income inequality indicators: the mean logarithmic deviation (see Shorrocks, 1980).

conditions. (a) The racial composition of every cluster k in Y is the same as the one for the original population as a whole, that is, $p_{g|k} = p_g$ for all k and g, so that there is no between-groups segregation in Y. In this case, overall segregation in Y coincides with the within-groups term. (b) The level of segregation within each cluster remains as in the original city, so that the within-groups term in Y coincides with the one in X. Hence, overall segregation in Y coincides with within-groups segregation in X. If this operation were possible it is easy to see that, as in the income inequality case, the difference between overall segregation in X and in Y would be equal to the between-groups term. However, under condition (a) within-group segregation in Y results from the comparison between the racial distributions at school level with the racial distribution in the original city; but this comparison is what is involved in computing overall segregation in X. Therefore, within-groups segregation in Y is equal to overall segregation in the original city, which contradicts the fact that overall segregation in Y coincides with within-groups segregation in X. This contradiction arises because it is generally impossible in the segregation context to eliminate the between-groups segregation maintaining the existing within-groups segregation as the former affects the latter. Nevertheless, this does not preclude the investigation of the original question about which geographical level accounts for a greater percentage of overall segregation. For any segregation measure satisfying ***D1***, the size of the between-groups term at each geographical level provides a clear answer, if only in the sense of interpretation (i).

III.2. Strong Group Decomposability

In many research situations it is useful to partition demographic groups into supergroups. Consider a partition of the G groups in a city $X \in \Xi(G, N)$ into $L < G$ supergroups, indexed by l

$= 1,\ldots, L$. Let X_l be supergroup l, and G_l its cardinal with $\sum_{l=1}^{L} G_l = G$. The data available in X_l can be organized into the following G_l x N matrix:

$$X_l = t_{g_l n} = \begin{bmatrix} t_{1_l 1} & \cdots & t_{1_l N} \\ \vdots & \ddots & \vdots \\ t_{G_l 1} & \cdots & t_{G_l N} \end{bmatrix}.$$

where $t_{g_l n}$ denotes the number of individuals of racial group g_l in supergroup X_l attending school n. School and race frequencies at city level simply result from vertical grouping of the frequencies from all L supergroups, $X = \left[\tilde{X}_1^T \quad \ldots \quad \tilde{X}_L^T\right]^T$, where superscript T stands for the transpose operator.

Suppose that the G_l groups in supergroup l have the same distribution over organizational units as the supergroup as a whole, i. e. let $\tilde{X}_l$ be the supergroup in which $p_{n|g_l} = p_{n|l}$ for all g_l and n, or the supergroup in which the G_l original groups have been combined into a single group with conditional school distribution $P_{n|l} = p_{n|l}$.[13] Then $S(\tilde{X}_l) = 0$ for every $l = 1,\ldots, L$, according to any sensible segregation index S. Would this mean that city segregation should be equal to zero? As long as the spatial distribution of at least two supergroups would differ from each other, it is to be expected that overall city segregation should be positive and equal to "between-supergroups" segregation, or $S(X) = S(\tilde{X}_1,\ldots,\tilde{X}_L)$ where $S(\tilde{X}_1,\ldots,\tilde{X}_L) \equiv S\left[\tilde{X}_1^T \quad \ldots \quad \tilde{X}_L^T\right]^T$.

[13] Alternatively, suppose that all organizational units have the same racial composition within each supergroup.

This motivates a decomposability property for a segregation index according to which, for any partition of the G racial groups into supergroups, overall city segregation can be expressed as the sum of two terms, one that captures between-supergroups segregation, and another that captures within-supergroups segregation and is equal to the weighted average of segregation within each of the supergroups. Again, it is natural to require that the weights coincide with the supergroups' demographic importance. Thus, we have

Definition 2. A school segregation index S is said to be *strongly group decomposable*, ***D2***, if and only if for any partition of the G groups into $L < G$ supergroups so that $X = \left[\tilde{X}_1^T \quad \ldots \quad \tilde{X}_L^T\right]^T \in \Xi(G,N)$, $X_l \in \Xi(G_l, N)$, $l = 1, \ldots, L$, and $\sum_{l=1}^{L} G_l = G$, overall segregation, $S(X)$, can be written as

$$S(X) = S(\tilde{X}_1, \ldots, \tilde{X}_L) + \sum_{l=1}^{L} p_l S(X_l), \tag{7}$$

where $\tilde{X}_l$ refers to the supergroup in which $p_{n|g_l} = p_{n|l}$ for all g_k and n, and p_l is the proportion of students in supergroup l, $p_l = (1/t) \sum_{g_l \in X_l} \sum_{n=1}^{N} t_{g_l n}$.

This definition also implies a satisfactory way of assigning segregation contributions to the supergroups. For, when equation (7) holds, the definition $C_l = p_l\, S(X_l)$, is consistent with all the obvious interpretations of the concept "contribution to segregation by supergroup l": the amount by which overall segregation falls if the segregation within supergroup l is eliminated, or the amount by which overall segregation increases if segregation within supergroup l is introduced starting from the position of zero segregation within each supergroup. Similarly to the case of the partition of schools into clusters, it is impossible to eliminate the between-supergroups segregation maintaining the existing within-supergroups segregation as the latter is affected by the former.

III.3. Decomposability Properties of the *M* Index

It is easy to show that the *M* index satisfies both ***D1*** and ***D2*** in the multigroup case. First, Equation (5) takes the form:

$$M = M^B + \sum_{k=1}^{K} p_k M_k^W \tag{8}$$

where

$$M^B = \sum_{k=1}^{K} p_k \left[E(P_g) - E(P_{g|k}) \right] = \sum_{g=1}^{G} p_g \left[E(P_k) - E(P_{k|g}) \right]$$

is the between-groups term that captures what we will refer to as cluster segregation, and

$$M_k^W = \sum_{n \in X^k}^{N_k} p_n \left[E(P_{g|k}) - E(P_{g|n \in X^k}) \right] = \sum_{g=1}^{G} p_{g|k} \left[E(P_{n|n \in X^k}) - E(P_{n|g, n \in X^k}) \right]$$

captures school segregation within cluster *k*. Given that the *M* index satisfies ***D1***, the contribution $CM_k^W = p_k M_k^W$ is consistent with all the obvious interpretations of the concept "contribution to segregation by cluster *k*". Similarly, *M* admits the following decomposition

$$M = M_B + \sum_{l=1}^{L} p_l M_l^W \tag{9}$$

where

$$M_B = \sum_{n=1}^{N} p_n \left[E(P_l) - E(P_{l|n}) \right] = \sum_{l=1}^{L} p_l \left[E(P_n) - E(P_{n|l}) \right]$$

is the between-groups term that captures school segregation by supergroup, and

$$M_l^W = \sum_{n=1}^{N} p_n \left[E(P_{g|g \in X_l}) - E(P_{g|n, g \in X_l}) \right] = \sum_{g \in X_l} p_{g|g \in X_l} \left[E(P_{n|g \in X_l}) - E(P_{n|g, g \in X_l}) \right]$$

captures school segregation within supergroup l. Given that the M index satisfies ***D2***, the contribution $CM_l^W = p_l M_l^W$ is consistent with all the obvious interpretations of the concept "contribution to segregation by supergroup l".

IV. WEAK DECOMPOSABILITY PROPERTIES

IV.1. The Properties

Although the H and H^* indices violate ***D1*** and ***D2***, it can be seen that they satisfy some weaker decomposability properties. Firstly, consider any partition of the N schools into $K < N$ clusters, and recall that H can be computed by dividing the M index by the racial entropy, $E \;\; P_g$. On the one hand, starting from the representativeness representation of decomposition (8) we have:

$$H = \frac{M^B}{E \;\; P_g} + \sum_{k=1}^{K} p_k \frac{M_k^W}{E \;\; P_g}.$$

Multiplying and dividing each summand of the second term by the within-group's racial entropy, $E \;\; P_{g|k}$, and using the relation between the un-normalized and the normalized indexes, we have:

$$H = H^B + \sum_{k=1}^{K} p_k \frac{E \;\; P_{g|k}}{E \;\; P_g} H_k^W, \qquad (10)$$

where H^B captures cluster segregation, and H_k^W captures school segregation within cluster k. On the other hand, starting from the evenness representation of equation (8), for the H^* index we have

$$H^{*} = \frac{M^{B}}{E \;\; P_{n}} + \sum_{k=1}^{K} p_{k} \frac{M_{k}^{W}}{E \;\; P_{n}}.$$

Multiplying and dividing the between-groups term by $E \;\; P_{k}$ and each summand of the second term by $E \;\; P_{n|k}$, we have:

$$H^{*} = \frac{E \;\; P_{k}}{E \;\; P_{n}} H^{*B} + \sum_{k=1}^{K} p_{k} \frac{E \;\; P_{n|k}}{E \;\; P_{n}} H_{k}^{*W} \qquad (11)$$

where H^{*B} captures cluster segregation, and H_{k}^{*W} captures school segregation within cluster *k*.

Secondly, consider any partition of the *G* groups into $L < G$ supergroups. Starting from the representativeness representation of equation (9), for the *H* index we have

$$H = \frac{M_{B}}{E \;\; P_{g}} + \sum_{l=1}^{L} p_{l} \frac{M_{l}^{W}}{E \;\; P_{g}}.$$

Multiplying and dividing the between-groups term by $E \;\; P_{l}$ and each summand of the second term by $E \;\; P_{g|l}$, we have:

$$H = \frac{E \;\; P_{l}}{E \;\; P_{g}} H_{B} + \sum_{l=1}^{L} p_{l} \frac{E \;\; P_{g|l}}{E \;\; P_{g}} H_{l}^{W} \qquad (12)$$

where H_{B} captures school segregation by supergroup, and H_{l}^{W} captures school segregation within supergroup *l*. [14] Finally, starting from the evenness representation of decomposition (9), we have:

$$H^{*} = \frac{M_{B}}{E \;\; P_{n}} + \sum_{l=1}^{L} p_{l} \frac{M_{l}^{W}}{E \;\; P_{n}}.$$

Multiplying and dividing each summand of the second term by $E \;\; P_{n|l}$, we have:

[14] Equation (12) figures prominently in Reardon *et al.* (2000) -see their expression (4).

$$H^* = H_B^* + \sum_{l=1}^{L} p_l \frac{E(P_{n|l})}{E(P_n)} H_l^{*W}, \qquad (13)$$

where H_B^* captures school segregation by supergroup, and H_l^{*W} captures school segregation within supergroup l.

IV.2. Ambiguities in the Interpretation of the Contributions to Segregation

It should be noted at the outset that the contributions of the between-groups and within-groups terms expressed as a percentage of the H and the H^* indices in expressions (10)-(11), and (12)-(13) pose no problem because they coincide with those same relative contributions for the M index in expressions (8) and (9), respectively. Thus, for example, in the case of decomposition (10) we have:

$$\begin{aligned}\frac{H^B}{H} + \sum_{k=1}^{K} \left(p_k \frac{E(P_{g|k})}{E(P_g)} \right) \left(\frac{H_k^W}{H} \right) &= \frac{M^B / E(P_g)}{M / E(P_g)} + \sum_{k=1}^{K} \left(p_k \frac{E(P_{g|k})}{E(P_g)} \right) \left(\frac{M_k^W / E(P_{g|k})}{M / E(P_g)} \right). \\ &= \frac{M^B}{M} + \sum_{k=1}^{K} p_k \frac{M_k^W}{M}.\end{aligned}$$

Similarly, for decomposition (11) we have:

$$\begin{aligned}\left(\frac{E(P_k)}{E(P_n)} \right) \left(\frac{H^{*B}}{H^*} \right) + \sum_{k=1}^{K} \left(p_k \frac{E(P_{n|k})}{E(P_n)} \right) \left(\frac{H_k^{*W}}{H^*} \right) &= \left(\frac{E(P_k)}{E(P_n)} \right) \left(\frac{M^B / E(P_k)}{M / E(P_n)} \right) + \sum_{k=1}^{K} \left(p_k \frac{E(P_{n|k})}{E(P_n)} \right) \left(\frac{M_k^W / E(P_{n|k})}{M / E(P_n)} \right). \\ &= \frac{M^B}{M} + \sum_{k=1}^{K} p_k \frac{M_k^W}{M}.\end{aligned}$$

On the other hand, it is important to recognize that the terms in decompositions (10) and (13) admit the same interpretations as those terms in any ***D1*** and ***D2*** index. Firstly, define cluster k's contribution to overall segregation as $CH_k^W = p_k \frac{E(P_{g|k})}{E(P_g)} H_k^W$. It is easy to show that CH_k^W can be interpreted both as the amount by which overall segregation falls if the segregation within cluster *k* is eliminated, and the amount by which overall segregation increases if segregation within cluster *k* is introduced starting from the position of zero segregation within each cluster. Likewise, define the contribution of all clusters to segregation as $CH^W = \sum_{k=1}^{K} CH_k^W$. It turns out that CH^W equals the reduction in segregation that would arise if the segregation within all clusters were eliminated. Finally, the interpretation of the between-groups term in decomposition (10), H^B, is subject to the same conceptual limitation pointed out in the previous Section III.1 in relation to the decomposition of any ***D1*** index. Namely, H^B can be interpreted as the level of segregation if racial differences across clusters were the only source of school segregation so that $H_k^W = 0$ for all *k* = 1,..., *K*. However, it cannot be interpreted as the decrease in segregation if racial differences at the cluster level were eliminated. For reasons of brevity, the properties of decomposition (13) are not discussed in detail. Nevertheless, similar arguments to those provided for decomposition (10) can be used to see that the terms in decomposition (13) can be interpreted as those in the decomposition of any ***D2*** index for any partition of ethnic groups into supergroups.

However, as indicated in the Introduction, decompositions (11) and (12) present serious problems of interpretation. The next example illustrates that equation (12) does not provide the *H* index with a decomposition that admits the same interpretation as that of any ***D2*** index. It first shows that supergroup *l*'s contribution to overall segregation, $CH_l^W = p_l \dfrac{E(P_{g|l})}{E(P_g)} H_l^W$, cannot generally be interpreted as the amount by which overall segregation falls if the segregation within supergroup *l* is eliminated. The reason is that in this case the overall racial entropy $E\ P_g$ will usually change, and this may induce changes in the weights of the contributions by other supergroups. The example also shows that the term CH_l^W cannot be always interpreted as the amount by which overall segregation increases if segregation within supergroup l is introduced starting from the position of zero segregation within each racial supergroup. Finally, it is illustrated that $CH_B = \frac{E(P_l)}{E(P_g)} H_B$ cannot be interpreted as the level of segregation if differences in the supergroup distributions across schools were the only source of school segregation.

<u>Example 1</u>: Consider two cities, *X* and *Y*, with students from three racial groups, *white*, *Asian*, and *black*, and two schools, s1 and s2. The joint frequencies of students across schools and racial groups are summarized in the following two matrices:

$$\mathrm{X} = \underset{\substack{\text{s1} \quad \text{s2} \\ \text{Schools}}}{\begin{bmatrix} 7 & 38 \\ 3 & 2 \\ 20 & 5 \end{bmatrix}} \qquad \mathrm{Y} = \underset{\substack{\text{s1} \quad \text{s2} \\ \text{Schools}}}{\begin{bmatrix} 7 & 28 \\ 3 & 12 \\ 20 & 5 \end{bmatrix}} \qquad \overset{\text{Ethnic groups}}{\begin{bmatrix} \text{white} \\ \text{Asian} \\ \text{black} \end{bmatrix}}$$

Suppose that we group together white and Asian students, referring to the resulting supergroup as *wa*. To begin with, according to index *H* school segregation within supergroup *wa* is zero in city *Y*, but positive in *X*, $H_{wa}^W(X) = 10.28.$[15] However, the contribution of within-supergroups segregation in city *X*, $CH_{wa}^W(X) = 3.45,$ is not equal to the fall in overall segregation when eliminating segregation within supergroup *wa*, i.e. moving from city *X* to city *Y*, $H(Y) - H(X) = -7.14.$ The reason is that the overall racial entropy has increased: $E\ \ P_g(Y)\ = 104.38$ vs. $E\ \ P_g(X)\ = 85.32$. Secondly, it is immediate to note that $CH_{wa}^W(X) = 3.45$

[15] All entropy and index calculations reported hereafter are computed using natural logarithms and are multiplied by 100.

does not equal the amount by which overall segregation increases if segregation within supergroup l is introduced starting from the position of zero segregation within each racial supergroup, i.e. moving from city Y to city X, $H(X)-H(Y)=7.14$. Finally, the term $CH_B(X)=20.23$ does not equal the level of segregation if differences in the supergroup distributions across schools were the only source of school segregation, $H(Y)=16.54$.[16]

IV.3. Further Problems of Interpretation Due to the Nature of the Weights

In the motivation of strong school and group decomposability, it has been noted that it would be desirable that the weights in any decomposition should be invariant to changes in the within-groups distributions. Clearly, all decompositions (10) to (13) violate this property, leading to several problems of interpretation. Consider decomposition (12) for H. The nature of the weights $\frac{E\ P_l}{E\ P_g}$ and $p_l \frac{E\ P_{g|l}}{E\ P_g}$ leads to the following two problems. Firstly, we may have two cities with the same H_B but different contribution $CH_B = \frac{E(P_l)}{E(P_g)} H_B$ to overall segregation due to differences in the entropy ratio $\frac{E\ P_l}{E\ P_g}$. Secondly, for a given joint distribution of supergroups and schools, P_{ln}, the weights $p_l \frac{E\ P_{g|l}}{E\ P_g}$ generally change in response to exogenous changes in the joint distribution of groups and schools within supergroups. Thus, although supergroup demographic shares, p_l, remain constant, the overall racial entropy at group level, $E\ P_g$, or the racial entropy at group level in supergroup l, $E\ P_{g|l}$, may change.

[16] Analogous to the situation considered in remark 3, the contributions of the between- and the within-supergroup terms expressed as a percentage of the H (H^*) indices in expression (11) (expression 12) pose no interpretability problem because they coincide with those same relative contributions for the M index.

Consequently, the contribution to within-groups segregation, $CH_W = \sum_{l=1}^{L} CH_l^W$, may change in a direction contrary to what the terms H_l^W would indicate. Both problems are illustrated in the following example.

Example 2: Consider two cities, *X* and *Y*, with students from four racial groups, *white*, *Asian*, *black*, *and Hispanic* and two schools, s1 and s2. The relative frequencies (in %) of students across schools and racial groups can be summarized in the following two matrices:

$$X = \begin{bmatrix} 9 & 36 \\ 3 & 2 \\ 20 & 5 \\ 20 & 5 \end{bmatrix} \qquad Y = \begin{bmatrix} 9.05 & 35.95 \\ 2.95 & 2.05 \\ 36 & 9 \\ 4 & 1 \end{bmatrix} \qquad \text{Ethnic groups} \begin{bmatrix} \text{white} \\ \text{Asian} \\ \text{black} \\ \text{Hispanic} \end{bmatrix}$$

s1 s2 (Schools) — for both X and Y

Suppose that we group together, on the one hand, white and Asian students, referring to the resulting supergroup as *wa*, and, on the other hand, black and Hispanic students, referring to the resulting supergroup as *bh*. Firstly, the joint distribution of supergroups and schools is the same in both cities *X* and *Y* and, consequently, so is the value for school segregation by supergroup, $H_B(X) = H_B(Y) = 24.03$. However the contribution of between-groups segregation to overall segregation, CH_B, is larger in *Y* than in *X* $(CH_B(Y) = 16.36$ vs. $CH_B(X) = 13.86)$ simply because the entropy ratio is larger there. Secondly, measured by H_l^W, supergroup *wa* experiences slightly more school segregation in *X* than in *Y* ($H_{wa}^W(X) = 10.28$ vs. $H_{wa}^W(Y) = 9.74$), while supergroup *bh* has no school segregation in both cities $(H_{bh}^W(X) = H_{bh}^W(Y) = 0)$. Since the difference in the shares of black and Hispanic students is much smaller in X than in Y, both the overall racial entropy and the racial entropy within supergroup *bh* are larger in X than in Y: $E\ P_g(X) = 120.23$ vs. $E\ P_g(Y) = 101.82$, and $E\ P_{g|bh}(X) = 34.66$ vs. $E\ P_{g|bh}(Y) = 9.48$. As a result, even though the joint frequency of supergroups and schools is the same for both cities, the weights $p_l \dfrac{E\ P_{g|l}}{E\ P_g}$ are so much larger in city Y –the city with less segregation within supergroup *wa*– that the contribution of within-groups segregation is also larger there: $CH_W(Y) = 1.55$ vs. $CH_W(X) = 1.39$.

Decomposition (10) for H presents analogous problems of interpretation for the within-groups term as $CH^W = \sum_{k=1}^{K} p_k \dfrac{E(P_{g|k})}{E(P_g)} H_k^W$ may change in a direction contrary to what the terms H_k^W would indicate. Also, the decompositions (11) and (13) for H^* have similar problems of interpretation. For decomposition (11), we may have two cities with the same between-groups segregation, H^{*B}, but different contributions to overall segregation due to differences in the entropy ratio $\dfrac{E\ P_k}{E\ P_n}$. Finally, the contributions to within-groups segregation, $CH^{*W} = \sum_{k=1}^{K} p_k \dfrac{E(P_{n|k})}{E(P_n)} H_k^{*W}$, and $CH_W^* = \sum_{l=1}^{L} p_l \dfrac{E(P_{n|l})}{E(P_n)} H_l^{*W}$, may change in a direction contrary to what the terms H_k^{*W} and H_l^{*W} would indicate, respectively.[17]

IV.4. Decomposability Properties in Practice: the *M* versus the *H* Index

It will be illustrative to compare how the decomposability properties of the *M* and the *H* indices fair in practice with data about the evolution of the U.S. student population enrolled in

[17] For the sake of brevity, proofs of the statements in this paragraph using illustrative examples will be only available upon request.

public schools in Core-Based Statistical Areas (CBSAs) –urban clusters of 10,000 or more inhabitants, referred to in the sequel as *cities*– during the 1989-90 and 2005-06 academic years.[18] Table 1 informs about two issues. Firstly, the evolution of ethnic diversity of the student population. Minorities (namely, Native Americans, blacks, Asians, and Hispanics) already represent 34.8% of the total population of 24.8 million in 1989. Since all of them grew more rapidly than whites during this period, they represent as much as 48.1% of the total population of 25.5 million in 2005. Secondly, the segregation levels achieved by the different entropy indices. In particular, the change in the M index during this period is ΔM = 48.90 – 43.92 = 4.98. Suppose that we group together Asian, black, Hispanic, and Native American students, referring to the resulting "minorities" supergroup as m. Consider now the evolution of segregation between whites vs. minorities and the evolution of segregation within minorities. Since only one supergroup is considered, equation (9) simplifies to $M = M_B + p_m M_m^W$, where p_m denotes the share of minorities in the student population, M_m^W is the M index within minorities, and M_B is the M index of school segregation for whites vs. all minorities together. The observed increase in overall segregation is due, first, to the increase in M_B, $\Delta M_B = 1.83$. In addition, the share of the minorities (who are highly segregated among themselves) increases substantially, $\Delta p_m = 0.13$. Thus, in spite of the fact that school segregation within minorities is decreasing, $\Delta M_m^W = -8.25$,

[18] Results pertain to those schools for which racial and ethnic information is available both in 1989 and in 2005. Given that a small proportion of schools did not report results in 1989, focusing on the schools which did probably gives a fairer comparison between the distributions observed in 1989 and in 2005 because it does not include those schools that did report in 2005 but failed to do so in 1989. However, interpretability of the results presented here is potentially compromised by the fact that some schools have been created whilst others have disappeared between 1989 and 2005. Nevertheless, results using all observations are qualitatively similar, suggesting that the selection mechanisms at work are not essential to our analysis. Results obtained using the full sample are available upon request.

the contribution of segregation within minorities to overall segregation is positive, $\Delta CM_m^W = 3.15$. Consequently, $\Delta M = 1.83 + 3.15 = 4.98$.

Table 1 around here

Given equation (2), it is seen that H decreases because the racial entropy is increasing (119.07 – 101.27 = 17.80) faster than M:

$$\Delta H = (48.90/119.07) - (43.92/101.27)$$

$$= 41.07 - 43.37 = -2.30$$

But how does H account for the trends in the minorities' partition? Note that, with only one supergroup, decomposition (12) simplifies to $H = \frac{E\ P_l}{E\ P_g} H_B + p_m \frac{E\ P_{g|m}}{E\ P_g} H_m^W$. The H index does also find a decrease in segregation within minorities, $\Delta H_m^W = -7.13$, and a very small increase in school segregation between whites and minorities, $\Delta H_B = 0.03$. In spite of the increasing importance of minorities in the student population, the within-minorities weight only slightly increases (from 0.36 to 0.42) as a combined result of the decrease in the racial entropy within minorities (from 105.40 in 1989 to 103.71 in 2005), together with the increase in the overall racial entropy (from 101.27 to 119.07). The small increase in the weight does not offset the large decrease in segregation within minorities, and, hence, the contribution of segregation within minorities to overall segregation is negative, $\Delta CH_m^W = -0.11$. Moreover, the contribution of between-groups segregation is also affected by the evolution of the ratio $\frac{E\ P_l}{E\ P_g}$. It turns out that, simply because the racial entropy is growing relatively more than the supergroup entropy between whites and minorities, most of the reported decrease in the entropy index,

$\Delta H = -\ 2.30$, stems from the decrease in the contribution of the between-groups term, $\Delta CH_B = -\ 2.19$, in spite of the reported increase in H_B.

V. INVARIANCE PROPERTIES

V. 1. The Invariance Question

Consider for a moment the special but important case of occupational segregation by gender, and assume that segregation in 1950 and 2000 are being compared in a given country. The following two questions are often asked. Firstly, should the measurement of occupational segregation be independent of the fact that female labour participation has greatly increased over time? Many people would agree that, as long as the male and female distributions over occupations remain constant, the degree of segregation should be the same in the two situations. This is known as composition invariance, or invariance 1 (*I1* hereafter). In the school segregation case with several ethnic groups, the question becomes: should segregation be invariant to changes in the ethnic composition of the population as long as the distribution of each group over the schools remains constant? Secondly, should occupational segregation be independent from the fact that agricultural and industrial occupations are much more important in 1950 than in 2000, while services occupations carry much more weight in 2000 than in 1950? Many people would agree that, as long as the gender composition of each occupation remains constant, the degree of segregation should be the same in the two situations. This is known as occupational invariance, or invariance 2 (*I2* hereafter). In the multigroup case, the question becomes: should segregation be invariant to changes in the size distribution of schools as long as the ethnic composition of each school remains constant?

As indicated in the Introduction, the three entropy-based measures *M*, *H*, and H^* violate both properties, that is, they mix up segregation changes with changes in the marginal distributions in segregation comparisons over time or across space. However, Mora and Ruiz-Castillo (2009a) argue that both invariance properties have strong implications, and provide examples to defend that there are good reasons for recognizing the demographic importance of racial groups and schools in a measure of segregation at a given moment in time. Nevertheless, it is quite clear that in segregation comparisons across time or space it becomes extremely useful to evaluate those changes in segregation that do not result from changes in the marginal distributions.

Mora and Ruiz-Castillo (2009a) present two decompositions of the *M* index in pair wise comparisons over time or across space that isolate the effects of the changes in the marginal distributions. In the first place, to identify an ***I1*** term in a decomposition of a pair wise comparison, the differences in the *M* index can be written as:

$$\Delta M = \Delta \text{Net}(\boldsymbol{I1}) + \Delta M(P_g) + \Delta E(P_n), \tag{14}$$

where $\Delta E(P_n)$ is the change in the school entropy, $\Delta M(P_g)$ isolates changes in *M* due to changes in the racial marginal distribution, P_g, while $\Delta \text{Net}(\boldsymbol{I1})$ is an ***I1*** term in the sense that it equals zero as long as $P_{n|g}$ remains constant. In the second place, to identify an ***I2*** term in a decomposition of a pair wise comparison, the differences in the *M* index can be written as:

$$\Delta M = \Delta \text{Net}(\boldsymbol{I2}) + \Delta M(P_n) + \Delta E(P_g), \tag{15}$$

where $\Delta M(P_n)$ isolates changes in *M* due to changes in P_n, $\Delta E(P_g)$ is the change in the racial entropy, while $\Delta \text{Net}(\boldsymbol{I2})$ is an ***I2*** term in the sense that it equals zero as long as $P_{g|n}$ remains constant.

Decompositions (14) and (15) are not available for the H, and H^* indexes. However, it is sometimes argued that since normalization makes complete segregation as defined in H independent of P_g, then the notion of segregation captured by H "is independent of the population's diversity."[19] Clearly, H is neither ***I1*** nor ***I2***, but to what extent does H reduce the invariance problems in M? Taking into account equation (14) and the linear approximation to changes in H, $\Delta H \approx \frac{1}{E(P_g)} \left[\Delta M - \Delta E(P_g)\right]$, it is obvious that as long as $\Delta M(P_g) \approx 0$ and $E(P_g) \approx 1$, then $\Delta H \approx \Delta \text{Net}(\boldsymbol{I2})$. However, it will be presently seen that changes in H can be a very inadequate approximation to isolate ***I2*** changes in deviations from representativeness. Firstly, by means of a numerical example it will be shown that changes in H (and also changes in H^*) may be unduly influenced by changes in P_g and changes in P_n when the racial and school entropies do not change. Secondly, in the case of the evolution of the U.S. student population enrolled in public schools, it will be seen how a large increase in the racial entropy coupled with a relatively smaller change in the school marginal distribution leads both to H greatly undervaluing the improvement in representativeness, and H^* missing the improvement in evenness.

V. 2. Changes in the Marginal Distributions Without Changes in the Entropies

The next example illustrates how neither H nor H^* correct for the lack of invariance in M if the marginal distributions of schools and races change but the entropies do not.

Example 3: Consider two cities, X and Y, with students from three racial groups, *white*, *black*, and *Hispanic*, and three schools, s1, s2, and s3. The joint absolute frequencies of students across schools and racial groups are summarized in the following two matrices:

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[19] See, inter alia, Reardon *et al.*, 2000, pp. 354.

$$\boldsymbol{X} = \begin{bmatrix} 30 & 10 & 5 \\ 5 & 15 & 5 \\ 5 & 10 & 15 \end{bmatrix} \quad \boldsymbol{Y} = \begin{bmatrix} 10 & 10 & 10 \\ 5 & 15 & 25 \\ 10 & 10 & 5 \end{bmatrix} \quad \begin{bmatrix} \textit{white} \\ \textit{black} \\ \textit{Hispanic} \end{bmatrix}$$

$$\begin{matrix} \text{s1} & \text{s2} & \text{s3} \\ & \text{Schools} & \end{matrix} \qquad \begin{matrix} \text{s1} & \text{s2} & \text{s3} \\ & \text{Schools} & \end{matrix} \qquad \text{Ethnic groups}$$

City X is predominantly white, while city Y is predominantly black. Hispanics are the second largest group in X and the smallest group in Y. However, racial entropies in both cities (multiplied by 100) are the same: $E\left(P_g(X)\right) = E\left(P_g(Y)\right) = 106.71$. On the other hand, school 1 is the largest and school 3 the smallest in city X, while the order is reversed in city Y. However, these changes in the school marginal distribution do not affect the school entropy (multiplied by 100): $E\left(P_n(X)\right) = E\left(P_n(Y)\right) = 108.05$. Moreover, both the school entropy and the racial entropy are close to 1. Consequently, changes in H and H^* are very similar to changes in M: $M(X) - M(Y) = 6.56$ vs. $H(X) - H(Y) = 6.15$ vs. $H^*(X) - H^*(Y) = 6.07$. However, according to decomposition (14), net segregation as deviations from evenness is lower in X than in Y, $\Delta\text{Net}(\boldsymbol{I1}) = -7.98$, and the change in the racial distribution increases M in X, $\Delta M(P_g) = 17.19$. Similarly, according to decomposition (15), net segregation as deviations from representativeness is lower in X than in Y, $\Delta\text{Net}(\boldsymbol{I2}) = -5.98$, and the change in the school distribution increases segregation in X, $\Delta M(P_n) = 12.54$. Hence, neither H nor H^* correct for the lack of invariance in M if the marginal distributions of schools and races change but the entropies do not.

V. 3. The Effects of an Increase in the Racial Entropy: Invariance Properties in Practice

The case of the evolution of the U.S. student population enrolled in public schools already studied in section III.2 is retaken here to evaluate whether, in practice, changes in either the H or the H^* index could be seen as reasonable approximations of ***I2*** or ***I1*** terms, respectively. In section III.2 it was reported that in the period 1989-2005 the M index increased by 4.98, the H index decreased by -2.30 because the racial entropy increased relatively more than M, while the H^* index slightly increased by 0.50 because the school entropy decreased. However, in equation (15) the change in the M index due to the change in the racial entropy is 17.80, while the change due to the change in the marginal distribution of schools is – 0.59. Therefore, the change in net segregation independent of these effects is

$$\Delta \text{Net}(\boldsymbol{I2}) = 4.98 - (-0.59) - 17.80 = -12.23.$$

The term ΔNet(***I2***) reflects changes in the schools' racial mix and, therefore, can be interpreted as changes in deviations from representativeness. Hence, the change in the normalized entropy index H greatly undervalues the improvement in representativeness.

On the other hand, in equation (13) the change in the M index due to the change in the schools entropy is – 4.53, while the change due to the change in the marginal distribution of racial groups is 10.63. Therefore, the change in net segregation independent of these effects is

$$\Delta \text{Net}(\boldsymbol{I1}) = 4.98 - (-4.53) - 10.63 = -1.11.$$

The term ΔNet(***I1***) reflects changes in the groups' conditional distributions over schools and, therefore, can be interpreted as changes in deviations from evenness. Hence, the change in the normalized entropy index H^* misses the improvement in evenness.

VI. THE NORMALIZATION ISSUE

Normalization properties concern the bounds for the range of admissible values for an index of segregation. Most researchers would identify the absence of segregation with the situation where organizational units have the same racial composition or, equivalently, where demographic groups have the same distribution across organizational units. Similarly, most researchers would accept that demographic groups are completely segregated whenever they do not mix at all within organizational units. A segregation index is said to be *normalized in the unit interval* –or possess the ***NOR*** property– if it takes value 0 whenever there is no segregation and it takes value 1 whenever it reaches complete segregation as defined above.

It is important to understand that requiring the subsidiary property ***NOR*** has larger implications than simply rescaling the measure of segregation. As has been repeatedly seen, the un-normalized and the normalized entropy-based indices do not generally give the same segregation ordering. In particular, both H and H^* rank all cities with no racial mixing within schools as equally segregated, while M assigns a higher segregation level to cities in which there is more initial uncertainty about a student's racial group. Following an example in Frankel and Volij (2009b), consider city $\boldsymbol{A}$ with three schools and three racial groups and city $\boldsymbol{B}$ with two schools and two racial groups, such that

$$\boldsymbol{A} = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix}.$$

Given each city's marginal distributions, segregation is at a maximum in both cities according to the three indexes. Both H and H^* assign to each city a segregation value of 1. However, learning a student's school (racial group) in $\boldsymbol{A}$ conveys more information about a student's race (school) than in ***B***. Consequently, segregation in ***A*** is larger than in ***B*** according to the *M* index: *M(A)* = 1.10 and *M(B)* = 0.69. Consider now a third completely segregated city ***C***:

$$C = \begin{bmatrix} 99 & 0 \\ 0 & 1 \end{bmatrix}$$

Both H and H^* assign again to ***C*** a segregation value of 1. However, since there is much less uncertainty about a student's racial group (school) in ***C*** than in either ***A*** or ***B***, segregation in ***C*** according to *M* is much smaller than before: *M(C)* = 0.06.

As was pointed out in Clotfelter (1979), a critical problem with segregation indices that satisfy ***NOR*** is that they fail to capture well changes in inter-racial contact. Compare the effect of merging the two schools in city ***C***, yielding the one-school city represented by column vector [99 1]', with the effect of merging the two schools in ***B***, yielding the one-school city represented by [50 50]'. The first merger has a very small effect on the inter-racial exposure of the average student, while the second one has a much larger effect: each student switches from a completely segregated school to one that is completely integrated. The *M* index reflects this difference, falling by 0.06 in ***C*** versus 0.69 in ***B***. In contrast, H and H^* miss the difference because the segregation value they both assign decreases by 1 in the two cases.

It can be concluded that there are conceptual reasons for not requiring subsidiary property ***NOR*** from a segregation index. Furthermore, Frankel and Volij (2009b) establish the incompatibility of ***NOR*** and decomposability properties ***D1*** and ***D2***, providing an argument in empirical studies for avoiding indexes that satisfy ***NOR***.

Finally, it should be noted that all segregation indices that are bounded above can be weakly normalized, in the sense that they can be expressed as proportions of maximum segregation, by simply dividing them by its maximum value. In particular, the *M* index reaches its maximum at the smallest value between log(G) and log(N) because, as a measure of departure from evenness, it cannot be larger than log(N), and, as a measure of departure from representativeness, it cannot be larger than log(G). Given that in most empirical applications log(G) $<$ log(N), normalizing M in this weak sense is simply equivalent to computing the logarithm in base G. The resulting measure can be interpreted as the proportion of maximum deviation from representativeness. However, this exercise is not very useful for two reasons. Firstly, the most robust feature of the index, namely the ranking it induces, is still the same and captures both departures from representativeness and evenness. Secondly, although the resulting index takes values in the unit interval it does not satisfy ***NOR***.

VII. CONCLUSIONS

This paper adopts the methodological criterion that, as in the income inequality literature, one way to select an adequate segregation measure is to study which basic and subsidiary but useful properties different indices satisfy. This is important because, as one of the leading advocates of this approach indicates, "If this search is not undertaken, there is a tendency to continue using those measures that have been popular in the past. The index is then chosen by default, or historical accident, rather than by any assessment of its merits." (Shorrocks, 1988, p. 433).[20] We have discussed three types of subsidiary properties as they apply to three entropy-based segregation indices, M, H, and H^*.

[20] Grusky and Charles (1998, p. 497) complain that this situation has indeed been prevalent in the history of research on occupational segregation by gender: "For all its faddishness, the concept of path dependency proves useful in understanding the history of sex segregation research, and not merely because the index of dissimilarity (hereafter, D) has shaped and defined the methodology of segregation analysis over the last 25 years. It is perhaps more important that D has been so dominant during this period that it undermined all independent conceptual development. Indeed, segregation scholars have effectively assumed that sex segregation is simply whatever D measures."

First, it is often convenient to have segregation measures with the subsidiary property of additive decomposability. In a decomposition context, consider the notion of "contribution to overall segregation by a subgroup *k*, or by all subgroups together in a certain partition", or consider the question of "how much segregation can be attributed to a given discrete variable". As in the income inequality or the economic poverty literature, it is not always possible that all intuitive interpretations of these questions coincide under a certain decomposability property. For the first time in the literature, in this paper it has been shown that these questions receive the more unambiguous answers that are possible in a segregation context under the decomposability properties ***D1*** and ***D2*** that are only satisfied by the *M* index. The *H* and the H^* indices satisfy some weaker decomposition properties. However, numerical examples and actual data have been used to establish that the dependence of the weights in these decompositions on *both* demographic information about the marginal distributions *and* school and racial entropies pose serious problems of interpretation, specially in the decomposition of the *H* index for partitions of groups into supergroups, and the decomposition of the H^* index for partitions of schools into clusters.

Second, the invariance properties that require a segregation measure to be independent from changes in the relative importance of demographic groups or organizational units have also greatly concerned many authors in the segregation field. The *M* index is not invariant in this sense but changes in overall segregation according to the *M* index can be decomposed in two complementary ways to isolate terms that capture changes in net segregation independent of variations in the marginal distributions of schools and racial groups. No such decompositions are available to the *H* and the H^* indices. When such demographic changes are important, as it has been shown to be the case in an example and when assessing the change in school segregation in the U.S. during 1989-2005, this is a serious limitation.

Finally, many authors have insisted on the convenience of a third subsidiary property, namely, normalization. This can be easily achieved in our case by dividing the *M* index into the appropriate population entropy. If the racial entropy is chosen, then the *H* index is obtained. Similarly, if the entropy of the schools is chosen, then the H^* index is obtained. However, the cost of either normalization is very high indeed. On one hand, at a conceptual or intuitive level, it can be argued that neither the *H* nor the H^* index captures well changes in inter-racial or inter-group exposure. On the other hand, all normalized indices, including the *H* and the H^* indices, violate the strong decomposability properties ***D1*** and ***D2*** with the consequences already analyzed.

In conclusion, applied researchers have available three segregation indices based on the entropy notion first advocated by Theil and his co-author Finizza: the *M* index on one hand, and the *H* and H^* indices on the other hand. However, the advantages of the *M* index are inescapable. In the first place, Frankel and Volij (2009a) have formally characterized the ranking induced by the *M* index in terms of eight ordinal axioms –a result that allows us to know exactly which value judgments are invoked when using this ranking rather than the ones induced by remaining entropy-based indices for which no such characterization result is available.[21] But beyond this convenient situation, one selects which index to use in practice

[21] Very few segregation indices have been similarly characterized. In the two groups case, Chakravarty and Silber (1992) characterize an index of absolute segregation, while Chakravarty and Silber (2007) axiomatically derive a class of numerical indices of relative segregation that parallel the multidimensional Atkinson inequality indices. Two members of that class are monotonically related to the square root index, independently characterized by Hutchens (2004), and the *M* index. In the multigroup case, Frankel and Volij (2009a) provide an ordinal characterization of two families of Atkinson indices.

taking also into account its cardinal properties. In this respect, this paper has shown that when decomposability properties are desired in the empirical work there is much to be gained by focusing exclusively on the un-normalized *M* index. When, in addition, invariance properties are also thought to be useful, it has been seen that applied researchers would do better using the *M* index and its invariant decompositions rather than using either H or H^*. Finally, the significance of the segregation differences and levels can only be studied under alternative hypothesis if the measure is explicitly embedded in a statistical framework. Researchers with these considerations in mind can exploit the statistical properties established in Mora and Ruiz-Castillo (2009b) for the *M* index. No comparanle statistical framework has been yet provided for the *H* and H^* indices.

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Table 1. School Enrolment, Ethnic Mix, Entropies and School Segregation in the U.S., 1989:2005

	No. of students (millions)			Racial Shares (%)		
	1989	2005	Change (%)	1989	2005	Change
Minorities	8.61	12.24	42.10	34.78	48.05	13.27
Native American	0.17	0.23	33.77	0.68	0.89	0.20
Asian	1.03	1.40	36.11	4.15	5.49	1.34
Black	3.99	4.53	13.70	16.10	17.80	1.70
Hispanic	3.43	6.08	77.33	13.85	23.87	10.02
White	16.14	13.23	-18.06	65.22	51.95	-13.27
Total	**24.76**	**25.47**	**2.87**	100	100	0

Entropies and Segregation Indexes

	$E\ P_g$	$E\ P_n$	$\sum_{n=1}^{N} p_n E\ P_{g\mid n}$	$\sum_{g=1}^{G} p_g E\ P_{n\mid g}$	M	H	H^*
1989	101.27	1040.25	57.35	996.32	43.92	43.37	4.22
2005	119.07	1035.72	70.17	986.82	48.90	41.07	4.72
Change	17.80	-4.53	12.82	-9.50	4.98	-2.30	0.50

Notes: Ethnic shares are the percentages of students from every race/ethnic group. The terms Native American, Asian, Black, and White refer to non-Hispanic members of these racial groups. Asian includes Native Hawaiians and Pacific Islanders; Native American includes American Indians and Alaska Natives (Innuit or Aleut). The term Hispanic is an ethnic rather than a racial category since Hispanic persons may belong to any race. Minorities include all categories except White.