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STABILITY OF THE VOLUME GROWTH RATE UNDER QUASI-ISOMETRIES

ANA GRANADOS[(1)], DOMINGO PESTANA[(1)], ANA PORTILLA[(1)], JOSÉ M. RODRÍGUEZ[(1)], AND EVA TOURÍS[(1)]

Abstract. Kanai proved the stability under quasi-isometries of numerous global properties (including the volume growth rate) between Riemannian manifolds of bounded geometry. Unfortunately, Kanai's hypotheses are not usually satisfied in the context of Riemann surfaces endowed with the Poincaré metric. In this work we prove the stability of the volume growth rate by quasi-isometries, under hypotheses that many Riemann surfaces (and even Riemannian surfaces with pinched negative curvature) satisfy. Although Kanai just deals with non-bordered Riemannian manifolds, here manifolds with border are allowed. In order to get our results, it is shown that many bordered Riemannian surfaces with pinched negative curvature are bilipschitz equivalent to bordered surfaces with constant negative curvature.

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1. Introduction

An interesting problem in geometric function theory is to consider general transformations between manifolds and to study which geometric properties they preserve. Among these transformations, quasi-isometries are of special interest since they preserve Gromov hyperbolicity of geodesic metric spaces (see, e.g., [19], [20]). Intuitively, two metric spaces are quasi-isometric if their large-scale metric structures are the same, ignoring fine details. Informally, quasi-isometries allow stretching and contracting distances.

By nature, quasi-isometries represent a flexible class of maps that behave well on a global scale, but that produce a large distortion on the local properties of the manifolds involved. Note that even though they form a large class of maps which do not need to be continuous, they do have invariance properties. In [24], [25], [26], M. Kanai studied several geometric properties (such as isoperimetric inequalities, Poincaré-Sobolev inequalities, recurrence and transience of the Brownian motion, growth rate of the volume of balls, and Liouville type theorems) for a large class of Riemannian manifolds whose Ricci curvature is bounded from below, and proved that such properties were preserved under quasi-isometries. They also preserve the parabolic Harnack inequality [13] and various estimates on transition probabilities of random walks, such as heat kernel estimates.

Since the local geometry of a manifold is not transferred onto another manifold by a quasi-isometry, Kanai needed an additional condition which governs local geometries of Riemannian manifolds: the Ricci curvature and the injectivity radius must be bounded from below. Subsequently, other authors such as Holopainen and Soardi (see [21], [23], [41]) proved that the existence of non-trivial solutions of a wide class of partial differential equations is also preserved under quasi-isometries.

Recall that the *injectivity radius* $\iota(p)$ *of* $p \in X$ is defined as the supremum of those $r > 0$ such that the ball $B_X(p, r)$ is simply connected or, equivalently, as half the infimum of the lengths of the (homotopically non-trivial) loops based at p in X. The *injectivity radius* $\iota(X)$ *of* X is the infimum over $p \in X$ of $\iota(p)$.

Our interest will lay not only on these results but also on the ideas behind their proofs. Discrete structures approximating Riemannian manifolds are a convenient way of studying several potential theoretical aspects of manifolds. Concretely, there are some ideas in the proofs of these results relating the manifold with a

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particular graph (an ε-net of the manifold). Several authors have followed Kanai's work either to study the stability of other properties, or to prove the equivalence between a manifold and a different associated graph (see, e.g., [1], [9], [16], [21], [23], [34], [35], [38], [39], [41], [42]).

Riemann surfaces endowed with their Poincaré metrics are a natural context to apply Kanai's results since they have constant negative curvature. However, these surfaces usually have isolated singularities which are cusps, and thus, injectivity radius equal to zero. That means that, unfortunately, it is not possible to apply Kanai's results in this case. Recall that the Poincaré metric plays a fundamental role in geometric function theory, since if R, S are Riemann surfaces with their Poincaré metrics, then $d_S(f(w), f(z)) \leq d_R(w, z)$ for every holomorphic map $f : R \to S$ and every $z, w \in R$.

Hence, a natural open problem consists of formulating different hypotheses that allow to obtain similar conclusions to Kanai's results, but which could be applied to Riemann surfaces with cusps. In particular, the linear isoperimetric inequality is preserved by quasi-isometries on surfaces with cusps and genus zero (recall that plane domains are the most important class of Riemann surfaces); besides, it is also preserved when the topological hypotheses on the genus are weakened in some sense (see [10] and [17]).

As usual, $d_X(x, y)$ will denote the distance in the Riemannian metric between the points $x, y \in X$, and $L_X(\gamma)$ the length of a curve $\gamma \subset X$ with respect to the intrinsic metric in X.

A function between two metric spaces $f : X \to Y$ is said to be an (a, b)-*quasi-isometric embedding* with constants $a \geq 1$, $b \geq 0$, if

$$\frac{1}{a}\, d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a\, d_X(x_1, x_2) + b\,, \qquad \text{for every } x_1, x_2 \in X.$$

Such a quasi-isometric embedding f is a *quasi-isometry* if, furthermore, there exists a constant $c \geq 0$ such that f is *c-full*, i.e., if for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leq c$.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry between them. It is well-known that to be quasi-isometric is an equivalence relation (see, e.g., [24]).

A quasi-isometry can drastically change the local topology (for example, any compact Riemannian manifold is quasi-isometric to a single point). Furthermore, important global properties, like the dimension, are not preserved by quasi-isometries: if X and Y are Riemannian manifolds and Y is compact, then X and $X \times Y$ are quasi-isometric. Nevertheless, some results on the stability of the injectivity radius for Riemann surfaces with some hypotheses on their genus are shown in [10] and [17]. In particular, points with small injectivity radius are shown to be mapped onto points with small injectivity radius.

In this paper, the stability of the volume growth rate is studied. Although the main focus of this work are Riemann surfaces, variable negative curvature surfaces and manifolds will also be considered.

Volume growth rate is related with other important topics in Riemannian manifolds. In [12], necessary conditions for the existence of a positive Green's function (a fundamental solution to the Poisson equation) were obtained; such conditions just involved the volume growth of the manifold (see also [29], [30]). These results are extended to the p-Laplace equation in [22]. In terms of Brownian motion, a complete manifold has Green's function if and only if the Brownian motion on the manifold is transient (i.e., the Brownian motion eventually escapes from any compact set with probability 1). Green's function is also related to other topics as the heat kernel and isoperimetric inequalities (see, e.g., [14], [15]).

[28] studies the connections between volume growth, spectral properties and stochastic completeness of locally finite weighted graphs. This allows to obtain some comparison results for both stochastic completeness and estimates on the bottom of the spectrum for general locally finite weighted graphs; see also [27]. In [2], [6] and [31] there are more relations between the volume growth rate and other properties of Riemannian manifolds.

Let X be a (bordered or non-bordered) Riemannian manifold, and $x \in X$. Define the *lower and upper polynomial growth orders* of X as

$$\operatorname{lpgo}(X) = \sup\big\{k \geq 0 : \liminf_{r\to\infty} r^{-k} \operatorname{vol}_X\big(B_X(x, r)\big) > 0\big\},$$
$$\operatorname{upgo}(X) = \inf\big\{k \geq 0 : \limsup_{r\to\infty} r^{-k} \operatorname{vol}_X\big(B_X(x, r)\big) < \infty\big\},$$

respectively (if $\limsup_{r\to\infty} r^{-k} \operatorname{vol}_X\big(B_X(x,r)\big) = \infty$ for every $k \geq 0$, then define $\operatorname{upgo}(X) = \infty$; if $\liminf_{r\to\infty} r^{-k} \operatorname{vol}_X\big(B_X(x,r)\big) = 0$ for every $k \geq 0$, then $\operatorname{lpgo}(X) := 0$). Thus, $0 \leq \operatorname{lpgo}(X) \leq \operatorname{upgo}(X) \leq \infty$. The surface X is said to have *polynomial growth* if $\operatorname{upgo}(X) < \infty$, and to have *exponential growth* if $\liminf_{r\to\infty} r^{-1} \log \operatorname{vol}_X\big(B_X(x,r)\big) > 0$.

Some other volume growth rates will be considered. A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said *admissible* if it is a non-decreasing function with $\lim_{r\to\infty} \varphi(r) = \infty$, $\lim_{r\to\infty} \varphi(r)/r^k = 0$ for every $k > 0$, and $\lim_{r\to\infty} \varphi(ar)/\varphi(r) = 1$ for some $a > 1$. Note that every admissible function satisfies $\lim_{r\to\infty} \varphi(ar + b)/\varphi(r) = 1$ for every $a > 0$ and $b \in \mathbb{R}$. Given any admissible function, let us define

$$\operatorname{lgo}_\varphi(X) = \sup\big\{k \geq 0 : \liminf_{r\to\infty} \varphi(r)^{-k} \operatorname{vol}_X\big(B_X(x,r)\big) > 0\big\},$$
$$\operatorname{ugo}_\varphi(X) = \inf\big\{k \geq 0 : \limsup_{r\to\infty} \varphi(r)^{-k} \operatorname{vol}_X\big(B_X(x,r)\big) < \infty\big\},$$

(if $\limsup_{r\to\infty} \varphi(r)^{-k} \operatorname{vol}_X\big(B_X(x,r)\big) = \infty$ for every $k \geq 0$, then define $\operatorname{ugo}_\varphi(X) := \infty$;
if $\liminf_{r\to\infty} \varphi(r)^{-k} \operatorname{vol}_X\big(B_X(x,r)\big) = 0$ for every $k \geq 0$, then $\operatorname{lgo}_\varphi(X) := 0$). Thus, $0 \leq \operatorname{lgo}_\varphi(X) \leq \operatorname{ugo}_\varphi(X) \leq \infty$.

It is clear that these definitions do not depend on the choice of the point $x \in X$. It is well-known that a complete n-dimensional Riemannian manifold with non-negative Ricci curvature has upper polynomial growth order of at most n, and that a simply connected complete Riemannian manifold with negative sectional curvature bounded away from zero has exponential growth.

Proposition 5.2 shows that there exist non-exceptional Riemann surfaces with area growth of balls as small as desired, and so, it makes sense to consider the growth rates $\operatorname{lgo}_\varphi$ and $\operatorname{ugo}_\varphi$.

In [24, Theorem 3.3], Kanai proves that for complete Riemannian manifolds with a lower bound for Ricci curvature and positive injectivity radius, the upper polynomial growth order and the exponential growth are preserved by quasi-isometries. One can check that the argument in the proof of [24, Theorem 3.3] also gives that the lower polynomial growth order, $\operatorname{lgo}_\varphi$ and $\operatorname{ugo}_\varphi$ (for every admissible function φ) are preserved by quasi-isometries, for this kind of manifolds.

One of the main results of this paper states that these volume growth orders are "almost preserved" by quasi-isometries, without hypotheses on the injectivity radius.

A bordered or non-bordered Riemannian manifold is said to be *complete* if every Cauchy sequence is convergent, i.e., every geodesic can be prolonged (unless it reaches the boundary, provided that the manifold is bordered).

A bordered Riemannian manifold X has *totally geodesic border* if ∂X is a (finite or countable) union of totally geodesic submanifolds.

A bordered Riemannian surface X is *geodesically bordered* if ∂X is a (finite or countable) union of simple closed geodesics; X is *finitely geodesically bordered* if ∂X is a finite union of simple closed geodesics. It is clear that if a bordered Riemannian surface X is geodesically bordered, then it has totally geodesic border.

A Riemannian surface has *pinched negative curvature* if there exist positive constants k_1, k_2 such that its Gaussian curvature K satisfies $-k_2 \leq K \leq -k_1$.

The main results on this paper are the following.

Theorem 1.1. *Let X and Y be quasi-isometric complete Riemannian manifolds. Assume that Y does not have border or has totally geodesic border, and that it has a lower bound on its Ricci curvature. Suppose that X has dimension 2, pinched negative curvature and it is either non-bordered or geodesically bordered. If X has polynomial growth, then Y also does. If Y has exponential growth, then X also does. Also,* $\operatorname{lpgo}(Y) \leq \operatorname{lpgo}(X) + 1$ *and* $\operatorname{upgo}(Y) \leq \operatorname{upgo}(X) + 1$, *and both inequalities are sharp.*

Note that the conclusion of Theorem 1.1 also holds if Y is a complete Riemannian manifold satisfying properties $(P1)$, $(P2)$ and $(P3)$ stated before Lemma 3.6.

Theorem 1.1 and Proposition 5.1 have the following direct consequence.

Corollary 1.2. *Let X and Y be quasi-isometric complete (geodesically bordered or non-bordered) Riemannian surfaces with pinched negative curvature. Then X has polynomial (respectively, exponential) growth if and only if Y does. Also, $|\operatorname{lpgo}(Y) - \operatorname{lpgo}(X)| \le 1$ and $|\operatorname{upgo}(Y) - \operatorname{upgo}(X)| \le 1$, and both inequalities are sharp.*

Theorem 1.3. *Let X and Y be quasi-isometric complete Riemannian surfaces with pinched negative curvature. Assume that Y has finite genus and it is either finitely geodesically bordered or non-bordered and X is geodesically bordered or non-bordered. Then X has polynomial (respectively, exponential) growth if and only if Y does. Also, $\operatorname{lpgo}(X) - 1 \le \operatorname{lpgo}(Y) \le \operatorname{lpgo}(X)$, $\operatorname{upgo}(X) - 1 \le \operatorname{upgo}(Y) \le \operatorname{upgo}(X)$, $\operatorname{lgo}_\varphi(Y) \le \operatorname{lgo}_\varphi(X)$ and $\operatorname{ugo}_\varphi(Y) \le \operatorname{ugo}_\varphi(X)$ for every admissible function φ. Furthermore, every inequality is sharp.*

Theorem 1.3 has the following direct consequence.

Corollary 1.4. *Let X and Y be quasi-isometric complete (finitely geodesically bordered or non-bordered) Riemannian surfaces with pinched negative curvature and finite genus. Then $\operatorname{lpgo}(Y) = \operatorname{lpgo}(X)$, $\operatorname{upgo}(Y) = \operatorname{upgo}(X)$, $\operatorname{lgo}_\varphi(Y) = \operatorname{lgo}_\varphi(X)$ and $\operatorname{ugo}_\varphi(Y) = \operatorname{ugo}_\varphi(X)$ for every admissible function φ.*

An (open or non-bordered) non-exceptional Riemann surface S is a Riemann surface whose universal covering space is the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, endowed with its Poincaré metric (also called the hyperbolic metric), i.e., the metric obtained by projecting the Poincaré metric of the unit disk

$$ds^2 = \Big(\frac{2}{1-|z|^2}\Big)^2 (dx^2 + dy^2).$$

With this metric, S is a complete Riemannian manifold with constant curvature -1. The only Riemann surfaces which are left out (the exceptional Riemann surfaces) are the sphere, the plane, the punctured plane and the tori.

Non-exceptional bordered Riemann surfaces always arise as closed subsets of some open non-exceptional Riemann surface, and they are given the metric induced by the Poincaré metric of the host open surface. Such induced metric has, of course, $K = -1$. Since this work always considers geodesically bordered Riemann surfaces, every orientable Riemannian surface S with $K = -1$ can be isometrically embedded into an open non-exceptional Riemann surface.

Corollary 1.5. *Let Ω_1 and Ω_2 be quasi-isometric non-exceptional plane domains (or Riemann surfaces with finite genus) with their Poincaré metrics. Then $\operatorname{lpgo}(\Omega_2) = \operatorname{lpgo}(\Omega_1)$, $\operatorname{upgo}(\Omega_2) = \operatorname{upgo}(\Omega_1)$, $\operatorname{lgo}_\varphi(\Omega_2) = \operatorname{lgo}_\varphi(\Omega_1)$ and $\operatorname{ugo}_\varphi(\Omega_2) = \operatorname{ugo}_\varphi(\Omega_1)$ for every admissible function φ.*

Summarizing, the results in this paper improve Kanai's result (in our context of Riemannian surfaces) in different ways: by removing the hypothesis on the injectivity radius, considering several volume growth rates and allowing bordered manifolds.

Note that the results on stability of volume growth rates are also useful in order to prove that two manifolds are not quasi-isometric, since usually it is not difficult to compute volume growth rates. As an elementary example, the result of Kanai shows that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not quasi-isometric if $n \ne m$, since $\operatorname{lpgo}(\mathbb{R}^n) = \operatorname{upgo}(\mathbb{R}^n) = n$.

The proofs here presented are based on two results that are interesting by themselves. The first one says that each geodesically bordered Riemannian surface with pinched negative curvature is bilipschitz equivalent to a surface with constant negative curvature (see Theorem 3.1). This result provides a new tool that allows to relate these geometries in a global way, and hopefully it will be useful in the future to generalize properties of bordered surfaces with constant negative curvature to surfaces with pinched negative curvature. The second result, Theorem 3.22, states that for each non-exceptional Riemann surface X there exists a graph P associated to it such that X and P have the same growth rates (see also Lemma 4.4); note that a main ingredient in Kanai's proof is this equivalence between Riemannian manifolds and graphs, but his proof needs, in an essential way, a lower bound of the injectivity radius.

Theorem 3.11 is also interesting by itself, since it is a general result on Riemann surfaces. Besides, some technical lemmas in this paper (such as 3.13, 3.14 and Corollary 3.15) help to understand the behavior of geodesics in Riemann surfaces. Also, some technical lemmas state universal results on collars in Riemann surfaces, as Lemma 3.3.

An example is provided in Section 5 which shows that the inequalities in Theorems 1.1 and 1.3 are sharp (see Proposition 5.1). Also, Proposition 5.2 shows that there exist non-exceptional Riemann surfaces with growth rate as small as desired.

The structure of the paper is as follows. Section 2 contains the necessary background. Section 3 contains the proofs of Theorems 1.1 and 3.1. In Section 4, the proof of Theorem 1.3 is given. Finally, some examples are constructed in Section 5.

2. Definitions and background

If $\gamma : [a, b] \to X$ is a continuous curve in a metric space (X, d), Define the *length* of γ as

$$L_X(\gamma) = L(\gamma) := \sup \Big\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : \, a = t_0 < t_1 < \cdots < t_n = b \Big\} .$$

If X_0 is an arc-connected subset of X, then define the *inner distance*

$$d_{X_0}(x, y) := \inf \big\{ L_X(\gamma) : \, \gamma \subset X_0 \text{ is a continuous curve joining } x \text{ and } y \big\} ,$$

for every $x, y \in X_0$. It is clear that $d_X|_{X_0} \le d_{X_0}$, i.e., $d_X(x, y) \le d_{X_0}(x, y)$ for every $x, y \in X_0$.

Nets associated to surfaces will play an important role in this paper.

Following Kanai's idea (see [24]), a subset G of X is said to be *ε-separated* for $\varepsilon > 0$, if $d_X(p, q) \ge \varepsilon$ whenever p and q are distinct points in G. Such set is called *maximal* if it is maximal with respect to the order relation of inclusion. Note that a maximal ε-separated set G is also ε-full in X; in fact, for each $x \in X$, there exists $p \in G$ such that $d_X(x, p) < \varepsilon$. Let G be a maximal ε-separated subset of X. A *net* (or graph) structure Γ of G is given by $N(p) = \{q \in G : \, 0 < d_X(p, q) \le 4\varepsilon\}$, i.e., Γ is the graph with $V(\Gamma) = G$ and $E(\Gamma) = \{pq : \, p, q \in G, \, 0 < d_X(p, q) \le 4\varepsilon\}$. Such net structure described above will be called an *ε-net* in X. (By a technical reason the definition of ε-net has been slightly modified with respect to the one given in [24], changing 2ε by 4ε; however, every result in [24] holds with this slightly different definition of ε-net. By the way, Kanai in [25] also changed this definition, with 3ε instead of 2ε.) A net Γ is *Δ-uniform* if $\deg_\Gamma(p) = |N(p)| \le \Delta$ for every $p \in G$; Γ is *uniform* if there exists a constant Δ such that it is Δ-uniform.

Since the only points in Γ are its vertices, any connected graph Γ has a natural distance defined on its points induced by taking shortest paths in Γ. Such distance will be denoted by d_Γ.

Given two Riemannian metrics $\sigma, \widetilde{\sigma}$ in a surface, one says that $\widetilde{\sigma} \le \sigma$ if there is a function $\rho \le 1$ satisfying $\widetilde{\sigma} = \rho\,\sigma$.

The following result appears in [18, Theorem 5.5].

Theorem 2.1. *Let X be an orientable complete Riemannian surface with a C^2 metric σ and curvature $-k_2 \le K \le -k_1 < 0$. Then there is a unique C^2 scalar function ρ on X, such that the Riemannian metric $\sigma_0 = \rho\,\sigma$ is complete, has constant curvature -1, and $k_1 \le \rho \le k_2$, i.e., $\frac{1}{k_2}\sigma_0 \le \sigma \le \frac{1}{k_1}\sigma_0$.*

Although the statement of Theorem 5.5 in [18] requires σ to be not only C^2 but also a smooth function, this condition can be relaxed: in the mentioned theorem, smoothness is required only to apply [44, Theorem 1(ii)], which, in turn uses [33, Theorem A'], and this last theorem requires σ to be just C^2.

The following slightly improvement of Theorem 2.1 can be deduced.

Corollary 2.2. *Let X be an orientable complete Riemannian surface with a C^m $(2 \le m \le \infty)$ metric σ and curvature $-k_2 \le K \le -k_1 < 0$. Then there is a unique C^m scalar function ρ on X such that the Riemannian metric $\sigma_0 = \rho\,\sigma$ is complete, has constant curvature -1, and $k_1 \le \rho \le k_2$, i.e., $\frac{1}{k_2}\sigma_0 \le \sigma \le \frac{1}{k_1}\sigma_0$.*

Corollary 2.2 is a consequence of Theorem 2.1, since it gives that there exists a unique C^2 scalar function ρ on X, such that the Riemannian metric $\sigma_0 = \rho\,\sigma$ is complete, has constant curvature -1, and $k_1 \le \rho \le k_2$. Now, since σ_0 (the Poincaré metric) is C^∞, it suffices to remark that ρ is a quotient of positive C^m functions.

Theorem 2.1 is interesting by itself and it will also be needed in order to prove Theorem 3.1 below. It is well-known that surfaces with pinched and constant negative curvature usually behave alike. In spite of this, to show that some specific properties satisfied by constant curvature surfaces are also satisfied by pinched ones, might be difficult to prove.

Given a Riemannian surface S, a geodesic γ in S, and a continuous unit vector field ξ along γ, orthogonal to γ, define the *Fermi coordinates* based on γ as the map $Y(r,\theta) := \exp_{\gamma(\theta)} r\xi(\theta)$.

It is well-known that the Riemannian metric can be expressed in Fermi coordinates as $ds^2 = dr^2 + G(r,\theta)^2\, d\theta^2$, where $G(r,\theta)$ is the solution of the scalar equation

$$\frac{\partial^2 G}{\partial r^2}(r,\theta) + K(r,\theta)G(r,\theta) = 0\,, \qquad G(0,\theta) = 1\,, \qquad \frac{\partial G}{\partial r}(0,\theta) = 0\,, \tag{2.1}$$

(see, e.g., [11, p. 247]).

A *collar* in S about a simple closed geodesic γ is a doubly connected domain in S "bounded" by two Jordan curves β_1, β_2, (called the boundary curves of the collar) orthogonal to the pencil of geodesics emanating from γ; such collar can be written as $\mathcal{C}_{\gamma,t} = \{p \in S : \, d_S(p,\gamma) < t\}$, for some positive constant t. The constant t is called the *width* of the collar. The two doubly connected subsets of $\mathcal{C}_{\gamma,t}$ containing γ "bounded" by γ and β_1, and by γ and β_2, respectively, are called *half-collars.* Obviously, every collar is the union of its two half-collars. Given two geodesics σ_1 and σ_2 in the closure of $\mathcal{C}_{\gamma,t}$ with length $2t$ joining β_1 and β_2 such that $\sigma_1 \cap \gamma$ and $\sigma_2 \cap \gamma$ are antipodal points in γ, the *transverse half-collars* of γ (relative to σ_1, σ_2) are the two simply connected subsets of $\mathcal{C}_{\gamma,t}$ containing $(\sigma_1 \cup \sigma_2) \cap \mathcal{C}_{\gamma,t}$, and whose union is $\mathcal{C}_{\gamma,t}$.

The following result, known as Collar Lemma [36], will be used several times along this work. It is generalized for surfaces with variable negative curvature in [7].

Lemma 2.3. *If γ is a simple closed geodesic in a non-exceptional Riemann surface S, then there exists a collar about γ of width t, for every $0 < t \le w$, where $\cosh w = \coth(L_S(\gamma)/2)$ or, equivalently, $\sinh w = 1/\sinh(L_S(\gamma)/2)$.*

Remark 2.4. *Denote by $\mathcal{C}_\gamma$ the collar about γ of width w given by the above lemma. It is well-known that if γ_1 and γ_2 are disjoint simple closed geodesics, then $\mathcal{C}_{\gamma_1} \cap \mathcal{C}_{\gamma_2} = \emptyset$.*

Let S be a non-exceptional Riemann surface with a cusp q (if $S \subset \mathbb{C}$, every isolated point in ∂S is a cusp). A *collar* in S about q is a doubly connected domain in S "bounded" both by q and a Jordan curve (called the boundary curve of the collar) orthogonal to the pencil of geodesics emanating from q. It is well-known that the length of the boundary curve is equal to the area of the collar (see, e.g., [5]). A collar of area β is called a β-collar. Thus, the length of the boundary of a β-collar is also β. For each cusp there exists a 2-collar and 2-collars of different cusps are disjoint. Besides, the collar $\mathcal{C}_\gamma$ of the simple closed geodesic γ does not intersect the 2-collar of a cusp (see [36], [40] and [8, Chapter 4]).

Let us recall the thick-thin decomposition of (non-bordered) Riemann surfaces given by Margulis Lemma (see, e.g., [3, p.107]). Concretely, for any $0 < \delta < \operatorname{arcsinh} 1$, any non-exceptional Riemann surface S can be partitioned into a thick part, $S_\delta := \{z \in S : \iota(z) \ge \delta\}$, and a thin part, $S \setminus S_\delta$. The components of the thin part are either collars of cusps or collars of closed geodesics of length less than 2δ. In fact, the following holds (see [10, Lemma 4.9]):

Lemma 2.5. *Let S be a non-exceptional (non-bordered) Riemann surface and $z \in S$. If $\iota(z) < \operatorname{arcsinh} 1$, then the shortest geodesic loop with base point z is contained either in the 2-collar of a cusp or in the collar $\mathcal{C}_\gamma$ of a simple closed geodesic γ.*

Given a geodesically bordered Riemann surface S and $z \in S$, define $\iota(z)$ as follows: if $\widetilde{S}$ is the Schottky double of S , then define $\iota(z)$ (in S) as the injectivity radius of z in the (non-bordered) Riemann surface $\widetilde{S}$. (Recall that the Schottky double of S is obtained from S and an isometric copy of itself by identifying the corresponding simple closed geodesics in the border of S).

By applying Margulis Lemma to $\widetilde{S}$, the components of the thin part of S are either collars of cusps, collars of closed geodesics, half-collars of closed geodesics or transverse half-collars of closed geodesics of length less than 2δ.

Recall a well-known hyperbolic trigonometric formula which will be useful (see, e.g., [8, p.454]).

Lemma 2.6. *The following formula holds on the unit disk (and then for simply connected quadrilaterals on any non-exceptional Riemann surface). Let us consider a geodesic quadrilateral with three right angles and let ϕ be the other angle. If α, β are the lengths of the sides which meet with angle ϕ and a is the length of the opposite side to the side with length α, then* $\sinh \alpha = \sinh a \cosh \beta$.

If $\iota(z) < \operatorname{arcsinh} 1$ and z belongs to a collar, a half-collar or a transverse half-collar of a simple closed geodesic γ of length l in a non-exceptional Riemann surface, then Collar Lemma and Lemmas 2.5 and 2.6 give

$$\sinh \iota(z) = \sinh(l/2) \cosh d_S(z, \gamma). \tag{2.2}$$

3. Proof of Theorem 1.1

Since the proof of Theorem 1.1 is long and technical, in order to make the arguments more transparent, let us collect some results needed along the proof in technical lemmas.

Let us start with the following result, which is interesting by itself.

Theorem 3.1. *Let X be an orientable complete geodesically bordered Riemannian surface with a C^m $(2 \leq m \leq \infty)$ metric σ and curvature $-k_2 \leq K \leq -k_1 < 0$. Then there is a unique C^m scalar function ρ on X, such that the Riemannian metric $\sigma_0 = \rho\, \sigma$ is complete, has constant curvature -1, $k_1 \leq \rho \leq k_2$, i.e., $\frac{1}{k_2}\sigma_0 \leq \sigma \leq \frac{1}{k_1}\sigma_0$, and X is geodesically bordered with the metric σ_0 as well.*

Proof. Let X' be a Riemannian surface isometric to X, and let $\widetilde{X}$ the Schottky double of X, i.e., the Riemannian surface obtained from X and X', by identifying the corresponding points in ∂X and $\partial X'$. Thus, $\widetilde{X}$ is an orientable complete (non-bordered) Riemannian surface with curvature satisfying $-k_2 \leq K \leq -k_1 < 0$. Let $\mathcal{A}$ be the atlas in X and $\mathcal{A}'$ the corresponding atlas in X'. Let us obtain now local charts for the points in $\partial X = \partial X'$. For each simple closed geodesic γ in $\partial X = \partial X'$ consider the Fermi coordinates $Y(r, \theta)$ based on γ in X defined in $[0, w) \times \mathbb{R}$ with

$$w = \frac{1}{\sqrt{k_2}} \operatorname{arccosh} \coth \frac{\sqrt{k_2}\, L_X(\gamma)}{2} = \frac{1}{\sqrt{k_2}} \operatorname{arcsinh} \frac{1}{\sinh(\sqrt{k_2}\, L_X(\gamma)/2)}\,.$$

Denote by $\overline{\sigma}$ the Riemannian metric in $\widetilde{X}$.

When defining Y on $(-w, w) \times \mathbb{R}$ by $Y(r, \theta) = Y(-r, \theta)$, one gets that $Y\big((-w, 0] \times \mathbb{R}\big)$ is contained in X', and by the Collar Lemma, Y provides a parametrization of a collar about γ in $\widetilde{X}$. Thus, when considering the atlas $\overline{\mathcal{A}}$ containing $\mathcal{A}$, $\mathcal{A}'$ and the local inverses of these parameterizations, $\widetilde{X}$ is a C^∞ manifold with this atlas $\overline{\mathcal{A}}$. The metric $\overline{\sigma}$ in $(-w, w) \times \mathbb{R}$ can be expressed by $ds^2 = dr^2 + G(r, \theta)^2\, d\theta^2$, where $G(r, \theta)$ satisfies (2.1), and $G(r, \theta) = G(-r, \theta)$. Besides, $K(r, \theta) = K(-r, \theta)$; thus, $K(r, \theta)$ is a continuous function on $(-w, w) \times \mathbb{R}$, and (2.1) gives that $G(r, \theta)$ is a C^2 function on $(-w, w) \times \mathbb{R}$. Hence, the metric $\overline{\sigma}$ in $\widetilde{X}$ belongs to $C^m(X) \cap C^m(X') \cap C^2(\widetilde{X})$.

By Theorem 2.1, there is a unique C^2 scalar function $\overline{\rho}$ on X, such that the Riemannian metric $\overline{\sigma}_0 = \overline{\rho}\, \overline{\sigma}$ is complete, has constant curvature -1, and $k_1 \leq \overline{\rho} \leq k_2$. Since $\overline{\sigma}_0$ (the Poincaré metric) is C^∞ and $\overline{\sigma}$ belongs to $C^m(X) \cap C^m(X') \cap C^2(\widetilde{X})$, one has that $\overline{\rho}$ belongs to $C^m(X) \cap C^m(X') \cap C^2(\widetilde{X})$ as a quotient of positive functions in this class.

Hence, $\sigma := \overline{\sigma}|_X$ and $\rho := \overline{\rho}|_X$ satisfy the desired properties.

Since the canonical reflection $R : \widetilde{X} \to \widetilde{X}$ with $R(X) = X'$ is an isometry with respect to $\overline{\sigma}_0$ and $R(\partial X) = \partial X$, the curves in ∂X are also simple closed geodesics for this metric.

Assume that there exist two functions ρ_1 and ρ_2 with the required properties, and consider their symmetric extensions $\overline{\rho}_1$ and $\overline{\rho}_2$ to $\widetilde{X}$. Thus, $\overline{\rho}_j\overline{\sigma}$ is a complete metric in $\widetilde{X}$ with curvature -1, for $j = 1, 2$, and Theorem 2.1 gives $\overline{\rho}_1\overline{\sigma} = \overline{\rho}_2\overline{\sigma}$ on $\widetilde{X}$, and so, $\rho_1 = \rho_2$ on X. □

A map is *bilipschitz* if it is an onto $(a, 0)$-quasi-isometry for some $a \ge 1$. Theorem 3.1 has direct applications in the study of concepts which are invariant under bilipschitz maps, as the volume growth rates.

Let us recall now the concept of *metric graph.* The triple $G = (V(G), E(G), L)$ is a metric graph if $G = (V(G), E(G))$ is a graph and L is a length function $L : E(G) \to (0, \infty]$. There is an edge with infinite length if and only if there exists a vertex v_∞ such that $L(uv) = \infty$ just if either u or v is equal to v_∞. One must identify any edge $uv \in E(G)$ with the real interval $[0, l]$ (if $l := L(uv)$); hence, if one consider the edge uv as a graph with just one edge, then it is isometric to the real interval $[0, l]$. Therefore, any point in the interior of any edge will be considered a point of G. The graph G is naturally equipped with a distance defined on its points, induced by taking shortest paths in G.

Given $0 < \delta < \operatorname{arcsinh} 1$ and a geodesically bordered or non-bordered Riemann surface S, it has been shown that S can be partitioned into a thick part, $S_\delta := \{z \in S : \iota(z) \ge \delta\}$, and a thin part, $S \setminus S_\delta$, whose connected components $\{C_\alpha\}_\alpha$ are either collars of cusps or collars, half-collars or transverse half-collars of simple closed geodesics of length less than 2δ. Note that if the simple closed geodesic has length l, (2.2) gives that the width of C_α is

$$\operatorname{arccosh} \frac{\sinh \delta}{\sinh(l/2)} \,. \tag{3.3}$$

The following result in [32, Lemma 2.14] is needed.

Lemma 3.2. *Let S be a non-exceptional (geodesically bordered or non-bordered) Riemann surface,* $0 < \delta < \operatorname{arcsinh} 1$ *and C a connected component of $S \setminus S_\delta$. Let K be the collar given by the Collar Lemma corresponding to C. Then*

$$d_S(C, \partial K) \ge \log \frac{1}{\sinh \delta} \,.$$

Furthermore, if C is a collar of a simple closed geodesic γ, then $d_S(C, \partial K)$ is an increasing function in $L_S(\gamma)$.

Lemma 3.3. *Let S be a non-exceptional (geodesically bordered or non-bordered) Riemann surface,* $0 < \delta < \operatorname{arcsinh} 1$ *and C a connected component of $S \setminus S_\delta$. Let K be the collar given by the Collar Lemma corresponding to C. Then*

$$\log \frac{1}{\sinh \delta} \le d_S(C, \partial K) < \log \frac{\cosh \delta + 1}{\sinh \delta} \,.$$

Proof. Lemma 3.2 gives the lower bound, let us prove the upper one.

Since $0 < \delta < \operatorname{arcsinh} 1$, Margulis Lemma gives that C is a collar of either a cusp or a simple closed geodesic.

Assume first that C is a β-collar of a cusp r ($0 < \beta \le 2$). As usual, consider a fundamental domain D for S in the upper half-plane $\mathbb{H}$ contained in $\{x + iy \in \mathbb{H} : 0 \le x \le 1\}$ and such that $\{x + iy \in \mathbb{H} : 0 \le x \le 1,\, y > 1/2\}$ corresponds to the 2-collar K of r given by the Collar Lemma. Thus, $C_\beta = \{x + iy \in \mathbb{H} : 0 \le x \le 1,\, y > 1/\beta\}$ corresponds to C. Denote by z the point in ∂C corresponding to i/β. Since $\iota(z) = \delta < \operatorname{arcsinh} 1$, Lemma 2.5 gives that the shortest geodesic loop σ with base point z is contained in K. A geodesic in the upper half-plane with endpoints i/β and $1 + i/\beta$ can be a corresponding curve to σ. One gets

$$\sinh \delta = \sinh \iota(z) = \sinh \frac{L_S(\sigma)}{2} = \sinh \frac{d_{\mathbb{H}}(i/\beta, 1 + i/\beta)}{2} = \frac{\beta}{2} \,,$$

and $\beta = 2 \sinh \delta$. Thus,

$$d_S(C, \partial K) = d_{\mathbb{H}}(i/\beta, i/2) = \log \frac{2}{\beta} = \log \frac{1}{\sinh \delta} < \log \frac{\cosh \delta + 1}{\sinh \delta} \,.$$

Assume now that C is a collar of a simple closed geodesic γ, and denote the length of γ by $l = L_S(\gamma)$. By Collar Lemma and equation (3.3),

$$d_S(C, \partial K) = \operatorname{arccosh} \frac{\cosh(l/2)}{\sinh(l/2)} - \operatorname{arccosh} \frac{\sinh \delta}{\sinh(l/2)} = F(l, \delta)$$

is an increasing function in $l \in (0, 2\delta)$ by Lemma 3.2. Hence,

$$F(l, \delta) < \lim_{t \to 2\delta} F(t, \delta) = \operatorname{arccosh} \coth \delta$$
$$= \log \big(\coth \delta + \sqrt{\coth^2 \delta - 1} \big) = \log \frac{\cosh \delta + 1}{\sinh \delta} .$$

□

Lemma 3.4. *Let S be a non-exceptional (geodesically bordered or non-bordered) Riemann surface, $0 < \delta < \operatorname{arcsinh} 1$ and C a connected component of $S \setminus S_\delta$. Let K be the collar given by the Collar Lemma corresponding to C. Fix*

$$0 < \varepsilon \le \varepsilon_0 := \frac{1}{2} \log \frac{1}{\sinh \delta} ,$$

and an ε-net P_0 of S. Then

$$\{z \in S_\delta : d_S(z, \partial C) < 2\varepsilon\} \subset K.$$

Furthermore, if K_0 is a connected component of $K \setminus C \subset S_\delta$ and $\zeta \in \partial C \cap K_0$, then there exists $u \in P_0 \cap K_0$ with $d_S(u, \partial C) \le d_S(u, \zeta) < 2\varepsilon$.

Proof. Since $2\varepsilon \le \log \frac{1}{\sinh \delta}$, Lemma 3.2 gives $\{z \in S_\delta : d_S(z, \partial C) < 2\varepsilon\} \subset K$.

Let us choose $z \in K_0$ satisfying $d_S(z, \zeta) = d_S(z, \partial C) = \varepsilon$. Then there exists $u \in P_0$ with $d_S(u, z) < \varepsilon$, and so, $u \in P_0 \cap K_0 \subset S_\delta$ and $d_S(u, \partial C) \le d_S(u, \zeta) < 2\varepsilon$. □

Denote by $\lceil t \rceil$ the upper integer part of t, i.e., the smallest integer greater than or equal to t.

Given fixed δ and ε so that $0 < \delta < \operatorname{arcsinh} 1$ and $0 < \varepsilon \le \varepsilon_0$, where ε_0 is the constant in Lemma 3.4, consider an ε-net P_0 of S. Denote by $\widetilde{S}$ the Schottky double of S. A (δ, ε)-*metric-net* P in S associated to P_0 will be defined. In order to do it, first one needs to define a metric graph P_1 containing $P_0 \cap S_\delta$, as follows:

For each α such that C_α is a collar of a cusp, consider a geodesic $\gamma_\alpha : [0, \infty) \to \overline{C_\alpha}$ starting at ∂C_α and denote by $p_\alpha := \gamma_\alpha \cap \partial C_\alpha$ its starting-point. Choose γ_α in such a way that $p_\alpha \notin P_0$. Consider p_α as a new vertex, with $N(p_\alpha) = \{p \in P_0 \cap S_\delta : d_S(p, \partial C_\alpha) < 2\varepsilon\} \cup \{v_\infty\}$, where v_∞ is a new vertex with $L(p_\alpha v_\infty) = \infty$. Note that Lemma 3.4 gives $N(p_\alpha) \setminus \{v_\infty\} \neq \emptyset$.

For each α such that C_α is a half-collar of a simple closed geodesic γ in ∂S with length $l < 2\delta$, consider a geodesic γ_α in $\overline{C_\alpha}$ joining γ and ∂C_α, and let $p_\alpha^1 := \gamma_\alpha \cap \partial C_\alpha$ and $p_\alpha^2 := \gamma_\alpha \cap \gamma$ be its endpoints. Choose γ_α in such a way that $p_\alpha^1, p_\alpha^2 \notin P_0$. Consider p_α^1, p_α^2 as new vertices, with $N(p_\alpha^1) = \{p \in P_0 \cap S_\delta : d_S(p, \partial C_\alpha) < 2\varepsilon\} \cup \{p_\alpha^2\}$ and $N(p_\alpha^2) = \{p_\alpha^1\}$. Also, Lemma 3.4 gives $N(p_\alpha^1) \setminus \{p_\alpha^2\} \neq \emptyset$. Define the length of $p_\alpha^1 p_\alpha^2$ as $\lceil L_S(\gamma_\alpha) \rceil$, i.e., the upper integer part of (3.3).

For each α such that C_α is either a collar of a simple closed geodesic γ in S or a transverse half-collar of a simple closed geodesic γ in the Schottky double $\widetilde{S}$, with length $l < 2\delta$, consider a geodesic γ_α in $\overline{C_\alpha}$ joining the connected components β_α^1 and β_α^2 of ∂C_α. Let K_α be the collar (or the transverse half-collar) of γ given by the Collar Lemma, and let K_α^1, K_α^2 be the connected components of $K_\alpha \setminus C_\alpha \subset S_\delta$ containing $\beta_\alpha^1, \beta_\alpha^2$, respectively. Let $p_\alpha^1 := \gamma_\alpha \cap \beta_\alpha^1$ and $p_\alpha^2 := \gamma_\alpha \cap \beta_\alpha^2$ be the endpoints of γ_α. Choose γ_α in such a way that $p_\alpha^1, p_\alpha^2 \notin P_0$. Consider p_α^1, p_α^2 as new vertices, with $N(p_\alpha^1) = \{p \in P_0 \cap K_\alpha^1 : d_S(p, \beta_\alpha^1) < 2\varepsilon\} \cup \{p_\alpha^2\}$ and $N(p_\alpha^2) = \{p \in P_0 \cap K_\alpha^2 : d_S(p, \beta_\alpha^2) < 2\varepsilon\} \cup \{p_\alpha^1\}$. Define the length of $p_\alpha^1 p_\alpha^2$ as $\lceil L_S(\gamma_\alpha) \rceil$, i.e., the upper integer part of twice (3.3).

The vertices of the new graph P_1 are those of $P_0 \cap S_\delta$ and the union of $p_\alpha, p_\alpha^1, p_\alpha^2$, for every α (and v_∞ if there is some cusp in S). Two vertices $u, v \in P_0 \cap S_\delta$ are connected by an edge if and only if $0 < d_S(u, v) \le 4\varepsilon$ and they do not lie on different connected components of $C_\alpha \setminus \gamma$, where C_α is either a collar of a simple closed geodesic γ in S or a transverse half-collar of a simple closed geodesic γ in the Schottky double $\widetilde{S}$, with length $l < 2\delta$.

Recall that if C_α is not the collar of a cusp, connect the vertices p_α^1, p_α^2 by an edge with length $\lceil L_S(\gamma_\alpha)\rceil$. Thus, apply a homogeneous dilation in order to consider γ_α as an edge of P, i.e., if $x, y \in \gamma_\alpha$ and γ_{xy} denotes the arc in γ_α joining x and y, then $L_P(\gamma_{xy}) = L_S(\gamma_{xy})\lceil L_S(\gamma_\alpha)\rceil / L_S(\gamma_\alpha)$.

Every edge with at least an endpoint in $P_0 \cap S_\delta$ is considered to have length 1. Hence, $L(e) \geq 1$ for every $e \in E(P_1)$ (in fact, $L : E(P_1) \to \mathbb{Z}^+ \cup \{\infty\}$). Define P as the metric space obtained from P_1 by deleting the single point v_∞ if S has some cusp; otherwise, $P = P_1$. In this case P is said to be a (δ, ε)-metric-net in S.

Given any graph, metric graph or metric-net G, define

$$\operatorname{vol}_G\big(B_G(p,r)\big) = \big|\{v \in V(G) : \, d_G(v,p) < r\}\big| = \big|V(G) \cap B_G(p,r)\big|.$$

Thus, $\operatorname{lpgo}(G)$, $\operatorname{upgo}(G)$, $\operatorname{lgo}_\varphi(G)$ and $\operatorname{ugo}_\varphi(G)$, for any admissible function φ, can be defined as in the case of manifolds.

If X is a complete (without border or with totally geodesic border) n-dimensional Riemannian manifold with Ricci curvature bounded from below by a constant $-(n-1)k^2$ $(k > 0)$, then the following facts hold:

$(P1)$ for every $x, y \in X$ there exists a geodesic in X joining x and y,

$(P2)$ $\operatorname{vol}_X\big(B_X(x,r)\big) \leq V(r)$ for every $x \in X$ and $r > 0$,

$(P3)$ the function $r \to \operatorname{vol}_X\big(B_X(x,r)\big)/V(r)$ is monotone non-increasing for every $x \in X$.

where $V(r)$ denotes the volume of a ball in the simply connected n-dimensional Riemannian manifold of constant curvature $-k^2$. These facts are well-known; the second and third one follow from standard comparison theorems. In particular, $V(r) = 4\pi \sinh^2(r/2)$ if X is a non-exceptional Riemann surface. We also have $A_X\big(B_X(x,r)\big) = 4\pi \sinh^2(r/2)$ for every non-exceptional Riemann surface$X$, $x \in X$ and $0 < r \leq \min\{\iota(x), d_X(x, \partial X)\}$; furthermore, $A_X\big(B_X(x,r)\big) \geq 2\pi \sinh^2(r/2)$ for every $x \in X$ and $0 < r \leq \iota(p)$.

These facts have the following consequences (see [24, Lemmas 2.3 and 2.5]):

Lemma 3.5. *Let X be a complete (without border or with totally geodesic border) n-dimensional Riemannian manifold with Ricci curvature bounded from below by a constant $-(n-1)k^2$ $(k > 0)$, and let P be an ε-net in X. Then there is a constant ν, which just depends on n, k, ε, r, such that $|\{p \in P : \, x \in B_X(p,r)\}| \leq \nu$, for every $r > 0$ and $x \in X$. Consequently, P is uniform.*

Lemma 3.6. *Let X be a complete (without border or with totally geodesic border) n-dimensional Riemannian manifold with Ricci curvature bounded from below by a constant $-(n-1)k^2$ $(k > 0)$, and let P be an ε-net in X. Then there are constants $a \geq 1$ and $b \geq 0$, which just depend on n, k, ε, such that*

$$\frac{1}{4\varepsilon}\, d_X(p_1,p_2) \leq d_P(p_1,p_2) \leq a\, d_X(p_1,p_2) + b \tag{3.4}$$

for every $p_1, p_2 \in P$. Consequently, the inclusion of P into X is a quasi-isometry.

Although Kanai requires in [24, Lemmas 2.3 and 2.5] X to be non-bordered, the proofs there use just properties $(P1)$, $(P2)$ and $(P3)$, that also hold for complete manifolds with totally geodesic border.

The following result in [24, Lemma 2.2] is also needed.

Lemma 3.7. *Let G_1 and G_2 be graphs with edges of length 1, let G_1 be uniform, and let $f : G_1 \to G_2$ be a quasi-isometry. Then there is a constant μ such that $|S| \leq \mu\, |f(S)|$ for any finite subset S of G_1.*

Lemma 3.8. *Let G_1 and G_2 be quasi-isometric uniform metric graphs such that $c_1 \leq L(e) \leq c_2$ for every $e \in E(G_1) \cup E(G_2)$ and some positive constants c_1, c_2. Then G_2 is of polynomial (resp.exponential) growth if and only if G_1 is; moreover, $\operatorname{lpgo}(G_2) = \operatorname{lpgo}(G_1)$, $\operatorname{upgo}(G_2) = \operatorname{upgo}(G_1)$, $\operatorname{lgo}_\varphi(G_2) = \operatorname{lgo}_\varphi(G_1)$ and $\operatorname{ugo}_\varphi(G_2) = \operatorname{ugo}_\varphi(G_1)$ for every admissible function φ.*

Proof. Since the growth rates are invariant by bilipschitz maps, one can assume that $L(e) = 1$ for every $e \in E(G_1) \cup E(G_2)$. Let $f : G_1 \to G_2$ be an (a, b)-quasi-isometry. Fix $p_1 \in G_1$, and define $p_2 = f(p_1)$. By Lemma 3.7,

$$\big|\{p \in G_1 : d_{G_1}(p_1, p) < r\}\big| \le \mu \, \big|f(\{p \in G_1 : d_{G_1}(p_1, p) < r\})\big| \le \mu \, \big|\{q \in G_2 : d_{G_2}(p_2, q) < ar + b\}\big|.$$

In a similar way, if $f' : G_2 \to G_1$ is an (a', b')-quasi-isometry, $p_2 \in G_2$ and $p_1 = f'(p_2)$, then

$$\big|\{p \in G_2 : d_{G_2}(p_2, p) < r\}\big| \le \mu' \, \big|\{q \in G_1 : d_{G_1}(p_1, q) < a'r + b'\}\big|.$$

These inequalities imply the lemma. □

Lemma 3.9. *Let X be a complete (without border or with totally geodesic border) n-dimensional Riemannian manifold with a lower bound on its Ricci curvature, $\varepsilon > 0$ and P an ε-net of X. If P has polynomial growth, then X also does. If X has exponential growth, then P also does. Also,* $\operatorname{lpgo}(X) \le \operatorname{lpgo}(P)$, $\operatorname{upgo}(X) \le \operatorname{upgo}(P)$, $\operatorname{lgo}_\varphi(X) \le \operatorname{lgo}_\varphi(P)$ *and* $\operatorname{ugo}_\varphi(X) \le \operatorname{ugo}_\varphi(P)$ *for every admissible function φ.*

Proof. By Lemma 3.6, there are constants $a \ge 1$ and $b \ge 0$, such that

$$\frac{1}{4\varepsilon}\, d_X(u, v) \le d_P(u, v) \le a\, d_X(u, v) + b$$

for every $u, v \in P$. Fix a point w in P. Since P is an ε-net of X, the ball $B_X(w, (r-b)/a - \varepsilon)$ is covered by

$$\{B_X(v, \varepsilon) : v \in P,\, d_X(w, v) < (r-b)/a\} \subseteq \{B_X(v, \varepsilon) : v \in P,\, d_P(w, v) < r\},$$

and property $(P2)$ gives

$$\operatorname{vol}_X\big(B_X(w, (r-b)/a - \varepsilon)\big) \le V(\varepsilon)\big|\{v \in P : d_P(w, v) < r\}\big| = V(\varepsilon)\operatorname{vol}_P\big(B_P(w, r)\big).$$

This inequality implies the lemma. □

Denote by $\lfloor t \rfloor$ the lower integer part of t, i.e., the largest integer not greater than t.

Lemma 3.10. *Let G be a uniform metric graph such that $L(e) \ge 1$ for every $e \in E(G)$. Let G' be a metric graph isometric to G obtained by choosing a set A of interior points of edges in G in such a way that $V(G') = V(G) \cup A$, $L(e') \ge 1$ for every $e' \in E(G')$, and for each $e' \in E(G')$ there is $e \in E(G)$ with $e' \subseteq e$. Then G' is uniform and G has polynomial (respectively, exponential) growth if and only if G' does. Also,* $\operatorname{lpgo}(G) \le \operatorname{lpgo}(G') \le \operatorname{lpgo}(G) + 1$ *and* $\operatorname{upgo}(G) \le \operatorname{upgo}(G') \le \operatorname{upgo}(G) + 1$.

Proof. Note that, by construction, if $v \in V(G)$, then $\deg_{G'}(v) = \deg_G(v)$ and, if $v \in V(G') \setminus V(G) = A$, then $\deg_{G'}(v) = 2$. Hence, if G is a ν-uniform graph, then G' is $\max\{\nu, 2\}$-uniform.

Let r be a positive number and $v \in V(G)$. Define

$$V_0 := V(G) \cap B_G(v, r), \quad E_0 := \big\{e \in E(G) : e \cap B_G(v, r) \ne \emptyset\big\}, \quad E_0' := \big\{e' \in E(G') : e' \cap B_{G'}(v, r) \ne \emptyset\big\}.$$

Thus,

$$|E_0| \le \sum_{v \in V_0} \deg_{G'}(v) \le \nu |V_0| = \nu \operatorname{vol}_G\big(B_G(v, r)\big).$$

Every $e \in E_0$ satisfies $L(e \cap B_G(v, r)) \le r$. Since $L(e') \ge 1$ for every $e' \in E(G')$, each $e \in E_0$ give rise to at most $\lfloor r \rfloor$ edges in E_0' and at most $\lfloor r - 1 \rfloor$ vertices in $B_{G'}(v, r) \setminus V(G)$. Then,

$$\begin{aligned}
\big|V(G') \cap B_{G'}(v, r) \setminus V(G)\big| &\le \lfloor r-1 \rfloor |E_0| \le \lfloor r-1 \rfloor \nu \operatorname{vol}_G\big(B_G(v, r)\big),\\
\operatorname{vol}_{G'}\big(B_{G'}(v, r)\big) &= \operatorname{vol}_G\big(B_G(v, r)\big) + \big|V(G') \cap B_{G'}(v, r) \setminus V(G)\big|\\
&\le \big(1 + \lfloor r-1 \rfloor \nu\big) \operatorname{vol}_G\big(B_G(v, r)\big)\\
&\le \nu\, r \operatorname{vol}_G\big(B_G(v, r)\big).
\end{aligned}$$

Thus, $\operatorname{vol}_G\big(B_G(v, r)\big) \le \operatorname{vol}_{G'}\big(B_{G'}(v, r)\big) \le \nu\, r \operatorname{vol}_G\big(B_G(v, r)\big)$, and the statement trivially follows from these inequalities. □

The next technical result is interesting by itself.

Theorem 3.11. *Let S be a non-exceptional (geodesically bordered or non-bordered) Riemann surface, $r > 0$, $0 < \delta \le \delta_0 := \operatorname{arcsinh}(\exp(-3 \operatorname{arccosh} 3))$ and $p \in S_\delta$. Then*

$$\frac{1}{2} A_S\big(B_S(p,r)\big) < A_S\big(B_S(p,r) \cap S_\delta\big) \le A_S\big(B_S(p,r)\big).$$

Proof. The second inequality is trivial, let us prove the first one.

Note that choosing δ this way gives $\delta < \operatorname{arcsinh} 1$, needed in order to apply Margulis Lemma.

Let γ be a simple closed geodesic contained in $S \setminus S_\delta$ with $L_S(\gamma) = l$, let C_γ be the collar of γ defined by Margulis Lemma (i.e., the connected component of $S \setminus S_\delta$ containing γ) and let w_δ denote its width, that is, $\cosh w_\delta = \sinh \delta / \sinh(l/2)$. Let $\partial C_\gamma = \eta_1 \cup \eta_2$, where η_1 and η_2 are closed curves. Let K_γ denote the collar of γ given by the Collar Lemma, with w being its width.

Since $\delta \le \delta_0$, Lemma 3.2 gives $d_S(C_\gamma, \partial K_\gamma) = w - w_\delta \ge -\log \sinh \delta \ge 3 \operatorname{arccosh} 3 > \operatorname{arccosh} 3 =: h$. Let K_γ^j be the connected component of $K_\gamma \setminus C_\gamma$ containing η_j and $H_\gamma^j := \{z \in K_\gamma^j : d_S(z, C_\gamma) \le h\}$, for $j = 1, 2$. We have $H_\gamma^j \ne \emptyset$, since $d_S(C_\gamma, \partial K_\gamma) = w - w_\delta \ge 3h$.

Note that $A_S(H_\gamma^j) - A_S(C_\gamma) = (l \cosh(w_\delta + h) - l \cosh w_\delta) - 2l \cosh w_\delta > l \cosh w_\delta \cosh h - 3l \cosh w_\delta = 0$ since $h = \operatorname{arccosh} 3$, and so, $A_S(H_\gamma^j) > A_S(C_\gamma)$.

Also, one has

$$\begin{aligned} L_S(\partial H_\gamma^j \setminus \eta_j) &= l \cosh(w_\delta + \operatorname{arccosh} 3) \le 2l \cosh w_\delta \cosh(\operatorname{arccosh} 3) \\ &= 6\, l\, \frac{\sinh \delta}{\sinh(l/2)} < 12 \sinh \delta \le 12 \exp(-3 \operatorname{arccosh} 3) < 2 \operatorname{arccosh} 3 = 2h. \end{aligned}$$

Let p be a point in S_δ and let $B_S(p,r)$ be the ball with radius $r > 0$.

Assume first that $p \in H_\gamma^1 \cup H_\gamma^2$ for some simple closed geodesic γ with $L_S(\gamma) < 2\delta$. Without loss of generality one can assume that $p \in H_\gamma^1$. If $r \ge 2 \operatorname{arccosh} 3 = 2h$, then $r > h + L_S(\partial H_\gamma^1 \setminus \eta_1)/2$ and therefore $H_\gamma^1 \subset B_S(p,r)$. Since $H_\gamma^1 \subset S_\delta$ and $A_S(H_\gamma^1) > A_S(C_\gamma)$, one gets

$$\begin{aligned} A_S\big(B_S(p,r) \cap C_\gamma\big) &< A_S\big(B_S(p,r) \cap K_\gamma \cap S_\delta\big), \\ A_S\big(B_S(p,r) \cap K_\gamma\big) &< 2 A_S\big(B_S(p,r) \cap K_\gamma \cap S_\delta\big). \end{aligned} \tag{3.5}$$

If $r < 2 \operatorname{arccosh} 3 = 2h$, since by the choice of δ one gets that $w - w_\delta \ge 3h$, then $r < w - w_\delta - h$ and so, $B_S(p,r) \subset K_\gamma$ (since $p \in H_\gamma^1$ gives $d_S(p, C_\gamma) \le h$ and $d_S(p, \partial K_\gamma) \ge w - w_\delta - h > r$), which trivially gives (3.5).

If p does not belong to $H_\gamma^1 \cup H_\gamma^2$, then one gets $B_S(p,r) \cap C_\gamma = \emptyset$ for $r \le d_S(p, C_\gamma)$, and therefore (3.5) holds; for $r > d_S(p, C_\gamma)$, an argument similar to the one given when $p \in H_\gamma^1 \cup H_\gamma^2$ gives (3.5).

If K is the collar of a cusp given by the Collar Lemma, then a similar argument gives

$$A_S\big(B_S(p,r) \cap K\big) < 2 A_S\big(B_S(p,r) \cap K \cap S_\delta\big). \tag{3.6}$$

By applying (3.5) and (3.6) to each connected component of $B_S(p,r) \cap (S \setminus S_\delta)$ one obtains the first inequality. □

The following result in [32, Lemma 2.11] will be needed.

Lemma 3.12. *Let S be a non-exceptional (geodesically bordered or non-bordered) Riemann surface, $0 < \delta < \operatorname{arcsinh} 1$ and C a connected component of $S \setminus S_\delta$. If η is a connected component of ∂C, then $L_S(\eta) \le 2 \sinh \delta < 2$.*

Lemma 3.13. *Let S be a non-exceptional (geodesically bordered or non-bordered) Riemann surface, $0 < \delta \le \delta_1 := \operatorname{arcsinh}(\sqrt{2}/2)$, C a connected component of $S \setminus S_\delta$ and K the collar given by the Collar Lemma corresponding to C. Let γ be a geodesic in S joining $z_1, z_2 \in S \setminus C$ such that a connected component of $\gamma \cap \overline{C}$ joins two points in the same connected component of ∂C. Then $\gamma \cap \overline{C}$ is connected, γ is contained in K,*

z_1, z_2 are in the same connected component of $K \setminus C$, and $d_S(z_1, z_2) \le c$, where c is a constant which just depends on δ.

Proof. Since $0 < \delta \le \operatorname{arcsinh}(\sqrt{2}/2) < \operatorname{arcsinh} 1$, Margulis Lemma gives that C is a collar of either a cusp or a simple closed geodesic.

Assume first that C is a β-collar of a cusp r $(0 < \beta \le 2)$. As usual, consider a fundamental domain D for S in the upper half-plane $\mathbb{H}$ contained in $\{x + iy \in \mathbb{H} : 0 \le x \le 1\}$ and such that $\{x + iy \in \mathbb{H} : 0 \le x \le 1,\, y > 1/2\}$ corresponds to the 2-collar K of r given by the Collar Lemma. Thus, $C_\beta = \{x + iy \in \mathbb{H} : 0 \le x \le 1,\, y > 1/\beta\}$ corresponds to C. Let $z \in \partial C$ be the corresponding point to $i/\beta \in \partial C_\beta$. Since $\iota(z) = \delta < \operatorname{arcsinh} 1$, Lemma 2.5 gives that the shortest geodesic loop σ with base point z is contained in K. Let us represent σ in the upper half-plane by means of a geodesic with endpoints i/β and $1 + i/\beta$. Then

$$\sinh \delta = \sinh \iota(z) = \sinh \frac{L_S(\sigma)}{2} = \sinh \frac{d_{\mathbb{H}}(i/\beta, 1 + i/\beta)}{2} = \frac{\beta}{2}\,,$$

and $\beta = 2 \sinh \delta$.

Denote by γ_0 a connected component of $\gamma \cap \overline{C}$ joining two points in ∂C (in this case, ∂C is connected). The curve γ_0 corresponds to an arc γ_1 in the fundamental domain D joining two points $w_1 = a_1 + i\beta$ and $w_2 = a_2 + i\beta$ contained in a geodesic $\gamma_2 = \{x + iy \in \mathbb{H} : (x - x_0)^2 + y^2 = R^2\}$ for some $x_0 \in \mathbb{R}$ and $R > 0$. Any geodesic joining the points $x_1 + iy_1$ and $x_2 + iy_2$ with $y_1, y_2 > 1/2$ corresponds to a geodesic in S if and only if $|x_1 - x_2| \le 1/2$. (Note that if $x_2 = x_1 + 1/2 + s$ for some $0 < s < 1/2$, then $x_1 + 1 + iy_1$ and $x_1 + iy_1$ represent the same point in S, and if η_1 and η_2 are the geodesics in $\mathbb{H}$ joining $x_1 + iy_1$ and $x_2 + iy_2$, and $x_1 + 1 + iy_1$ and $x_2 + iy_2$, respectively, then $L_{\mathbb{H}}(\eta_1) > L_{\mathbb{H}}(\eta_2)$. A similar argument also works if $s \ge 1/2$.) Hence, without loss of generality one can assume that $0 \le a_1 < a_2 \le 1/2$. We are going to obtain a lower bound E_{γ_3} for the imaginary part of the points in the set $\gamma_3 = \{x + iy \in \mathbb{H} : (x - x_0)^2 + y^2 = R^2,\, 0 \le x \le 1/2\} \subset \gamma_2$. One can check that $E_{\gamma_3} > E_g$ if g is the geodesic starting at the point $1/2 + i/\beta$ and tangent to ∂C_β, i.e., $g = \{x + iy \in \mathbb{H} : (x - 1/2)^2 + y^2 = 1/\beta^2,\, 0 \le x \le 1/2\}$. Hence, $E_{\gamma_3} > E_g = \sqrt{1/\beta^2 - 1/4}$. Since $\delta \le \operatorname{arcsinh}(\sqrt{2}/2)$, one gets $\beta = 2 \sinh \delta \le \sqrt{2}$ and $\sqrt{1/\beta^2 - 1/4} \ge \sqrt{1/2 - 1/4} = 1/2$. Therefore, $\gamma_3 \subset \{x + iy \in \mathbb{H} : y > 1/2\}$, and conclude that γ is contained in the 2-collar K. Thus, $\gamma \cap \overline{C}$ is connected, and z_1, z_2 are in $K \setminus C$.

Assume now that C is a collar of a simple closed geodesic σ in S. Hence, the length ℓ of σ verifies $\ell < 2\delta$. As before, consider a fundamental domain D for S in $\mathbb{H}$ such that $\{x + iy \in \mathbb{H} :\ x = 0\}$ corresponds to σ and $\{x + iy \in \mathbb{H} :\ e^{-\ell/2} \le r = \sqrt{x^2 + y^2} \le e^{\ell/2}\,,\ 0 \le \theta = \arctan(x/y) < \theta_c\}$ corresponds to one half of the collar K of σ given by the Collar Lemma (see Lemma 2.3). It is known (see, e.g., [4, p. 162]) that θ_c is related with the width w of the collar K via the formula

$$\cosh w = \sec \theta_c\,.$$

By the Collar Lemma, $\cosh w = \coth(\ell/2)$. Therefore,

$$\cos \theta_c = \tanh(\ell/2)\,.$$

By (3.3), the width w_δ of the collar C verifies

$$\cosh w_\delta = \frac{\sinh \delta}{\sinh(\ell/2)}\,.$$

One half of the collar C can be represented in $\mathbb{H}$ as $\{x + iy \in \mathbb{H} :\ e^{-\ell/2} \le r = \sqrt{x^2 + y^2} \le e^{\ell/2}\,,\ 0 \le \theta = \arctan(x/y) < \theta_M\}$, where $\theta_M \le \theta_c$, and θ_M is related to w_δ via the formula (see, e.g., [4, p.162])

$$\cosh w_\delta = \sec \theta_M\,,$$

and as a consequence,

$$\cos \theta_M = \frac{\sinh(\ell/2)}{\sinh \delta}\,.$$

In order to prove the lemma in this case, it suffices to consider the extreme case where the geodesic γ_1 representing γ in $\mathbb{H}$ is tangent to the ray $y = m_M x$, with $m_M = \operatorname{cotan}\theta_M$, $x > 0$, in the intersection with the circumference $x^2 + y^2 = e^\ell$. Let R be the Euclidean radius of this geodesic γ_1 and $(x_0, 0)$ its Euclidean center. The tangency point has rectangular coordinates $x = e^{\ell/2}\sin\theta_M$, $y = e^{\ell/2}\cos\theta_M$. The intersection of the real axis with the straight line passing through this point with slope $-1/m_M$ is the center $(x_0, 0)$ of the geodesic. An easy computation gives that $x_0 = e^{\ell/2}/\sin\theta_M$. Also, R is the Euclidean distance from the center to the tangency point, and it is easy to get that $R = e^{\ell/2}\operatorname{cotan}\theta_M$. The point (x_1, y_1) which is the intersection of the geodesic $(x - x_0)^2 + y^2 = R^2$ with the circumference $x^2 + y^2 = 1$ verifies

$$x_1 = \frac{x_0^2 - R^2 + 1}{2x_0} = \frac{\sin\theta_M}{2e^{\ell/2}}\Big(\frac{e^\ell}{\sin^2\theta_M} - e^\ell\operatorname{cotan}^2\theta_M + 1\Big) = \frac{e^\ell + 1}{2e^{\ell/2}}\sin\theta_M = \cosh(\ell/2)\sin\theta_M .$$

The "co-argument" θ of (x_1, y_1) verifies $\tan\theta = x_1/y_1 = x_1/\sqrt{1 - x_1^2}$ and, in order to finish the proof in this case, it is enough to show that $\theta < \theta_c$ or, equivalently, that $\tan^2\theta < \tan^2\theta_c = 1/\sinh^2(\ell/2)$. This is equivalent to show that

$$\sinh^2(\ell/2) = \operatorname{cotan}^2\theta_c < \frac{1}{x_1^2} - 1 = \frac{1}{\cosh^2(\ell/2)\sin^2\theta_M} - 1 \qquad \Leftrightarrow \qquad \cosh^4(\ell/2)\sin^2\theta_M < 1$$

or

$$\cosh^4(\ell/2)(\sinh^2\delta - \sinh^2(\ell/2)) < \sinh^2\delta\,.$$

But this inequality follows from the fact that the function $f(t) = (1 + t)^2(\sinh^2\delta - t)$, defined for $0 < t \le \sinh^2\delta$, is strictly decreasing since $\sinh\delta \le \sqrt{2}/2$, and therefore $f(t) < f(0) = \sinh^2\delta$.

Therefore, one concludes that γ is contained in the collar K, its endpoints z_1, z_2 are contained in the same connected component of $K \setminus C$, and $\gamma \cap \overline{C}$ is connected.

In both cases (collar of cusp or geodesic), if η is a connected component of ∂C, then

$$d_S(z_1, z_2) \le d_S(\partial K, \partial C) + \frac{1}{2}\, L_S(\eta) + d_S(\partial K, \partial C).$$

Thus, Lemmas 3.3 and 3.12 give

$$d_S(z_1, z_2) \le 2\log\frac{\cosh\delta + 1}{\sinh\delta} + \sinh\delta\,.$$

□

Lemma 3.14. *Let S be a non-exceptional (geodesically bordered or non-bordered) Riemann surface, $0 < \delta \le \delta_1 := \operatorname{arcsinh}(\sqrt{2}/2)$, C a connected component of $S \setminus S_\delta$ that is the collar of a simple closed geodesic in S, and γ a geodesic in S joining $z_1, z_2 \in S$. If $\gamma \cap \overline{C}$ is not connected, then the connected components of $\gamma \cap \overline{C}$ do not join the two connected components of ∂C. Furthermore, if $d_S(z_1, z_2) > c$, with c the constant in Lemma 3.13, then $z_1, z_2 \in C$, and $\gamma \cap \overline{C}$ has exactly two connected components, the first one joining z_1 and a connected component of ∂C, and the second one joining z_2 with the other connected component of ∂C.*

Proof. Denote by σ the simple closed geodesic in S whose collar is C. Seeking for a contradiction assume that a connected component γ_0 of $\gamma \cap \overline{C}$ joins the two connected components of ∂C. Let ζ_1 be a point in the closure of other connected component of $\gamma \cap \overline{C}$ contained in ∂C, and denote by F the connected component of ∂C containing ζ_1. Let ζ_2 be a point of the closure of γ_0 contained in F. Let γ_1 be the curve contained in γ joining ζ_1 and ζ_2. Thus, γ_1 contains two curves joining ∂C and ∂K, and Lemma 3.2 gives

$$d_S(\zeta_1, \zeta_2) = L_S(\gamma_1) \ge 2d_S(C, \partial K) \ge 2\log\frac{1}{\sinh\delta}\,.$$

Also, (3.3) gives

$$\begin{aligned} d_S(\zeta_1, \zeta_2) &\le \frac{1}{2}\, L_S(F) = \frac{1}{2}\, L_S(\sigma)\cosh\Big(\operatorname{arccosh}\frac{\sinh\delta}{\sinh(L_S(\sigma)/2)}\Big) \\ &= \frac{1}{2}\, L_S(\sigma)\,\frac{\sinh\delta}{\sinh(L_S(\sigma)/2)} \le \sinh\delta. \end{aligned}$$

But this is a contradiction, since $0 < \delta \le \operatorname{arcsinh}(1/2)$ gives

$$2\log\frac{1}{\sinh\delta} \ge 2\log 2 > \frac{\sqrt{2}}{2} \ge \sinh\delta.$$

Thus, a connected component of $\gamma \cap \overline{C}$ can not join the two connected components of ∂C.

If $d_S(z_1, z_2) > c$ with c the constant in Lemma 3.13, then Lemma 3.13 gives that a connected component of $\gamma \cap \overline{C}$ can not join two points in the same connected component of ∂C. Hence, a connected component of $\gamma \cap \overline{C}$ can not join two points in ∂C, and the conclusion of the lemma follows. □

Lemmas 3.13 and 3.14 have the following consequence.

Corollary 3.15. *Let S be a non-exceptional (geodesically bordered or non-bordered) Riemann surface, $0 < \delta \le \delta_1 := \operatorname{arcsinh}(\sqrt{2}/2)$, and γ a geodesic in S joining $z_1, z_2 \in S_\delta$. If C is a connected component of $S \setminus S_\delta$ with $\gamma \cap \overline{C} \ne \emptyset$, then $\gamma \cap \overline{C}$ is connected and it joins two points in ∂C.*

Remark 3.16. *In the coming results, ε_0, δ_1 and δ_0 are the constant in Lemmas 3.4, 3.13 and Theorem 3.11, respectively.*

Lemma 3.17. *Let S be a non-exceptional (geodesically bordered or non-bordered) Riemann surface, $0 < \delta < \operatorname{arcsinh} 1$, $0 < \varepsilon \le \varepsilon_0$, and P_0 any ε-net in S. Then, for each $z \in S_\delta$ there exists $v \in P_0 \cap S_\delta$ with $d_S(v, z) < 2\varepsilon$. Furthermore, v and z belong to the same connected component of S_δ.*

Proof. If $z \in S_\delta$ with $d_S(z, \partial S_\delta) \ge \varepsilon$, then there exists $v \in P_0$ with $d_S(v, z) < \varepsilon$; hence, $v \in P_0 \cap S_\delta$.

Consider now $z \in S_\delta$ with $d_S(z, \partial S_\delta) < \varepsilon$. Let C be a connected component of $S \setminus S_\delta$ with $d_S(z, C) = d_S(z, \partial S_\delta) < \varepsilon$, and K the collar given by the Collar Lemma corresponding to C. Since $\varepsilon \le \varepsilon_0$, Lemma 3.2 gives that there is a geodesic η starting orthogonal to ∂C, containing z, with length ε and contained in K. Let z_0 be the endpoint of η which does not belong to ∂C. Thus, $z_0 \in S_\delta$, $d_S(z_0, \partial S_\delta) = \varepsilon$ and $d_S(z_0, z) \le \varepsilon$, and there exists $v \in P_0$ with $d_S(v, z_0) < \varepsilon$; hence, $v \in P_0 \cap S_\delta$ and $d_S(v, z) < 2\varepsilon$. □

Recall that if G is a graph and $V_0 \subset V(G)$, then the *induced subgraph* G_0 by V_0 is the graph with $V(G_0) = V_0 \subset V(G)$ and $E(G_0) = \{uv \in E(G) : u, v \in V_0\} \subset E(G)$.

Lemma 3.18. *Let S be a non-exceptional (geodesically bordered or non-bordered) Riemann surface, $0 < \delta < \operatorname{arcsinh} 1$, $0 < \varepsilon \le \varepsilon_0$, P_0 be an ε-net in S and P a (δ, ε)-metric-net in S associated to P_0. Let S^0 be a connected component of S_δ. Then the subgraph of P (and of P_0) induced by $P_0 \cap S^0$ is connected. Furthermore, if $z_1, z_2 \in S^0$, η is a geodesic in S joining them, and $\eta \subset S^0$, then there exist constants $a \ge 1, b \ge 0$, which just depend on ε, such that*

$$d_{P_0\cap S^0}(u_1, u_2) \le a d_S(z_1, z_2) + b,$$

for every $u_1, u_2 \in P_0 \cap S^0$ with $d_S(u_1, z_1) < 2\varepsilon$ and $d_S(u_2, z_2) < 2\varepsilon$.

Proof. Consider the set $\Lambda := \{w \in P_0 \cap S^0 : B_S(w, 2\varepsilon) \cap \eta \ne \emptyset\}$. Since $\{B_S(w, 2\varepsilon) \cap \eta\}_{w\in\Lambda}$ is an open covering of η by Lemma 3.17, and $ww' \in E(P_0 \cap S^0)$ when $B_S(w, 2\varepsilon) \cap B_S(w', 2\varepsilon) \ne \emptyset$, one has that Λ is a connected set. Since z_1 and z_2 are arbitrary points in S^0, then $P_0 \cap S^0$ is connected. Since Λ is connected and $u_1, u_2 \in \Lambda$, one obtains $d_{P_0\cap S^0}(u_1, u_2) \le |\Lambda| - 1$.

Let m be the unique integer such that $m - 1 < d_S(z_1, z_2)/(4\varepsilon) \le m$ and set $s_0 = z_1, s_1, \dots, s_{m-1}, s_m = z_2$ in η with $d_S(s_{i-1}, s_i) = d_S(z_1, z_2)/m \le 4\varepsilon$. Note that if $w \in \Lambda$, then $w \in \cup_{j=0}^m B_S(s_j, 4\varepsilon)$, and thus $\Lambda \subset \cup_{j=0}^m \{w \in P_0 \cap S^0 : s_j \in B_S(w, 4\varepsilon)\}$. By Lemma 3.5, there exists a constant ν, which just depends on ε, such that $\big|\{w \in P_0 \cap S^0 : z \in B_S(w, 4\varepsilon)\}\big| \le \nu$ for every $z \in S^0$. Therefore,

$$\begin{aligned} d_{P_0\cap S^0}(u_1, u_2) &\le |\Lambda| - 1 \le \sum_{j=0}^m \big|\{w \in P_0 \cap S^0 : s_j \in B_S(w, 4\varepsilon)\}\big| - 1 \\ &\le \nu(m+1) - 1 < \nu\Big(\frac{d_S(z_1, z_2)}{4\varepsilon} + 2\Big) - 1. \end{aligned}$$

□

Lemma 3.19. *Let S be a non-exceptional (geodesically bordered or non-bordered) Riemann surface, $0<\delta\le\delta_1$, $0<\varepsilon\le\varepsilon_0$, and P any (δ,ε)-metric-net in S. Then the inclusion $i: P\cap S\to S$ is a quasi-isometry.*

Proof. Let P_0 be an ε-net in S such that P is a (δ,ε)-metric-net associated to P_0.

Denote by $\{S^n_\delta\}$ the connected components of S_δ, and by $\{C^m\}$ the connected components of $S\setminus S_\delta$.

Let P^n be the subgraph of P induced by $(P_0\cap S^n_\delta)\cup A^n$, where A^n is the set of vertices in $P\setminus P_0$ that are neighbors of some vertex in $P_0\cap S^n_\delta$. Thus, every $e\in E(P^n)$ has length 1. Note that $V(P)=\cup_n V(P^n)$.

Consider two points $p,q\in P\cap S$ and let $\sigma:[0,\ell]\to S$ be a geodesic in S joining p and q. We consider σ as an oriented curve from p to q.

Case A. Assume first that $p,q\in P\cap P_0$.

Note that σ meets at most a finite number of S^n_δ and C^m, since it is a compact curve; in order to simplify the notation, denote them by $S^1_\delta, C^1, S^2_\delta, C^2,\dots, C^{r-1}, S^r_\delta$, where $p\in S^1_\delta$, $q\in S^r_\delta$, and the geodesic σ meets C^k after S^k_δ, and S^{k+1}_δ after C^k (recall that we consider σ as an oriented curve). If σ is contained in a connected component of S_δ, then Lemma 3.18 gives $d_P(p,q)\le a\,d_S(p,q)+b$, for constants $a\ge1$ and $b\ge0$ which just depend on ε. Hence, one can assume that $r\ge2$.

Corollary 3.15 gives that $\sigma\cap\overline{C^k}$ is connected, and it joins two points in ∂C^k, for $1\le k<r$. Hence, $C^j\ne C^k$ for every $1\le j<k<r$. However, it is possible to have $S^j_\delta=S^k_\delta$ with $1\le j<k\le r$.

Case A.1. Assume that $\sigma\cap\overline{C^k}$ joins two points in different connected components of ∂C^k (and so, C^k is a collar of a geodesic), for every $1\le k<r$.

Denote by v^1_k, v^2_k the vertices in P corresponding to the collar C^k, in such a way that v^1_k (respectively, v^2_k) is the corresponding vertex in P to the connected component η^1_k (respectively, η^2_k) of ∂C^k which intersects σ in the first (respectively, second) time, for $1\le k<r$.

Let us define the points $x^1_k:=\sigma\cap\partial S^k_\delta\cap\partial C^k$ and $x^2_k:=\sigma\cap\partial C^k\cap\partial S^{k+1}_\delta$ for every $1\le k<r$ with $S^k_\delta\ne S^{k+1}_\delta$. If $S^k_\delta=S^{k+1}_\delta$, then $\sigma\cap\partial S^k_\delta\cap\partial C^k$ is a set with two points; define x^1_k (respectively, x^2_k) as the first (respectively, second) point in σ that meets $\partial S^k_\delta\cap\partial C^k$.

Let us denote by $[x,y]$ any geodesic joining x and y (since uniqueness of geodesics is not required, this notation is ambiguous, but it is convenient).

Since $\sigma:=[p,x^1_1]\cup[x^1_1,x^2_1]\cup[x^2_1,x^1_2]\cup\cdots\cup[x^1_{r-1},x^2_{r-1}]\cup[x^2_{r-1},q]$, it holds

$$d_S(p,q)=L_S(\sigma)=d_{S^1_\delta}(p,x^1_1)+d_{C^1}(x^1_1,x^2_1)+\sum_{k=2}^{r-1}\Big(d_{S^k_\delta}(x^2_{k-1},x^1_k)+d_{C^k}(x^1_k,x^2_k)\Big)+d_{S^r_\delta}(x^2_{r-1},q).$$

Also, one has

$$d_{C^k}(x^1_k,x^2_k)\ge d_{C^k}(v^1_k,v^2_k)\ge L_P(v^1_kv^2_k)-1.$$

By Lemma 3.4, there exist $u^1_k\in P_0\cap S^k_\delta$ and $u^2_k\in P_0\cap S^{k+1}_\delta$ with $d_S(u^1_k,\eta^1_k)\le d_S(u^1_k,x^1_k)<2\varepsilon$ and $d_S(u^2_k,\eta^2_k)\le d_S(u^2_k,x^2_k)<2\varepsilon$ for $1\le k<r$. Thus, $u^1_kv^1_k, u^2_kv^2_k\in E(P)$ and $L(u^1_kv^1_k)=L(u^2_kv^2_k)=1$.

Now, by Lemma 3.18, and taking into account that $d_S(x^2_{k-1},x^1_k)=d_{S^k_\delta}(x^2_{k-1},x^1_k)$, for $2\le k\le r-1$, there exist constants $a\ge1$ and $b\ge0$ which just depend on ε, such that

$$d_{P^k}(u^2_{k-1},u^1_k)\le d_{P_0\cap S^k_\delta}(u^2_{k-1},u^1_k)\le a\,d_{S^k_\delta}(x^2_{k-1},x^1_k)+b.$$

Therefore, it holds

$$\begin{aligned}
d_P(p,q) &\le d_{P^1}(p,u_1^1) + d_P(u_1^1,v_1^1) + L_P(v_1^1v_1^2) \\
&\quad + \sum_{k=2}^{r-1}\Big(d_P(v_{k-1}^2,u_{k-1}^2) + d_{P^k}(u_{k-1}^2,u_k^1) + d_P(u_k^1,v_k^1) + L_P(v_k^1v_k^2)\Big) + d_P(v_{r-1}^2,u_{r-1}^2) + d_{P^r}(u_{r-1}^2,q) \\
&\le a\, d_{S_\delta^1}(p,x_1^1) + b + 2 + d_{C^1}(x_1^1,x_1^2) \\
&\quad + \sum_{k=2}^{r-1}\Big(a\, d_{S_\delta^k}(x_{k-1}^2,x_k^1) + b + 3 + d_{C^k}(x_k^1,x_k^2)\Big) + a\, d_{S_\delta^r}(x_{r-1}^2,q) + b + 1 \\
&\le a\, d_S(p,q) + 2b + 3 + (r-2)(b+3).
\end{aligned}$$

Since Lemma 3.2 gives $d_{S_\delta^k}(x_{k-1}^2,x_k^1) \ge 2\log\frac{1}{\sinh\delta} =: \kappa^{-1}$ for $2 \le k \le r-1$, then

$$(r-2)\kappa^{-1} \le \sum_{k=2}^{r-1} d_{S_\delta^k}(x_{k-1}^2,x_k^1) \le d_S(p,q),$$

and therefore

$$\begin{aligned}
d_P(p,q) &\le a\, d_S(p,q) + 2b + 3 + (r-2)(b+3) \\
&\le \big(a + \kappa(b+3)\big)\, d_S(p,q) + 2b + 3 = a' d_S(p,q) + b'.
\end{aligned} \tag{3.7}$$

Case A.2. Assume that the hypothesis in Case A.1 does not hold. Therefore, there exists $1 \le k < r$ such that $\gamma \cap \overline{C^k}$ joins two points in the same connected component of ∂C^k. Hence, Lemma 3.13 gives $r = 2$ and $S_\delta^1 = S_\delta^2$. Thus, a similar (and simpler) argument to the previous one, with the three components $S_\delta^1, C^1, S_\delta^2 = S_\delta^1$, gives (3.7).

Case B. Assume now that $p, q \in P \setminus P_0$.

Denote by C^1 (respectively, C^2) the connected component of $S \setminus S_\delta$ containing p (respectively, q). Hence, p (respectively, q) belongs to a geodesic $\gamma^1 \subset e^1 \in E(P)$ (respectively, $\gamma^2 \subset e^2 \in E(P)$) joining it with a connected component of ∂C^1 (respectively, ∂C^2).

Assume that $\sigma \subset C^1$. Thus, $q \in C^1$, $p, q \in \sigma \subset \gamma^1$, and

$$d_P(p,q) = \frac{\lceil L_S(e^1)\rceil}{L_S(e^1)}\, d_S(p,q).$$

If $L_S(e^1) \ge 1$, then $\lceil L_S(e^1)\rceil / L_S(e^1) \le 2$ and $d_P(p,q) \le 2 d_S(p,q)$. If $L_S(e^1) < 1$, then $d_P(p,q) \le 1 \le d_S(p,q) + 1$. Hence, in both cases $d_P(p,q) \le 2 d_S(p,q) + 1$.

Assume that σ is not contained in C^1. Denote by σ_1 (respectively, σ_2) the closure of the connected component of $\sigma \cap (S \setminus S_\delta)$ containing p (respectively, q), and define $z_1 := \sigma_1 \cap \partial C^1$ and $z_2 := \sigma_2 \cap \partial C^2$. Let us denote by v_1 (respectively, v_2) the point of γ^1 (respectively, γ^2) in the connected component of ∂S_δ containing z_1 (respectively, z_2). Thus, Lemma 3.12 gives $d_S(v_1,z_1), d_S(v_2,z_2) < \sinh\delta$. By Lemma 3.17, one can choose $u_1, u_2 \in P_0 \cap S_\delta$ with $d_S(z_1,u_1), d_S(z_2,u_2) < 2\varepsilon$ and $u_1 \in N(v_1), u_2 \in N(v_2)$. Therefore, (3.7) gives, by defining $A = \max\{a', 2\}$ and $B = 4\sinh\delta + 4\varepsilon a' + b' + 4$,

$$\begin{aligned}
d_P(p,q) &\le d_P(p,v_1) + d_P(v_1,u_1) + d_P(u_1,u_2) + d_P(u_2,v_2) + d_P(v_2,q) \\
&\le 2 d_S(p,v_1) + 1 + 1 + a' d_S(u_1,u_2) + b' + 1 + 2 d_S(v_2,q) + 1 \\
&\le 2\big(d_S(p,z_1) + \sinh\delta\big) + a'\big(d_S(z_1,z_2) + 4\varepsilon\big) + 2\big(d_S(z_2,q) + \sinh\delta\big) + b' + 4 \\
&\le A\big(d_S(p,z_1) + d_S(z_1,z_2) + d_S(z_2,q)\big) + 4\sinh\delta + 4\varepsilon a' + b' + 4 \\
&= A\, d_S(p,q) + B.
\end{aligned}$$

Case C. If $p \in P \setminus P_0$ and $q \in P \cap P_0$ (or viceversa), the previous argument also gives, in a simpler way, $d_P(p,q) \le A\, d_S(p,q) + B$.

In order to get the other inequality let us follow a similar (and simpler) argument.

Let $p, q \in P \cap S$ and g an oriented geodesic in P from p to q.

Assume that $p, q \in P \cap P_0$. Given a collar C^k, denote by e^k the edge in P contained in C^k. Thus, g meets $P^1, e^1, P^2, e^2, \dots, e^{s-1}, P^s$, where $p \in P^1$, $q \in P^s$, and the geodesic g meets e^k after P^k, and P^{k+1} after e^k.

Let us denote by v_k^1 (respectively, v_k^2) the first (respectively, second) vertex of g in e^k (thus, $e^k = v_k^1 v_k^2$ and $L(e^k) = \lceil d_{C^k}(v_k^1, v_k^2) \rceil \geq 1$).

Let $S_\delta^1, S_\delta^2, \dots, S_\delta^s$ be the components of S_δ associated to $P^1, P^2, \dots, P^s$, respectively (therefore, $p \in S_\delta^1$ and $q \in S_\delta^s$). Let us denote by u_k^i, for $i = 1, 2$, the points $u_k^1 \in g \cap S_\delta^k$ and $u_k^2 \in g \cap S_\delta^{k+1}$ such that $u_k^1 v_k^1 \in E(P^k)$, $u_k^2 v_k^2 \in E(P^{k+1})$, and so, Lemma 3.12 gives $d_{S_\delta^k}(u_k^i, v_k^i) \leq 2\varepsilon + \sinh \delta$ for every $1 \leq k < s$.

The definition of P_0 gives $d_{S_\delta^k}(u_{k-1}^2, u_k^1) \leq 4\varepsilon\, d_{P^k}(u_{k-1}^2, u_k^1)$ for every $2 \leq k \leq s-1$, $d_{S_\delta^1}(p, u_1^1) \leq 4\varepsilon\, d_{P^1}(p, u_1^1)$ and $d_{S_\delta^s}(u_{s-1}^2, q) \leq 4\varepsilon\, d_{P^s}(u_{s-1}^2, q)$, and so,

$$
\begin{aligned}
d_S(p,q) &\leq d_{S_\delta^1}(p, u_1^1) + d_{S_\delta^1}(u_1^1, v_1^1) + d_{C^1}(v_1^1, v_1^2) \\
&\quad + \sum_{k=2}^{s-1} \Big(d_{S_\delta^k}(v_{k-1}^2, u_{k-1}^2) + d_{S_\delta^k}(u_{k-1}^2, u_k^1) + d_{S_\delta^k}(u_k^1, v_k^1) + d_{C^k}(v_k^1, v_k^2) \Big) \\
&\quad + d_{S_\delta^s}(v_{s-1}^2, u_{s-1}^2) + d_{S_\delta^s}(u_{s-1}^2, q) \\
&\leq 4\varepsilon\, d_{P^1}(p, u_1^1) + 2\varepsilon + \sinh\delta + L(e^1) + \sum_{k=2}^{s-1} \Big(2\varepsilon + \sinh\delta + 4\varepsilon\, d_{P^k}(u_{k-1}^2, u_k^1) + 2\varepsilon + \sinh\delta + L(e^k) \Big) \\
&\quad + 2\varepsilon + \sinh\delta + 4\varepsilon\, d_{P^s}(u_{s-1}^2, q) \\
&\leq 4\varepsilon\, d_{P^1}(p, u_1^1) + L(e^1) + \sum_{k=2}^{s-1} \Big(4\varepsilon\, d_{P^k}(u_{k-1}^2, u_k^1) + (1 + 4\varepsilon + 2\sinh\delta) L(e^k) \Big) \\
&\quad + 4\varepsilon\, d_{P^s}(u_{s-1}^2, q) + 4\varepsilon + 2\sinh\delta\,.
\end{aligned}
$$

Thus,

$$
d_S(p,q) \leq (1 + 4\varepsilon + 2\sinh\delta)\, d_P(p,q) + 4\varepsilon + 2\sinh\delta\,. \tag{3.8}
$$

If $p, q \in e^m$ for some m and there is a geodesic joining them contained in e^m, then $d_S(p,q) \leq d_P(p,q)$. This inequality and the previous argument give that (3.8) also holds for every $p, q \in P \cap S$. □

Lemma 3.19 has the following consequence.

Corollary 3.20. *Let S be a non-exceptional (geodesically bordered or non-bordered) Riemann surface, $0 < \delta \leq \delta_1$, $0 < \varepsilon \leq \varepsilon_0$, and P any (δ, ε)-metric-net in S. Then P and S are quasi-isometric.*

Proof. Lemma 3.19 gives that the inclusion $i : P \cap S \to S$ is a quasi-isometry. It suffices to check that the function $I : P \to S$ is a quasi-isometry, where I is defined as follows: $I(z) = z$ for every $z \in P \cap S$; if $z \in P \setminus S$, then choose $z_0 \in P \cap S$ with $d_P(z_0, z) \leq 1/2$ and define $I(z) = z_0$. □

Next, let us introduce some new definitions. Namely, if X is either a manifold, a graph, a metric graph or a metric-net, $x \in X$ and $A \subseteq X$, then set

$$
\begin{aligned}
\operatorname{lpgo}(X, A) &:= \sup \big\{ k \geq 0 : \liminf_{r \to \infty} r^{-k} \operatorname{vol}_X \big(B_X(x, r) \cap A \big) > 0 \big\}, \\
\operatorname{upgo}(X, A) &:= \inf \big\{ k \geq 0 : \limsup_{r \to \infty} r^{-k} \operatorname{vol}_X \big(B_X(x, r) \cap A \big) < \infty \big\}, \\
\operatorname{lgo}_\varphi(X, A) &:= \sup \big\{ k \geq 0 : \liminf_{r \to \infty} \varphi(r)^{-k} \operatorname{vol}_X \big(B_X(x, r) \cap A \big) > 0 \big\}, \\
\operatorname{ugo}_\varphi(X, A) &:= \inf \big\{ k \geq 0 : \limsup_{r \to \infty} \varphi(r)^{-k} \operatorname{vol}_X \big(B_X(x, r) \cap A \big) < \infty \big\},
\end{aligned}
$$

for every admissible function φ.

It is clear that $\operatorname{lpgo}(X,X) = \operatorname{lpgo}(X)$ and $\operatorname{lpgo}(X,A) = \operatorname{lpgo}(X)$ for every A such that $X \setminus A$ has finite volume (and the same holds for upgo, $\operatorname{lgo}_\varphi$ and $\operatorname{ugo}_\varphi$).

Lemma 3.21. *Let G be a graph, a metric graph or a metric-net, $A \subset V(G)$ and an injective map $j : A \to V(G) \setminus A$ such that $d_G(a, j(a)) \le c$ for every $a \in A$ and some constant c. Then $\operatorname{upgo}(G) = \operatorname{upgo}(G, V(G) \setminus A)$, $\operatorname{lpgo}(G) = \operatorname{lpgo}(G, V(G) \setminus A)$, $\operatorname{lgo}_\varphi(G) = \operatorname{lgo}_\varphi(G, V(G) \setminus A)$ and $\operatorname{ugo}_\varphi(G) = \operatorname{ugo}_\varphi(G, V(G) \setminus A)$ for every admissible function φ.*

Proof. Inequalities $\operatorname{upgo}(G) \ge \operatorname{upgo}(G, V(G) \setminus A)$, $\operatorname{lpgo}(G) \ge \operatorname{lpgo}(G, V(G) \setminus A)$, $\operatorname{lgo}_\varphi(G) \ge \operatorname{lgo}_\varphi(G, V(G) \setminus A)$ and $\operatorname{ugo}_\varphi(G) \ge \operatorname{ugo}_\varphi(G, V(G) \setminus A)$ for every admissible function φ, are direct.

Let $w \in G$; if $a \in B_G(w, r) \cap A$, then $j(a) \in B_G(w, r + c)$; since $j(a) \notin A$, one has $j(a) \in B_G(w, r + c) \setminus A$. Hence, $j\big(B_G(w, r) \cap A\big) \subseteq B_G(w, r + c) \setminus A$. Since j is an injective map,

$$\begin{aligned}
\operatorname{vol}_G\big(B_G(w,r) \cap A\big) &= \operatorname{vol}_G\big(j\big(B_G(w,r) \cap A\big)\big) \le \operatorname{vol}_G\big(B_G(w, r+c) \setminus A\big), \\
\operatorname{vol}_G\big(B_G(w,r)\big) &= \operatorname{vol}_G\big(B_G(w,r) \cap A\big) + \operatorname{vol}_G\big(B_G(w,r) \setminus A\big) \\
&\le 2 \operatorname{vol}_G\big(B_G(w, r+c) \setminus A\big),
\end{aligned}$$

and therefore $\operatorname{upgo}(G) \le \operatorname{upgo}(G, V(G) \setminus A)$, $\operatorname{lpgo}(G) \le \operatorname{lpgo}(G, V(G) \setminus A)$, $\operatorname{lgo}_\varphi(G) \le \operatorname{lgo}_\varphi(G, V(G) \setminus A)$ and $\operatorname{ugo}_\varphi(G) \le \operatorname{ugo}_\varphi(G, V(G) \setminus A)$ for every admissible function φ. □

Theorem 3.22. *Let S be a non-exceptional (geodesically bordered or non-bordered) Riemann surface, $0 < \delta \le \delta_0$, $0 < \varepsilon \le \min\{\varepsilon_0, 2\delta\}$ and P any (δ, ε)-metric-net in S. Then $\operatorname{upgo}(P) = \operatorname{upgo}(S)$, $\operatorname{lpgo}(P) = \operatorname{lpgo}(S)$, $\operatorname{lgo}_\varphi(P) = \operatorname{lgo}_\varphi(S)$ and $\operatorname{ugo}_\varphi(P) = \operatorname{ugo}_\varphi(S)$ for every admissible function φ.*

Proof. Let P_0 be an ε-net in S such that P is a (δ, ε)-metric-net in S associated to P_0. By Lemma 3.19, there exist constants a, b, such that the inclusion $i : P \cap S \to S$ is an (a, b)-quasi-isometry.

Fix a point w in $P_0 \cap S_\delta$ and $r > 0$.

Given any $v \in P_0 \cap S_\delta$ with $d_P(w, v) < r$, $B_S(v, \varepsilon/2)$ is contained in $B_S(w, ar + b + \varepsilon/2)$. Since $P_0 \cap S_\delta$ is an ε-separated set in S and $\varepsilon/2 \le \delta$, $\{B_S(v, \varepsilon/2)\}_{v \in P_0 \cap S_\delta}$ are pairwise disjoint simply connected balls, and

$$A_S\big(B_S(w, ar + b + \varepsilon/2)\big) \ge 2\pi \sinh^2\Big(\frac{\varepsilon}{4}\Big) \big|\{v \in P_0 \cap S_\delta :\, d_P(w, v) < r\}\big| .$$

Hence, $\operatorname{lpgo}(S) \ge \operatorname{lpgo}(P, P_0 \cap S_\delta)$, $\operatorname{upgo}(S) \ge \operatorname{upgo}(P, P_0 \cap S_\delta)$, $\operatorname{lgo}_\varphi(S) \ge \operatorname{lgo}_\varphi(P, P_0 \cap S_\delta)$ and $\operatorname{ugo}_\varphi(S) \ge \operatorname{ugo}_\varphi(P, P_0 \cap S_\delta)$, for every admissible function φ.

By Lemma 3.17, the set $B_S(w, r/a - b - 2\varepsilon) \cap S_\delta$ is covered by

$$\{B_S(v, 2\varepsilon) :\, v \in P_0 \cap S_\delta,\, d_S(w, v) < r/a - b\} \subseteq \{B_S(v, 2\varepsilon) :\, v \in P_0 \cap S_\delta,\, d_P(w, v) < r\},$$

and property $(P2)$ gives

$$A_S\big(B_S(w, r/a - b - 2\varepsilon) \cap S_\delta\big) \le 4\pi \sinh^2\varepsilon \,\big|\{v \in P_0 \cap S_\delta :\, d_P(w, v) < r\}\big| .$$

Therefore, Theorem 3.11 gives

$$A_S\big(B_S(w, r/a - b - 2\varepsilon)\big) < 8\pi \sinh^2\varepsilon \,\big|\{v \in P_0 \cap S_\delta :\, d_P(w, v) < r\}\big| ,$$

and one obtains $\operatorname{lpgo}(S) \le \operatorname{lpgo}(P, P_0 \cap S_\delta)$, $\operatorname{upgo}(S) \le \operatorname{upgo}(P, P_0 \cap S_\delta)$, $\operatorname{lgo}_\varphi(S) \le \operatorname{lgo}_\varphi(P, P_0 \cap S_\delta)$ and $\operatorname{ugo}_\varphi(S) \le \operatorname{ugo}_\varphi(P, P_0 \cap S_\delta)$, for every admissible function φ.

In order to finish the proof of the lemma, it suffices to prove that the hypotheses of Lemma 3.21 on j hold with $G = P$ and $A = V(P) \setminus P_0$. Consider different points $a, a' \in A$ and the collars C, C' in S with $a \in \partial C$ and $a \in \partial C'$, and let K, K' be the collars given by the Collar Lemma corresponding to C, C', respectively. If $C \ne C'$, since collars are pairwise disjoint, then Lemma 3.4 gives that $N(a) \cap N(a') \cap S_\delta = \emptyset$. If $C = C'$, then the definition of the (δ, ε)-metric-net P gives that $N(a) \cap N(a') \cap S_\delta = \emptyset$. By defining $j : V(P) \setminus P_0 \to P_0 \cap S_\delta$ and by choosing $j(a)$ as any point in $N(a) \cap S_\delta$, then the map j is injective.

This finishes the proof of the lemma. □

Let us now proceed with the proof of Theorem 1.1.

Proof. Since the volume growth rates are invariant by bilipschitz maps, one can assume by Theorem 3.1 that X has constant curvature -1.

Recall the values for δ_0 and ε_0 (the constants in Theorem 3.11 and Lemma 3.4 respectively). Let $0 < \delta \le \delta_0$, $0 < \varepsilon \le \min\{\varepsilon_0, 2\delta\}$, P any (δ, ε)-metric-net in X, and Q any ε-net in Y. Theorem 3.22 gives $\operatorname{upgo}(P) = \operatorname{upgo}(X)$ and $\operatorname{lpgo}(P) = \operatorname{lpgo}(X)$. Recall that, by definition of (δ, ε)-metric-net, $L(e) \in \mathbb{Z}^+ \cup \{\infty\}$ for every $e \in E(P)$.

The *normalized graph* P' of P is an isometric graph to P defined by (possibly) adding some new vertices to $V(P)$ as follows:

For every edge $uv \in E(P)$ with length greater than 1 (and $u, v \neq v_\infty$), the normalized graph of P will contain the vertices $u_0 = u, u_1, \dots, u_{l-1}, u_l = v \in uv$, where $l = L(uv) \in \mathbb{Z}^+$ and $d_{uv}(u_j, u) = j$ for every $0 \le j < l$, and $u_j u_{j+1} \in E(P')$ for every $0 \le j < l$. For every edge $uv_\infty \in E(P)$, P' will contain infinitely many vertices $u_0 = u, u_1, u_2, \dots \in uv_\infty$, where $d_G(u_j, u) = j$ for every $j \ge 0$, and $u_j u_{j+1} \in E(P')$ for every $j \ge 0$.

Obviously, the length of all the edges of P' is 1. Thus, Lemma 3.10 gives $\operatorname{lpgo}(P') \le \operatorname{lpgo}(P) + 1$ and $\operatorname{upgo}(P') \le \operatorname{upgo}(P) + 1$; also, P' is uniform (since P is uniform by Lemma 3.5), and P has polynomial (respectively, exponential) growth if and only if P' does.

By Corollary 3.20, P and X are quasi-isometric, and therefore, by composition of quasi-isometries and isometries, P' and Q are uniform quasi-isometric graphs. Thus, Lemma 3.8 gives $\operatorname{lpgo}(Q) = \operatorname{lpgo}(P')$ and $\operatorname{upgo}(Q) = \operatorname{upgo}(P')$.

Finally, Lemma 3.9 gives $\operatorname{lpgo}(Y) \le \operatorname{lpgo}(Q)$ and $\operatorname{upgo}(Y) \le \operatorname{upgo}(Q)$ (and the appropriate statements about polynomial and exponential growth), and this gives the conclusion $\operatorname{lpgo}(Y) \le \operatorname{lpgo}(X) + 1$ and $\operatorname{upgo}(Y) \le \operatorname{upgo}(X) + 1$ (and the statements about polynomial and exponential growth).

Proposition 5.1 shows that the inequalities $\operatorname{lpgo}(Y) \le \operatorname{lpgo}(X) + 1$ and $\operatorname{upgo}(Y) \le \operatorname{upgo}(X) + 1$ are sharp. □

4. Proof of Theorem 1.3

In [17, Theorem 3.5] appears the following result.

Theorem 4.1. *Let S_1 and S_2 be non-exceptional (finitely geodesically bordered or non-bordered) Riemann surfaces and $g : S_1 \to S_2$ a quasi-isometry. If S_2 has finite genus, then for each $\varepsilon_2 > 0$ there exists $\varepsilon_1 > 0$ such that $\iota(g(z)) < \varepsilon_2$ if $\iota(z) < \varepsilon_1$.*

In fact, the proof of [17, Theorem 3.5] gives the following stronger result. To see this, notice that the argument in the proof relies on [17, Corollary 3.2], and it just needs that the local geodesics in appropriated collars in S_1 are in fact geodesics in S_1; obviously, this holds when replacing the hypothesis of S_1 from being finitely geodesically bordered to be geodesically bordered; see the paragraphs previous to Corollary 3.2 in [17].

Theorem 4.2. *Let S_1 and S_2 be non-exceptional (geodesically bordered or non-bordered) Riemann surfaces and $g : S_1 \to S_2$ a quasi-isometry. If S_2 has finite genus and is finitely geodesically bordered or non-bordered, then for each $\varepsilon_2 > 0$ there exists $\varepsilon_1 > 0$ such that $\iota(g(z)) < \varepsilon_2$ if $\iota(z) < \varepsilon_1$.*

Lemma 4.3. *Let G be a uniform graph whose edges have all the same length $\ell > 0$. Let $V_0, A \subset V(G)$ and $c \ge 0$ satisfy the following property: for every $a \in A \cap V_0$ there exists $v_a \in V_0 \setminus A$ such that $d_G(a, v_a) \le c$. Then $\operatorname{upgo}(G, V_0) = \operatorname{upgo}(G, V_0 \setminus A)$, $\operatorname{lpgo}(G, V_0) = \operatorname{lpgo}(G, V_0 \setminus A)$, $\operatorname{lgo}_\varphi(G, V_0) = \operatorname{lgo}_\varphi(G, V_0 \setminus A)$ and $\operatorname{ugo}_\varphi(G, V_0) = \operatorname{ugo}_\varphi(G, V_0 \setminus A)$ for every admissible function φ.*

Proof. Since the volume growth rates are invariant by bilipschitz maps, one can assume $\ell = 1$.

If $a \in B_G(w,r) \cap A \cap V_0$, then $v_a \in B_G(w,r+c) \cap V_0 \setminus A$. Let μ be a constant so that G is μ-uniform. If $a, a' \in A \cap V_0$ and $v_a = v_{a'}$, then $d_G(a,a') \le 2c$. Since all the edges in G have length 1, one has

$$\left|\{a' \in A : v_{a'} = v_a\}\right| \le \sum_{j=0}^{2c} \mu^j =: M.$$

Since $v_a \notin A$, one has

$$\begin{aligned}\operatorname{vol}_G\big(B_G(w,r) \cap A \cap V_0\big) &\le M \operatorname{vol}_G\big(B_G(w,r+c) \cap V_0 \setminus A\big),\\ \operatorname{vol}_G\big(B_G(w,r) \cap V_0\big) &= \operatorname{vol}_G\big(B_G(w,r) \cap V_0 \cap A\big) + \operatorname{vol}_G\big(B_G(w,r) \cap V_0 \setminus A\big)\\ &\le (M+1) \operatorname{vol}_G\big(B_G(w,r+c) \cap V_0 \setminus A\big).\end{aligned}$$

Since $\operatorname{vol}_G\big(B_G(w,r) \cap V_0 \setminus A\big) \le \operatorname{vol}_G\big(B_G(w,r) \cap V_0\big)$, the conclusion holds. □

Lemma 4.4. *Let S be a non-exceptional (geodesically bordered or non-bordered) Riemann surface, $0 < \delta \le \delta_0$ (the constant in Theorem 3.11), $0 < \varepsilon \le \min\{\varepsilon_0, 2\delta\}$, and P any ε-net in S. Then $\operatorname{lpgo}(S) = \operatorname{lpgo}(P, P \cap S_\delta)$, $\operatorname{upgo}(S) = \operatorname{upgo}(P, P \cap S_\delta)$, $\operatorname{lgo}_\varphi(S) = \operatorname{lgo}_\varphi(P, P \cap S_\delta)$ and $\operatorname{ugo}_\varphi(S) = \operatorname{ugo}_\varphi(P, P \cap S_\delta)$, for every admissible function φ.*

Proof. By Lemma 3.6, there are constants $a \ge 1$ and $b \ge 0$, such that

$$\frac{1}{4\varepsilon}\, d_S(u,v) \le d_P(u,v) \le a\, d_S(u,v) + b$$

for every $u, v \in P$. Fix a point w in $P \cap S_\delta$ and $r > 0$.

Given any $v \in P$ with $d_P(w,v) < r$, $B_S(v, \varepsilon/2)$ is contained in $B_S(w, 4\varepsilon r + \varepsilon/2)$. Since P is an ε-separated set in S and $\varepsilon/2 \le \delta$, $\{B_S(v,\varepsilon/2)\}_{v \in P \cap S_\delta}$ are pairwise disjoint simply connected balls, $A_S\big(B_S(v,\varepsilon/2)\big) \ge 2\pi \sinh^2(\varepsilon/4)$ and

$$A_S\big(B_S(w, 4\varepsilon r + \varepsilon/2)\big) \ge 2\pi \sinh^2\Big(\frac{\varepsilon}{4}\Big) \left|\{v \in P \cap S_\delta : d_P(w,v) < r\}\right|.$$

Hence, $\operatorname{lpgo}(S) \ge \operatorname{lpgo}(P, P \cap S_\delta)$, $\operatorname{upgo}(S) \ge \operatorname{upgo}(P, P \cap S_\delta)$, $\operatorname{lgo}_\varphi(S) \ge \operatorname{lgo}_\varphi(P, P \cap S_\delta)$ and $\operatorname{ugo}_\varphi(S) \ge \operatorname{ugo}_\varphi(P, P \cap S_\delta)$, for every admissible function φ.

By Lemma 3.17, the set $B_S(w, (r-b)/a - 2\varepsilon) \cap S_\delta$ is covered by

$$\{B_S(v, 2\varepsilon) : v \in P \cap S_\delta,\, d_S(w,v) < (r-b)/a\} \subseteq \{B_S(v, 2\varepsilon) : v \in P \cap S_\delta,\, d_P(w,v) < r\},$$

and property $(P2)$ gives

$$A_S\big(B_S(w, (r-b)/a - 2\varepsilon) \cap S_\delta\big) \le 4\pi \sinh^2\varepsilon \left|\{v \in P \cap S_\delta : d_P(w,v) < r\}\right|.$$

Therefore, Theorem 3.11 gives

$$A_S\big(B_S(w, (r-b)/a - 2\varepsilon)\big) < 8\pi \sinh^2\varepsilon \left|\{v \in P \cap S_\delta : d_P(w,v) < r\}\right|,$$

and thus $\operatorname{lpgo}(S) \le \operatorname{lpgo}(P, P \cap S_\delta)$, $\operatorname{upgo}(S) \le \operatorname{upgo}(P, P \cap S_\delta)$, $\operatorname{lgo}_\varphi(S) \le \operatorname{lgo}_\varphi(P, P \cap S_\delta)$ and $\operatorname{ugo}_\varphi(S) \le \operatorname{ugo}_\varphi(P, P \cap S_\delta)$, for every admissible function φ.

This finishes the proof of the lemma. □

If P is an ε-net of a Riemannian manifold, denote by P^D the graph isomorphic to P whose edges have all the same length 4ε. In particular, one can consider that the vertices in both graphs are the same.

The argument in the proof of [24, Lemma 2.5] gives the following.

Lemma 4.5. *Let X be a complete (without border or with totally geodesic border) n-dimensional Riemannian manifold with Ricci curvature bounded from below by a constant $-(n-1)k^2$ $(k > 0)$, and let P be an ε-net in X. Then there are constants $a \ge 1$ and $b \ge 0$, which just depends on n, k (in particular, they do not depend on ε), such that*

$$d_X(p_1,p_2) \le d_{P^D}(p_1,p_2) \le a\, d_X(p_1,p_2) + b \tag{4.9}$$

for every $p_1, p_2 \in P^D$. *Consequently, the inclusion of* P^D *into* X *is a quasi-isometry.*

Since the volume growth rates are invariant by bilipschitz maps, Lemma 4.4 has the following consequence.

Lemma 4.6. *Let* S *be a non-exceptional (geodesically bordered or non-bordered) Riemann surface,* $0 < \delta \le \delta_0$, $0 < \varepsilon \le \min\{\varepsilon_0, 2\delta\}$, *and* P *any* ε*-net in* S. *Then* $\operatorname{lpgo}(S) = \operatorname{lpgo}(P^D, P^D \cap S_\delta)$, $\operatorname{upgo}(S) = \operatorname{upgo}(P^D, P^D \cap S_\delta)$, $\operatorname{lgo}_\varphi(S) = \operatorname{lgo}_\varphi(P^D, P^D \cap S_\delta)$ *and* $\operatorname{ugo}_\varphi(S) = \operatorname{ugo}_\varphi(P^D, P^D \cap S_\delta)$, *for every admissible function* φ.

Let us now proceed with the proof of Theorem 1.3.

Proof. Corollary 1.2 gives that X has polynomial (respectively, exponential) growth if and only if Y does. Corollary 1.2 also gives $\operatorname{lpgo}(X) - 1 \le \operatorname{lpgo}(Y)$ and $\operatorname{upgo}(X) - 1 \le \operatorname{upgo}(Y)$. Let us prove $\operatorname{lpgo}(Y) \le \operatorname{lpgo}(X)$, $\operatorname{upgo}(Y) \le \operatorname{upgo}(X)$, $\operatorname{lgo}_\varphi(Y) \le \operatorname{lgo}_\varphi(X)$ and $\operatorname{ugo}_\varphi(Y) \le \operatorname{ugo}_\varphi(X)$ for every admissible function φ.

Since the volume growth rates are invariant by bilipschitz maps, one can assume by Theorem 3.1 that X and Y have constant curvature -1. Fix a c_0-full (a_0, b_0)-quasi-isometry f_0 from X to Y. Let $0 < \delta \le \delta_0 = \operatorname{arcsinh}(\exp(-3 \operatorname{arccosh} 3))$. Also, let $\delta_1 = \delta_1(\delta)$ given by Theorem 4.2, where $\delta/2$ plays the role of ε_2, and δ_1 that of ε_1, and $S_1 = X$, $S_2 = Y$, $g = f_0$. Thus, $\iota(f_0(z)) < \delta/2$ if $\iota(z) < \delta_1$. Without loss of generality one can assume that $\delta_1 \le \delta_0$. (Note that choosing δ, δ_1 this way gives $\delta, \delta_1 < \operatorname{arcsinh} 1$ which is needed in order to apply Margulis Lemma). Let $0 < \varepsilon < \min\{\varepsilon_0, \delta/2, 2\delta_1\}$, where ε_0 is the constant in Lemma 3.4. Note that, in particular, $0 < \varepsilon < \delta/2 < 1/2$.

Let P and Q be ε-nets associated to X and Y, respectively. By Lemma 4.5, P^D is quasi-isometric to X and Q^D is to Y (with constants which do not depend either on ε, δ or δ_1), and, moreover, by Lemma 4.6, one gets $\operatorname{upgo}(X) = \operatorname{upgo}(P^D, P^D \cap X_{\delta_1})$, $\operatorname{upgo}(Y) = \operatorname{upgo}(Q^D, Q^D \cap Y_\delta)$, $\operatorname{ugo}_\varphi(X) = \operatorname{ugo}_\varphi(P^D, P^D \cap X_{\delta_1})$ and $\operatorname{ugo}_\varphi(Y) = \operatorname{ugo}_\varphi(Q^D, Q^D \cap Y_\delta)$ for every admissible function φ (the same holds for the lower growth rates). Therefore, it will be enough to show the inequalities above replacing the surfaces X, Y by the nets $P^D \cap X_{\delta_1}$ and $Q^D \cap Y_\delta$, respectively. For example, $\operatorname{upgo}(Q^D, Q^D \cap Y_\delta) \le \operatorname{upgo}(P^D, P^D \cap X_{\delta_1})$.

Let us define a map $j : Y \longrightarrow Q^D$ so that j is a quasi-inverse of the inclusion map in a way that $j(Y \setminus Y_{\delta/2}) \subset Q^D \setminus Q^D_\delta$, where, for the sake of simplicity, denote $Q^D_\delta := Q^D \cap Y_\delta$.

To this end, let us define $j(y) := y$ for every $y \in Q^D$. Consider now $y \notin Q^D$. If $y \in Y_{\delta/2}$, then there exists $q \in Q^D$ so that $d_Y(y, q) < \varepsilon$, and one sets $j(y) := q$. Otherwise, $y \in Y \setminus Y_{\delta/2}$. If y also satisfies the extra condition that $d_Y(y, Y_{\delta/2}) \ge \varepsilon$, then there exists $q \in Q^D$ with $d_Y(y, q) < \varepsilon$, and one sets $j(y) := q$; thus, $q = j(y) \in Q^D \setminus Q^D_{\delta/2} \subset Q^D \setminus Q^D_\delta$. Assume now that y does not satisfy the extra condition, that is, if $d_Y(y, Y_{\delta/2}) < \varepsilon$. If the connected component of $Y \setminus Y_{\delta/2}$ containing y is the collar of a simple closed geodesic with length l, then $l/2 < \delta/2$ and so, the width of the connected component of $Y \setminus Y_\delta$ containing y is

$$w_\delta = \operatorname{arccosh} \frac{\sinh \delta}{\sinh(l/2)} > \operatorname{arccosh} \frac{\sinh \delta}{\sinh(\delta/2)} = \operatorname{arccosh}(2\cosh(\delta/2)) > \delta/2 > \varepsilon.$$

Therefore, since $w_\delta > \varepsilon$, in this case there exists $y' \in Y \setminus Y_\delta$ so that $d_Y(y', Y_\delta) \ge \varepsilon$ and $d_Y(y, y') < \varepsilon$. Note that y' trivially exists if the connected component of $Y \setminus Y_{\delta/2}$ containing y is a collar of a cusp. Hence, there exists $q \in Q^D \setminus Q^D_\delta$ so that $d_Y(y', q) < \varepsilon$, and one sets $j(y) := q$ and gets that $d_Y(y, j(y)) = d_Y(y, q) < 2\varepsilon$. This concludes that $j(Y \setminus Y_{\delta/2}) \subset Q^D \setminus Q^D_\delta$.

Let us show that j is a quasi-isometry. By definition of j, notice that for any $y \in Y$ one gets $d_Y(j(y), y) < 2\varepsilon < \delta < 1$. If $y, z \in Y$, then by Lemma 4.5,

$$d_{Q^D}(j(y), j(z)) \ge d_Y(j(y), j(z)) > d_Y(y, z) - 4\varepsilon > d_Y(y, z) - 2.$$

On the other hand, also by Lemma 4.5, where a, b are the constants given there (which do not depend either on ε, δ or δ_1),

$$d_{Q^D}(j(y), j(z)) \le a d_Y(j(y), j(z)) + b \le a(d_Y(y, z) + 4\varepsilon) + b < a d_Y(y, z) + 2a + b.$$

This concludes that j is an onto $(a, 2a + b)$-quasi-isometry.

Next, consider the map $f := j \circ f_0 \circ i$, where i is the inclusion from P^D into X, the map f_0 is the fixed c_0-full (a_0, b_0)-quasi-isometry from X to Y, and j is defined above. Note that, by Lemma 4.5, i is also a quasi-isometry with universal constants (it is (1/2)-full, since $\varepsilon < 1/2$). Clearly, f is a c-full (α, β)-quasi-isometry, for some α, β, c which only depend on a, b, a_0, b_0, c_0, although f depends on ε and δ.

Our aim is to prove that f satisfies $f(P^D \setminus P^D_{\delta_1}) \subset Q^D \setminus Q^D_\delta$, where $P^D_{\delta_1} := P^D \cap X_{\delta_1}$, that is, f maps the *thin* part of P^D into the *thin* part of Q^D. Let p be a point $p \in P^D \setminus P^D_{\delta_1}$, thus $p = i(p) \in X \setminus X_{\delta_1}$, that is, its injectivity radius satisfies $\iota(p) < \delta_1$ and thus one gets $y := f_0(p) \in Y \setminus Y_{\delta/2}$, and so $f(p) = j(y) \in Q^D \setminus Q^D_\delta$. Hence, $f(P^D \setminus P^D_{\delta_1}) \subset Q^D \setminus Q^D_\delta$.

It has been shown that the map $f_0 \circ i$ is a c_1-full quasi-isometry, where c_1 is a constant which does not depend either on δ or ε. So, without loss of generality one can assume also that $\delta < \operatorname{arcsinh} e^{-c_1-1/2}$.

Consider $y \in Y_\delta$ with $d_Y(y, Y \setminus Y_\delta) \geq c_1 + \varepsilon$, and a point $p \in P^D$ with $d_Y((f_0 \circ i)(p), y) \leq c_1$. Thus, $d_Y((f_0 \circ i)(p), Y \setminus Y_\delta) \geq \varepsilon$ and there exists a point $q \in Q^D$ with $q = j((f_0 \circ i)(p)) = f(p)$ and $d_Y(q, (f_0 \circ i)(p)) < \varepsilon$. Hence, $q \in Y_\delta$ and $d_Y(y, f(P^D) \cap Q^D_\delta) < c_1 + \varepsilon$. Consider now $y \in Y_\delta$ with $d_Y(y, Y \setminus Y_\delta) < c_1 + \varepsilon$. Since $\delta < \operatorname{arcsinh} e^{-c_1-1/2}$, one gets

$$\log \frac{1}{\sinh \delta} > c_1 + \frac{1}{2} > c_1 + \varepsilon,$$

and Lemma 3.2 gives that y belongs to a collar K given by the Collar Lemma. Let σ be a geodesic in Y orthogonal to ∂K and containing y, and let y_0 be the closest point to y in $\sigma \cap \{z \in Y : d_Y(z, Y \setminus Y_\delta) = c_1 + \varepsilon\}$. Since $d_Y(y, Y \setminus Y_\delta) < c_1 + \varepsilon$ and $d_Y(y_0, Y \setminus Y_\delta) = c_1 + \varepsilon$, one has $d_Y(y, y_0) \leq c_1 + \varepsilon$. Since $d_Y(y_0, Y \setminus Y_\delta) \geq c_1 + \varepsilon$, the previous argument gives $d_Y(y_0, f(P^D) \cap Q^D_\delta) < c_1 + \varepsilon$ and so, $d_Y(y, f(P^D) \cap Q^D_\delta) < 2c_1 + 2\varepsilon < 2c_1 + 1$.

One concludes that given any $q \in Q^D_\delta$, $d_Y(q, f(P^D) \cap Q^D_\delta) < 2c_1 + 1$ and so,

$$d_{Q^D}(q, f(P^D) \cap Q^D_\delta) < a(2c_1 + 1) + 2a + b. \tag{4.10}$$

Fix $p_0 \in P^D$ and $r > 0$. If $p \in P^D$ and $f(p) \in B_{Q^D}(f(p_0), r)$, then

$$\frac{1}{\alpha} d_{P^D}(p, p_0) - \beta \leq d_{Q^D}(f(p), f(p_0)) < r, \qquad d_{P^D}(p, p_0) < \alpha r + \alpha\beta,$$

$$B_{Q^D}(f(p_0), r) \cap f(P^D) \subseteq f\big(B_{P^D}(p_0, \alpha r + \alpha\beta)\big).$$

Since $f(P^D \setminus P^D_{\delta_1}) \subset Q^D \setminus Q^D_\delta$, one concludes

$$B_{Q^D}(f(p_0), r) \cap Q^D_\delta \cap f(P^D) \subseteq f\big(B_{P^D}(p_0, \alpha r + \alpha\beta) \cap P^D_{\delta_1}\big),$$

and so,

$$\big|B_{Q^D}(f(p_0), r) \cap Q^D_\delta \cap f(P^D)\big| \leq \big|f\big(B_{P^D}(p_0, \alpha r + \alpha\beta) \cap P^D_{\delta_1}\big)\big| \leq \big|B_{P^D}(p_0, \alpha r + \alpha\beta) \cap P^D_{\delta_1}\big|.$$

Hence, $\operatorname{upgo}(Q^D, Q^D_\delta \cap f(P^D)) \leq \operatorname{upgo}(P^D, P^D_{\delta_1})$. (4.10) allows to apply Lemma 4.3, with $V_0 = Q^D_\delta$ and $A = Q^D \setminus f(P^D)$ (thus, $A \cap V_0 = Q^D_\delta \setminus f(P^D)$ and $V_0 \setminus A = f(P^D) \cap Q^D_\delta$), and so, $\operatorname{upgo}(Q^D, Q^D_\delta \cap f(P^D)) = \operatorname{upgo}(Q^D, Q^D_\delta)$. Therefore,

$$\operatorname{upgo}(Y) = \operatorname{upgo}(Q^D, Q^D_\delta) = \operatorname{upgo}(Q^D, Q^D_\delta \cap f(P^D)) \leq \operatorname{upgo}(P^D, P^D_{\delta_1}) = \operatorname{upgo}(X),$$

and the same holds for the other growth rates.

Proposition 5.1 gives that the inequalities $\operatorname{lpgo}(X) - 1 \leq \operatorname{lpgo}(Y)$ and $\operatorname{upgo}(X) - 1 \leq \operatorname{upgo}(Y)$ are sharp. In order to check that the other inequalities can be attained, it suffices to consider $Y = X$. □

5. Examples

In order to construct the examples in this section, some preliminaries are needed.

A *Y-piece* (or a pair of pants) is a compact bordered Riemann surface which is topologically a sphere without three disks and whose border is the union of three simple closed geodesics. Given three positive numbers a, b, c, there is a unique (up to conformal mapping) Y-piece such that its boundary curves have lengths a, b, c (see, e.g., [37, p.410]). Y-pieces are a standard tool for constructing Riemann surfaces (see [11, Chapter X.3] and [8, Chapter 1]).

The following result shows that the inequalities in Theorems 1.1 and 1.3 and Corollary 1.2 are sharp.

Proposition 5.1. *There exist quasi-isometric non-exceptional Riemann surfaces S_1 and S_2 with* $0 < \operatorname{lpgo}(S_1) = \operatorname{upgo}(S_1) < \infty$*,* $0 < \operatorname{lpgo}(S_2) = \operatorname{upgo}(S_2) < \infty$ *and* $\operatorname{upgo}(S_2) = \operatorname{upgo}(S_1) + 1$.

Proof. Let G_1 be a rooted dyadic tree such that the length of the first generation edges is a, the length of the second generation edges is a^2, and in general, the length of the n-th generation edges is a^n with $a \in \mathbb{Z}$, $a > 1$. Let v_0 be the root of G_1 and let $R_k = \sum_{j=1}^k a^j$. Then

$$\operatorname{vol}_{G_1}\big(B_{G_1}(v_0, R_k)\big) = \sum_{j=0}^{k-1} 2^j \asymp 2^k = (a^k)^{\log 2/\log a} \asymp R_k^{\log 2/\log a},$$

where the symbol $\asymp$ means that both quantities are comparable up two constants which are independent of k. If $R_k < R \le R_{k+1}$ one concludes that

$$\operatorname{vol}_{G_1}\big(B_{G_1}(v_0, R)\big) = \operatorname{vol}_{G_1}\big(B_{G_1}(v_0, R_{k+1})\big) \asymp R_{k+1}^{\log 2/\log a}$$

and, since $R_{k+1} \asymp R_k$, for certain constants c_1 and c_2, which are independent of k,

$$c_1\, R^{\log 2/\log a} \le \operatorname{vol}_{G_1}\big(B_{G_1}(v_0, R)\big) \le c_2\, R^{\log 2/\log a}.$$

Therefore,

$$\operatorname{lpgo}(G_1) = \operatorname{upgo}(G_1) = \frac{\log 2}{\log a}.$$

Let now G_2 be the isometric graph to G_1 obtained from G_1 by adding vertices in all the edges in such a way that all the edges of G_2 have length 1. Then

$$\operatorname{vol}_{G_2}\big(B_{G_2}(v_0, R_k)\big) = \sum_{j=0}^{k-1}(2a)^j + 2^k(a^k - 1) \asymp 2^k a^k = (a^k)^{\log 2/\log a} a^k \asymp R_k^{1+\log 2/\log a}.$$

If $R_k < R \le R_{k+1}$, let us write $R = R_k + r$ with $0 < r \le a^{k+1}$. Denote by $[r]$ the greatest integer which is strictly less than r. Then

$$\operatorname{vol}_{G_2}\big(B_{G_2}(v_0, R)\big) = \sum_{j=0}^{k}(2a)^j + [r]2^k \asymp 2^k(a^k + r) \asymp 2^k a^k \asymp R_{k+1}^{1+\log 2/\log a}.$$

Since $R_{k+1} \asymp R_k$, for certain constants c_3 and c_4 which are independent of k,

$$c_3\, R^{1+\log 2/\log a} \le \operatorname{vol}_{G_1}\big(B_{G_1}(v_0, R)\big) \le c_4\, R^{1+\log 2/\log a}.$$

Therefore,

$$\operatorname{lpgo}(G_2) = \operatorname{upgo}(G_2) = 1 + \frac{\log 2}{\log a}.$$

Let us consider a Y-piece P such that the length of every simple closed geodesic on its boundary is equal to 1. If P_1, P_2 are Y-pieces isometric to P, let Q be the bordered Riemann surface with genus one whose border is the union of two simple closed geodesics, obtained from P_1 and P_2 by identifying two simple closed geodesics in ∂P_1 with two simple closed geodesics in ∂P_2.

Let us construct now a Riemann surface S_2 by using the graph G_2 as an skeleton. Substitute each bifurcation by a Y-piece isometric to P, and each edge of length 1 by a piece isometric to Q, and identify the simple closed geodesics in the boundary of any of these pieces (recall that all these geodesics have length equal to 1). Then S_2 is quasi-isometric to G_2, $\iota(S_2) > 0$ and, by the results of Kanai, one concludes that $\operatorname{lpgo}(S_2) = \operatorname{lpgo}(S_2)$ and $\operatorname{upgo}(S_2) = \operatorname{upgo}(G_2)$. Therefore,

$$\operatorname{lpgo}(S_2) = \operatorname{upgo}(S_2) = 1 + \frac{\log 2}{\log a}.$$

In a similar way, a Riemann surface S_1 can be constructed by using the graph G_1 as its skeleton. To do this, substitute the first bifurcation by a Y-piece with simple closed geodesics of lengths ℓ_0, ℓ_1, ℓ_1 and such that the distance from the geodesic with length ℓ_0 to each geodesic of length ℓ_1 is $a + a^2/2$. This Y-piece is the union of two isometric right-angled hexagons with three consecutive sides of length $\ell_0/2$, $a + a^2/2$ and $\ell_1/2$. Divide each one of these hexagons into two isometric pentagons with three consecutive sides of length $\ell_0/4$, $a+a^2/2$ and $\ell_1/2$. Using now a well-known hyperbolic trigonometric formula (see, e.g., [4, p.159]), one obtains

$$\tanh\frac{\ell_0}{4}\,\cosh\frac{2a+a^2}{2}\,\tanh\frac{\ell_1}{2} = 1\,, \tag{5.11}$$

and this formula allows to get ℓ_1 from ℓ_0 once it is shown that $\tanh(\ell_0/4)\,\cosh((2a+a^2)/2) > 1$. To this end, notice that $f(t) = \tanh t/t$ is a strictly decreasing function in $(0,\infty)$, and, moreover, by restricting the t values to be so that $0 < t \le 1/2 < \operatorname{arctanh}(1/2) = (\log 3)/2 =: t_0$, it follows that

$$\frac{1}{2} < \frac{1}{\log 3} = \frac{\tanh t_0}{t_0} < \frac{\tanh t}{t}\,.$$

Also,

$$\frac{\tanh t}{t} < \lim_{t\to 0}\frac{\tanh t}{t} = 1\,, \qquad \text{if } t > 0\,.$$

Choose $\ell_0 = 1/2$ and $a \ge 3$. Since $\tanh(1/8)\cosh(15/2) > 16\,\coth(1/4) > \coth(1/4)$, there exists $\ell_1 < 1/2$ satisfying (5.11).

Next, in an inductive way, for $n \ge 2$, substitute each bifurcation of the n-th generation by a Y-piece with simple closed geodesics of lengths ℓ_{n-1}, ℓ_n, ℓ_n, and such that the distance from the geodesic with length ℓ_{n-1} to each geodesic of length ℓ_n is $(a^n + a^{n+1})/2$. In a similar way, the following can be obtained,

$$\tanh\frac{\ell_{n-1}}{4}\,\cosh\frac{a^n+a^{n+1}}{2}\,\tanh\frac{\ell_n}{2} = 1 \tag{5.12}$$

and so, as above, ℓ_n can be obtained from ℓ_{n-1} once it is shown that $\tanh(\ell_{n-1}/4)\,\cosh((a^n+a^{n+1})/2) > 1$ for $n \ge 2$. Let us show this by induction. In fact, it will be proved as well that $\ell_{n-1} < 1/2$ and that $\ell_{n-1}e^{(a^n+a^{n+1})/2} > 16\,\coth(1/4)$.

The case $n = 2$, was shown above, concluding that $\ell_1 < 1/2$. Also, by the choices made of $\ell_0 = 1/2$ and $a \ge 3$, the equality (5.11) gives

$$\Big(\tanh\frac{1}{8}\Big)\,e^{(2a+a^2)/2}\,\frac{\ell_1}{2} > 1.$$

Since $a^3 \ge 2a + 21$ for $a \ge 3$, one further gets

$$\ell_1\,e^{(a^2+a^3)/2} \ge \ell_1 e^{21/2}e^{(2a+a^2)/2} > 2\,e^{21/2}\coth(1/8) > 16\,\coth(1/4)\,.$$

And, finally, since $\ell_1 < 1/2$,

$$\tanh\frac{\ell_1}{4}\,\cosh\frac{a^2+a^3}{2} > \frac{\ell_1}{8}\,\frac{1}{2}\,e^{(a^2+a^3)/2} > \coth(1/4) > 1\,.$$

Next, for some $n \ge 2$, let us assume that $\ell_{n-1} < 1/2$ and that $\ell_{n-1}e^{(a^n+a^{n+1})/2} > 16\,\coth(1/4)$.

Equality (5.12) gives

$$\frac{\ell_{n-1}}{4}\,e^{(a^n+a^{n+1})/2}\frac{\ell_n}{2} > 1 \qquad \Longrightarrow \qquad \ell_n > \frac{8}{\ell_{n-1}}\,e^{-(a^n+a^{n+1})/2}\,.$$

As $a \ge 3$ it further follows that $a^2 - a - 2 > 0$ and thus $a^{n+2} + a^{n+1} > 2(a^{n+1} + a^n)$. Therefore, since $\ell_{n-1} < 1/2 < \sqrt{8}$,

$$\begin{aligned} \ell_n e^{(a^{n+1}+a^{n+2})/2} &> \frac{8}{\ell_{n-1}}\,e^{-(a^n+a^{n+1})/2}e^{(a^{n+1}+a^{n+2})/2} > \frac{8}{\ell_{n-1}}\,e^{(a^n+a^{n+1})/2} \\ &> \ell_{n-1}e^{(a^n+a^{n+1})/2} > 16\,\coth(1/4)\,. \end{aligned}$$

Using again (5.12) and $\ell_{n-1} < 1/2$, one also obtains

$$\coth \frac{\ell_n}{2} = \tanh \frac{\ell_{n-1}}{4} \cosh \frac{a^n + a^{n+1}}{2} > \frac{\ell_{n-1}}{8} \frac{1}{2} e^{(a^n+a^{n+1})/2} > \coth(1/4) > 1 \,,$$

and so $\ell_n < 1/2$.

This finishes the construction of the Riemann surface S_1.

Because all the Y-pieces have area equal to 2π one can check that S_1 and G_1 are quasi-isometric and that

$$\operatorname{lpgo}(S_1) = \operatorname{upgo}(S_1) = \operatorname{lpgo}(G_1) = \operatorname{upgo}(G_1) = \frac{\log 2}{\log a} \,.$$

Since G_1 and G_2 are isometric, S_1 and S_2 are quasi-isometric, and, by the above arguments,

$$\operatorname{lpgo}(S_2) = \operatorname{lpgo}(S_1) + 1 \,, \qquad \operatorname{upgo}(S_2) = \operatorname{upgo}(S_1) + 1 \,.$$

□

The surfaces S_1, S_2 in the proof of Proposition 5.1 show that the inequalities in Theorems 1.1 and 1.3 and Corollary 1.2 are sharp.

We finish this work by showing that it makes sense to consider the growth rates $\operatorname{lgo}_\varphi$ and $\operatorname{ugo}_\varphi$.

Proposition 5.2. *There exist non-exceptional Riemann surfaces with area growth of balls as small as desired.*

Proof. Let $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a strictly increasing admissible function, set $r_0 = 0$ and let $r_n = \varphi^{-1}(n+1)$ for every $n \geq 1$. Define the sequence $\{a_n\}_{n=1}^{\infty}$ to be $a_n := r_n - r_{n-1}$.

Let G be an infinite rooted path graph with vertices $\{v_n\}_{n=0}^{\infty}$ and edges $\{v_{n-1}v_n\}_{n=1}^{\infty}$ such that the length of the edge $v_{n-1}v_n$ is a_n. Thus, $d_G(v_0, v_n) = r_n$ for every $n \geq 1$. If $r_{n-1} < r \leq r_n$ and $n \geq 2$, then

$$\operatorname{vol}_G(B_G(v_0, r)) = n = \varphi(r_{n-1}) \leq \varphi(r) \qquad \Longrightarrow \qquad \operatorname{ugo}_\varphi(G) \leq 1 \,.$$

Let G' be an isomorphic graph to G with edges $v_{n-1}v_n$ of length $a'_n \geq a_n$. It is clear that $\operatorname{vol}_{G'}(B_{G'}(v_0, r)) \leq \operatorname{vol}_G(B_G(v_0, r))$ and so,

$$\operatorname{ugo}_\varphi(G') \leq \operatorname{ugo}_\varphi(G) \leq 1 \,.$$

Let us construct now a Riemann surface S by using the graph G' as an skeleton. Let us recall that a *generalized Y-piece* is a bordered or non-bordered hyperbolic Riemann surface which is topologically a sphere without n open disks and m points, with integers $n, m \geq 0$ and $n + m = 3$, so that the n boundary curves are simple closed geodesics and the m deleted points are cusps. Observe that a generalized Y-piece is topologically the union of a Y-piece and m cylinders, with $0 \leq m \leq 3$.

In the graph G', substitute each edge $v_{n-1}v_n$ of length a'_n by a *generalized Y-piece* with one cusp and so that the two boundary simple closed geodesics have respective lengths $2b_{n-1}$ and $2b_n$. Both sequences $\{b_n\}$ and $\{a'_n\}$ will be defined by induction in what follows. Each one of these generalized Y-pieces is the union of two isometric (degenerated) right-angled hexagons with three consecutive sides of lengths b_{n-1}, a'_n and b_n. Each hexagon is the union of two isometric (degenerated) right-angled quadrilaterals. One of them has two sides with infinite length and the other two sides of lengths b_{n-1} and x_n. The other quadrilateral also has two sides with infinite length and the other two sides of lengths $a'_n - x_n$ and b_n.

By a well-known trigonometric formula (see, e.g., [4, pp.157-158]),

$$\sinh b_{n-1} \sinh x_n = 1 \,, \qquad \sinh(a'_n - x_n) \sinh b_n = 1 \,.$$

The first equation gives $\cosh x_n = \operatorname{cotanh} b_{n-1}$ and the second one,

$$\sinh b_n = \frac{1}{\sinh a'_n \cosh x_n - \sinh x_n \cosh a'_n} = \frac{\sinh b_{n-1}}{\sinh a'_n \cosh b_{n-1} - \cosh a'_n} \,. \tag{5.13}$$

Define now

$$a'_n := \max \left\{ a_n \,,\ \operatorname{arcsinh} \left(2 \operatorname{cosech} b_{n-1} \operatorname{cotanh} b_{n-1} \right) \right\} \,. \tag{5.14}$$

Then, it is easy to check that

$$\sinh a'_n \cosh b_{n-1} - \cosh a'_n \geq 1 \quad \Longrightarrow \quad 0 < \sinh b_n \leq \sinh b_{n-1} \quad \Longrightarrow \quad 0 < b_n \leq b_{n-1} \, .$$

Therefore, by fixing $b_0 > 0$, formulas (5.13) and (5.14) allow to inductively define a'_n and b_n from b_{n-1} for $n \geq 1$.

Since all the generalized Y-pieces have area equal to 2π, one can check that

$$\operatorname{ugo}_\varphi(S) = \operatorname{ugo}_\varphi(G') \leq 1 \, .$$

□

References

[1] Alvarez, V., Rodríguez, J. M., Yakubovich, V. A., Subadditivity of p-harmonic "measure" on graphs, *Michigan Math. J.* **49** (2001), 47–64.

[2] Avez, A., Variétés riemanniennes sans points focaux, *C. R. Acad. Sci. Paris Ser. A* **270** (1970), 188–191.

[3] Ballman, W., Gromov, M., Schroeder, V., Manifolds of Non-positive Curvature, Birkhauser, Boston, 1985.

[4] Beardon, A. F., The geometry of discrete groups. Springer-Verlag, New York, 1983.

[5] Bers, L., An Inequality for Riemann Surfaces. Differential Geometry and Complex Analysis. H. E. Rauch Memorial Volume. Springer-Verlag, 1985.

[6] Brooks, R., A relation between growth and the spectrum of the laplacian, *Math. Z.* **178** (1981), 501–508.

[7] Buser, P., The collar theorem and examples, *Manuscr. Math.* **25** (1978), 349–357.

[8] Buser, P., Geometry and Spectra of Compact Riemann Surfaces. Birkhäuser, Boston, 1992.

[9] Cantón, A., Fernández, J. L., Pestana, D., Rodríguez, J. M., On harmonic functions on trees, *Potential Analysis* **15** (2001), 199–244.

[10] Cantón, A., Granados, A., Portilla, A., Rodríguez, J. M., Quasi-isometries and isoperimetric inequalities in planar domains, *J. Math. Soc. Japan* **67** (2015), 127–157.

[11] Chavel, I., Eigenvalues in Riemannian Geometry. Academic Press, New York, 1984.

[12] Cheng, S. Y., Yau, S.-T., Differential equations on Riemannian manifolds and their geometric applications, *Comm. Pure Appl. Math.* **28** (1975), 333–354.

[13] Delmotte, T., Parabolic Harnack inequality and estimates of Markov chains on graphs, *Rev. Mat. Iberoamericana* **15** (1999), 181–232.

[14] Fernández, J. L., On the existence of Green's Function in Riemannian Manifolds, *Proc. Amer. Math. Soc.* **96** (1986), 284–286.

[15] Fernández, J. L., Rodríguez, J. M., The exponent of convergence of Riemann surfaces. Bass Riemann surfaces. *Annales Acad. Scient. Fenn. A. I.* **15** (1990), 165–183.

[16] Fernández, J. L., Rodríguez, J. M., Area growth and Green's function of Riemann surfaces, *Arkiv för matematik* **30** (1992), 83–92.

[17] Granados, A., Pestana, D., Portilla, A., Rodríguez, J. M., Tourís, E., Stability of the injectivity radius under quasi-isometries and applications to isoperimetric inequalities, *RACSAM*, in press. DOI 10.1007/s13398-017-0417-4

[18] Granados, A., Pestana, D., Portilla, A., Rodríguez, J. M., Stability of p-parabolicity under quasi-isometries. Submitted.

[19] Gromov, M., Hyperbolic groups, in "Essays in group theory". Edited by S. M. Gersten, M. S. R. I. Publ. **8**. Springer, 1987, 75–263.

[20] Ghys, E., de la Harpe, P., Sur les Groupes Hyperboliques d'après Mikhael Gromov. Progress in Mathematics, Volume 83. Birkhäuser. 1990.

[21] Holopainen, I., Rough isometries and p-harmonic functions with finite Dirichlet integral, *Rev. Mat. Iberoam.* **10** (1994), 143–176.

[22] Holopainen, I., Volume growth, Green's functions, and parabolicity of ends, *Duke Math. J.* **97:2** (1999), 319–346.

[23] Holopainen, I., Soardi, P. M., p-harmonic functions on graphs and manifolds, *Manuscr. Math.* **94** (1997), 95–110.

[24] Kanai, M., Rough isometries and combinatorial approximations of geometries of non-compact Riemannian manifolds, *J. Math. Soc. Japan* **37** (1985), 391–413.

[25] Kanai, M., Rough isometries and the parabolicity of Riemannian manifolds, *J. Math. Soc. Japan* **38** (1986), 227–238.

[26] Kanai, M., Analytic inequalities and rough isometries between non-compact Riemannian manifolds. Curvature and Topology of Riemannian manifolds (Katata, 1985). Lecture Notes in Math. **1201**. Springer (1986), 122–137.

[27] Keller, M., Lenz, D., Dirichlet forms and stochastic completeness of graphs and subgraphs, *J. Reine Angew. Math.* **666** (2012), 189-223.

[28] Keller, M., Lenz, D., Wojciechowski, R. K., Volume growth, spectrum and stochastic completeness of infinite graphs, *Math. Z.* **274** (2013), 905-932.

[29] Li, P., Tam, L. F., Positive harmonic functions on complete manifolds with nonnegative curvature outside a compact set, *Ann. of Math.* **(2) 125** (1987), 171–207.

[30] Li, P., Tam, L. F., Green's functions, harmonic functions, and volume comparison, *J. Differential Geom.* **41** (1995), 277–318.
[31] Manning, A., Topological entropy for geodesic flows, *Ann. of Math.* **110** (1979), 567–573.
[32] Martínez-Pérez, A., Rodríguez, J. M., Isoperimetric inequalities in Riemann surfaces and graphs. Submitted. ArXiv:1806.04619.
[33] Omori, H., Isometric immersions of Riemannian manifolds, *J. Math. Soc. Japan* **19** (1967), 205–214.
[34] Portilla, A., Rodríguez, J. M., Tourís, E., Gromov hyperbolicity through decomposition of metric spaces II, *J. Geom. Anal.* **14** (2004), 123–149.
[35] Portilla, A., Tourís, E., A characterization of Gromov hyperbolicity of surfaces with variable negative curvature, *Publ. Mat.* **53** (2009), 83–110.
[36] Randol, B., Cylinders in Riemann surfaces, *Comment. Math. Helv.* **54** (1979), 1–5.
[37] Ratcliffe, J.G., Foundations of Hyperbolic Manifolds. Springer-Verlag, New York, 1994.
[38] Rodríguez, J. M., Isoperimetric inequalities and Dirichlet functions of Riemann surfaces, *Publ. Mat.* **38** (1994), 243–253.
[39] Rodríguez, J. M., Two remarks on Riemann surfaces, *Publ. Mat.* **38** (1994), 463–477.
[40] Shimizu, H., On discontinuous groups operating on the product of upper half-planes, *Ann. of Math.* **77** (1963), 33–71.
[41] Soardi, P. M., Potential Theory in infinite networks. Lecture Notes in Math., Volume 1590, Springer-Verlag, 1994.
[42] Tourís, E., Graphs and Gromov hyperbolicity of non-constant negatively curved surfaces, *J. Math. Anal. Appl.* **380** (2011), 865–881.
[43] Wolpert, S., A generalization of the Ahlfors-Schwarz lemma, *Proc. A. M. S.* **84:3** (1982), 377–378.
[44] Yau, S.-T., Remarks on conformal transformations, *J. Diff. Geom.* **8** (1973), 369–381.

Saint Louis University, Madrid Campus Avenida del Valle, 34 - 28003 Madrid, Spain
E-mail address: ana.granados@slu.edu

Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain
E-mail address: dompes@math.uc3m.es

Saint Louis University, Madrid Campus Avenida del Valle, 34 - 28003 Madrid, Spain
E-mail address: ana.portilla@slu.edu

Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain
E-mail address: jomaro@math.uc3m.es

Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, Campus de Cantoblanco, 28049 Madrid, Spain
E-mail address: eva.touris@uam.es