

# Symmetries shape the current in ratchets induced by a bi-harmonic force.

## Supplementary Material

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Let us analyze the following evolution equations  $E[x(t), f(t)] = 0$  for the variables  $x(t)$  (position) and  $u(t)$  (velocity) of a relativistic particle of mass  $M > 0$

$$\begin{aligned} M \frac{du}{dt} &= -f(t)(1-u^2)^{3/2} - \gamma u(1-u^2), \\ \frac{dx}{dt} &= u(t), \quad u(0) = u_0, \quad x(0) = x_0, \end{aligned} \quad (1)$$

where  $x_0$  and  $u_0$  are the initial conditions,  $\gamma > 0$  represents the damping coefficient and  $f(t)$  is a  $T$ -periodic driving force [1]. Notice that defining the momentum

$$P(t) = \frac{Mu(t)}{\sqrt{1-u^2(t)}}, \quad (2)$$

we can transform Eq. (1) into the linear equation

$$\frac{dP}{dt} = -\beta P - f(t), \quad (3)$$

where  $\beta = \gamma/M$ , whose solution is given by

$$P(t) = P(0)e^{-\beta t} - \int_0^t dz f(z)e^{-\beta(t-z)}. \quad (4)$$

Equation (1) is invariant under *time shift* ( $\mathcal{S} : t \mapsto t + T/2$ ) along with the change  $x \mapsto -x$ , provided  $(\mathcal{S}f)(t) = f(t + T/2) = -f(t)$ . The bi-harmonic force

$$f(t) = \epsilon_1 \cos(q\omega t + \phi_1) + \epsilon_2 \cos(p\omega t + \phi_2), \quad (5)$$

preserves this symmetry if, both,  $p$  and  $q$  are odd integer numbers, so in this case the average velocity

$$v = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t u(\tau) d\tau, \quad (6)$$

is zero. In contrast, if  $p + q$  is odd and  $p$  and  $q$  are coprimes, a nonzero average current can appear. For the sake of simplicity we will take  $p = 2$  and  $q = 1$  in Eq. (5) [2]. Then the solution to (4) for the chosen force (5) will be

$$\begin{aligned} P(t) &= \tilde{P}_0 \exp(-\beta t) - \frac{\epsilon_1}{\sqrt{\beta^2 + \omega^2}} \cos(\omega t + \phi_1 - \chi_1) \\ &\quad - \frac{\epsilon_2}{\sqrt{\beta^2 + 4\omega^2}} \cos(2\omega t + \phi_2 - \chi_2), \end{aligned} \quad (7)$$

with  $\tilde{P}_0 = P(0) + (\epsilon_1/\sqrt{\beta^2 + \omega^2}) \cos(\phi_1 - \chi_1) + (\epsilon_2/\sqrt{\beta^2 + 4\omega^2}) \cos(\phi_2 - \chi_2)$ ,  $\chi_1 = \arctan(\omega/\beta)$ , and  $\chi_2 = \arctan(2\omega/\beta)$ . From (2), one obtains

$$u(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (1/2)_k}{k! M^{2k+1}} [P(t)]^{2k+1}, \quad (8)$$

where  $(1/2)_k \equiv (1/2)(1/2 + 1) \cdots (1/2 + k - 1)$ . From (6) and (8) it follows that the time-average velocity,  $v$ , cannot be expressed as a function of the odd moments of  $f(t)$ , unless  $P(t)$  is proportional to  $f(t)$ . Indeed, it is only in the overdamped case [in which the inertial term in (1) is neglected] that the evolution equation is given by  $P(t) = -(1/\beta)f(t)$  and then  $v$  do admit an expansion in odd moments of  $f(t)$ .

Moreover, for small amplitudes  $\epsilon_1$  and  $\epsilon_2$ , the leading term of the time-average velocity (8) reads

$$v = B \epsilon_1^2 \epsilon_2 \cos(2\phi_1 - \phi_2 + \theta_0), \quad (9)$$

where  $B = 3/(8M^3(\beta^2 + \omega^2)\sqrt{\beta^2 + 4\omega^2})$  and  $\theta_0 = -2\chi_1 + \chi_2$ . This expression is in agreement with the prediction of our theory. Furthermore, in the limit  $\beta \rightarrow 0$  we have  $-2\chi_1 + \chi_2 \rightarrow \pi/2$ , and in the combined limit  $M \rightarrow 0$  and  $\beta \rightarrow \infty$ , with  $\gamma = \text{const.}$ ,  $-2\chi_1 + \chi_2 \rightarrow 0$ . One can check that in the former case Eq. (1) is invariant under *time reversal* ( $\mathcal{R} : t \mapsto -t$ ) provided  $(\mathcal{R}f)(t) = f(-t) = f(t)$ , and therefore  $\theta_0 = \pi/2$  is the prediction of our theory. In the latter case, however, it is  $(\mathcal{R}f)(t) = f(-t) = -f(t)$  that leaves Eq. (1) invariant and then our theory predicts  $\theta_0 = 0$ .

[1] O. H. Olsen and M. R. Samuelsen, Phys. Rev. B **28**, 210 (1983).

[2] M. Salerno and Y. Zolotaryuk, Phys. Rev. E **65**, 056603 (2002).