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MÉTODOS GEOMÉTRICOS EN TEORÍAS CLÁSICAS DE CAMPOS E INTEGRACIÓN NUMÉRICA

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Introduction

Symplectic Geometry has historically proven to be the natural frame to describe the motion of bodies, the principia that originate them, and all the concepts that take part in the description, and that had their origin in the Analytical Mechanics as stated by Newton, Lagrange, D'Alembert, Poincaré, Hamilton and many of the most relevant scientists of the past three centuries, such as the energy, the constraints, the symmetries, the forces, and so on.

In particular, we have that

- The admissible configurations of the system is described by a manifold Q whose dimension equals the degrees of freedom of the system. For instance, this is the situation that arises when one considers some holonomic constraints to $\mathbb{R}^n$.
- The phase space of the velocities is considered to be TQ (or respectively $\mathbb{R} \times TQ$ for the time-dependent case), where a first order Lagrangian function is defined, and a variational principle gives the equations of motion. As we shall see, the dual T^*Q (respectively $\mathbb{R} \times T^*Q$) also takes part in the description. For the case of non-holonomic constraints, one additionally considers a distribution $D \subseteq TQ$.
- The concepts and tools of symplectic geometry (respectively, cosymplectic geometry for the time-dependent case) are used to intrinsically define the evolution equations of the system, from where we can naturally study the presence of symmetries, preserved quantities, and an eventual reduction to a simpler problem. The geometric description of mechanics in terms of symplectic (and cosymplectic) geometry is very briefly exposed in the first chapter of this work.
- Finally, the theory of generating function arises in a very natural way of defining numerical methods, for performing the effective computation of evolution paths, as is explained at the beginning of chapter 6.

Most of other Classical Field Theories are known to come from a variational principle obtained from a first order Lagrangian function defined on certain space. One of the models to describe this, that has recently been particularly interesting for a large amount of researchers is the multisymplectic geometry defined over jet manifolds, which is one possible natural extension to symplectic geometry, and which is explained in chapters 2 and 3 of this work. More precisely,

- A field is represented by a section of certain fibration $\pi : Y \longrightarrow X$
- A Lagrangian function L is defined on the first order jet manifold $J^1\pi$, that replaces naturally the phase space of the velocities. The evolution equations arise from a variational principle involving this Lagrangian function.
- The concepts and tools of multisymplectic geometry are used to intrinsically define the evolution equations of the system, from where we can naturally study the presence of symmetries, preserved quantities, and an eventual reduction to a simpler problem. The second chapter is devoted to multisymplectic geometry and jet manifolds, whereas the third chapter describes the Classical Field Theories using this formalism.

In particular, with this work it is my intention to expose the results of the research works on which I have been involved during the last years in order to comply with the project described above. In particular:

- In the chapter 2, devoted to multisymplectic geometry over jet manifolds, I describe results aimed to produce Darboux coordinates on certain type of multisymplectic manifolds, with which the multisymplectic form can, at least locally, be assimilated to the standard multisymplectic form.
- In the chapter 3, devoted to the description of the model for the various classical field theories in terms of multisymplectic geometry on jet manifolds, we obtain, in the regular case, an analogous result to the Tulczyjew in [154, 155] (which identifies TT^*M with T^*TM and T^*T^*M) for the Classical Field Theories.
- In the chapter 4, we analyze the presence of symmetries in the Field equations in their different approaches, and use a Noether theorem to obtain preserved quantities.
- Chapter 5 describes the geometry of the Cauchy surfaces in the classical field theory, with special attention to the case in which the base manifold X can be decomposed into a product of a time-like 1-dimensional manifold, and a space-like manifold, that arises into simpler equations.
- Finally, chapter 6 explains how the theory of generating functions can be used as a source of a new family of numerical methods having better geometrical properties. We apply these ideas to two particular cases, the case of Mechanics with non-holonomic constraints, and the case of optimal control theory, and we expose in the last chapter how this could be used to eventually produce numerical methods for the classical field theories based on the same ideas.

Throughout all this work, some basic knowledge of Differential Geometry is assumed. We shall also use the following notation: $\mathcal{L}_X$ to denote the Lie derivative respect to the vector field X,

$\mathfrak{X}^k(M)$ to denote the k-multivectors on M (that is, the sections of $\Lambda^k TM$), and by $\Lambda^k M$ the k-forms on M (that is, sections of $\Lambda^k T^*M$); $\tau_Q : TQ \longrightarrow Q$, and $\pi_Q : T^*Q \longrightarrow Q$ will denote the canonical projections. If G is a Lie group acting on a manifold M, and $\mathfrak{g}$ is its Lie algebra, then for $\xi \in \mathfrak{g}$, by ξ_M we shall denote the fundamental vector field (or infinitesimal generator) determined by ξ.

CHAPTER 1

Symplectic geometry and mechanics

This introductory chapter aims to be a brief review of the well known concepts of symplectic and cosymplectic geometry, and how these theories become a natural model to describe the behaviour and properties of time-independent and time-dependent Mechanics, respectively. We shall only center in the time-dependent Mechanics described in terms of cosymplectic geometry.

The results mentioned in this chapter are well known, and the proofs can be profusely found in the literature (see for instance [1]), so they will be often omitted.

For both the symplectic and cosymplectic geometry sections, both the algebraic and differential aspects are considered, as well as a brief digression on Hamiltonian functions.

1.1 Symplectic geometry

This section deals with symplectic geometry. It is something to be remarked that finite or infinite dimension will be considered when possible, whenever finite dimensionality is not explicitly asked.

1.1.1 Symplectic algebra

Definition 1.1.1. *Let V be a (finite or infinite dimensional) vector space, and let ω be a 2-form on V. One defines the following map $\omega^\flat : V \longrightarrow V^*$ given by*

$$\langle \omega^\flat(v_1) | v_2 \rangle := \omega(v_1, v_2)$$

*We say that ω is **weakly** (resp. **strongly**) **non-degenerate** whenever $\omega^\flat$ is injective, or in other words, $\iota_v\omega = 0 \Leftrightarrow v = 0$ (resp. an isomorphism).*

Obviously, if V is of finite dimension both concepts coincide, and we simply say that ω is **non-degenerate**. This is not the case whenever V is an infinite-dimensional vector space.

Definition 1.1.2. *A (weak, strong)* ***symplectic form*** *on a vector space V is a (weakly, strongly) non-degenerate 2-form.*

A (weak, strong) ***symplectic space*** *is a vector space V equipped with a (weak, strong) symplectic form.*

We have the following properties on every symplectic vector space.

Proposition 1.1.3. *Let ω be a 2-form on a finite dimensional vector space V, then*

(i) ω is non-degenerate if and only if V is even dimensional (say $\dim V = 2n$) and ω^n is a volume form on V.

(ii) If ω is symplectic, then there exists a basis of V with respect to which the matrix associated to ω has the following expression in blocks

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

where 0 denotes the $n \times n$ null matrix, and I denotes the n-dimensional identity matrix.

A symplectic form may be used to define orthogonality conditions and complements.

Definition 1.1.4. *If W is a subspace of a symplectic vector space (V, ω), then we define the* ***orthogonal complement*** *of W by*

$$W^{\perp} := \{v \in V \mid \iota_{v \wedge w}\omega = 0 \quad \text{for all} \quad w \in W\}$$

Definition 1.1.5. *A vector subspace W of a symplectic vector space is said to be*
(i) ***isotropic*** *if $W \subseteq W^{\perp}$*
(ii) ***coisotropic*** *if $W^{\perp} \subseteq W$*
(iii) ***Lagrangian*** *if it is isotropic and coisotropic, or in other words, $W^{\perp} = W$*
(iv) ***symplectic*** *if $W \cap W^{\perp} = \{0\}$ (and thus $(W, \omega|_W)$ is a symplectic vector space)*

We have that

Proposition 1.1.6. *Given a vector subspace $W \leq V$,*

(i) W is Lagrangian if and only if W is maximally isotropic

(ii) if V is finite dimensional, then W is Lagrangian if and only if W is isotropic and $\dim V = 2 \dim W$

Example 1.1.7. Let V be an arbitrary vector space and consider the direct product $V \times V^*$. Define a 2-form ω_V on $V \times V^*$ as follows:

$$\omega_V((v, \alpha), (w, \beta)) := \beta(v) - \alpha(w)$$

A direct computation shows that ω_V is indeed symplectic. Furthermore, $V \times \{0\}$ and $\{0\} \times V^*$ are lagrangian subspaces.

1.1.2 Symplectic manifolds

The preceding definitions can be extended to differentiable manifolds.

Definition 1.1.8. *Let ω be a 2-form on a smooth (finite or infinite dimensional) manifold M modelled over a Banach space. For each $x \in M$, consider the mapping $\omega_x^\flat : T_xM \longrightarrow T_x^*M$ by*

$$\omega_x^\flat(V) := \iota_V \omega_x.$$

*We say that ω is **weakly non-degenerate** at x whenever $\omega_x^\flat$ is injective, in other words, $\iota_V\omega_x = 0 \Leftrightarrow V = 0$, and **strongly non-degenerate** whenever it is an isomorphism of vector spaces.*

*ω is said to be a (weak, strong) **symplectic form** on M whenever it is closed and (weakly, strongly) non-degenerate everywhere. Therefore, ω_x is a symplectic form on T_xM for each $x \in M$. In the case of weak symplectic manifolds, we have that the induced map $\omega^\flat : \mathfrak{X}(M) \longrightarrow \Lambda(M)$ defined by*

$$\omega^\flat(X) := \iota_X \omega_x.$$

is one to one, but needs not be surjective in general.

*A **symplectic manifold** is a manifold M endowed with a symplectic form.*

In such case, we have that if M is finite-dimensional, then it is an even dimensional manifold (say $\dim M = 2n$), and the form ω^n is a volume form on M.

We also have the following theorem by Darboux:

Theorem 1.1.9. ***(Darboux)*** *If ω is a non-degenerate 2-form on a finite-dimensional manifold M ($\dim M = 2n$), then ω is closed if and only if there is a chart $(U, \boldsymbol{x})$ around each $x \in M$ such that $\boldsymbol{x}(x) = 0$, and if $\boldsymbol{x}(u) = (q^1(u), \ldots, q^n(u), p_1(u), \ldots, p_n(u))$ for $u \in U$, then we have that*

$$\omega|_U = dq^i \wedge dp_i$$

The coordinates given by the system of coordinates $(U, \boldsymbol{x})$ are usually called **Darboux coordinates**.

Definition 1.1.10. *A submanifold N of a symplectic manifold (M, ω) is called **isotropic** (resp. **coisotropic**, **Lagrangian**, **symplectic**) whenever T_xN is an isotropic (resp. coisotropic, Lagrangian, symplectic) subspace of (T_xM, ω_x), for each $x \in N$.*

Finally we have:

Definition 1.1.11. *A diffeomorphism Φ between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is said to be a **symplectomorphism** or a **canonical transformation** whenever $\Phi^*\omega_2 = \omega_1$.*

Example 1.1.12. Let Q be an arbitrary manifold. We define the following 1-form θ_Q on T^*Q, called the **Liouville form**, where if $W \in T_\alpha T^*Q$,

$$\langle (\theta_Q)_\alpha | W \rangle := \langle \alpha | T\pi_Q W \rangle$$

where π_Q is the canonical projection.

If Q is finite dimensional, and (q^i, p_i) are canonical coordinates on T^*Q, then we have that $\theta_Q = p_i dq^i$.

Now consider $\omega_Q := -d\theta_Q$. It is a closed 2-form, and an easy computation shows that it is actually a symplectic form. If Q is finite dimensional, and (q^i, p_i) are canonical coordinates on T^*Q, then we have that $\omega_Q = dq^i \wedge dp_i$. This form is called the **canonical symplectic form** of T^*Q.

It is called canonical because Darboux theorem guarantees that, at every point of a symplectic manifold, the symplectic form can be assimilated to the canonical symplectic form for certain manifold Q . When a manifold is globally symplectomorphic to (T^*Q, ω) then we say that it is a **special symplectic manifold** (according to the terminology introduced by W. Tulczyjew in [154, 155]).

1.1.3 Hamiltonian vector fields and functions for symplectic manifolds

Along this section, (M, ω) will be a symplectic manifold. The results in this section are well known for finite dimensional manifolds, and for infinite dimensional manifolds can be found for example in [130, 151].

Definition 1.1.13. *A vector field $X \in \mathfrak{X}(M)$ will be said to be*
(i) ***Hamiltonian*** *if there exists a function h (called* ***Hamiltonian function****) such that*

$$\iota_X \omega = dh$$

(ii) ***locally Hamiltonian*** *if $\mathfrak{L}_X \omega = 0$*

Notice in particular that the flow of locally Hamiltonian vector field preserves the symplectic form, that is, the flow F_t is a canonical transformation.

The non-degeneracy of ω guarantees that for a given function h, if there exists an associated Hamiltonian vector field, then it is unique, and will be denoted by X_h. We denote by

$$\mathcal{H}(M) := \{h \in \mathcal{C}^\infty(M) | dh = \iota_X \omega \text{ for some } X \in \mathfrak{X}(M)\}$$

If M is finite-dimensional, then $\mathcal{H}(M) = \mathcal{C}^\infty(M)$, but for infinite dimensional manifolds, this cannot be guaranteed.

In local Darboux coordinates of the modelling Banach space, X_h is given by

$$X_h(u, w) = \left(\frac{\partial h}{\partial u}, -\frac{\partial h}{\partial w} \right)$$

(for more details see [151]).

Definition 1.1.14. *In $\mathcal{H}(M)$ we can define the following* ***Poisson bracket:***

$$\{f, g\} := \iota_{X_g} \iota_{X_f} \omega = \omega(X_f, X_g)$$

Furthermore, we have that

Proposition 1.1.15. *The following properties hold for the Poisson bracket for $f, g, h \in \mathcal{H}(M)$:*

(i) $\{f,g\} = - \{g,f\}$

(ii) $\{f+g,h\}=\{f,h\}+\{g,h\}$

(iii) $\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0$

Proposition 1.1.16. *If $f, g \in \mathcal{H}(M)$, then $\{f, g\} \in \mathcal{H}(M)$, and*

$$X_{\{f,g\}} = -[X_f, X_g]$$

Therefore, the set of Hamiltonian vector fields is a Lie subalgebra of the Lie algebra of vector fields on M with the Lie bracket.

Similarly, it is shown that $\mathcal{H}(M)$ is a Lie algebra with the Poisson bracket defined above.

1.1.4 Tulczyjew's triples

In [154, 155], Tulczyjew introduced an identification of TT^*M with T^*TM and T^*T^*M (see [123] for an in-depth analysis of these isomorphisms, and also [30]). For a direct study see [106].

The manifolds TT^*M and T^*T^*M are identified by using the isomorphism given by contraction with the canonical symplectic form on T^*M. It is locally expressed by $\beta(q, p, \dot{q}, \dot{p}) = (q, p, -\dot{p}, \dot{q})$.

To define the isomorphism $\alpha : TT^*M \longrightarrow T^*TM$ we need two extra ingredients.

One of them is the canonical involution of TTM, which is defined as follows: $\kappa_M : TTM \longrightarrow TTM$ given by

$$\kappa_M(\left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} \chi(s,t)) = \left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} \tilde{\chi}(s,t)$$

where for $\chi : \mathbb{R}^2 \longrightarrow M$ we define $\tilde{\chi} : \mathbb{R}^2 \longrightarrow M$ by $\tilde{\chi}(s,t) = \chi(t,s)$. In local coordinates it gets $\kappa_M(q, v, \dot{q}, \dot{v}) = (q, \dot{q}, v, \dot{v})$.

The second ingredient is the tangent pairing. Given two manifolds M and N, and a pairing $\langle\cdot|\cdot\rangle$ between them, it is given by $\langle\cdot|\cdot\rangle^T : TM \times TN \longrightarrow \mathbb{R}$ by

$$\langle \left.\frac{d}{dt}\right|_{t=0} \gamma(t) | \left.\frac{d}{dt}\right|_{t=0} \delta(t)\rangle^T = \left.\frac{d}{dt}\right|_{t=0} \langle\gamma(t), \delta(t)\rangle$$

Thus, we define α by $\langle\alpha(z)|w\rangle = \langle z|\kappa_M(w)\rangle^T$ for $z \in TT^*M$ and $w \in TTM$, which in local coordinates is given by $\alpha(q, p, \dot{q}, \dot{p}) = (q, \dot{q}, \dot{p}, p)$.

1.2 Cosymplectic geometry

1.2.1 Cosymplectic algebra

Definition 1.2.1. *Let V be a $2n+1$-dimensional vector space, and let Ω and η be a 2-form and a 1-form on V, respectively. We say that (V, Ω, η) is a **cosymplectic vector space** whenever*

$$\Omega^n \wedge \eta \neq 0$$

One can define the following map $\flat : V \longrightarrow V^*$ given by

$$\flat(v) := \iota_v \Omega + \eta(v)\eta$$

On a cosymplectic vector space, one proves that $\flat$ is an isomorphism. The vector $R := \flat^{-1}(\eta)$ is called the **Reeb vector**, and is given by the equations:

$$\iota_R \Omega = 0, \qquad \iota_R \eta = 1$$

Cosymplectic vector spaces can be coordenatised in a good manner, as shows the following

Proposition 1.2.2. *On every cosymplectic vector space (V, Ω, η) there exists a basis of V (that we shall call a **Darboux basis**, with some abuse of notation) (u^i, v^i, w), with dual basis $((u^i)^*, (v^i)^*, w^*)$, such that $\Omega = (u^i)^* \wedge (v^i)^*$, $\eta = w^*$.*

Example 1.2.3. The canonical example of a cosymplectic vector space is given by $V \times V^* \times \mathbb{R}$ for any real vector space V, where Ω is the canonical symplectic form inherited from $V \times V^*$, and $\eta = dt$ inherited from $\mathbb{R}$.

1.2.2 Cosymplectic manifolds

Definition 1.2.4. *A **cosymplectic manifold** is a triple (M, Ω, η) where M is a $(2n+1)$-dimensional manifold, Ω is a closed 2-form, η is a closed 1-form, and for each $x \in M$, $(T_x M, \Omega_x, \eta_x)$ is a cosymplectic vector space.*

In a cosymplectic manifold we can define the following map $\flat : \mathfrak{X}(M) \longrightarrow \Lambda M$ given by

$$\flat(V) := \iota_V \Omega + \eta(V)\eta,$$

and one can easily prove that it is an isomorphism of $\mathcal{C}^\infty(M)$-modules.

The vector field $R := \flat^{-1}(\eta)$ is called the **Reeb vector field**, and is given by the equations:

$$\iota_R \Omega = 0, \qquad \iota_R \eta = 1$$

Cosymplectic manifolds can also be locally coordenatised in a good manner, as shown by

Proposition 1.2.5. *On every cosymplectic manifold (M, Ω, η), every point admits a coordinate chart (q^i, p_i, t) (that we shall call **Darboux coordinates**, with some abuse of notation) such that $\Omega = dq^i \wedge dp_i$, $\eta = dt$.*

Finally we have the following definition:

Definition 1.2.6. *A diffeomorphism* Φ *between two cosymplectic manifolds* (M_1, ω_1, η_1) *and* (M_2, ω_2, η_2) *is said to be a* ***cosymplectomorphism*** *whenever* $\Phi^*\omega_2 = \omega_1$*, and* $\Phi^*\eta_2 = \eta_1$*.*

Example 1.2.7. The canonical example of a cosymplectic manifold is given by $T^*Q \times \mathbb{R}$ for any manifold Q, where Ω is the canonical symplectic form ω_Q inherited from T^*Q, and η is the form dt inherited from $\mathbb{R}$.

Extending the terminology introduced by Tulczyjew, we shall say that a cosymplectic manifold (M, Ω, η) is an **special cosymplectic manifold** whenever there exists a manifold Q and a diffeomorphism $\Phi : M \longrightarrow T^*Q \times \mathbb{R}$ such that $\Phi^*\omega_Q = \Omega$ and $\Phi^*(dt) = \eta$.

Any special cosymplectic manifold also inherits a form $\Theta = \Phi^*\theta_Q$ such that $-d\Theta = \Omega$.

1.2.3 Hamiltonian vector fields and functions for cosymplectic manifolds

Along this section, (M, Ω, η) will be a cosymplectic manifold with Reeb vector field R. For a wider description of the topics covered in this section, see [18].

Definition 1.2.8. *For a given function* $h : M \longrightarrow \mathbb{R}$ *defined on* M*, we can define:*

(i) The ***gradient vector field*** $\operatorname{grad} h := \flat^{-1}(dh)$*, given by the equations*

$$\begin{aligned} \iota_{\operatorname{grad} h}\eta &= R(h) \\ \iota_{\operatorname{grad} h}\Omega &= dh - R(h)\eta \end{aligned}$$

and canonical local coordinate expression

$$\operatorname{grad} h = \frac{\partial h}{\partial t}\frac{\partial}{\partial t} + \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} + \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}$$

(ii) The ***Hamiltonian vector field*** $X_h := \flat^{-1}(dh - R(h)\eta)$*, given by the equations*

$$\begin{aligned} \iota_{X_h}\eta &= 0 \\ \iota_{X_h}\Omega &= dh - R(h)\eta \end{aligned}$$

and canonical local coordinate expression

$$X_h = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} + \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}$$

(iii) The ***evolution vector field*** $E_h := R + X_h$*, given by the equations*

$$\begin{aligned} \iota_{E_h}\eta &= 0 \\ \iota_{E_h}\Omega_h &= 1 \end{aligned}$$

where $\Omega_h = dh \wedge \eta + \Omega$*, and canonical local coordinate expression*

$$E_h = \frac{\partial}{\partial t} + \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} + \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}$$

The preceding definition allows us to define a Poisson structure on $\mathcal{C}^\infty(M)$, given by

$$\{f,g\} := \Omega(\mathrm{grad} f, \mathrm{grad} g)$$

such that the mapping $\mathcal{C}^\infty(M) \longrightarrow \mathfrak{X}(M)$ given by $f \mapsto X_f$ is a Lie algebra antihomomorphism, that is,

$$X_{\{f,g\}} = -[X_f, X_g]$$

Notice that we also have that

(i) $\iota_{X_h}\Omega = \iota_{\mathrm{grad} h}\Omega$

(ii) $X_h = \mathrm{grad} h \Leftrightarrow R(h) = 0$

Finally, we notice that if we have a function $h : M \longrightarrow \mathbb{R}$ defined on M, we can define another cosymplectic structure for M, given by (M, Ω_h, η), where $\Omega_h := dh \wedge \eta + \Omega$. And thus, the evolution vector field for h is precisely the Reeb vector field for this new cosymplectic structure as in definition 1.2.8. Furthermore, its flow preserves the cosymplectic structure given by Ω_h and η. That is, if we denote by F_t the flow of E_h then $F_t^*\Omega_h = \Omega_h$ and $F_t^*\eta = \eta$.

1.3 Geometric structure of the Tangent Bundle

This section defines the concepts of vertical and complete lifts of vector fields on a manifold to its tangent bundle, and introduces the vertical endomorphism. These concepts will be used later to describe the geometric setting for Mechanics. We shall denote by $(q^i, \dot{q}^i)$ a chart of local bundle coordinates in TQ, the tangent bundle of certain configuration manifold Q, that will be assumed to be fixed through the whole section. Let also $\tau_Q : TQ \longrightarrow Q$ be the canonical projection, in coordinates $\tau_Q(q^i, \dot{q}^i) = (q^i)$.

If X is a vector field on Q with flow Φ_t, then $T\Phi_t$ is a 1-parameter family of transformations on TQ.

Definition 1.3.1. *If $X \in \mathfrak{X}(Q)$ with flow Φ_t, then the infinitesimal generator X^c of $T\Phi_t$ on TQ is called the* ***complete lift*** *of X to TQ.*

In local coordinates, if $X = X^i \frac{\partial}{\partial q^i}$ then we have

$$X^c = X^i \frac{\partial}{\partial q^i} + \dot{q}^j \frac{\partial X^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i}$$

We also define

Definition 1.3.2. *For a fibration $\pi : N \longrightarrow M$, a tangent vector $v \in T_yN$ such that $\pi(v) = 0$ is called a* ***vertical tangent vector****. The subspace of vertical tangent vectors at y will be denoted by $\mathcal{V}_y\pi$, and the subbundle of vertical vectors by $\mathcal{V}\pi$.*

We define the vertical lift of tangent vectors and vector fields as follows:

Definition 1.3.3. *Let $V \in T_qQ$, we define the linear map $T_qQ \longrightarrow \mathcal{V}_V\tau_Q$, for $X \in T_qQ$ by*

$$X^v := \frac{d}{dt}(V + tX)|_{t=0}$$

which is called the ***vertical lift*** *of X to TQ at V. The vertical lift of vector fields is defined pointwise.*

In local coordinates, if $X = X^i \frac{\partial}{\partial q^i}$ then we have

$$X^v = X^i \frac{\partial}{\partial \dot{q}^i}$$

Finally, with these ingredients we can define the vertical endomorphism S:

Definition 1.3.4. *We define the following endomorphism S on TQ. Let $V \in TQ$, then for $Y \in T_VTQ$ by*

$$S_V(Y) := ((T\tau_Q)_V(Y))^v$$

In local coordinates, S is given by

$$S = dq^i \otimes \frac{\partial}{\partial \dot{q}^i}$$

The tangent bundle TQ is also equipped with another fundamental geometrical object [118]:

Definition 1.3.5. *the* ***Liouville vector field*** *or* ***dilation vector field*** *Δ, intrinsically defined as a vector field on TQ given by*

$$\Delta(V) = (V^v)_V$$

In natural bundle coordinates it is expressed as

$$\Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}$$

Finally, some vector fields ξ on TQ could be regarded as second order differential equations over Q, which are those verifying the **second order differential equation**:

$$S(\xi) = \Delta$$

1.4 Geometric setting for Mechanics

Symplectic and cosymplectic manifolds have shown to be the natural ambient to describe time independent and time dependent mechanics, respectively. In this section, we shall focus our attention in describing the time-dependent case, thus centering our attention in the cosymplectic description.

The description begins with a manifold Q of the possible configurations (not subject to holonomic constraints). The tangent bundle TQ is the phase space of the velocities of the manifold Q.

1.4.1 Lagrangian setting

We choose a **Lagrangian function**, L defined on $\mathbb{R} \times TQ$. Let us introduce the following local notation that we shall often use: $\hat{p}_i := \frac{\partial L}{\partial \dot{q}^i}$.

We also define the **Lagrangian energy** $E_L = \Delta(L) - L$ in terms of the Liouville vector field, with local expression

$$E_L = -\hat{p} := \dot{q}^i \hat{p}_i - L$$

In some cases, we shall need to assume some regularity conditions on the Lagrangian function:

Definition 1.4.1. *For a Lagrangian function $L : TQ \times \mathbb{R} \longrightarrow \mathbb{R}$, it is defined its **Hessian matrix***

$$\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right)_{i,j}.$$

*The Lagrangian is said to be **regular at** a point whenever such matrix is regular at that point, and **regular** whenever it is regular everywhere. Otherwise, the Lagrangian is said to be **singular**.*

Definition 1.4.2. *For a given Lagrangian L we define the **Poincaré-Cartan** 1**-form** as*

$$\Theta_L := L dt + S^*(dL) \tag{1.1}$$

In fibred coordinates, it has the following expression

$$\begin{aligned} \Theta_L &= \Big(L - \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} \Big) dt + \frac{\partial L}{\partial \dot{q}^i} dq^i \\ &= \hat{p} dt + \hat{p}_i dq^i \end{aligned}$$

From this form, we can also define its differential

Definition 1.4.3. *The **Poincaré-Cartan** 2**-form** is defined as*

$$\Omega_L := -d\Theta_L.$$

In fibred coordinates is expressed as follows

$$\Omega_L = dt \wedge d\hat{p} + dq^i \wedge d\hat{p}_i,$$

from where we immediately see that if L is regular, then $(\mathbb{R} \times TQ, \Omega_L, dt)$ is a cosymplectic manifold, and conversely.

The dynamics of the system is given by curves $c(t) = (t, q(t), \dot{q}(t))$ which extremise the following integral, called the **action**

$$S(c) = \int_{t_0}^{t_1} L(t, q(t), \dot{q}(t)) dt$$

for each compact interval $[t_0, t_1]$.

Variations of such curves are introduced by small perturbations of a curve $c(t)$, and the solution for the variational problem are the well known Euler-Lagrange equations.

Theorem 1.4.4. *Let $c(t) = (t, q(t), \dot{q}(t))$ be a curve. Then c is an extremal of L, if and only if*

$$0 = \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

An easy computation shows that, in the regular case, for a curve $c(t)$, satisfying the Euler-Lagrange equations is equivalent to being an integral curve of the Reeb vector field for the cosymplectic system $(\mathbb{R} \times TQ, \Omega_L, dt)$. Also notice that the tangent vector field to the corresponding curve $\tilde{c}(t) = (q(t), \dot{q}(t)$ on TQ verifies the second order differential equation.

1.4.2 The singular case

For a singular Lagrangian L, one cannot expect to find globally defined solutions; in general, if such a solution exists, it does so only along a submanifold Z_f of $\mathbb{R} \times TQ$.

This final submanifold is computed using the Dirac-Bergmann-Gotay-Nester-Hinds algorithm for Mechanics (see [25, 101] and also [34, 63, 66, 67, 68]).

Put $Z_1 = \mathbb{R} \times TQ$. We then consider the subset

$$Z_2 \;=\; \{z \in Z_1 \mid \exists R \in T_z Z_1 \quad \text{such that } \iota_R \Omega_L = 0,\, \iota_R dt = 1\}.$$

If Z_2 is a submanifold, then there are solutions but we have to include the tangency condition, and consider a new step:

$$Z_3 \;=\; \{z \in Z_2 \mid \exists R \in T_z Z_2 \quad \text{such that } \iota_R \Omega_L = 0,\, \iota_R dt = 1\}.$$

If Z_3 is a submanifold of Z_2, but there is a point for which there is no solution to the equations above tangent to Z_3, we go to the third step, and so on. In the favourable case, we would obtain a final constraint submanifold Z_f of non-zero dimension (which projects onto an open subset of $\mathbb{R}$ through the second projection) where the equations

$$(\iota_X \Omega_L = 0, \qquad \iota_X dt = 1)|_{Z_f} \tag{1.2}$$

have a solution.

1.4.3 Legendre transformation and equivalence theorem

The Legendre transformation connects the Lagrangian and the Hamiltonian description. The definition depends on the Poincaré-Cartan 1-form, which as we noted, depends on the Lagrangian that we chose for modelling the theory.

Definition 1.4.5. *We define the* ***Legendre transformation*** $leg_L : \mathbb{R} \times TQ \longrightarrow \mathbb{R} \times T^*Q$ *by* $leg_L(v,t) = ((leg_L)_t(v), t)$ *for each* $(v,t) \in \mathbb{R} \times TQ$*. Here,* $(leg_L)_t : TQ \longrightarrow T^*Q$ *denotes the usual fiber derivative of the restriction of* L *to each value of* t *(see [1]). In coordinates,* $leg_L(t, q^i, \dot{q}^i) = (t, q^i, \frac{\partial L}{\partial \dot{q}^i})$.

We have another alternative way of constructing the Legendre transformation. Consider the natural projection $\pi_{\mathbb{R}\times Q} : T^*(\mathbb{R}\times Q) \longrightarrow \mathbb{R}\times Q$, then we define $\mu : T^*(\mathbb{R}\times Q) \longrightarrow \mathbb{R}\times T^*Q$ as the canonical projection such that $p\circ\mu = \pi_{\mathbb{R}\times Q}$ (where $p : \mathbb{R}\times T^*Q \longrightarrow \mathbb{R}\times Q$). Now we define the extended Legendre transformation $Leg_L : \mathbb{R}\times TQ \longrightarrow T^*(\mathbb{R}\times Q)$ by

$$\langle Leg_L(t, v_q)|X\rangle := \langle(\Theta_L)_{(t,v_q)}|\tilde{X}\rangle$$

where $\tilde{X}$ is any vector tangent to $\mathbb{R}\times TQ$ at (t, v_q) which projects onto X, tangent vector to $\mathbb{R}\times Q$ at (t, q). The Legendre transformation is, in this case, $leg_L := \mu\circ Leg_L$.

We also have the following trivial proposition:

Proposition 1.4.6. *L is a regular Lagrangian if and only if leg_L is a local diffeomorphism*

Definition 1.4.7. *The Lagrangian L is said to be **hyper-regular** whenever leg_L is a global diffeomorphism.*

In such case, $H := (leg_L^{-1})^* E_L$ defines a function on $\mathbb{R}\times T^*Q$. Furthermore, define $\tilde{M}_1 := Leg_L(\mathbb{R}\times TQ)$, and denote by $\tilde{j} : \tilde{M}_1 \longrightarrow T^*(\mathbb{R}\times Q)$ the canonical inclusion, and by Leg_1 the co-restriction of Leg_L to $\tilde{M}_1$ (that is, $Leg_L = \tilde{j}\circ Leg_1$). Notice that when L is hyper-regular, then the restriction μ_1 of μ to $\tilde{M}_1$ is a diffeomorphism, and has an inverse $h_1 := \mu_1^{-1} = Leg_1\circ leg_L^{-1}$. Thus we can also define $h := \tilde{j}\circ h_1 : \mathbb{R}\times T^*Q \longrightarrow T^*(\mathbb{R}\times Q)$. If we write h in coordinates, we precisely have that $h(t, q^i, p_i) = (t, q^i; -H(t, q^i, p_i), p_i)$.

If we denote by $\tilde{\eta}_1 = (pr_1\circ\nu)^*(dt)$, where $pr_1 : \mathbb{R}\times Q \longrightarrow \mathbb{R}$ is the natural projection, then $(\tilde{M}_1, \tilde{j}^*\omega_{\mathbb{R}\times Q}, \tilde{j}^*\tilde{\eta}_1)$ is a cosymplectic manifold.

In addition, we can define $\Theta_h := h^*\theta_{\mathbb{R}\times Q}$, and $\Omega_h := h^*\omega_{\mathbb{R}\times Q} = -d\Theta_h$, which precisely coincide with $\Theta_H = -Hdt + \Theta$ and $\Omega_H = dH\wedge dt + \Omega$, respectively, and also $\eta_1 := h^*\tilde{\eta}_1$. This way, $(\mathbb{R}\times T^*Q, \Omega_h, \eta_1)$ becomes a cosymplectic manifold.

After some little computation,

Proposition 1.4.8. *We have that*

$$(leg_L)^*\Theta_h = \Theta_L, \qquad (leg_L)^*\Omega_h = \Omega_L,$$

Theorem 1.4.9. *(**equivalence theorem**). Suppose that the Lagrangian is hyper-regular. Then R is a Reeb vector field for $(\mathbb{R}\times TQ, \Omega_L, dt)$, if and only if $Tleg_L(R)$ is a Reeb vector field for $(\mathbb{R}\times T^*Q, \Omega_h, dt)$. Furthermore, the mappings Leg_1, μ_1 and leg_L are cosymplectomorphisms.*

1.4.4 Almost regular Lagrangians

When the Lagrangian is not regular then to develop a Hamiltonian counterpart, we need some weak regularity condition on the Lagrangian L, the almost-regularity assumption. The results in this section are widely explored in [25] and [101].

Definition 1.4.10. *A Lagrangian $L : \mathbb{R}\times TQ \longrightarrow \mathbb{R}$ is said to be **almost regular** if $Leg_L(\mathbb{R}\times TQ) = \tilde{M}_1$ is a submanifold of $T^*(\mathbb{R}\times Q)$, and $Leg_L : \mathbb{R}\times TQ \longrightarrow \tilde{M}_1$ is a submersion with connected fibers.*

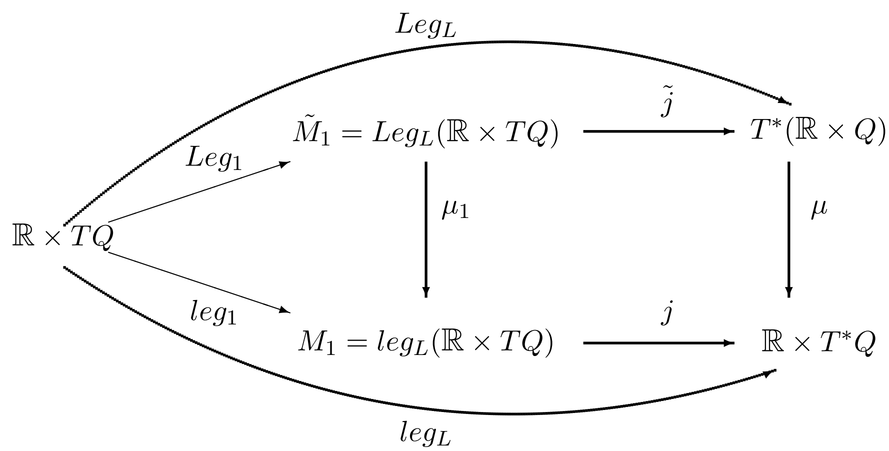

Figure 1.1: Almost regular Lagrangians and Legendre transformation

If L is almost regular, we deduce that:

- $M_1 = leg_L(\mathbb{R} \times TQ)$ is a submanifold of $\mathbb{R} \times T^*Q$.
- The restriction $\mu_1 : \tilde{M}_1 \longrightarrow M_1$ of μ is a diffeomorphism.
- The mapping $leg_L : \mathbb{R} \times TQ \longrightarrow M_1$ is a submersion with connected fibers.

On the hypothesis of almost regularity, we can define a mapping $h_1 = (\mu_1)^{-1} : M_1 \longrightarrow \tilde{M}_1$, and a 2-form Ω_{M_1} on M_1 by $\Omega_{M_1} = h_1^*(j^*\Omega)$ considering the inclusion map $j : \tilde{M}_1 \hookrightarrow T^*(\mathbb{R} \times Q)$. Obviously, we have $leg_1^*\Omega_{M_1} = \Omega_L$, where $j \circ leg_1 = leg_L$ (see Figure 1.1).

The Hamiltonian description is now based in the equation

$$\iota_Y \Omega_{M_1} = 0, \qquad \iota_Y \eta_{M_1} = 1$$

and we can apply the constraint algorithm as follows. We define

$$M_2 = \{y \in M_1 | \exists Y \in T_y M_1 \,\text{such that}\, \iota_Y \Omega_{M_1} = 0,\, \iota_Y \eta_{M_1} = 1\}$$

If M_2 is a submanifold then there are solutions at each point but we have to include the condition of tangency to M_2, and consider a new step:

$$M_3 = \{y \in M_2 | \exists Y \in T_y M_2 \,\text{such that}\, \iota_Y \Omega_{M_1} = 0,\, \iota_Y \eta_{M_1} = 1\}$$

and subsequent steps. If we assume that each M_i is a submanifold of $\mathbb{R} \times T^*Q$, we proceed further to obtain a sequence of embedded submanifolds

$$\ldots \hookrightarrow M_3 \hookrightarrow M_2 \hookrightarrow M_1 \hookrightarrow \mathbb{R} \times T^*Q$$

If this constraint algorithm stabilizes, we shall obtain a final constraint submanifold M_f of non-zero dimension where the equations

$$(\iota_X \Omega_{M_f} = 0, \qquad \iota_X \eta_{M_f} = 1)|_{M_f}$$

$$
\begin{array}{llllll}
Z_1 = Z & \xrightarrow{leg_1} & leg_L(Z) = M_1 & \xhookrightarrow{j} & \mathbb{R} \times T^*Q \\
\uparrow i_1 & & \uparrow j_1 & & \\
Z_2 & \xrightarrow{leg_2} & M_2 & & \\
\uparrow i_2 & & \uparrow j_2 & & \\
Z_3 & \xrightarrow{leg_3} & M_3 & & \\
\uparrow i_3 & & \uparrow j_3 & & \\
\vdots & & \vdots & & \\
\uparrow i_{k-2} & & \uparrow j_{k-2} & & \\
Z_{k-1} & \xrightarrow{leg_{k-1}} & M_{k-1} & & \\
\uparrow i_{k-1} & & \uparrow j_{k-1} & & \\
Z_k & \xrightarrow{leg_k} & M_k & &
\end{array}
$$

Figure 1.2: Relating the constraint algorithms

have a well-defined solution.

We can now compare the algorithms on both sides. A direct computation shows that $leg_1(Z_a) = M_a$ for each integer (see the figure 1.2)

In consequence, both algorithms have the same behaviour; in particular, if one of them stabilizes, so does the other, and at the same step. In particular, we have $leg_1(Z_f) = M_f$. In such a case, the restriction $leg_f : Z_f \longrightarrow M_f$ is a surjective submersion (that is, a fibration) and $leg_f^{-1}(leg_f(z)) = leg_1^{-1}(leg_1(z))$, for all $z \in Z_f$ (that is, its fibres are the ones of leg_1). Therefore, the Lagrangian and Hamiltonian sides can be compared through the fibration $leg_f : Z_f \longrightarrow M_f$.

Finally, if we have obtained a final constraint submanifold Z_f, the constraint algorithm is not sufficient to assure that solutions of (1.2) verify the second order differential equation condition. If the Lagrangian verifies extra regularity assumptions (admissible and degenerate, in particular if it is almost regular), then the authors in [25] prove the existence of a submanifold of Z_f where there exists a unique solution of (1.2) which verifies the second order differential equation condition, and it is such that the restriction of leg_f to S becomes a diffeomorphism.

CHAPTER 2

Multisymplectic geometry and jet manifolds

Multisymplectic geometry is a possible natural extension of symplectic geometry, that as we shall try to show in following chapters, proves to be a natural model to describe the Classical Field Theories, the same way symplectic geometry was used to describe Classical Mechanics.

This section describes the multisymplectic geometry in both its algebraic and differential aspects, and finishes with the description of jet manifolds, that are a convenient generalisation of the tangent bundle, where multisymplectic forms appear in a natural way, and that can be conveniently used to describe the various Classical Field Theories.

For further references on multisymplectic geometry, see for instance [16, 17, 50, 51, 84, 44].

Finally, notice that in this section, vector spaces and manifolds are all of them considered to have finite dimensional.

2.1 Multisymplectic algebra

2.1.1 Multisymplectic vector spaces

Definition 2.1.1. *A* ***multisymplectic form*** Ω *on a real vector space* V *is a form of degree* k *for* $1 < k \leq \dim V$ *such that verifies the following non-degeneracy property*

$$\iota_v \Omega = 0 \quad \Leftrightarrow \quad v = 0$$

Using the following notation,

$$Ker\Omega := \{v \in V \mid \iota_v \Omega = 0\},$$

the non-degeneracy property is rewritten as $Ker\Omega = 0$.
A ***multisymplectic vector space*** *of order* k *is a vector space endowed with a multisymplectic*

k-form.
*A **multisymplectomorphism** $\phi : (V_1, \Omega_1) \longrightarrow (V_2, \Omega_2)$ between two multisymplectic vector spaces is a linear isomorphism which preserves the multisymplectic form $\phi^*\Omega_2 = \Omega_1$.*

Notice that a multisymplectic 2-form is a symplectic form.

Definition 2.1.2. *Any k-form Ω on V induces the following homomorphisms*

$$\hat{\Omega}_j : \bigwedge^j V \longrightarrow \bigwedge^{k-j} V^*$$
$$U \longmapsto \iota_U \Omega$$

for $1 \leq j \leq k$. Multisymplecticity implies that $\hat{\Omega}_1$ is injective, and $\hat{\Omega}_{k-1}$ is surjective.

A multisymplectic form may be used to define orthogonality conditions and complements.

Definition 2.1.3. *If W is a subspace of a multisymplectic vector space (V, Ω) of degree k, then for each l such that $1 \leq l \leq k-1$ we define the l**-th orthogonal complement** of W by*

$$W^{\perp,l} := \{v \in V | \iota_{v\wedge w_1 \wedge \ldots \wedge w_l}\Omega = 0 \quad \textit{for all} \quad w_i \in W, \, i \in \{1, 2, \ldots, l\}\}$$

In particular, we have the following filtration of orthogonal complements

$$W^{\perp,1} \subseteq W^{\perp,2} \subseteq \ldots \subseteq W^{\perp,k}$$

We obviously have that $W^{\perp,l} = V$ whenever $l > \dim W$, and $W \cap W^{\perp,k} = Ker(\Omega|_W)$.

Definition 2.1.4. *A vector subspace W of a multisymplectic vector space is said to be*
*(i) l**-isotropic** if $W \subseteq W^{\perp,l}$*
*(ii) l**-coisotropic** if $W^{\perp,l} \subseteq W$*
*(iii) l**-Lagrangian** if it is l-isotropic and l-coisotropic, or in other words, $W^{\perp,l} = W$*
*(iv) **multisymplectic** if $W \cap W^{\perp,k} = \{0\}$*

Notice that for symplectic forms, the definitions of 1-isotropy, 1-coisotropy, 1-Lagrangian and multisymplecticity coincide with those of the symplectic geometry.

Example 2.1.5. Let V be an arbitrary vector space and consider the direct product $\mathcal{V}_V = V \times \Lambda^k V^*$. Define a $(k+1)$-form Ω_V on $\mathcal{V}_V$ as follows:

$$\Omega_V((v_1, \gamma_1), \ldots, (v_{k+1}, \gamma_{k+1})) = \sum_{i=1}^{k+1} (-1)^i \gamma_i(v_1, \ldots, \hat{v}_i, \ldots, v_{k+1}),$$

for all $(v_i, \gamma_i) \in \mathcal{V}_V$, $i = 1, \ldots, k+1$, where a hat over a letter means that it is omitted. A direct computation shows that Ω_V is indeed multisymplectic, and it is usually called the canonical multisymplectic form on $\mathcal{V}_V$.

If E is a vector subspace of V, we consider the subspace $\mathcal{V}_V^r = V \times \Lambda^k_r V^*$, where $\Lambda^k_r V^*$ denotes the space of k-forms on V vanishing when applied to at least r of their arguments from E. Of course, $\mathcal{V}_V^r$ equipped with the restriction Ω_V^r of Ω_V to $\mathcal{V}_V^r$ is a multisymplectic vector space. If $E = \{0\}$ we recover $\mathcal{V}_V$.

Let $(\mathcal{V}_1, \Omega_1)$ and $(\mathcal{V}_2, \Omega_2)$ be two multisymplectic vector spaces of order $k+1$. Take the direct product $\mathcal{V}_1 \times \mathcal{V}_2$ endowed with the $(k+1)$-form $\Omega_1 \ominus \Omega_2 = \pi_1^*\Omega_1 - \pi_2^*\Omega_2$, where $\pi_1 : \mathcal{V}_1 \times \mathcal{V}_2 \longrightarrow \mathcal{V}_1$ and $\pi_2 : \mathcal{V}_1 \times \mathcal{V}_2 \longrightarrow \mathcal{V}_2$ are the canonical projections. Then $(\mathcal{V}_1 \times \mathcal{V}_2, \Omega_1 \ominus \Omega_2)$ is a multisymplectic vector space.

Proposition 2.1.6. *Let $(\mathcal{V}_1, \Omega_1)$ and $(\mathcal{V}_2, \Omega_2)$ be two multisymplectic vector spaces of order $(k+1)$ and $\phi : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ a linear isomorphism. Then ϕ is a multisymplectomorphism if and only if its graph is a k-Lagrangian subspace of the multisymplectic vector space $(\mathcal{V}_1 \times \mathcal{V}_2, \Omega_1 \ominus \Omega_2)$.*

Proof. We recall that

$$\begin{aligned}(\operatorname{Graph}\phi)^{\perp,k} \quad = \quad & \{(x,y) \in \mathcal{V}_1 \times \mathcal{V}_2 \,|\, (\Omega_1 \ominus \Omega_2)((x,y),(x_1,\phi(x_1)),\ldots,(x_k,\phi(x_k))) = 0,\\ & \forall x_1,\ldots,x_k \in \mathcal{V}_1\}\end{aligned}$$

Assume that $\phi^*\Omega_2 = \Omega_1$, then if $(x,\phi(x)) \in \operatorname{Graph}\phi$, we have

$$\begin{aligned}&(\Omega_1 \ominus \Omega_2)((x,\phi(x)),(x_1,\phi(x_1)),\ldots,(x_k,\phi(x_k)))\\ &= \Omega_1(x,x_1,\ldots,x_k) - \Omega_2(\phi(x),\phi(x_1),\ldots,\phi(x_k))\\ &= \Omega_1(x,x_1,\ldots,x_k) - \phi^*\Omega_2(x,x_1,\ldots,x_k)\\ &= 0\end{aligned}$$

which implies that $\operatorname{Graph}\phi \subseteq (\operatorname{Graph}\phi)^{\perp,k}$.

Conversely, if $\operatorname{Graph}\phi$ is k-isotropic, we have $(x,\phi(x)) \in (\operatorname{Graph}\phi)^{\perp,k}$ for all $x \in \mathcal{V}_1$, and therefore $\phi^*\Omega_2 = \Omega_1$.

In addition, if $\operatorname{Graph}\phi$ is k-isotropic, it is also k-Lagrangian. In fact, if $(x,y) \in (\operatorname{Graph}\phi)^{\perp,k}$ then we have

$$\Omega_2(\phi(x) - y, \phi(x_1), \ldots, \phi(x_k)) = 0$$

for all $x_1,\ldots,x_k \in \mathcal{V}_1$ and therefore $y = \phi(x)$ because of the regularity of the multisymplectic form Ω_2 and the fact that ϕ is an isomorphism. ∎

2.1.2 Darboux basis for multisymplectic vector spaces

The results in this section has been developed as part of our work in [116], although some results are sketched in [132, 133].

Proposition 2.1.7. *Let V be an arbitrary vector space. Then:*

(i) V is a k-Lagrangian subspace of $\mathcal{V}_V$ and $\mathcal{V}_V^r$, for all r;

(ii) $\Lambda^k V^*$ (resp. $\Lambda^k_r V^*$) is a 1-isotropic subspace of $\mathcal{V}_V$ (resp. $\mathcal{V}_V^r$).

Proof.

(i) A direct computation shows that

$$V^{\perp,k} = \{(x,\gamma) \mid \Omega_V((x,\gamma),(x_1,0),\ldots,(x_k,0)) = 0, \text{ for all } x_1,\ldots,x_k\}$$

which is equivalent to the condition $\gamma(x_1, \ldots, x_k) = 0$ for all $x_1, \ldots, x_k \in V$, and therefore $\gamma = 0$. Hence $V^{\perp,k} = V$.

The same proof holds for $\mathcal{V}_V^r$.

(ii) We have to prove that

$$\Lambda^k V^* \subset (\Lambda^k V^*)^{\perp,1}$$

which is obvious because

$$i_{(0,\gamma_1)\wedge(0,\gamma_2)}\,\Omega_V = 0.$$

The same argument works for $\mathcal{V}_V^r$. ∎

Remark 2.1.8. In addition, notice that

$$(\Lambda^k V^*)^{\perp,1} = \Lambda^k V^*$$

which implies that $\Lambda^k V^*$ is in fact 1-Lagrangian.

Theorem 2.1.9. *[132, 133] Let $(\mathcal{V}, \Omega)$ be a multisymplectic vector space and $\mathcal{W} \subset \mathcal{V}$ a 1-isotropic subspace such that $\dim \mathcal{W} = \dim \Lambda^k(\mathcal{V}/\mathcal{W})^*$ and $\dim \mathcal{V}/\mathcal{W} > k$. Then there exists a k-Lagrangian subspace V of $\mathcal{V}$ which is transversal to $\mathcal{W}$ (i.e. $V \cap \mathcal{W} = \{0\}$) such that $(\mathcal{V}, \Omega)$ is multisymplectomorphic to the model $(\mathcal{V}_V, \Omega_V)$.*

Proof. First step: Define the mapping

$$\begin{array}{rcl} \iota : \mathcal{W} & \longrightarrow & \Lambda^k(\mathcal{V}/\mathcal{W})^* \\ v & \mapsto & \iota(v) = \widetilde{i_v\Omega} \end{array}$$

where $\widetilde{i_v\Omega}$ is the induced linear form from $i_v\Omega \in \Lambda^k\mathcal{V}^*$. Notice that $\widetilde{i_v\Omega}$ is well-defined because the isotropic character of $\mathcal{W}$. In addition, ι is a linear isomorphism because of the regularity of Ω.

Second step: Such a subspace $\mathcal{W}$ is unique. First of all, we shall prove that if $u, v \in \mathcal{V}$ are linearly independent vectors satisfying $i_{u\wedge v}\,\Omega = 0$, then $\mathrm{span}\,(u, v) \cap \mathcal{W} \neq \{0\}$. Otherwise, we could choose $v_1, \ldots, v_{k-2} \in \mathcal{V}$ with $v_i \notin \mathcal{W}$ such that $\{u, v, v_1, \ldots, v_{k-2}\}$ are linearly independent and $\mathrm{span}\,(u, v, v_1, \ldots, v_{k-2}) \cap \mathcal{W} = \{0\}$, because the codimension of $\mathcal{W}$ is at least k. But for any $w \in \mathcal{W}$ we would have $i_{w\wedge u\wedge v\wedge v_1\wedge\cdots\wedge v_{k-2}}\,\Omega = 0$ which contradicts the fact that $\iota : \mathcal{W} \longrightarrow \Lambda^k(\mathcal{V}/\mathcal{W})^*$ is an isomorphism.

Next, let $\mathcal{W}$ and $\mathcal{W}'$ be two subspaces of $\mathcal{V}$ satisfying the hypothesis of the theorem. Assume that $\mathcal{W} \neq \mathcal{W}'$; then, there exists $v \in \mathcal{W}'$ such that $v \notin \mathcal{W}$. Using the argument above, we deduce that $\mathcal{W} \cap \mathcal{W}'$ has dimension at least 1. Consider the subspace $Z = \pi(v) \wedge \Lambda_{k-1}(\mathcal{V}/\mathcal{W})$ of $\Lambda_k(\mathcal{V}/\mathcal{W})$, where $\Lambda_r\mathcal{V}$ is the space of r-vectors on $\mathcal{V}$, and $\pi : \mathcal{V} \longrightarrow \mathcal{V}/\mathcal{W}$ is the canonical projection. Of course, $\dim Z > 1$, and we have $\iota(w)(z) = 0$ for any $w \in \mathcal{W} \cap \mathcal{W}'$ and for any $z \in Z$. Hence we would have $w \in \ker \iota$.

Third step: There exists a k-Lagrangian subspace V such that $\mathcal{V} = \mathcal{W} \oplus V$. Obviously, there are k-isotropic subspaces U such that $U \cap \mathcal{W} = \{0\}$. To show this last assertion, one could take a vector $v \in \mathcal{V}$ such that $u \notin \mathcal{W}$. It is obvious that $\mathrm{span}\,(u)$ is k-isotropic.

Assume that $U \oplus \mathcal{W} = \mathcal{V}$. Then $\mathcal{W} \cap U^{\perp,k} \subset \ker \iota$ and hence $\mathcal{W} \cap U^{\perp,k} = \{0\}$. Therefore $U = U^{\perp,k}$, and U is k-Lagrangian.

Suppose now that $U \oplus \mathcal{W} \neq \mathcal{V}$, then $U \neq U^{\perp,k}$; indeed, if $U = U^{\perp,k}$ (that is, if U were k-Lagrangian) then there would be a vector $x \in \mathcal{V}$ such that $x \notin U \oplus \mathcal{W}$, and then $U \oplus \operatorname{span}(x)$ would be k-isotropic in contradiction with the maximality of U. Therefore, there is a vector $v \in U^{\perp,k}$ such that $v \notin U \cup \mathcal{W}$, and we would have a k-isotropic subspace $U' = U \oplus \operatorname{span}(u)$ such that $U' \cap \mathcal{W} = \{0\}$. If $U' \oplus \mathcal{W} \neq \mathcal{V}$, we can repeat the argument and will eventually arrive at a k-isotropic subspace V which is complementary to $\mathcal{W}$. And using the argument above, we conclude that V is in fact k-Lagrangian.

Fourth step: Define a linear mapping

$$\begin{aligned} \phi \ : \ & \mathcal{W} \longrightarrow \Lambda^k V^* \\ & \phi(w) = -\frac{1}{k+1}(i_w\Omega)_{|V} \end{aligned}$$

A direct computation shows that ϕ is an isomorphism. Next, we define

$$\begin{aligned} \psi \ : \ & \mathcal{V} \longrightarrow V \times \Lambda^k V^* \\ & \psi(v,w) = (v, \phi(w)) \end{aligned}$$

which is also an isomorphism such that $\psi^*\Omega_V = \Omega$. ∎

Remark 2.1.10. A direct application of Theorem 2.1.9 shows that there exists a basis (a Darboux basis) $\{e_1, \dots, e_n, f_{\alpha_1 \dots \alpha_k}\}$ such that $\{e_i\}$ is a basis of V and $\{f_{\alpha_1 \dots \alpha_k}\}$ is a basis of $\mathcal{W}$ satisfying the relations

$$i_{f_{\alpha_1 \dots \alpha_k}} \Omega = e^*_{\alpha_1} \wedge \cdots \wedge e^*_{\alpha_k}$$

where $\{e^*_1, \dots, e^*_n\}$ denotes the dual basis of $\{e_1, \dots e_n\}$. Therefore we have

$$\Omega = \sum_{\alpha} f^*_{\alpha_1 \dots \alpha_k} \wedge e^*_{\alpha_1} \wedge \cdots \wedge e^*_{\alpha_k} \tag{2.1}$$

where $\{f^*_{\alpha_1 \dots \alpha_k}\}$ is the dual basis of $\{f_{\alpha_1 \dots \alpha_k}\}$.

Definition 2.1.11. *A triple $(\mathcal{V}, \Omega, \mathcal{W})$ satisfying the hypothesis in Theorem 2.1.9 will be called a* ***multisymplectic vector space of type*** $(k+1, 0)$.

Theorem 2.1.12. *Let $(\mathcal{V}, \Omega)$ be a multisymplectic vector space and $\mathcal{W} \subset \mathcal{V}$ a 1-isotropic subspace. Assume that $\mathcal{E} \subset \mathcal{V}/\mathcal{W}$ is a vector subspace of the quotient vector space $\mathcal{V}/\mathcal{W}$. Let us denote by $\pi : \mathcal{V} \longrightarrow \mathcal{V}/\mathcal{W}$ the canonical projection. Assume that*

(i) $i_{v_1 \wedge \cdots \wedge v_r} \Omega = 0$ *if $\pi(v_i) \in \mathcal{E}$, for all $i = 1, \dots, r$;*

(ii) $\dim \mathcal{W} = \dim \Lambda^k_r (\mathcal{V}/\mathcal{W})^*$, *where the horizontal forms are considered with respect to the subspace $\mathcal{E}$;*

(iii) $\dim(\mathcal{V}/\mathcal{W}) > k$.

Then there exists a k-Lagrangian subspace V of $\mathcal{V}$ which is transversal to $\mathcal{W}$ (i.e., $V \cap \mathcal{W} = \{0\}$) such that $(\mathcal{V}, \Omega)$ is multisymplectomorphic to the model $(\mathcal{V}_V^r, \Omega_V^r)$.

Proof. First, we define the linear isomorphism

$$\begin{aligned} \iota \quad : \quad & \mathcal{W} \longrightarrow \Lambda_r^k(\mathcal{V}/\mathcal{W})^* \\ & w \mapsto \iota(w) = \widetilde{i_w \Omega} \end{aligned}$$

where $\widetilde{i_w \Omega}$ is the induced k-form using that $\mathcal{W}$ is isotropic and that Ω satisfies the first condition above.

Next, one follows the arguments given in the proof of Theorem 2.1.9. ∎

Remark 2.1.13. A direct application of Theorem 2.1.12 shows that the multisymplectic form Ω can be written as the canonical multisymplectic form Ω_V^r on $\mathcal{V}_V^r$ by choosing a convenient Darboux basis.

Definition 2.1.14. *A triple $(\mathcal{V}, \Omega, \mathcal{W}, \mathcal{E})$ satisfying the hypothesis in Theorem 2.1.12 will be called a **multisymplectic vector space of type** $(k+1, r)$.*

2.2 Multisymplectic manifolds

2.2.1 Definition

The preceding definitions can be naturally extended to the realm of differentiable manifolds.

Definition 2.2.1. *[60, 16] A **multisymplectic form** on a (smooth, real, finite-dimensional) manifold M is a closed k-form, with $1 < k \leq \dim M$, which verifies the following non-degeneracy property, for any $v \in T_xM$*

$$\iota_v \Omega_x = 0 \Leftrightarrow v = 0$$

Therefore, Ω_x is a multisymplectic form on T_xM for each $x \in M$.
*A **multisymplectic manifold** of order k is a manifold M endowed with a multisymplectic k-form.*

A multisymplectic form on M induces the morphisms of $\mathcal{C}^\infty(M)-$modules

$$\begin{aligned} \hat{\Omega}_j : \mathfrak{X}^j(M) &\longrightarrow \Lambda^{k-j} M \\ U &\longmapsto \iota_U \Omega \end{aligned}$$

Definition 2.2.2. *A diffeomorphism Φ between two multisymplectic manifolds (M_1, Ω_1) and (M_2, Ω_2) of the same degree is said to be a **multisymplectomorphism** whenever $\Phi^*\Omega_2 = \Omega_1$.*

Example 2.2.3. For an n-dimensional manifold M, the bundle $\Lambda^k M$ of k-forms is equipped with the following canonical k-form Θ_M^k defined by

$$(\Theta_M^k)_\alpha(X_1, \ldots, X_k) := \alpha(T\nu_k(X_0), \ldots, T\nu_k(X_n))$$

where $\nu_k : \Lambda^k M \longrightarrow M$ is the canonical projection and $X_1, \ldots, X_k \in T_\alpha \Lambda^k M$. This form is called the **Liouville form**, and has local expression

$$\Theta_M^k = p_{i_1 \ldots i_k} dq^{i_1} \wedge \ldots \wedge dq^{i_k}$$

for adapted coordinates $\{q^i, p_{i_1 \ldots i_k}\}$ on $\Lambda^k M$.

We also define the **canonical multisymplectic form** on $\Lambda^k M$ by

$$\Omega_M^k := -d\Theta_M^k$$

with expression in local coordinates

$$\Omega_M^k = -dp_{i_1 \ldots i_k} \wedge dq^{i_1} \wedge \ldots \wedge dq^{i_k}.$$

A computation shows that this form is in fact multisymplectic. In a later section, we shall introduce the analogous concept to Darboux coordinates for certain multisymplectic manifolds, for which the model is precisely given by this example.

Assume now that M is a fibred manifold over a manifold N, say $\pi : M \longrightarrow N$ is a fibration. Consider the bundle $\Lambda_r^k M$ of k-forms on M which are r-horizontal with respect to the fibration $\pi : M \longrightarrow N$, that is, those k-forms γ on M such that $i_{X_1 \wedge \cdots \wedge X_r} \gamma = 0$ when $X_1, \ldots, X_r$ are π-vertical. The space $\Lambda_r^k M$ is a submanifold of $\Lambda^k M$, and hence we have the restriction $(\Theta_M)_r^k$ of Θ_M^k to $\Lambda_r^k M$. A simple computation shows that the pair $(\Lambda_r^k M, (\Omega_M)_r^k = -d(\Theta_M)_r^k)$ is also a multisymplectic manifold. Of course, we have $(\Omega_M^k)_{|\Lambda_r^k M} = (\Omega_M)_r^k$. The canonical projection will be denoted by $\rho_r^k : \Lambda_r^k M \longrightarrow M$

Following the notion of special symplectic manifold introduced by Tulczyjew we can give the following definition.

Definition 2.2.4. *A* ***special multisymplectic manifold*** $(\mathcal{P}, \Omega)$ *is a multisymplectic manifold which is multisymplectomorphic to a bundle of forms, as defined in the preceding example. More precisely,* $\Omega = -d\Theta$*, and there exists a diffeomorphism* $\alpha : \mathcal{P} \longrightarrow \Lambda^k M$ *(or* $\alpha : \mathcal{P} \longrightarrow \Lambda_r^k M$*) for some manifold* M*, and a fibration* $\pi : \mathcal{P} \longrightarrow M$ *such that* $\rho \circ \alpha = \pi$ *(resp.* $\rho_r \circ \alpha = \pi$*) and* $\Theta = \alpha^* \Theta_M^k$ *(resp.* $\Theta = \alpha^* (\Theta_M)_r^k$*).*

Definition 2.2.5. *A submanifold* N *of a multisymplectic manifold* (M, Ω) *is called* l-***isotropic*** *(resp.* l-***coisotropic***, l-***Lagrangian***, ***multisymplectic****) whenever* $T_x N$ *is a* l*-isotropic (resp.* l*-coisotropic,* l*-Lagrangian, multisymplectic) subspace of* $(T_x M, \Omega_x)$*, for each* $x \in N$*.*

Two interesting results related to Lagrangian submanifolds follow.

Proposition 2.2.6. *A subspace* $\mathcal{W}$ *is* l*-Lagrangian if and if it is maximally* l*-isotropic.*

Proof. If $W \subseteq V$ is l-Lagrangian, then it is isotropic. If $W \subseteq W'$ with W' isotropic, then we have

$$W \subseteq W' \subseteq W'^{\perp,l} \subseteq W^{\perp,l} = W$$

thus $W = W'$.

Conversely, suppose that W is maximally isotropic, and that $W \subset W^{\perp,l}$. Take $v \in W^{\perp,l} - W$, and consider $W' = W \oplus \langle v \rangle$. We have that $W \subset W'$, and W' is isotropic (because $v \in W^{\perp,l}$), which contradicts our supposition. ∎

Proposition 2.2.7. *On* $(\Lambda^k M, \Omega^k_M)$ *we have*

(i) The fibres of $\rho : \Lambda^k M \longrightarrow M$ *(and of* $\rho_r : \Lambda^k_r M \longrightarrow M$*) are 1-isotropic.*

(ii) The image of a k*-form* γ *on* M *(resp. a* r*-horizontal* k*-form) is* k*-Lagrangian if and only if* γ *is closed.*

Proof. It follows from Proposition 2.1.7. ∎

If γ is a (r-horizontal) closed k-form on M, then $(-d(\Theta_M)^k_r)_{|\mathrm{Im}\gamma} = 0$ (as happens in any other general Lagrangian submanifold) which implies that $((\Theta_M)^k_r)_{|\mathrm{Im}\gamma}$ is locally closed, say

$$((\Theta_M)^k_r)_{|\mathrm{Im}\gamma} = d\theta,$$

and θ is called a **generating** k**-form**. Also notice that M is diffeomorphic to $\mathrm{Im}\gamma$, so θ can be regarded as a $(k-1)$-form on M.

2.2.2 Hamiltonian vector fields and forms

Along this section, (M, Ω) will be a multisymplectic manifold of order $k+1$. The concepts shown in this section are widely explored in [142], [58] and [16], to which the reader is referred for an in-depth discussion of the concepts shown in this section.

Definition 2.2.8. *A vector field* $X \in \mathfrak{X}(M)$ *will be said to be*
(i) ***Hamiltonian*** *if there exists a* $(k-1)$*-form* α *(called* ***Hamiltonian form****) such that*

$$\iota_X \Omega = d\alpha$$

(ii) ***locally Hamiltonian*** *if* $\mathfrak{L}_X \Omega = 0$

The non-degeneracy of Ω guarantees that for a given form α, if there exists an associated Hamiltonian vector field, then it will be unique, and will be denoted by X_α.

We denote by

$$\mathcal{H}^{k-1}(M) := \{\alpha \in \Lambda^{k-1}(M) | d\alpha = \iota_X \Omega \text{ for some } X \in \mathfrak{X}(M)\}$$

Definition 2.2.9. *In* $\mathcal{H}^{k-1}(M)$ *we can define the following* ***Poisson bracket:***

$$\{\alpha, \beta\} := \iota_{X_\beta} \iota_{X_\alpha} \Omega$$

Furthermore, we have that

Proposition 2.2.10. *If and* $\alpha, \beta \in \mathcal{H}^{k-1}(M)$ *, then* $\{\alpha, \beta\}$ *is a Hamiltonian* $(k-1)$*-form which has associated Hamiltonian vector field* $[X_\alpha, X_\beta]$*. In other words,*

$$X_{\{\alpha,\beta\}} = -[X_\alpha, X_\beta]$$

Proof.

$$\begin{aligned}\iota_{[X_\alpha,X_\beta]}\Omega &= \pounds_{X_\alpha}\iota_{X_\beta}\Omega - \iota_{X_\beta}\pounds_{X_\alpha}\Omega\\ &= \pounds_{X_\alpha}d\beta - \iota_{X_\beta}d\iota_{X_\alpha}\Omega - \iota_{X_\beta}\iota_{X_\alpha}d\Omega\\ &= d\iota_{X_\alpha}d\beta - \iota_{X_\beta}dd\alpha\\ &= d\iota_{X_\alpha}\iota_{X_\beta}\Omega\\ &= -d\{\alpha,\beta\}\end{aligned}$$

by uniqueness, we obtain the desired result. ∎

Therefore, the set of Hamiltonian vector fields is a Lie subalgebra of the Lie algebra of vector fields on M with the Lie bracket.

Given a Hamiltonian vector field, its Hamiltonian form is determined up to a closed $(k-1)$-form. If we denote by $\mathcal{Z}^{k-1}(M)$ the set of closed $(k-1)$-forms, and by $\tilde{\mathcal{H}}^{k-1}(M) = \mathcal{H}^{k-1}(M)/\mathcal{Z}^{k-1}(M)$, then we can induce a well defined bracket operation there.

In a similar manner, we can define Hamiltonian forms of lower degrees by using the so-called Hamiltonian multivector fields, and relate the associated Poisson bracket to the Schouten-Nijenhuis bracket of multivectors. They define a graded algebra of Hamiltonian forms with certain new bracket. All the details may be found in [17].

2.2.3 Darboux coordinates

In this section, we shall find necessary and sufficient conditions for a multisymplectic manifold to be multisymplectomorphic to the models described in example 2.2.3. As a consequence, we shall be able to find Darboux coordinates for those manifolds. The results in this section has been developed as part of our work in [116], although some results were proven in [132, 133].

The section has two parts, where the first one is devoted to finding Darboux coordinates to multisymplectic manifolds of type $(k+1,0)$, and the second part extends the result to general multisymplectic manifolds of type $(k+1,r)$ (the types are defined at the beginning of each part).

Definition 2.2.11. *A triple $(\mathcal{P},\Omega,\mathcal{W})$, where $\mathcal{W}$ is a 1-isotropic involutive distribution on a multisymplectic manifold $(\mathcal{P},\Omega)$ such that the triple $(T_x\mathcal{P},\Omega_x,\mathcal{W}(x))$ is a multisymplectic vector space of type $(k+1,0)$, for all $x\in\mathcal{P}$, will be called a* ***multisymplectic manifold of type*** $(k+1,0)$.

From now on, $(\mathcal{P},\Omega,\mathcal{W})$ will be a multisymplectic manifold of type $(k+1,0)$. Furthermore, the distribution $\mathcal{W}$ and the corresponding vector bundle $\pi_0:\mathcal{W}\longrightarrow\mathcal{P}$ over $\mathcal{P}$ will be denoted by the same letter.

Theorem 2.2.12. *[132] Let $\mathcal{L}$ be a k-Lagrangian submanifold such that $T\mathcal{L}\cap\mathcal{W}_{|\mathcal{L}}=\{0\}$. Then there exists a tubular neighbourhood U of $\mathcal{L}$ in $\mathcal{P}$, a manifold $\mathcal{N}$ and a diffeomorphism $\Phi: U\longrightarrow V=\Phi(U)\subset\Lambda^k\mathcal{N}$ into an open neighbourhood V of the zero cross-section in $\Lambda^k\mathcal{L}$ such that $\Phi:\mathcal{L}\longrightarrow\mathcal{N}$ is an immersion and $\Phi^*((\Omega^k_{\mathcal{N}})_{|V})=\Omega_{|U}$, where $\Omega^k_{\mathcal{N}}$ is the canonical multisymplectic $(k+1)$-form on $\Lambda^k\mathcal{N}$.*

Proof. The proof is a direct consequence of Lemmas 2.2.14 and 2.2.15. ∎

First of all, we recall the relative Poincaré lemma, which will be very useful in what follows.

Lemma 2.2.13. (**Relative Poincaré lemma**) *Let N be a submanifold of a differentiable submanifold M, and let U be a tubular neigbourhood of N with bundle map $\pi_0 : U \longrightarrow N$. Notice that $\pi_0 : U \longrightarrow N$ is a vector bundle. Denote by Δ the dilation vector field of this vector bundle, and let φ_t be the multiplication by t. If we define an integral operator on forms on U as follows*

$$I(\Omega)_p = \int_0^1 i_{\Delta_t}\, \varphi_t^* \Omega_p dt$$

where $\Delta_t = \frac{1}{t}\Delta$, and $p \in U$, then we have

$$I(d\Omega) + d(I\Omega) = \Omega - \pi_0^*(\Omega_{|N})$$

where $\Omega_{|N}$ is the form on N obtained by restricting Ω pointwise to TN (observe that U can be taken as a normal bundle of TN in M).

Next, we shall prove the following result.

Lemma 2.2.14. *Let $\mathcal{L}$ be a k-Lagrangian submanifold of $\mathcal{P}$ which is complementary to $\mathcal{W}$ (that is, $T\mathcal{L} \oplus \mathcal{W}_{|\mathcal{L}} = T\mathcal{P}_{|\mathcal{L}}$). Then there is a tubular neighbourhood U of $\mathcal{L}$ and a diffeomorphism $\Phi : U \longrightarrow V \subset \Lambda^k\mathcal{L}$ where V is an neighbourhood of the zero section, such that $\Phi_{|\mathcal{L}}$ is the standard identification of $\mathcal{L}$ with the zero section of $\Lambda^k\mathcal{L}$, and*

$$\Phi^*((\Omega^k_{\mathcal{L}})_{|V}) = \Omega_{|U}.$$

Proof of Lemma 2.2.14.

Firstly, we define a vector bundle morphism over the identity of $\mathcal{L}$ by

$$\phi(w) = -\frac{1}{k+1}\, i_w\, \Omega.$$

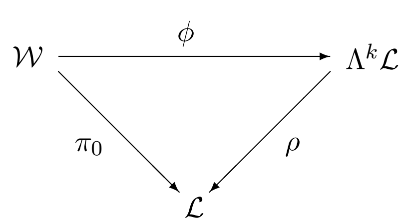

Figure 2.1: The vector bundle morphism ϕ

Obviously ϕ is injective, and since the dimensionality assumptions, we deduce that ϕ is in fact a vector bundle isomorphism (see the diagram 2.1).

Since $T\mathcal{P}_{|\mathcal{L}} = T\mathcal{L} \oplus \mathcal{W}_{|\mathcal{L}}$, then ϕ induces a diffeomorphism on a tubular neighbourhood defined by $\mathcal{W}$ onto a neighbourhood of $\mathcal{L}$ in $\Lambda^k\mathcal{L}$ (as usual, the latter embedding is understood as the identification of $\mathcal{L}$ with the zero section). We shall denote the restriction of ϕ to this tubular neigbourhood by f. Notice that the restriction of f to $\mathcal{L}$ is just the identity, so that Tf is also the identity on $T\mathcal{L}$; on the other hand, Tf restricted to $\mathcal{W}$ coincides with ϕ because it is fibrewise linear. Using the identifications $T\mathcal{P}_{|\mathcal{L}} = T\mathcal{L} \oplus \mathcal{W}_{|\mathcal{L}}$ and $T\Lambda^k\mathcal{L}_{|\mathcal{L}} = T\mathcal{L} \oplus \Lambda^k\mathcal{L}$, we have

$$\begin{aligned} f^*\Omega^k_{\mathcal{L}}((v_1, w_1), \ldots, (v_{k+1}, w_{k+1})) &= \Omega^k_{\mathcal{L}}((v_1, \phi(w_1), \ldots, (v_{k+1}, \phi(w_{k+1})) \\ &= \sum_{i=1}^{k+1} (-1)^i \, \phi(w_i)(v_1, \ldots, \hat{v}_i, \ldots, v_{k+1}) \\ &= \sum_{i=1}^{k+1} \frac{1}{k+1} \Omega(v_1, \ldots, w_i, \ldots, v_{k+1}) \\ &= \Omega((v_1, w_1), \ldots, (v_{k+1}, w_{k+1})) \end{aligned}$$

which implies $f^*\Omega^k_{\mathcal{L}} = \Omega$ on $\mathcal{L}$.

Next, we use f to pushforward Ω to obtain a $k+1$-form Ω_1 in a neighbourhood of $\mathcal{L}$ in $\Lambda^k\mathcal{L}$. Using Lemma 2.2.13 we deduce that $\Omega_1 = d\Theta_1$, where $\Theta_1 = I(\Omega_1)$. Recall that $\Omega^k_{\mathcal{L}} = -d\Theta^k_{\mathcal{L}}$, and

$$(\Theta^k_{\mathcal{L}})_{|\mathcal{L}} = (\Theta_1)_{|\mathcal{L}} = 0 \tag{2.2}$$

because of the definition of I. Define

$$\Omega_t = \Omega^k_{\mathcal{L}} + t(\Omega_1 - \Omega^k_{\mathcal{L}}), \qquad t \in [0, 1].$$

Since

$$(\Omega_t)_{|\mathcal{L}} = (\Omega^k_{\mathcal{L}})_{|\mathcal{L}} = (\Omega_1)_{|\mathcal{L}}$$

is non-singular, and this is an "open condition", we can find a neighbourhood of $\mathcal{L}$ in $\Lambda^k\mathcal{L}$ on which all Ω_t are non-singular for all $t \in [0, 1]$. In addition, $\mathcal{W}_{\mathcal{L}} = \ker\{T\rho : T\Lambda^k\mathcal{L} \longrightarrow T\mathcal{L}\}$ is 1-isotropic for all Ω_t, in such a way that $(\Lambda^k\mathcal{L}, \Omega_t, \mathcal{W}_{\mathcal{L}})$ is a multisymplectic manifold of type $(k+1, 0)$, for all t. Notice that $\Omega_1 - \Omega^k_{\mathcal{L}} = d(\Theta_1 + \Theta^k_{\mathcal{L}})$.

From (2.2) we deduce that there is a unique time-dependent vector field X_t taking values in $\mathcal{W}_{\mathcal{L}}$ (in other words, ρ-vertical) such that

$$i_{X_t}\,\Omega_t = -\Theta^k_{\mathcal{L}} + \Theta_1.$$

Since the vector field X_t vanishes on $\mathcal{L}$, we can find a neighbourhood of $\mathcal{L}$ in $\Lambda^k\mathcal{L}$ such that the flow φ_t of X_t is defined at least for all $t \leq 1$. Therefore we have

$$\begin{aligned} \frac{d}{dt}(\varphi_t^*\Omega_t) &= \varphi_t^*(L_{X_t}\,\Omega_t) + \varphi_t^*(\frac{d\Omega_t}{dt}) \\ &= \varphi_t^*(di_{X_t}\Omega_t) + \varphi_t^*(\Omega_1 - \Omega^k_{\mathcal{L}}) \\ &= \varphi_t^*(-d(\Theta_1 - \Theta^k_{\mathcal{L}}) + \Omega_1 - \Omega^k_{\mathcal{L}}) = 0. \end{aligned}$$

Then we obtain

$$\varphi_1^*\Omega_1 = \varphi_0^*\Omega^k_{\mathcal{L}} = \Omega^k_{\mathcal{L}}.$$

But $(X_t)_{|\mathcal{L}} = 0$ implies $(\varphi_t)_{|\mathcal{L}} = id_{|\mathcal{L}}$, and then we deduce that $\varphi_1 \circ f$ gives the desired local diffeomorphism. ∎

Lemma 2.2.15. *Let $\mathcal{L}'$ be a k-isotropic submanifold of $\mathcal{P}$ which is transversal to $\mathcal{W}$ (that is, $T\mathcal{L}' \cap \mathcal{W}_{|\mathcal{L}'} = \{0\}$). Then there is a k-Lagrangian submanifold $\mathcal{L}$ of $\mathcal{P}$ which is complementary to $\mathcal{W}$ and contains $\mathcal{L}'$.*

*P*roof of Lemma 2.2.15.

Since $\mathcal{L}'$ is transversal to $\mathcal{W}$ we can choose a submanifold $\mathcal{L}''$ of U' such that $\mathcal{L}'$ is a deformation retract of $\mathcal{L}''$, and $\mathcal{L}''$ is complementary to $\mathcal{W}$. As in the theorem above, since $T\mathcal{P}_{|\mathcal{L}''} = T\mathcal{L}'' \oplus \mathcal{W}_{|\mathcal{L}''}$, then $\mathcal{W}$ induces a tubular neighbourhood of $\mathcal{L}''$ in the usual way: $\pi_1 : U' \longrightarrow \mathcal{L}''$.

Next, we apply the relative Poincaré lemma to the restricted form Ω to this tubular neigbourhood. Therefore, there is a k-form μ on U' such that

$$d\mu = \Omega - \pi_1^*(\Omega_{|\mathcal{L}''})$$

(indeed, $\mu = I(\Omega)$).

Now, we can repeat the construction developed in the proof of Lemma 2.2.14 for the $k+1$-form $d\mu$. In fact, the mapping $\psi : \mathcal{W} \longrightarrow \Lambda^k\mathcal{L}''$ defined by $\psi(u) = -\dfrac{1}{k+1}(i_u\, d\mu)$ is a vector isomorphism, and it induces a local diffeomorphism $g : U'' \subset U' \longrightarrow g(U'') \subset \Lambda^k\mathcal{L}''$; g restricted to $\mathcal{L}''$ is the identity, and ψ on the fibers. Again we can prove

$$g^*\Omega^k_{\mathcal{L}''} = d\mu$$

since $(d\mu)_{|\mathcal{L}''} = 0$. Proceeding as in the proof of Lemma 2.2.14 we can find a local diffeomorphism Ψ from a tubular neigbourhood V of $\mathcal{L}''$ onto a neighbourhood of the zero section of $\Lambda^k\mathcal{L}''$ which maps $\mathcal{L}''$ onto the zero section, and such that

$$\Psi^*\Omega^k_{\mathcal{L}''} = \Omega$$

on V.

Now, if $j : \mathcal{L}' \longrightarrow \mathcal{L}''$ is the natural inclusion, we know that j induces an isomorphism in cohomology. Therefore $j^*(\Omega_{|\mathcal{L}''}) = \Omega_{|\mathcal{L}'} = 0$ implies $[\Omega_{|\mathcal{L}''}]_{DR} = 0$, and we deduce that $\Omega_{|\mathcal{L}''} = d\nu$, for some k-form ν on $\mathcal{L}''$. A direct computation shows now that

$$\mathcal{L} = \Psi^{-1} \circ (-\nu)(\mathcal{L}'')$$

is a k-Lagrangian submanifold in $(\mathcal{P}, \Omega)$, and in addition $T\mathcal{P}_{|\mathcal{L}} = T\mathcal{L} \oplus \mathcal{W}_{|\mathcal{L}}$. ∎

Corollary 2.2.16. *A multisymplectic manifold $(\mathcal{P}, \Omega, \mathcal{W})$ of type $(k+1, 0)$ is locally multisymplectomorphic to a canonical multisymplectic manifold $\Lambda^k M$ for some manifold M. Therefore, there are Darboux coordinates around each point of $\mathcal{P}$.*

Proof. We only need to choose a point in Lemma 2.2.15, and then apply Theorem 2.2.12. ∎

Definition 2.2.17. *Let $(\mathcal{P}, \Omega)$ be a multisymplectic manifold of order $k+1$. Assume that $\mathcal{W}$ is a 1-isotropic foliation of $(\mathcal{P}, \Omega)$, and $\mathcal{E}$ is a "generalised distribution" on $\mathcal{P}$ in the sense that $\mathcal{E}(x) \subset T_x\mathcal{P}/\mathcal{W}(x)$ is a vector subspace for all $x \in \mathcal{P}$. Assume that the quadruple $(T_x\mathcal{P}, \Omega_x, \mathcal{W}(x), \mathcal{E}(x))$ is a multisymplectic vector space of type $(k+1, r)$, for all $x \in \mathcal{P}$. A quadruple $(\mathcal{P}, \Omega, \mathcal{W}, \mathcal{E})$ satisfying these conditions will be called a* ***multisymplectic manifold of type*** *$(k+1, r)$.*

For the remaining of the section, $(\mathcal{P}, \Omega, \mathcal{W}, \mathcal{E})$ will be a multisymplectic manifold of type $(k+1, r)$.

Theorem 2.2.18. *Let $\mathcal{L}$ be a k-Lagrangian submanifold such that $T\mathcal{L} \cap \mathcal{W}_{\mathcal{L}} = \{0\}$. Then there exists a tubular neighbourhood U of $\mathcal{L}$ in $\mathcal{P}$, a manifold $\mathcal{N}$, and a diffeomorphism $\Phi : U \longrightarrow V = \Phi(U) \subset \Lambda^k_r \mathcal{N}$ into an open neighbourhood V of the zero cross-section in $\Lambda^k \mathcal{N}$ such that $\Phi : \mathcal{L} \longrightarrow \mathcal{N}$ is an immersion, and $\Phi^*(((\Omega_{\mathcal{N}})^k_r)_{|V}) = \Omega_{|U}$, where $(\Omega_{\mathcal{N}})^k_r$ is the canonical multisymplectic $(k+1)$-form on $\Lambda^k_r \mathcal{N}$.*

Proof. The proof is a consequence of the following two lemmas, which are proved in a similar way to Lemma 2.2.14 and Lemma 2.2.15.

Lemma 2.2.19. *Let $\mathcal{L}$ be a k-Lagrangian submanifold of $\mathcal{P}$ which is complementary to $\mathcal{W}$. Then there is a tubular neighbourhood U of $\mathcal{L}$ and a diffeomorphism $\Psi : U \longrightarrow V \subset \Lambda^k_r \mathcal{L}$, where V is an neighbourhood of the zero section, such that $\Psi_{|\mathcal{L}}$ is the standard identification of $\mathcal{L}$ with the zero section of $\Lambda^k_r \mathcal{L}$, and*

$$\Psi^*(((\Omega_{\mathcal{L}})^k_r)_{|V}) = \Omega_{|U}.$$

Lemma 2.2.20. *Let $\mathcal{L}'$ be a k-isotropic submanifold of $\mathcal{P}$ which is transversal to $\mathcal{W}$. Then there is a k-Lagrangian submanifold $\mathcal{L}$ of $\mathcal{P}$ which is complementary to $\mathcal{W}$ and contains $\mathcal{L}'$.*

Corollary 2.2.21. *A multisymplectic manifold $(\mathcal{P}, \Omega, \mathcal{W}, \mathcal{E})$ of type $(k+1, r)$ is locally multisymplectomorphic to a canonical multisymplectic manifold $\Lambda^k_r M$ for some fibration $M \longrightarrow N$. Therefore, there are Darboux coordinates around each point of $\mathcal{P}$.*

Proof. We only need to choose a point in Lemma 2.2.20, and then apply Theorem 2.2.18. ∎

2.3 Jet manifolds

The concept of jet of first order generalises the notion of tangent vector. While a curve in a manifold Q is seen as a mapping from some interval I of the real line into Q, or a section of the first projection bundle $I \times Q \longrightarrow I$, and tangent vectors are equivalence classes of such curves, first order jets will be classes of equivalences of sections on a fiber bundle $Y \longrightarrow X$.

Thus, for the remaining of this section, we shall fix a fiber bundle $\pi = \pi_{XY} : Y \longrightarrow X$, where X is an orientable $(n+1)$-dimensional manifold (possibly with boundary ∂X), and Y will be an $(n+1+m)$-dimensional manifold (with boundary $\partial Y := \pi^{-1}(\partial X)$). The set of sections of this fibration will be denoted by $\Gamma(\pi)$, and the sections of $\Gamma(\pi)$ around a point $x \in X$ will be denoted by $\Gamma_x(\pi)$. We shall also assume that X is oriented and has a fixed volume form η.

We can pick local fibred coordinates (x^μ) on X and (x^μ, y^i) on Y in such a way that the local expression of π gets $\pi(x^\mu, y^i) = (x^\mu)$. We shall also assume that η can be expressed $\eta = d^{n+1}x = dx^0 \wedge \ldots \wedge dx^n$. The indices vary in the range $0 \le \mu, \nu, ... \le n$ and $1 \le i, j, ... \le m$.

We shall also use the following notation:

$$d^n x_\mu := \iota_{\partial/\partial x^\mu} d^{n+1}x, \quad d^{n-1}x_{\mu\nu} := \iota_{\partial/\partial x^\mu} \iota_{\partial/\partial x^\nu} d^{n+1}x,$$

and so on.

2.3.1 Definition and notations

Definition 2.3.1. *The k-th order* ***jet manifold*** *for the fibration π is defined as the set of equivalence classes around points of X which have in local coordinates the same Taylor expansion up to order k. If $x \in X$ and $\phi : X \longrightarrow Y$ is a section of π defined in some neighbourhood of x, then such equivalence class is denoted by $j^k_x\phi$. Thus,*

$$J^k\pi := \{j^k_x\phi | x \in X, \phi \in \Gamma(\pi)\}$$

Such set is easily provided with a structure of manifold with the following charts:

$$\begin{aligned} x^\mu(j^k_x\phi) &= x^\mu(x) \\ y^i(j^k_x\phi) &= y^i(\phi(x)) = \phi^i(x) \\ z^i_\mu(j^k_x\phi) &= \frac{\partial \phi^i}{\partial x^\mu}\Big|_x \\ &\vdots \\ z^i_{\mu_1\ldots\mu_k}(j^k_x\phi) &= \frac{\partial^k \phi^i}{\partial x^{\mu_1}\ldots\partial x^{\mu_k}}\Big|_x \end{aligned}$$

The dimension is obtained by computing the different partial derivatives. Thus, $\dim J^1\pi = (n+1) + m + (n+1)m$, $\dim J^2\pi = (n+1) + m + (n+1)m + \begin{pmatrix} n+2 \\ 2 \end{pmatrix} m$, and so on.

For a section ϕ, the k-jet prolongation of ϕ is defined as $j^k\phi : X \longrightarrow J^k\phi$ by $j^k\phi(x) = j^k_x\phi$.

In what follows, we shall focus our attention in the first order jet manifold, that will be denoted by $Z = J^1\pi$. The natural projections will be denoted by $\pi_{YZ} : Z \longrightarrow Y$ and $\pi_{XZ} : Z \longrightarrow X$ (see Figure 2.2).

If X has a boundary ∂X, then Z is a manifold with a boundary $\partial Z := \pi^{-1}_{XZ}(\partial X)$. The same symbol η will be used to denote the volume form on X or its pullback to Y or Z.

Notice that if $\phi \in \Gamma(\pi)$, then $T\phi$ does only depend on its first derivative, and so $T_x\phi = T_x\phi'$ if and only if $j^1_x\phi = j^1_x\phi'$

As usual, one can define the concept of verticality, by defining the following vector subspaces and subbundles:

$$\begin{aligned} \mathcal{V}_y\pi &:= (T_y\pi)^{-1}(0_x) \\ \mathcal{V}_z\pi_{XZ} &:= (T_z\pi_{XZ})^{-1}(0_x) \\ \mathcal{V}\pi &:= \bigcup_{y\in Y} \mathcal{V}_y\pi \\ \mathcal{V}\pi_{XZ} &:= \bigcup_{z\in Z} \mathcal{V}_z\pi_{XZ} \end{aligned}$$

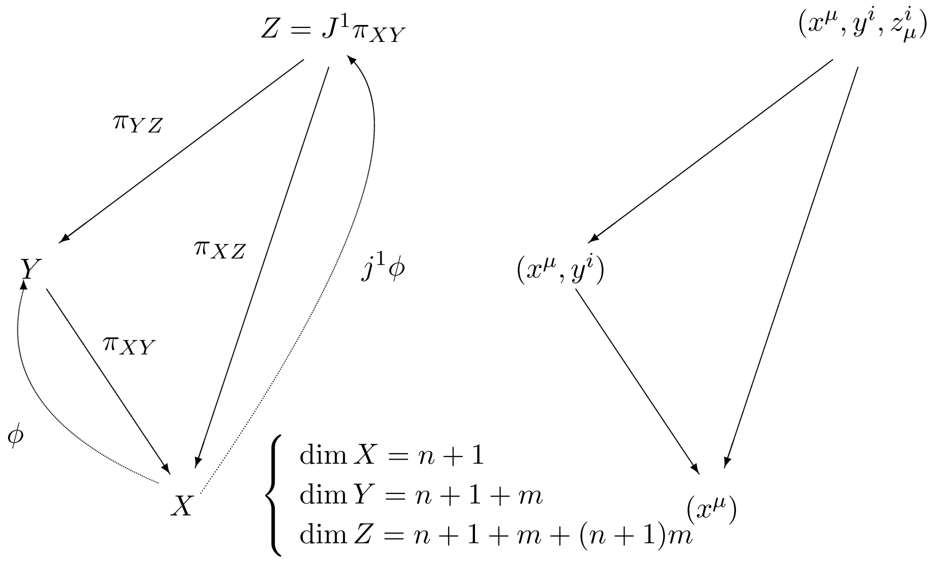

Figure 2.2: First order jet manifold

and local horizontality on the image of certain section $\phi \in \Gamma(\pi)$,

$$\mathcal{H}_y\pi_{YZ} := T_x\phi(T_xX)$$
$$\mathcal{H}\pi_{YZ} := \bigcup_{x\in X} T_x\phi(T_xX) = \bigcup_{x\in X} \mathcal{H}_{\phi(x)}\pi_{YZ}$$

Note that $\mathcal{H}_y\pi_{YZ}$ is the set of holonomic lifts of tangent vectors of T_xX by ϕ.

Every vector field along ϕ can be decomposed into a part on $\mathcal{V}\pi$ and a part on $\mathcal{H}\pi_{YZ}$, and those that only have a part on $\mathcal{H}\pi_{YZ}$ are called **total derivatives**. Locally, they are generated as $\mathcal{C}^\infty(Z)$-modules by the forms

$$\frac{d}{dx^\mu} = \frac{\partial}{\partial x^\mu} + z^i_\mu \frac{\partial}{\partial y^i}$$

usually called **coordinate total derivatives** (see [150] for further details). The definition can be extended to higher degrees, thus giving

$$\frac{d}{dx^\mu} = \frac{\partial}{\partial x^\mu} + z^i_\mu \frac{\partial}{\partial y^i} + z^i_{\mu\nu} \frac{\partial}{\partial z^i_\nu} + \dots$$

There are several other alternative (and equivalent) definitions of the first order jet bundle, such as considering the affine bundle over Y whose fibre over $y \in \pi^{-1}(x)$ consists of linear sections of $T\pi_{XY}$, modelled over the vector bundle on Y whose fibre over $y \in \pi^{-1}(x)$ is the space of linear maps of T_xX to $\mathcal{V}_y\pi$; in other words, Z is an affine bundle over Y modelled on the vector bundle $\pi^*T^*X \otimes_Y \mathcal{V}\pi$ (see [64, 150]).

The authors in [150] and [29] use that model to define another geometric object of interest associated to each volume form, which is the **vertical endomorphism**, which in this context will not be purely an endomorphism, but will extend the well-known concept for classical Mechanics (also see chapter 1 and [149]).

What follows is an alternative way of defining it. First of all, we construct the isomorphism vertical lift

$$v : \pi^*T^*X \otimes_Y \mathcal{V}\pi \longrightarrow \mathcal{V}\pi_{YZ}$$

as follows: given $f \in (\pi^*T^*X \otimes_Y \mathcal{V}\pi)|_{j^1\phi}$ consider the curve $\gamma_f : \mathbb{R} \longrightarrow \pi_{YZ}^{-1}(\pi_{YZ}(j_x^1\phi))$ given by

$$\gamma_f(t) = j_x^1\phi + t\ f\ ,$$

for all $t \in \mathbb{R}$. Now define

$$f^v = \frac{d}{dt}\gamma_f(t)|_{t=0}$$

to be the vertical lift of f to Z at $j_x^1\phi$.

If (x^μ, y^i) are fibred coordinates on Y and $f = f_\mu^i dx^\mu|_x \otimes \left.\frac{\partial}{\partial y^i}\right|_{\phi(x)}$ then

$$f^v = f_\mu^i \left.\frac{\partial}{\partial z_\mu^i}\right|_{j_x^1\phi} .$$

Let x be a point of X and $\phi \in \Gamma_x(\pi)$. If $V_0, \ldots, V_n$ are $n+1$ tangent vectors to $J^1\pi$ at the point $j_x^1\phi \in Z$, then we have that $T_{j_x^1\phi}\pi_{YZ}(V_\mu) - T_x\phi \circ T_{j_x^1\phi}\pi_{XZ}(V_\mu) \in (\mathcal{V}\pi)_{\phi(x)}$ (this is the vertical differential of a vector field on Z). From the volume form η, we also construct a family of 1-forms η_i as follows:

$$\eta_\mu(x) = (-1)^{n+1-\mu}\iota_{T_{j_x^1\phi}\pi_{XZ}(V_0)} \cdots \widehat{\iota_{T_{j_x^1\phi}\pi_{XZ}(V_\mu)}} \cdots \iota_{T_{j_x^1\phi}\pi_{XZ}(V_n)}\ \eta(x)\ ,$$

where the hat over a term means that the factor is omitted.

Next, we define the **vertical endomorphism** S_η as follows:

$$(S_\eta)_{j_x^1\phi}(V_0, \ldots, V_n) = \sum_{\mu=0}^{n} \Big(\eta_\mu(x) \otimes (T_{j_x^1\phi}\pi_{YZ}(V_\mu) - T_x\phi \circ T_{j_x^1\phi}\pi_{XZ}(V_\mu))\Big)^v .$$

Whenever we pick a different volume form $F\eta$, where F is a non-vanishing function on X, then $(F\eta)_\mu = F\eta_\mu$, whence we also get $S_{F\eta} = FS_\eta$.

The vertical endomorphism can be also written in local coordinates as follows

$$S_\eta = (dy^i - z_\mu^i dx^\mu) \wedge d^n x_\nu \otimes \frac{\partial}{\partial z_\nu^i}$$

2.3.2 Contact forms. Jet prolongation of vector fields

Contact forms are a very special kind of forms that can be used to distinguish sections of π_{XZ} that are 1-jet prolongations of sections of π, that is, the **holonomic sections**.

Definition 2.3.2. *A 1-form $\theta \in \Lambda^1(Z)$ is said to be a* ***contact 1-form*** *(or in short a contact form) whenever*

$$(j^1\phi)^*\theta = 0$$

for every section ϕ of π.

If (x^μ, y^i, z^i_μ) are a system of local coordinates on Z, then the contact forms are locally spanned by the family of 1-forms

$$\theta^i = dy^i - z^i_\mu \, dx^\mu$$

Indeed, if $\lambda = A_\mu dx^\mu + B_i dy^i + C^\mu_i dz^i_\mu$ is the local expression of a contact form, then $(j^1\phi)^*\lambda = 0$ results in

$$(j^1\phi)^* A_\mu + (j^1\phi)^* B_i \frac{\partial \phi^i}{\partial x^\mu} + (j^1\phi)^* C^\nu_i \frac{\partial^2 \phi^i}{\partial x^\mu \partial x^\nu} = 0$$

for all sections $\phi \in \Gamma(\pi)$. As $j^1\phi$ is expressed only in terms of the first derivatives, from the expression above we deduce that $C^\mu_i = 0$ and $A_\mu = -B_i z^i_\mu$, or in other words, $\lambda = B_i \theta^i$.

We shall denote by $\mathcal{C}$ the algebraic ideal of the contact forms, and by $\mathcal{I}(\mathcal{C})$ the differential ideal generated by the contact forms, in other words, the ideal of the exterior algebra generated by the contact forms and their differentials.

The distribution determined by the annihilation of the contact forms on Z is called the **Cartan distribution** and it plays a fundamental role, since it is the geometrical structure which distinguishes the holonomic sections (sections which are prolongations of sections of π_{XY}) from arbitrary sections of π_{XZ}. In fact, it is easy to see (from the coordinate expression) that

Proposition 2.3.3. *Given $\sigma \in \Gamma(\pi_{XZ})$, if $\sigma^*\theta = 0$ for all $\theta \in \mathcal{C}$, then σ is holonomic, that is $\sigma = j^1\phi$ for certain $\phi \in \Gamma(\pi)$.*

Proof. If σ is locally expressed as $\sigma(x) = (x, \sigma^i(x), \sigma^i_\mu(x))$, then

$$0 = \sigma^*\theta^i = d\sigma^i - \sigma^i_\mu dx^\mu = \left[\frac{\partial \sigma^i}{\partial x^\mu} - \sigma^i_\mu\right] dx^\mu$$

and thus $\sigma^i_\mu = \partial\sigma^i/\partial x^\mu$, for all $i \in \{1, 2, \ldots, m\}$. ∎

Contact forms may be defined for higher orders, and used to distinguish holonomic from non-holonomic sections. See [13, 87, 137] for more details.

The Cartan distribution may also be used to induce vector fields on Z from vector fields on Y in a natural manner, what is called the 1-jet prolongation of vector fields.

Proposition 2.3.4. *For any vector field X in Z, the following two conditions are equivalent:*

(i) For every Y vector field along the Cartan distribution $\pounds_X Y$ lies in the Cartan distribution; in other words, X preserves the Cartan distribution

(ii) X preserves $\mathcal{C}$, in other words, for every $\theta \in \mathcal{C}$, $\pounds_X\theta \in \mathcal{C}$.

If any of the preceding two holds, then X preserves $\mathcal{I}(\mathcal{C})$, in other words, for every $\alpha \in \mathcal{I}(\mathcal{C})$, $\pounds_X\alpha \in \mathcal{I}(\mathcal{C})$.

Proof. For all Y in the Cartan distribution, and for every $\theta \in \mathcal{C}$, $\theta(Y) = 0$ and we have that

$$(\pounds_X\theta)(Y) = \pounds_X(\theta(Y)) - \theta(\pounds_X Y) = -\theta(\pounds_X Y)$$

therefore, (i) and (ii) are equivalent.

Finally, $\pounds_X d\theta = d\pounds_X\theta$ gives the compatibility with respect to exterior differentiation, whence we have the preservation of the differential ideal $\mathcal{I}(\mathcal{C})$. ∎

Definition 2.3.5. *Given a vector field $\xi_Y \in \mathfrak{X}(Y)$, then its* ***1-jet prolongation*** *is defined as the unique vector field $\xi_Y^{(1)} \in \mathfrak{X}(Z)$ projectable onto ξ_Y by π_{YZ}, and which preserves the Cartan distribution (in other words, $\pounds_{\xi_Y^{(1)}}\theta \in \mathcal{C}$ for every contact form θ).*

If ξ_Y is locally expressed as

$$\xi_Y = \xi_Y^\mu \frac{\partial}{\partial x^\mu} + \xi_Y^i \frac{\partial}{\partial y^i}$$

then the 1-jet prolongation of ξ_Y has the following form

$$\xi_Y^{(1)} = \xi_Y^\mu \frac{\partial}{\partial x^\mu} + \xi_Y^i \frac{\partial}{\partial y^i} + \left(\frac{d\xi_Y^i}{dx^\mu} - z_\nu^i \frac{d\xi_Y^\nu}{dx^\mu} \right) \frac{\partial}{\partial z_\mu^i} \tag{2.3}$$

Assume that the local expression of $\xi_Y^{(1)}$ is

$$\xi_Y^{(1)} = \xi_Y^\mu \frac{\partial}{\partial x^\mu} + \xi_Y^i \frac{\partial}{\partial y^i} + \xi_{\mu Y}^i \frac{\partial}{\partial z_\mu^i} \tag{2.4}$$

In order to see that (2.4) has the form (2.3), pick $i \in \{1, 2, \ldots, m\}$, and impose the second condition $\pounds_{\xi_Y^{(1)}}\theta^i \in \mathcal{C}$. As we have

$$\begin{aligned}
\pounds_{\xi_Y^{(1)}}(dy^i) &= \frac{\partial \xi_Y^i}{\partial x^\mu} dx^\mu + \frac{\partial \xi_Y^i}{\partial y^j} dy^j \\
\pounds_{\xi_Y^{(1)}}(z_\mu^i dx^\mu) &= \xi_{\mu Y}^i dx^\mu + z_\mu^i \left(\frac{\partial \xi_Y^\mu}{\partial x^\nu} dx^\nu + \frac{\partial \xi_Y^\mu}{\partial y^j} dy^j\right)
\end{aligned}$$

then

$$\begin{aligned}
\pounds_{\xi_Y^{(1)}}\theta^i &= \frac{\partial \xi_Y^i}{\partial x^\mu} dx^\mu + \frac{\partial \xi_Y^i}{\partial y^j} dy^j - \xi_{\mu Y}^i dx^\mu - z_\mu^i \left(\frac{\partial \xi_Y^\mu}{\partial x^\nu} dx^\nu + \frac{\partial \xi_Y^\mu}{\partial y^j} dy^j\right) \\
&= \left(\frac{\partial \xi_Y^i}{\partial y^j} - z_\mu^i \frac{\partial \xi_Y^\mu}{\partial y^j}\right) dy^j - \left(-\frac{\partial \xi_Y^i}{\partial x^\nu} + \xi_{\nu Y}^i + z_\mu^i \frac{\partial \xi_Y^\mu}{\partial x^\nu}\right) dx^\nu
\end{aligned}$$

Therefore

$$-\frac{\partial \xi_Y^i}{\partial x^\nu} + \xi_{\nu Y}^i + z_\mu^i \frac{\partial \xi_Y^\mu}{\partial x^\nu} = z_\nu^j \left(\frac{\partial \xi_Y^i}{\partial y^j} - z_\mu^i \frac{\partial \xi_Y^\mu}{\partial y^j}\right)$$

and we get

$$\xi_{\mu Y}^i = \frac{d\xi_Y^i}{dx^\mu} - z_\nu^i \frac{d\xi_Y^\nu}{dx^\mu}.$$

The jet prolongation of vector field also has the following interesting property:

Proposition 2.3.6. *If ξ_Y is a vector field on Y which is tangent to a submanifold on Y defined by $F \equiv 0$ (for $F \in \mathcal{C}^\infty(Y)$), then $\xi_Y^{(1)}$ is tangent to the submanifolds of $J^1\pi$ defined by $F \circ \pi_{YZ} = 0$, and $G_\mu \equiv \frac{dF}{dx^\mu} = 0$.*

Proof. Within this proof, we shall the same letter F to denote both F and $F \circ \pi$ whenever the expressions are written on Z. We shall use local coordinates and denote

$$\xi_Y = \xi_Y^\mu \frac{\partial}{\partial x^\mu} + \xi_Y^i \frac{\partial}{\partial y^i}$$

also notice that

$$G_\mu = \frac{dF}{dx^\mu} = \frac{\partial F}{\partial x^\mu} + z^i_\mu \frac{\partial F}{\partial y^i}.$$

By hypothesis, we have that

$$\xi^\nu_Y \frac{\partial F}{\partial x^\nu} + \xi^i_Y \frac{\partial F}{\partial y^i} = 0.$$

We can compute the total derivative of the preceding expression, evaluated on the manifold defined by $F = 0$ and $G_\mu = 0$ on Z and have

$$\begin{aligned} 0 &= \frac{d\xi^\nu_Y}{dx^\mu}\frac{\partial F}{\partial x^\nu} + \xi^\nu_Y \frac{d}{dx^\mu}\frac{\partial F}{\partial x^\nu} + \frac{d\xi^i_Y}{dx^\mu}\frac{\partial F}{\partial y^i} + \xi^i_Y \frac{d}{dx^\mu}\frac{\partial F}{\partial y^i} \\ &= \frac{d\xi^\nu_Y}{dx^\mu}\frac{\partial F}{\partial x^\nu} + \xi^\nu_Y \frac{\partial G_\mu}{\partial x^\nu} + \frac{d\xi^i_Y}{dx^\mu}\frac{\partial F}{\partial y^i} + \xi^i_Y \frac{\partial G_\mu}{\partial y^i} \\ &= -\frac{d\xi^\nu_Y}{dx^\mu} z^i_\nu \frac{\partial F}{\partial y^i} + \xi^\nu_Y \frac{\partial G_\mu}{\partial x^\nu} + \frac{d\xi^i_Y}{dx^\mu}\frac{\partial F}{\partial y^i} + \xi^i_Y \frac{\partial G_\mu}{\partial y^i} \end{aligned}$$

where for the last equality we have evaluated on $G_\mu = 0$. Now notice that $\frac{\partial F}{\partial y^i} = \frac{\partial G_\mu}{\partial z^i_\gamma}$, therefore the last expression becomes precisely $\xi^{(1)}_Y(G_\mu) = 0$. ∎

Corollary 2.3.7. *If ξ_Y is a vector field on Y tangent to the image of certain $\phi \in \Gamma(\pi)$ then $\xi^{(1)}_Y$ is tangent to the image of $j^1\phi$.*

It is a direct consequence of the previous proposition, by using a function F locally described by $F = \phi^i(x) - y^i$.

1-jet prolongation of vector fields is a Lie algebra homomorphism, as we can see in

Proposition 2.3.8. *For every $\xi, \zeta \in \mathfrak{X}(Y)$,*

$$[\xi, \zeta]^{(1)} = [\xi^{(1)}, \zeta^{(1)}]$$

Proof. $[\xi^{(1)}, \zeta^{(1)}]$ obviously projects onto $[\xi, \zeta]$, and if α is a contact form, then

$$\pounds_{[\xi^{(1)},\zeta^{(1)}]}\alpha = \pounds_{\xi^{(1)}}\pounds_{\zeta^{(1)}}\alpha - \pounds_{\zeta^{(1)}}\pounds_{\xi^{(1)}}\alpha$$

which is obviously an element of $\mathcal{C}$. ∎

If ξ_Y is projectable onto a vector field $\xi_X \in \mathfrak{X}(X)$, there is a natural alternative way of defining its 1-jet prolongation, which will be used afterwards. If ξ_Y projects onto ξ_X, having flows Φ^Y_t and Φ^X_t respectively, then $\Phi^Z_t : Z \longrightarrow Z$ defined by $\Phi^Z_t(j^1_x(\phi)) = j^1_{\Phi^X_t(x)}(\Phi^Y_t \circ \phi \circ (\Phi^X_t)^{-1})$ is the flow of the 1-jet prolongation of ξ_Y (see [150] for further details).

Lemma 2.3.9. *For every π_{XY}-projectable vector field $\xi_Y \in \mathfrak{X}(Y)$ and for any form $\alpha \in \bigwedge Z$, and any section $\phi : X \longrightarrow Y$ of π, we have*

$$\left.\frac{d}{dt}\right|_{t=0} (j^1(\Phi^Y_t \circ \phi \circ (\Phi^X_t)^{-1}))^*\alpha = (j^1\phi)^*\pounds_{\xi^{(1)}_Y}\alpha$$

where Φ^Y_t and Φ^X_t are the flows induced by ξ_Y and its projection onto X, respectively.

For a proof of this lemma, see [150], p. 129; the proof of lemma 4.4.5. there can be generalised to prove this lemma.

2.3.3 Ehresmann connections and multivectors

Recall that, given a fibration $\pi : E \longrightarrow M$, a **connection in the sense of Ehresmann** is a complementary distribution to the vertical bundle. In other words, it is a distribution $\mathcal{H}$ on E such that we have the following Withney sum of vector bundles:

$$TE = \mathcal{H} \oplus V\pi$$

The connection is said to be **flat** whenever the horizontal distribution is integrable, and in such case, Frobenius' theorem guarantees the existence of horizontal local sections through each point of E.

Remember that given a section $\phi : M \longrightarrow E$, for every point $x \in M$ we have that $T_x\phi(T_xM)$ is a horizontal subspace of $T_{\phi(x)}E$, thus horizontal subspaces can be regarded as infinitesimal approximation to the sections, just as tangent vectors are linear approximation to trajectories. If we have that $T_x\phi(T_xM) = \mathcal{H}_{\phi(x)}$ then we shall say that σ is an integral local section of the connection at $\phi(x)$.

We relate horizontal subspace to multivectors. A multivector $Y \in \mathfrak{X}^k(E)$ is said to be **locally decomposable** if for every $p \in E$, there exists an open neighbourhood U_p of p, and $Y_1, \ldots, Y_k \in \mathfrak{X}(U_p)$ such that $Y = Y_1 \wedge \ldots \wedge Y_k$ on U_p.

We say that a distribution D of range k on E is locally associated to a non-vanishing k-multivector field Y on E whenever there exists a connected open subset U of E such as $Y|_U$ is a section of $\Lambda^m D|_U$. Two of such multivectors Y and Y' will be said to be equivalent if there is $f \in C^\infty(U)$ such that $Y' = fY$. In [39, 40, 41], it is proved that there is a one-to-one correspondence of equivalence classes of non-vanishing locally decomposable k-multivectors and k-dimensional orientable distributions in TE.

In later sections, we shall examine the relation between multivectors, Ehresmann connections of the fibration $\pi_{XZ} : Z \longrightarrow X$, and solution to Field equations.

Consider an Ehresmann connection in $\pi_{XZ} : Z \longrightarrow X$ with horizontal projector $\mathbf{h}$, which in local adapted coordinates is written as

$$\begin{cases} \mathbf{h}(\dfrac{\partial}{\partial x^\mu}) &= \dfrac{\partial}{\partial x^\mu} + \Gamma^i_\mu \dfrac{\partial}{\partial y^i} + \Gamma^i_{\mu\nu} \dfrac{\partial}{\partial z^i_\nu} \\ \mathbf{h}(\dfrac{\partial}{\partial y^i}) &= 0 \\ \mathbf{h}(\dfrac{\partial}{\partial z^i_\mu}) &= 0 \end{cases}$$

a horizontal local section $\sigma \in \Gamma(\pi_{XZ})$ verifies

$$\mathbf{h}(\frac{\partial}{\partial x^\mu}) = T\sigma(\frac{\partial}{\partial x^\mu}) \tag{2.5}$$

which is to say that $\Gamma^i_\mu = \dfrac{\partial \sigma^i}{\partial x^\mu}$ and $\Gamma^i_{\mu\nu} = \dfrac{\partial \sigma^i_\nu}{\partial x^\mu}$.

At this point we can introduce the concept of **first order jet field**, which is a section $\gamma : Z \longrightarrow J^1\pi_{XZ}$ of the projection $J^1\pi_{XZ} \longrightarrow Z$. A section $\sigma \in \Gamma(\pi_{XZ})$ is said to be an **integral**

local section of γ whenever $j^1\sigma = \gamma \circ \sigma$, which is locally expressed as follows, if $\gamma(x^\mu, y^i, z^i_\mu) = (x^\mu, y^i, z^i_\mu, \Gamma^i_\nu(x^\mu, y^i, z^i_\mu), \Gamma^i_{\nu\zeta}(x^\mu, y^i, z^i_\mu))$, then we get $\Gamma^i_\mu = \dfrac{\partial \sigma^i}{\partial x^\mu}$ and $\Gamma^i_{\mu\nu} = \dfrac{\partial \sigma^i_\nu}{\partial x^\mu}$.

There is no guarantee that for a given jet field, integral sections will exist, even locally. If there exist integral sections, the jet field is said to be **integrable**. Two sections of π_{XZ} are integral sections for the same connection at $\sigma(z)$ if and only if they are tangent to each other at that point. Not surprisingly, there is a bijection between connections on $X \longrightarrow Z$ and first order jet fields, and if σ is an integral local section of a jet field, the relation between them is given by

$$\mathcal{H}_z = T\sigma_x(T_xX)$$

A connection is said to be **flat** whenever it has a horizontal global section. Thus, from the computations above we obtain that a connection is flat if and only if its associated first order jet field is integrable.

We distinguish those connections whose integral sections are jet prolongations of sections of π, namely semiholonomic connections.

To start with, we notice that there are two special submanifolds of $J^1\pi_{XZ}$ that we describe below.

First, $J^2\pi$ can be canonically embedded in $J^1\pi_{XZ}$. If we choose coordinates $(x^\mu, y^i, z^i_\mu, y^i_{;\nu}, z^i_{\mu;\nu})$ on $J^1\pi_{XZ}$, then the embedding is locally given by $\iota(x^\mu, y^i, z^i_\mu, y^i_{;\nu}, z^i_{\mu;\nu}) = (x^\mu, y^i, z^i_\mu, y^i_{;\nu}, y^i_{;\nu}, z^i_{\mu;\nu})$. In fact,

$$\iota(J^2\pi) = \{j^1_x\sigma \in J^1\pi_{XZ} \mid \sigma = j^1(\pi_{YZ} \circ \sigma)\} = \{j^1_x(j^1\phi) \mid \phi \in \Gamma(\pi)\}$$

In practice, we shall identify $J^2\pi$ with its image, and consider it to be a submanifold of $J^1\pi_{XZ}$.

And second, we define

$$\hat{J}^2(\pi) = \{j^1_x\sigma \in J^1\pi_{XZ} \mid \sigma(x) = j^1_x(\pi_{YZ} \circ \sigma)\}$$

Definition 2.3.10. *We say that a connection is **semiholonomic** (resp. **holonomic**) whenever its associated first order jet field takes values in $\hat{J}^2\pi$ (resp. $J^2\pi$).*

We have the following characterisation:

Proposition 2.3.11. *A connection on $\pi_{XZ} : Z \longrightarrow X$, with horizontal projector $\mathbf{h}$ is semiholonomic if and only if*

$$S_\eta(\mathbf{h}, \ldots, \mathbf{h}) = 0$$

which means, in coordinates, that $\Gamma^i_\mu = z^i_\mu$ (see [100] for further details).

Finally, some words on connections defined on submanifolds. In the general setting, suppose that $\pi : E \longrightarrow M$ is a fibration, and $P \subseteq E$ is a submanifold, embedded via $i : P \longrightarrow E$.

Definition 2.3.12. *A section $\mathbf{h}$ of the fibration $\bigcup_{z\in P} Lin(T_zE, T_zP) \longrightarrow P$ which verifies*

$$\mathbf{h}^2_z = \mathbf{h}_z, \qquad ker\mathbf{h}_z = (\mathcal{V}\pi)_z$$

*for all $z \in P$ will be called a **connection** on P. The connection is said to be **flat** whenever the distribution $Im\mathbf{h}$ on P is completely integrable.*

We have the following results (see for example [35])

Proposition 2.3.13. *Let* $\mathbf{h}$ *be a connection on* P. *Then we have*

(i) $\pi(P)$ *is an open subset of* M, *and* $\pi_P : P \longrightarrow \pi(P)$ *is a fibration.*

(ii) $J^1(\pi \circ i)$ *is a submanifold of* $J^1\pi$.

(iii) The connection $\mathbf{h}$ *defines a first order jet field* ξ *in the fibration* $\pi \circ i : P \longrightarrow \pi(P)$.

(iv) The connection $\mathbf{h}$ *is flat if and only if* ξ *is integrable.*

2.3.4 Dual Jet Bundle, Liouville form and multisymplectic form

When we introduced the first order jet bundle, we briefly listed the different approaches to the notion of jet bundle, being one of these certain structure of affine bundle over Y.

The dual affine bundle of the jet bundle is called **dual jet bundle**, and it is usually denoted by $J^1\pi^*$, and for us it will be denoted by Z^*. An alternative construction of such bundle is given here.

Definition 2.3.14. *Consider the family of spaces of forms*

$$\Lambda^{n+1}_r Y := \{\sigma \in \Lambda^{n+1} Y \mid \iota_{V_1} \dots \iota_{V_r} \sigma = 0, \forall V_i \ \pi\text{-vertical}\ 1 \leq i \leq r\}$$

defining the following filtration:

$$\Lambda^{n+1} Y \subseteq \Lambda^{n+1}_1 Y \subseteq \Lambda^{n+1}_2 Y \subseteq \ldots \subseteq \Lambda^{n+1}_{n+1} Y.$$

In particular, the elements of $\Lambda^{n+1}_1 Y$ *are called* ***semibasic*** $(n+1)$***-forms***. *It is a fiber bundle over* Y *of rank* $(n+1+m+1)$, *and whose elements can be locally expressed as* $p(x,y)d^{n+1}x$.

Similarly, $\Lambda^{n+1}_2 Y$ *is a vector bundle over* Y *of rank* $(n+1+m+(n+1)m+1)$, *having* $\Lambda^{n+1}_1 Y$ *as subbundle, and which elements can be locally expressed as* $p(x,y)d^{n+1}x + p^\mu_i(x,y)dy^i \wedge d^n x_\mu$. *The natural projection will be called:*

$$\nu_r : \Lambda^{n+1}_r Y \longrightarrow Y$$

The quotient bundle

$$Z^* := \Lambda^{n+1}_2 Y / \Lambda^{n+1}_1 Y$$

is a vector bundle over Y *of rank* $n+1+m+(n+1)m$ *which elements can be locally expressed as* $p^\mu_i(x,y)dy^i \wedge d^n x_\mu$, *and that is called the* ***dual first order jet bundle***. *The canonical projection will be denoted by* $\mu : \Lambda^{n+1}_2 Y \longrightarrow Z^*$.

Locally, we can choose fibred coordinates (x^μ, y^i, p^μ_i, p) *for* $\Lambda^{n+1}_2 Y$ *and* (x^μ, y^i, p) *for* $\Lambda^{n+1}_1 Y$, *thus we can choose coordinates* (x^μ, y^i, p^μ_i) *in* Z^*.

We can define a projection $\pi_{XZ^*} : Z^* \longrightarrow X$, *which is induced by* ν_2 *into the quotient space* Z^*, *composed with* π_{XY}.

We recall that $\Lambda^{n+1}_2 Y$ is equipped with the Liouville $(n+1)$-form and with the canonical multisymplectic $(n+2)$-form, that we shall simply denote by Θ and Ω, respectively.

2.3.5 Lift of vector fields to the dual jet bundle

A vector field ξ_Y on Y, having flow ϕ_t, admit a natural lift to $\Lambda^k Y$ for any k, having flow $(\phi_t^{-1})^*$.

If the vector field ξ_Y is projectable, then the flow preserves $\Lambda_2^{n+1}Y$ and $\Lambda_1^{n+1}Y$, and therefore we can define on $\Lambda_2^{n+1}Y$ a vector field which projects onto a vector field on Z^*, which we shall denote by $\xi_Y^{(1*)}$.

In general, if α is the pull-back to $\Lambda_2^{n+1}Y$ of certain semibasic n-form on Y, locally expressed by

$$\alpha = \alpha^\nu(x^\mu, y^i) d^n x_\nu,$$

the additional condition $\pounds_{\xi_Y^\alpha}\Theta = d\alpha$ imposed to vector fields on $\Lambda^{n+1}Y$ which project to ξ_Y, determines a vector field on $\Lambda^{n+1}Y$ that can be defined on $\Lambda_2^{n+1}Y$.

In other words, we have the following definition:

Definition 2.3.15. *If α is the pull-back to $\Lambda_2^{n+1}Y$ of a π_{XY}-semibasic form, then the α-lift of a vector field ξ_Y on Y to $\Lambda_2^{n+1}Y$ is defined as the unique vector field ξ_Y^α satisfying:*

(1) ξ_Y^α projects onto ξ_Y

(2) $\pounds_{\xi_Y^\alpha}\Theta = d\alpha$

If $\xi_Y = \xi_Y^\mu \frac{\partial}{\partial x^\mu} + \xi_Y^i \frac{\partial}{\partial y^i}$, then $\xi_Y^\alpha = \xi_Y^\mu \frac{\partial}{\partial x^\mu} + \xi_Y^i \frac{\partial}{\partial y^i} + \xi_Y^p \frac{\partial}{\partial p} + \xi_Y^{p_i^\mu} \frac{\partial}{\partial p_i^\mu}$, where the components ξ_Y^p and $\xi_Y^{p_i^\mu}$ are determined by the equations (see also [64, 141]):

$$\begin{aligned}
\xi_Y^p &= -p\frac{\partial \xi_Y^\mu}{\partial x^\mu} - p_i^\mu \frac{\partial \xi_Y^i}{\partial x^\mu} - \frac{\partial \alpha^\mu}{\partial x^\mu} \\
\xi_Y^{p_i^\mu} &= p_i^\nu \frac{\partial \xi_Y^\mu}{\partial x^\nu} - p_j^\mu \frac{\partial \xi_Y^j}{\partial y^i} - p_i^\mu \frac{\partial \xi_Y^\nu}{\partial x^\nu} - \frac{\partial \alpha^\mu}{\partial y^i}
\end{aligned}$$

When ξ_Y is π_{XY}-projectable, with flow ϕ_t, then the flow of the 0-lift is precisely $(\phi_t^{-1})^*$.

CHAPTER 3

Classical Field Theory

Amongst the possible geometric formalisms to describe the Field Theories, we choose to use the multisymplectic framework. In this framework, a first order theory is modelled by finding an appropriate fibration $\pi : Y \longrightarrow X$ (X has possibly a boundary), where X is the space-time manifold, and the sections of π represent the fields. X is assumed to be oriented with a fixed volume form η.

For example, time-dependent mechanics can be regarded as one of these theories, where $X = \mathbb{R}$, and Y is chosen to be $Q \times \mathbb{R}$, where Q is the configuration manifold.

Historically, the multisymplectic description of Classical Field Theories goes back to the end of the sixties, when it was developed by the Polish school led by W. Tulczyjew (see [9, 83, 84, 85, 153]), and also independently by P.L García and A. Pérez-Rendón [52, 53, 54], and H. Goldschmidt and S. Sternberg [58]. From that time, this topic has continuously deserved a lot of attention mainly after the paper [21], and more recently in [8, 50, 51, 72, 73, 74, 92, 104, 49, 141, 142, 88]. A serious attempts to get a full development of the theory has been done in the monographs [64, 65] (see also [117] for higher order theories). In addition, multisymplectic setting is proving to be useful for numerical purposes [127]. Let us remark that there are alternative approaches using the so-called polysymplectic structures (see [55, 56, 82, 146, 147, 148]) or even n-symplectic structures (see [98] for a recent survey).

Finally, higher order theories have been studied in [158]. The ideas behind the study of the Euler-Poincaré equations that appear as reduced equations from equations on a principal bundle have been attempted to be ported to the classical field theories using jet manifolds in [22, 23].

In this chapter, the formalism of jet manifolds introduced in the previous chapter is used. If X has a boundary ∂X, so does Z, defining $\partial Z := \pi_{XZ}^{-1}(\partial X)$, and similarly for Y, $\partial Y := \pi_{XY}^{-1}(\partial X)$.

A boundary condition is introduced as a subfibration $B \subseteq \partial Z$ of the fibration $\partial Z \longrightarrow \partial X$ (and of $\partial Z \longrightarrow \partial Y$).

In this geometric context, one can present the field equations in two alternative ways: in terms of multivectors (see [36, 37, 38, 39, 40, 41, 42, 43, 46, 45, 50]), or in terms of Ehresmann connections [96, 99, 100, 116]. In what follows, we shall be dealing with the equations in terms of Ehresmann connections.

3.1 Lagrangian description

3.1.1 Lagrangian setting

We choose a **Lagrangian form** $\mathcal{L}$, which is a π_{XZ}-semibasic $(n+1)$-form on Z. Thus, $\mathcal{L} = L\eta$ for certain function $L \in \mathcal{C}^\infty(Z)$, which is called the **Lagrangian function**, or simply, the **Lagrangian**. In the case of mechanics, it corresponds to a time-dependent Lagrangian.

Let us introduce the following local notation, that we shall often use.

Definition 3.1.1. *We denote by*

$$\hat{p}_i^\mu := \frac{\partial L}{\partial z_\mu^i}$$

and by

$$\hat{p} := L - z_\mu^i \hat{p}_i^\mu$$

Definition 3.1.2. *For a given Lagrangian form $\mathcal{L}$ and a volume form η we define the* ***Poincaré-Cartan*** $(n+1)$***-form*** *as*

$$\Theta_L := \mathcal{L} + (S_\eta)^*(dL) \tag{3.1}$$

In induced coordinates, it has the following expression

$$\begin{aligned}\Theta_L &= \Big(L - z_\mu^i \frac{\partial L}{\partial z_\mu^i}\Big) d^{n+1}x + \frac{\partial L}{\partial z_\mu^i} dy^i \wedge d^n x_\mu \\ &= \mathcal{L} + \hat{p}_i^\mu \theta^i \wedge d^n x_\mu \\ &= (\hat{p} dx^\mu + \hat{p}_i^\mu dy^i) \wedge d^n x_\mu\end{aligned}$$

From this form, we can also define its differential

Definition 3.1.3. *The* ***Poincaré-Cartan*** $(n+2)$***-form*** *is defined as*

$$\Omega_L := -d\Theta_L.$$

In induced coordinates is expressed as follows

$$\begin{aligned}\Omega_L &= -(dy^i - z_\mu^i dx^\mu) \wedge \Big(\frac{\partial L}{\partial y^i} d^{n+1}x - d\Big(\frac{\partial L}{\partial z_\mu^i}\Big) \wedge d^n x_\mu\Big) \\ &= -\theta^i \wedge \Big(\frac{\partial L}{\partial y^i} d^{n+1}x - d\hat{p}_i^\mu \wedge d^n x_\mu\Big) \\ &= (d\hat{p} \wedge dx^\mu + d\hat{p}_i^\mu \wedge dy^i) \wedge d^n x_\mu\end{aligned}$$

Remark 3.1.4. A different choice for the volume form η does not produce changes in the Poincaré-Cartan forms. In fact, if we replace η with a new volume form $F\eta$ ($F > 0$), and we denote by $\tilde{L} = L/F$, we would have $\mathcal{L} = L\eta = \tilde{L}F\eta$ and using the preceding computations we finally get $\Theta_L = \Theta_{\tilde{L}}$. Thus, we could use the notation $\Theta_{\mathcal{L}}$ and $\Omega_{\mathcal{L}}$ (see [37]).

At this point, we have to introduce an extra hypothesis, which is the existence of an n-form Π on B such that

$$i_B^*\Theta_L = d\Pi$$

where $i_B : B \longrightarrow Z$ is the inclusion map (see [9]).

From the coordinate expressions above we can deduce the following properties

Proposition 3.1.5. *The following holds:*
(i) $(j^1\phi)^*\pounds_{\xi_Y^{(1)}}(\mathcal{L}) = (j^1\phi)^*\pounds_{\xi_Y^{(1)}}(\Theta_L)$
(ii) For any $z \in Z$ *and every two* π_{XZ}*-vertical tangent vectors* $v, w \in \mathcal{V}_z\pi_{XZ}$,

$$\iota_v\iota_w(\Theta_L)_z = 0$$

(iii) For any $z \in Z$ *and every three* π_{XZ}*-vertical tangent vectors* $u, v, w \in \mathcal{V}_z\pi_{XZ}$,

$$\iota_u\iota_v\iota_w(\Omega_L)_z = 0$$

The following proposition will be useful later.

Proposition 3.1.6. *If* σ *is a section of* π_{XZ} *and* ξ *is a vector field in* Z *tangent to* σ*, then*

$$\sigma^*(\iota_\xi\Omega_L) = 0$$

Proof. $\xi = T\sigma(\lambda)$ along σ for certain $\lambda \in \mathfrak{X}(X)$. Thus,

$$\sigma^*(\iota_\xi\Omega_L) = \sigma^*(\iota_{T\sigma(\lambda)}\Omega_L) = \iota_\lambda(\sigma^*\Omega_L) = 0$$

as $\sigma^*\Omega_L$ would be an $(n+2)$-form on an $(n+1)$-dimensional manifold. ∎

The dynamics of the system is given by sections ϕ of π_{XY} which verify the boundary condition $(j^1\phi)(\partial X) \subseteq B$ and that extremise the **action integral**

$$S(\phi) = \int_{(j^1\phi)(C)} \mathcal{L}$$

for each compact $(n+1)-$dimensional submanifold C of X.

Variations of such sections are introduced by small perturbations of certain section ϕ along the trajectories of projectable vector field ξ_Y; in other words, if Φ_t^Y is the flow of ξ_Y, it defines **variations** as the sections $\phi_t := \Phi_t^Y \circ \phi \circ \Phi_{-t}^X$.

Definition 3.1.7. *A section* $\phi \in \Gamma(\pi)$ *is an* ***extremal*** *of* S *if*

$$\frac{d}{dt}\Big|_{t=0} \int_{(j^1\phi_t)(C)} \mathcal{L} = \frac{d}{dt}\Big|_{t=0} \int_C (j^1\phi_t)^*\mathcal{L} = 0$$

for any compact $(n+1)$*-dimensional submanifold* C *of* X*, and for every projectable vector field* $\xi_Y \in \mathfrak{X}(Y)$

As C is chosen to be compact, the derivative respect to t commutes with the integral, and Lemma 2.3.9 allows us to rewrite to extremality condition as

$$\int_C (j^1\phi)^* \pounds_{\xi_Y^{(1)}}(\mathcal{L}) = 0 \tag{3.2}$$

Theorem 3.1.8. *If ϕ is an extremal of L, then for every compact submanifold C of X, such that $\phi(C)$ lies in a single coordinate domain (x^μ, y^i), and for every projectable vector field ξ_Y on Y we have*

$$\begin{aligned}0 = &\int_C (j^2\phi)^* \left[\frac{\partial L}{\partial y^i} - \frac{d}{dx^\mu}\frac{\partial L}{\partial z^i_\mu}\right](\xi^i_Y - z^i_\nu \xi^\nu_Y)\eta \\ &+ \int_{\partial C}(j^1\phi)^*(\iota_{\xi_Y^{(1)}}\Theta_L)\end{aligned}$$

*Whenever ϕ is an extremal for the variational problem with fixed value at the boundary of C, then ϕ satisfies the **Euler-Lagrange equations***

$$(j^2\phi)^*\left(\frac{\partial L}{\partial y^i} - \frac{d}{dx^\mu}\frac{\partial L}{\partial z^i_\mu}\right) = 0\,, 1 \le i \le m$$

Proof. A computation on the previous formula gives us

$$\begin{aligned}\int_C (j^1\phi)^*\pounds_{\xi_Y^{(1)}}(\mathcal{L}) &= \int_C (j^1\phi)^*\xi_Y^{(1)}(L)\eta + \int_C (j^1\phi)^* L(\pounds_{\xi_Y^{(1)}}(\eta)) \\ &= \int_C (j^1\phi)^*\xi^\mu_Y \frac{\partial L}{\partial x^\mu}\eta + \int_C (j^1\phi)^*\xi^i_Y\frac{\partial L}{\partial y^i}\eta \\ &+ \int_C (j^1\phi)^*\left[\frac{d}{dx^\mu}\xi^i_Y - z^i_\nu\frac{d}{dx^\mu}\xi^\nu_Y\right]\frac{\partial L}{\partial z^i_\mu}\eta + \int_C (j^1\phi)^* L(\pounds_{\xi_Y^{(1)}}(\eta)) \\ &= \int_C (j^1\phi)^*\xi^\mu_Y \frac{\partial L}{\partial x^\mu}\eta + \int_C (j^1\phi)^*\xi^i_Y\frac{\partial L}{\partial y^i}\eta \\ &+ \int_C (j^2\phi)^*\frac{d}{dx^\mu}\left[\xi^i_Y - z^i_\nu\xi^\nu_Y\right]\frac{\partial L}{\partial z^i_\mu}\eta + \int_C (j^2\phi)^*\xi^\nu_Y\frac{dz^i_\nu}{dx^\mu}\frac{\partial L}{\partial z^i_\mu}\eta \\ &+ \int_C (j^1\phi)^* L\frac{d\xi^\mu_Y}{dx^\mu}\eta \\ &= \int_C (j^1\phi)^*\xi^\mu_Y \frac{\partial L}{\partial x^\mu}\eta + \int_C (j^1\phi)^*\xi^i_Y\frac{\partial L}{\partial y^i}\eta \\ &+ \int_C (j^2\phi)^*\frac{d}{dx^\mu}\left[\xi^i_Y - z^i_\nu\xi^\nu_Y\right]\frac{\partial L}{\partial z^i_\mu}\eta + \int_C (j^2\phi)^*\xi^\nu_Y\frac{dz^i_\nu}{dx^\mu}\frac{\partial L}{\partial z^i_\mu}\eta \\ &+ \int_{\partial C}(j^1\phi)^* L\xi^\mu_Y d^n x_\mu - \int_C (j^1\phi)^*\xi^\mu_Y\frac{\partial L}{\partial x^\mu}\eta - \int_C (j^1\phi)^* z^i_\mu\frac{\partial L}{\partial y^i}\xi^\mu_Y\eta \\ &- \int_C (j^2\phi)^*\xi^\mu_Y\frac{dz^i_\nu}{dx^\mu}\frac{\partial L}{\partial z^i_\nu}\eta \\ &= \int_C (j^1\phi)^*\frac{\partial L}{\partial y^i}(\xi^i_Y - z^i_\mu\xi^\mu_Y)\eta + \int_C (j^2\phi)^*\frac{d}{dx^\mu}\left[\xi^i_Y - z^i_\nu\xi^\nu_Y\right]\frac{\partial L}{\partial z^i_\mu}\eta \\ &+ \int_{\partial C}(j^1\phi)^* L\xi^\mu_Y d^n x_\mu\end{aligned}$$

$$= \int_C (j^2\phi)^* \left[\frac{\partial L}{\partial y^i} - \frac{d}{dx^\mu} \frac{\partial L}{\partial z^i_\mu} \right] (\xi^i_Y - z^i_\nu \xi^\nu_Y)\eta$$
$$+ \int_{\partial C} (j^1\phi)^* \left[(\xi^i_Y - z^i_\nu \xi^\nu_Y) \frac{\partial L}{\partial z^i_\mu} + L\xi^\mu_Y \right] d^n x_\mu$$

The condition of fixed value at the boundary of C means $\xi^\mu_Y|_{\partial C} = \xi^i_Y|_{\partial C} = 0$, therefore we have

$$0 = \int_C (j^2\phi)^* \left[\frac{\partial L}{\partial y^i} - \frac{d}{dx^\mu} \frac{\partial L}{\partial z^i_\mu} \right] (\xi^i_Y - z^i_\nu \xi^\nu_Y)\eta$$

for all ξ^μ_Y and ξ^i_Y, whence we obtain the Euler-Lagrange equations. ∎

Lemma 3.1.9. *If ϕ is a section of π_{XY} and ξ is a π_{YZ}-vertical vector field in Z, then*

$$(j^1\phi)^*(\iota_\xi \Omega_L) = 0$$

Proof. ξ has components $(0, 0, w^i_\mu)$, and an easy computation shows that

$$\iota_\xi \Omega_L = -w^j_\nu \frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} (\theta^i \wedge d^n x_\mu) \in \mathcal{I}(\mathcal{C})$$

which vanishes when pulled back by a 1-jet prolongation of a section of π_{XY}. ∎

Proposition 3.1.10. ***(Intrinsic version of Euler-Lagrange equations)*** *A section $\phi \in \Gamma(\pi)$ is an extremal of S if and only if*

$$(j^1\phi)^*(\iota_\xi \Omega_L) = 0$$

for every vector field ξ on Z.

Proof. We have that

$$\int_C (j^1\phi)^* \mathcal{L}_{\xi^{(1)}_Y}(\mathcal{L}) = \int_C (j^1\phi)^* \mathcal{L}_{\xi^{(1)}_Y} \Theta_L = -\int_C (j^1\phi)^* \iota_{\xi^{(1)}_Y} \Omega_L + \int_{\partial C} (j^1\phi)^* \iota_{\xi^{(1)}_Y} \Theta_L.$$

Therefore,

$$-\int_C (j^1\phi)^* \iota_{\xi^{(1)}_Y} \Omega_L = \int_C (j^2\phi)^* \left[\frac{\partial L}{\partial y^i} - \frac{d}{dx^\mu} \frac{\partial L}{\partial z^i_\mu} \right] (\xi^i_Y - z^i_\nu \xi^\nu_Y)\eta$$

for every projectable vector field ξ_Y on Y. Then, Euler-Lagrange equations are satisfied in every C if and only if

$$(j^1\phi)^* \iota_{\xi^{(1)}_Y} \Omega_L = 0$$

for every projectable vector field ξ_Y on Y, in every compact C of X. Now different local solutions can be glued together using partitions of unity, so that we get that

$$(j^1\phi)^* \iota_{\xi^{(1)}_Y} \Omega_L = 0$$

is the expression for global sections ϕ.

Finally, any general vector field ξ_Z may be decomposed into a vector field tangent to $j^1\phi$, the lift of a π_{XY}-vertical vector field on Y and a π_{YZ}-vertical vector field. Using the preceding lemma, and Proposition 3.1.6, we get the result. ∎

3.1.2 Regular Lagrangians. De Donder equations

In some cases, we shall need to assume extra regularity conditions on the Lagrangian function:

Definition 3.1.11. *For a Lagrangian function $L : Z \longrightarrow \mathbb{R}$, it is defined its **Hessian matrix***

$$\left(\frac{\partial^2 L}{\partial z_i^\mu \partial z_j^\nu}\right)_{\mu,\nu,i,j}$$

*The Lagrangian is said to be **regular at** a point whenever such matrix is regular at that point, and **regular** whenever it is regular everywhere.*

When the Lagrangian is regular, the implicit function theorem allows us to introduce new coordinates for Z, called **Darboux coordinates** [116], namely $(x^\mu, y^i, \hat{p}_i^\mu)$, which will also be very convenient to relate the Lagrangian formalism to Hamiltonian formalism. In this case, and if $n > 0$, the form Ω_L is a multisymplectic form. To see this, notice that

$$\iota_{\partial/\partial x^\nu}\Omega_L = -\frac{\partial \hat{p}}{\partial x^\nu} d^{n+1}x + d\hat{p} \wedge d^n x_\nu + d\hat{p}_i^\mu \wedge dy^i \wedge d^{n-1}x_{\mu\nu} \tag{3.3}$$

$$= \frac{\partial \hat{p}}{\partial y^i} dy^i \wedge d^n x_\nu + \frac{\partial \hat{p}}{\partial \hat{p}_i^\mu} d\hat{p}_i^\mu \wedge d^n x_\nu + d\hat{p}_i^\mu \wedge dy^i \wedge d^{n-1}x_{\mu\nu} \tag{3.4}$$

$$\iota_{\partial/\partial y^j}\Omega_L = \frac{\partial \hat{p}}{\partial y^j} d^{n+1}x - d\hat{p}_j^\mu \wedge d^n x_\mu \tag{3.5}$$

$$\iota_{\partial/\partial \hat{p}_j^\nu}\Omega_L = \frac{\partial \hat{p}}{\partial \hat{p}_j^\nu} d^{n+1}x + dy^j \wedge d^n x_\nu \tag{3.6}$$

if we have $\xi = A^\nu \frac{\partial}{\partial x^\nu} + B^j \frac{\partial}{\partial y^j} + C_j^\nu \frac{\partial}{\partial p_j^\nu}$ then

$$\iota_\xi \Omega_L = \left(B^j \frac{\partial \hat{p}}{\partial y^j} - C_j^\nu \frac{\partial \hat{p}}{\partial \hat{p}_j^\nu}\right) d^{n+1}x + \left(A^\nu \frac{\partial \hat{p}}{\partial \hat{p}_j^\mu} - \delta_\mu^\nu B^j\right) d\hat{p}_j^\mu \wedge d^n x_\nu$$
$$+ \left(A^\nu \frac{\partial \hat{p}}{\partial y^j} - C_j^\nu\right) dy^j \wedge d^n x_\nu + A^\nu d\hat{p}_i^\mu \wedge dy^i \wedge d^{n-1}x_{\mu\nu}$$

Therefore, if $\iota_\xi \Omega_L = 0$ and $n > 0$, then from the last term of the expression above, $A^\nu = 0$, and we easily get that the rest of terms B^j and C_j^ν vanish as well.

We introduce the De Donder equations, closely related to the Euler-Lagrange equations.

Definition 3.1.12. *The following equation on sections σ of π_{XZ} is called the **De Donder equations**:*

$$\sigma^*(\iota_\xi \Omega_L) = 0 \qquad \forall \xi \in \mathfrak{X}(Z) \tag{3.7}$$

*Sections satisfying the De Donder equations and in addition the boundary condition $\sigma(\partial X) \subseteq B$ are called **solutions of the De Donder equations**.*

From proposition (3.1.6), we deduce that the De Donder equations can be equivalently restated in terms of π_{XZ}-vertical vector fields. In local coordinates, if $\sigma(x^\mu) = (x^\mu, \sigma^i(x^\mu), \sigma^i_\nu(x^\mu))$ for any $\xi = v^i \frac{\partial}{\partial y^i} + w^i_\mu \frac{\partial}{\partial z^i_\mu}$ the equations are written as

$$
\begin{aligned}
0 = &- v^i \left(\frac{\partial L}{\partial y^i} - \frac{\partial^2 L}{\partial x^\nu \partial z^i_\nu} - \frac{\partial \sigma^j}{\partial x^\mu} \frac{\partial^2 L}{\partial y^j \partial z^i_\mu} - \frac{\partial \sigma^j_\mu}{\partial x^\nu} \frac{\partial^2 L}{\partial z^j_\mu \partial z^i_\nu} + \left(\frac{\partial \sigma^j}{\partial x^\mu} - \sigma^j_\mu \right) \frac{\partial^2 L}{\partial y^i \partial z^j_\mu} \right) \\
&+ w^i_\mu \left(\left(\frac{\partial \sigma^j}{\partial x^\nu} - \sigma^j_\nu \right) \frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} \right),
\end{aligned}
$$

or in other words,

$$
\left.
\begin{aligned}
\frac{\partial L}{\partial y^i} - \frac{\partial^2 L}{\partial x^\nu \partial z^i_\nu} - \frac{\partial \sigma^j}{\partial x^\mu} \frac{\partial^2 L}{\partial y^j \partial z^i_\mu} - \frac{\partial \sigma^j_\mu}{\partial x^\nu} \frac{\partial^2 L}{\partial z^j_\mu \partial z^i_\nu} + \left(\frac{\partial \sigma^j}{\partial x^\mu} - \sigma^j_\mu \right) \frac{\partial^2 L}{\partial y^i \partial z^j_\mu} &= 0 \\
\left(\frac{\partial \sigma^j}{\partial x^\nu} - \sigma^j_\nu \right) \frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} &= 0
\end{aligned}
\right\}
$$

From the expression above, we immediately deduce that

Proposition 3.1.13. *If the Lagrangian is regular, then if a section $\sigma : X \longmapsto Z$ of π_{XZ} is a solution of the De Donder equations, then there is a local section $\phi : X \longrightarrow Y$ of π_{XY} such that, locally, $\sigma = j^1\phi$. Furthermore, ϕ is a solution of the Euler-Lagrange equations.*

Therefore, for regular Lagrangians, the solutions of the De Donder equations provide information about the dynamics of the system.

The De Donder equations in terms of Ehresmann connections

Suppose that we have a connection Γ in $\pi : Z \longrightarrow X$, with horizontal projector $\mathbf{h}$ having the following local expression:

$$
\begin{cases}
\mathbf{h}(\frac{\partial}{\partial x^\mu}) &= \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu \frac{\partial}{\partial y^i} + \Gamma^i_{\mu\nu} \frac{\partial}{\partial z^i_\nu} \\
\mathbf{h}(\frac{\partial}{\partial y^i}) &= 0 \\
\mathbf{h}(\frac{\partial}{\partial z^i_\mu}) &= 0
\end{cases}
$$

A direct computation shows that

$$
\begin{aligned}
\iota_{\mathbf{h}} \Omega_L = n\Omega_L - \sum_i \Bigg[&\frac{\partial L}{\partial y^i} - \sum_\nu \frac{\partial^2 L}{\partial x^\nu \partial z^i_\nu} - \sum_{\nu,j} \Gamma^j_\nu \frac{\partial^2 L}{\partial y^j \partial z^i_\nu} \\
&- \sum_{\nu,\mu,j} \Gamma^j_{\mu\nu} \frac{\partial^2 L}{\partial z^j_\mu \partial z^i_\nu} + \sum_{\nu,j} (\Gamma^j_\nu - z^j_\nu) \frac{\partial^2 L}{\partial y^i \partial z^j_\nu} \Bigg] dy^i \wedge d^{n+1}x \\
&- \sum_{\mu,i} \left(\sum_{\nu,j} (\Gamma^j_\nu - z^j_\nu) \frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} \right) dz^i_\mu \wedge d^{n+1}x
\end{aligned}
$$

from where we can state the following.

Proposition 3.1.14. *Let Γ be a connection with horizontal projector $\mathbf{h}$ verifying*

$$\iota_{\mathbf{h}}\Omega_L = n\Omega_L \tag{3.8}$$

If σ is a horizontal local integral section of Γ, then σ is a solution of the De Donder equations.

Therefore, one can think of equation (3.8) as an alternative integral approach to the De Donder equations.

Proof. $\mathbf{h}$ satisfies (3.8) if and only if

$$\left.\begin{array}{r} \dfrac{\partial L}{\partial y^i} - \dfrac{\partial^2 L}{\partial x^\nu \partial z^i_\nu} - \Gamma^j_\nu \dfrac{\partial^2 L}{\partial y^j \partial z^i_\nu} - \Gamma^j_{\mu\nu} \dfrac{\partial^2 L}{\partial z^j_\mu \partial z^i_\nu} + (\Gamma^j_\nu - z^j_\nu) \dfrac{\partial^2 L}{\partial y^i \partial z^j_\nu} = 0 \\ (\Gamma^j_\nu - z^j_\nu) \dfrac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} = 0 \end{array}\right\}$$

If $\sigma(x^\mu) = (x^\mu, \sigma^i(x^\mu), \sigma^i_\nu(x^\mu))$ is a horizontal local integral section of Γ, then we have that

$$\mathbf{h}(\frac{\partial}{\partial x^\mu}) = T\sigma(\frac{\partial}{\partial x^\mu}) \tag{3.9}$$

which means that $\Gamma^i_\mu = \dfrac{\partial \sigma^i}{\partial x^\mu}$ and $\Gamma^i_{\mu\nu} = \dfrac{\partial \sigma^i_\nu}{\partial x^\mu}$, and therefore (3.8) becomes the De Donder equations in coordinates.

Local solutions can be glued together using partitions of unity. ∎

If we consider boundary conditions, then the connection $\mathbf{h}$ induces a connection $\partial\mathbf{h}$ in the fibration $\pi_{\partial XB} : B \longrightarrow \partial X$, since we are considering sections $\sigma \in \Gamma(\pi_{XZ})$ such that $\sigma(\partial X) \subseteq B$.

In this way, the equation (3.8) becomes $\iota_{\mathbf{h}}\Omega_L = n\Omega_L$ with the additional condition that $\mathbf{h}$ induces $\partial\mathbf{h}$ (or equivalently $\mathbf{h}_z(T_zB) \subseteq T_zB$ for all $z \in B$).

In the regular case (or for semiholonomic connections, that is $\Gamma^i_\mu = z^i_\mu$), two of these solutions differ by a $(1,1)-$tensor field T, locally given by

$$T = T^i_{\mu\nu} dx^\nu \otimes \frac{\partial}{\partial z^i_\mu}$$

and verifying

$$T^i_{\mu\nu} \frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} = 0$$

Remark 3.1.15. An alternative approach may be considered if we express (3.8) on horizontal integrable distributions in terms of multivector fields generating those distributions. For further details, see [36, 38, 39, 40, 41, 42] and [50, 141, 142].

Finally, it is also remarkable that the case $n = 0$ has many differences with the case $n > 0$. In this case, Ω_L is not multisymplectic, and corresponds to the case of the time-dependent Lagrangian mechanics (see [118]). The regularity of L implies that (Y, Ω_L, dt) (where $dt = \eta$ is the volume form) is a cosymplectic manifold. The connection equation reduces to

$$\iota_{\mathbf{h}}\Omega_L = 0$$

where if we call $\tau = \frac{\partial}{\partial t}$ (so that $\langle \eta | \tau \rangle = 1$), then the horizontal projector $\mathbf{h}$ can be written in coordinates as follows

$$\mathbf{h}(\tau) = \tau + h^i \frac{\partial}{\partial q^i} + h'^i \frac{\partial}{\partial v^i}$$

(for $q^i = y^i, v^i = z_0^i$). Sections of π_{XY} are curves on Y, and Z can be embedded in TY.

One obtains from De Donder equations that $h'^i = \frac{\partial h^i}{\partial t}$, and that $h(\tau)$ verifies the time dependent Euler-Lagrange equations on $J^1\pi$. Furthermore, for a $(1,1)$-tensor field h on $J^1\pi$, being the horizontal projector of a distribution solution of

$$\iota_{\mathbf{h}}\Omega_L = 0$$

is equivalent to having $\xi = \mathbf{h}(\tau)$ which verifies

$$\iota_\xi \Omega_L = 0$$
$$\iota_\xi \eta = 1$$

3.1.3 The singular case

For a singular Lagrangian L, one cannot expect to find globally defined solutions; in general, if such connection $\mathbf{h}$ exists, it does so only along a submanifold Z_f of Z.

In [99, 100] the authors have developed a constraint algorithm which extends the Dirac-Bergmann-Gotay-Nester-Hinds algorithm for Mechanics (see [63, 66, 67, 68]). We have adapted it to introduce boundary conditions (see also [102]).

Put $Z_1 = Z$. We then consider the subset

$$\begin{aligned} Z_2 \;=\; & \{z \in Z \mid \exists \mathbf{h}_z : T_zZ \longrightarrow T_zZ \quad \text{linear such that} \quad \mathbf{h}_z^2 = \mathbf{h}_z, \\ & \ker \mathbf{h}_z = (\mathcal{V}\pi_{XZ})_z, i_{\mathbf{h}_z}\Omega_L(z) = n\Omega_L(z), \text{and for } z \in B, \mathbf{h}_z(T_zB) \subseteq T_zB\}. \end{aligned}$$

If Z_2 is a submanifold, then there are solutions but we have to include the tangency condition, and consider a new step (denoting $B_2 = B \cap Z_2$, and in general, $B_r = B \cap Z_r$):

$$\begin{aligned} Z_3 \;=\; & \{z \in Z_2 \mid \exists \mathbf{h}_z : T_zZ \longrightarrow T_zZ_2 \quad \text{linear such that} \quad \mathbf{h}_z^2 = \mathbf{h}_z, \\ & \ker \mathbf{h}_z = (\mathcal{V}\pi_{XZ})_z,\ i_{\mathbf{h}_z}\Omega_L(z) = n\Omega_L(z), \text{and for } z \in B_2, \mathbf{h}_z(T_zB_2) \subseteq T_zB_2\}. \end{aligned}$$

If Z_3 is a submanifold of Z_2, but $\mathbf{h}_z(T_zZ)$ is not contained in T_zZ_3, we go to the third step, and so on. In the favourable case, we would obtain a final constraint submanifold Z_f of non-zero dimension and a connection in the fibration $\pi_{XZ} : Z \longrightarrow X$ along the submanifold Z_f (in fact, a family of connections) with horizontal projector $\mathbf{h}$ which is a solution of equation (3.8), and which in addition satisfies the boundary conditions.

There is an additional problem, since our connection would be a solution of the De Donder problem, but not a solution of the Euler-Lagrange equations, which requires that such a solution be of second order. This problem is solved constructing a submanifold of Z_f where such a solution exists (see [99, 100] for more details).

3.2 Hamiltonian description and equivalence theorem

3.2.1 Hamiltonian setting

Definition 3.2.1. *A* ***Hamiltonian*** *is a section* $h : Z^* \longrightarrow \Lambda_2^{n+1}Y$ *of the natural projection* $\mu : \Lambda_2^{n+1}Y \longrightarrow Z^*$.

In local coordinates, h is given by

$$h(x^\mu, y^i, p_i^\mu) = (x^\mu, y^i, p = -H(x^\mu, y^i, p_i^\mu), p_i^\mu)$$

where H is called a ***Hamiltonian function****.*

Definition 3.2.2. *Given a Hamiltonian, we define the following forms in* Z^*

$$\Theta_h := h^*\Theta$$

having local expression

$$\begin{aligned}\Theta_h &= -Hd^{n+1}x + p_i^\mu dy^i \wedge d^n x_\mu \\ &= (-Hdx^\mu + p_i^\mu dy^i) \wedge d^n x_\mu\end{aligned}$$

and

$$\begin{aligned}\Omega_h &:= h^*\Omega = -d\Theta_h \\ &= (-dH \wedge dx^\mu + dp_i^\mu \wedge dy^i) \wedge d^n x_\mu\end{aligned}$$

Definition 3.2.3. *For a given Hamiltonian h, a section* $\sigma : X \longrightarrow Z^*$ *of* π_{XZ^*} *is said to satisfy the* ***Hamilton equations*** *if*

$$\sigma^*(\iota_\xi \Omega_h) = 0$$

for all vector field ξ *on* Z^*.

If σ *has local expression* $\sigma(x^\mu) = (x^\mu, \sigma^i(x^\mu), \sigma_i^\nu(x^\mu))$, *then the Hamilton equations are written in coordinates as follows*

$$\begin{aligned}\frac{\partial \sigma^i}{\partial x^\mu} &= \frac{\partial H}{\partial p_i^\mu} \\ \sum_\mu \frac{\partial \sigma_i^\mu}{\partial x^\mu} &= -\frac{\partial H}{\partial y^i}\end{aligned}$$

We can consider the case of having a boundary condition given by a subbundle $B^* \subseteq \partial Z^*$ of $\tilde{\pi}_{\partial X \partial Z}$, which imposes a restriction on the possible solutions for the Hamilton equations. The additional requirement is naturally that the solutions must satisfy $\sigma(\partial X) \subseteq B^*$, and we also need to assume that

$$i_{B^*}^* \Theta_h = d\Pi^*$$

for certain n-form Π^* on B^* (where $i_{B^*} : B^* \longrightarrow \partial Z^*$ denotes the canonical inclusion).

Finally, there is also another formulation of the Hamilton equations in terms of connections. Suppose that we have a connection Γ (in the sense of Ehresmann) in $\pi^* : Z^* \longrightarrow X$, with horizontal projector $\mathbf{h}$, and having a local expression

$$\begin{cases} \mathbf{h}(\frac{\partial}{\partial x^\mu}) & = \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu \frac{\partial}{\partial y^i} + \Gamma^\nu_{i\mu} \frac{\partial}{\partial p^\nu_i} \\ \mathbf{h}(\frac{\partial}{\partial y^i}) & = 0 \\ \mathbf{h}(\frac{\partial}{\partial p^\mu_i}) & = 0 \end{cases}$$

A direct computation shows that

$$\begin{aligned} \iota_{\mathbf{h}}\Omega_h = n\Omega_h & - \left(\frac{\partial H}{\partial y^i} + \sum_\mu \Gamma^\mu_{i\mu} \right) dy^i \wedge d^{n+1}x \\ & + \left(\frac{\partial H}{\partial p^\mu_i} - \Gamma^i_\mu \right) dp^\mu_i \wedge d^{n+1}x \end{aligned}$$

From where we can state the following.

Proposition 3.2.4. *Let Γ be a connection with horizontal projector $\mathbf{h}$ verifying*

$$\iota_{\mathbf{h}}\Omega_h = n\Omega_h \tag{3.10}$$

and also the boundary compatibility condition $\mathbf{h}_\alpha(T_\alpha B^*) \subseteq T_\alpha B^*$ for $\alpha \in Z^*$ (i.e., $\mathbf{h}$ induces a connection $\partial\mathbf{h}$ in the fibration $\pi_{\partial X B^*} : B^* \longrightarrow \partial X$).

If σ is a horizontal local integral section of Γ, then σ is a solution of the Hamilton equations.

Therefore, one can think of the preceding equation as an alternative approach to the Hamilton equations.

3.2.2 Legendre transformation

The Legendre transformation connects the Lagrangian and the Hamiltonian description (see [99] and [100]). The definition depends on the Poincaré-Cartan $(n+1)$-form, which as we noted, depends on the Lagrangian density, and thus on the Lagrangian, that we chosen for modelling the theory.

Definition 3.2.5. *We define the **Legendre transformation** $Leg_L : Z \longrightarrow \Lambda^{n+1}_2 Y$ as follows, given $\xi_1, \ldots, \xi_{n+1} \in (T_{\pi_{YZ}(z)})Y$,*

$$(Leg_L(z))(\xi_1, \ldots, \xi_{n+1}) = (\Theta_L)_z(\tilde{\xi}_1, \ldots, \tilde{\xi}_{n+1})$$

where $\tilde{\xi}_a$ is a tangent vector at $z \in Z$ which projects onto ξ_a (see proposition 3.1.5).

It is well defined, as $\iota_\xi \Theta_L = 0$ for π_{YZ}-vertical vector fields, and $\iota_\xi \iota_\zeta Leg_L(z) = 0$ for $\xi, \zeta \in \mathcal{V}\pi$, therefore, $Leg_L(z) \in \Lambda^{n+1}_2 Y$.

In local coordinates,

$$Leg_L(x^\mu, y^i, z^i_\mu) = \left(x^\mu, y^i, p = L - z^i_\mu \frac{\partial L}{\partial z^i_\mu}, p^\mu_i = \frac{\partial L}{\partial z^i_\mu}\right)$$

which shows that Leg_L is a fibred map over Y.

For an expression of the Legendre transformation in terms of affine duals, see [64].

Definition 3.2.6. *We also define the* ***Legendre map*** *$leg_L := \mu \circ Leg_L : Z \longrightarrow Z^*$, which in coordinates has the form:*

$$leg_L(x^\mu, y^i, z^i_\mu) = \left(x^\mu, y^i, p^\mu_i = \frac{\partial L}{\partial z^i_\mu} = \hat{p}^\mu_i\right)$$

From the local expression of Θ_L, the following proposition is obvious.

Proposition 3.2.7. *The following assertions are equivalent*

(i) L is a regular Lagrangian

(ii) $leg_L : Z \longrightarrow Z^*$ is a local diffeomorphism

(iii) $Leg_L : Z \longrightarrow \Lambda^{n+1}_2 Y$ is an immersion

Definition 3.2.8. *The Lagrangian L is said to be* ***hyper-regular*** *whenever leg_L is a global diffeomorphism.*

In such case, we have that Z, Z^* and $Im(Leg_L) \subseteq \Lambda^{n+1}_2 Y$ are diffeomorphic, and $h := Leg_L \circ leg_L^{-1}$ is a Hamiltonian section.

If we also put that $\Pi = leg_L^* \Pi^*$, after some little computation we have that

Proposition 3.2.9. *For the Hamiltonian section $h := Leg_L \circ leg_L^{-1}$, we have the following relations:*

$$(Leg_L)^*\Theta = \Theta_L, \qquad (Leg_L)^*\Omega = \Omega_L$$
$$(leg_L)^*\Theta_h = \Theta_L, \qquad (leg_L)^*\Omega_h = \Omega_L$$

3.2.3 The equivalence theorem

In this section, we shall assume that L is hyper-regular, and that $\Pi = leg_L^* \Pi^*$.

A boundary condition on $B \subseteq \partial Z$ on the Lagrangian side will also lead us to consider a boundary condition $B^* := leg_L(B) \subseteq \partial Z^*$. In particular, the preceding construction leads us to a Hamiltonian, and the solutions of the corresponding Hamilton equations will also be required to satisfy the boundary condition $\sigma(\partial X) \subseteq B^*$. We have:

Theorem 3.2.10. ***(equivalence theorem)****. If a section σ_1 of π_{XZ} satisfies the De Donder equations*

$$\sigma_1^*(\iota_\xi \Omega_L) = 0 \qquad \forall \xi \in \mathfrak{X}(Z)$$

then $\sigma_2 := leg_L \circ \sigma_1$ *verifies the Hamilton equations*

$$\sigma_2^*(\iota_\xi \Omega_h) = 0 \qquad\qquad \forall \xi \in \mathfrak{X}(Z^*)$$

Reciprocally, if σ_2 *verifies Hamilton equations, then (the locally defined)* $\sigma_1 := leg_L^{-1} \circ \sigma_2$ *verifies the De Donder equations. Therefore, De Donder equations are equivalent to Hamilton equations.*

Remark 3.2.11. A standard computation also shows that, for a hyper-regular Lagrangian, if Γ is a connection solution of (3.8) with horizontal projector $\mathbf{h}$ then $Tleg_L(\Gamma)$ (with horizontal projector $Tleg_L \circ \mathbf{h} \circ Tleg_L^{-1}$) is a solution for the equation in terms of connections on the Hamiltonian side.

Furthermore, $Tleg_L(T_zB) \subseteq T_{leg_L(z)}B^*$, and in turn proves that compatible connection projectors relate to each other via the Legendre map.

3.2.4 Almost regular Lagrangians

When the Lagrangian is not regular then to develop a Hamiltonian counterpart, we need some weak regularity condition on the Lagrangian L, the almost-regularity assumption.

Definition 3.2.12. *A Lagrangian* $L : Z \longrightarrow \mathbb{R}$ *is said to be **almost regular** if* $Leg_L(Z) = \tilde{M}_1$ *is a submanifold of* $\Lambda_2^{n+1}Y$*, and* $Leg_L : Z \longrightarrow \tilde{M}_1$ *is a submersion with connected fibers.*

If L is almost regular, we deduce that:

- $M_1 = leg_L(Z)$ is a submanifold of Z^*, and in addition, a fibration over X and Y.
- The restriction $\mu_1 : \tilde{M}_1 \longrightarrow M_1$ of μ is a diffeomorphism.
- The mapping $leg_L : Z \longrightarrow M_1$ is a submersion with connected fibers.

On the hypothesis of almost regularity, we can define a mapping $h_1 = (\mu_1)^{-1} : M_1 \longrightarrow \tilde{M}_1$, and a $(n+2)$-form Ω_{M_1} on M_1 by $\Omega_{M_1} = h_1^*(j^*\Omega)$ considering the inclusion map $j : \tilde{M}_1 \hookrightarrow \Lambda_2^{n+1}Y$. Obviously, we have $leg_1^*\Omega_{M_1} = \Omega_L$, where $j \circ leg_1 = leg_L$ (see Figure 3.1).

The Hamiltonian description is now based in the equation

$$i_{\tilde{\mathbf{h}}}\Omega_{M_1} = n\Omega_{M_1} \tag{3.11}$$

where $\tilde{\mathbf{h}}$ is a connection in the fibration $\pi_{XM_1} : M_1 \longrightarrow X$, and the additional boundary condition for $\tilde{\mathbf{h}}$.

Proceeding as before, we construct a constraint algorithm as follows. First, we denote by $B_1^* = B^* \cap M_1$, and will assume it to be a submanifold of B^* (and in general we shall denote $B_r^* = B^* \cap M_r$, which will also be assumed to be a submanifold of B_{r-1}^*), and we define

$$\begin{aligned} M_2 \;\; = \;\; & \{\tilde{z} \in M_1 \mid \exists \tilde{\mathbf{h}}_{\tilde{z}} : T_{\tilde{z}}M_1 \longrightarrow T_{\tilde{z}}M_1 \quad \text{linear such that} \;\; \tilde{\mathbf{h}}_{\tilde{z}}^2 = \tilde{\mathbf{h}}_{\tilde{z}}, \\ & \ker \tilde{\mathbf{h}}_{\tilde{z}} = (\mathcal{V}\pi_{XM_1})_{\tilde{z}}, i_{\tilde{\mathbf{h}}_{\tilde{z}}}\Omega_{M_1}(\tilde{z}) = n\Omega_{M_1}(\tilde{z}), \text{and for } \tilde{z} \in B_1^*, \tilde{\mathbf{h}}_{\tilde{z}}(T_{\tilde{z}}B_1^*) \subseteq T_{\tilde{z}}B_1^*\}. \end{aligned}$$

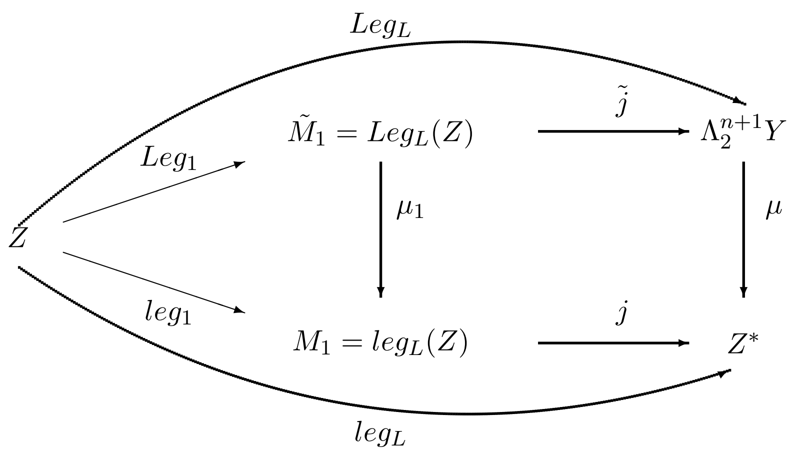

Figure 3.1: Almost regular Lagrangians and Legendre transformation

If M_2 is a submanifold (with boundary, possibly) then there are solutions but we have to include the tangency condition, and consider a new step:

$$\begin{aligned} M_3 &= \{\tilde{z} \in M_2 \mid \exists \tilde{\mathbf{h}}_{\tilde{z}} : T_{\tilde{z}}M_1 \longrightarrow T_{\tilde{z}}M_2 \quad \text{linear such that } \tilde{\mathbf{h}}_{\tilde{z}}^2 = \tilde{\mathbf{h}}_{\tilde{z}}, \\ &\ker \tilde{\mathbf{h}}_{\tilde{z}} = (\mathcal{V}\pi_{XM_1})_{\tilde{z}},\, i_{\tilde{\mathbf{h}}_{\tilde{z}}}\Omega_{M_1}(\tilde{z}) = n\Omega_{M_1}(\tilde{z}), \text{and for } \tilde{z} \in B_2^*, \tilde{\mathbf{h}}_{\tilde{z}}(T_{\tilde{z}}B_2^*) \subseteq T_{\tilde{z}}B_2^*\}. \end{aligned}$$

If M_3 is a submanifold of M_2, but $\tilde{\mathbf{h}}_{\tilde{z}}(T_{\tilde{z}}M_1)$ is not contained in $T_{\tilde{z}}M_3$ or $\tilde{\mathbf{h}}_{\tilde{z}}(T_{\tilde{z}}B_1^*)$ is not contained in $T_{\tilde{z}}B_2^*$ for $\tilde{z} \int B_2^*$, we go to the third step, and so on. Thus, we proceed further to obtain a sequence of embedded submanifolds

$$... \hookrightarrow M_3 \hookrightarrow M_2 \hookrightarrow M_1 \hookrightarrow Z^*$$

with boundaries

$$... \hookrightarrow B_3^* \hookrightarrow B_2^* \hookrightarrow B_1^* \hookrightarrow B^*$$

If this constraint algorithm stabilizes, we shall obtain a final constraint submanifold M_f of non-zero dimension and a connection in the fibration $\pi_{XM_1} : M_1 \longrightarrow X$ along the submanifold M_f (in fact, a family of connections) with horizontal projector $\tilde{\mathbf{h}}$ verifying the boundary compatibility condition, and which is a solution of equation (3.11). M_f projects onto an open submanifold of X (and B_f^* projects also onto an open submanifold of ∂X).

If M_f is the final constraint submanifold and $j_{f1} : M_f \longrightarrow M_1$ is the canonical immersion then we may consider the $(n+2)$-form $\Omega_{M_f} = j_{f1}^*\Omega_{M_1}$, and the $(n+1)$-form $\Theta_{M_f} = i_{f1}^*\Theta_{M_1}$, where $\Omega_{M_f} = -d\Theta_{M_f}$.

Denoting $leg_a := leg_L|_{Z_a}$, a direct computation shows that $leg_1(Z_a) = M_a$ for each integer (see 3.2).

In consequence, both algorithms have the same behaviour; in particular, if one of them stabilizes, so does the other, and at the same step. In particular, we have $leg_1(Z_f) = M_f$. In such case, the restriction $leg_f : Z_f \longrightarrow M_f$ is a surjective submersion (that is, a fibration) and $leg_f^{-1}(leg_f(z)) = leg_1^{-1}(leg_1(z))$, for all $z \in Z_f$ (that is, its fibres are the ones of leg_1).

$$
\begin{array}{lcll}
Z_1 = Z & \xrightarrow{leg_1} & leg_L(Z) = M_1 & \xhookrightarrow{j} \; Z^* \\
\uparrow i_1 & & \uparrow j_1 & \\
Z_2 & \xrightarrow{leg_2} & M_2 & \\
\uparrow i_2 & & \uparrow j_2 & \\
Z_3 & \xrightarrow{leg_3} & M_3 & \\
\uparrow i_3 & & \uparrow j_3 & \\
\vdots & & \vdots & \\
\uparrow i_{k-2} & & \uparrow j_{k-2} & \\
Z_{k-1} & \xrightarrow{leg_{k-1}} & M_{k-1} & \\
\uparrow i_{k-1} & & \uparrow j_{k-1} & \\
Z_k & \xrightarrow{leg_k} & M_k &
\end{array}
$$

Figure 3.2: Relating the constraint algorithms

Therefore, the Lagrangian and Hamiltonian sides can be compared through the fibration $leg_f : Z_f \longrightarrow M_f$. Indeed, if we have a connection in the fibration $\pi_{XZ} : Z \longrightarrow X$ along the submanifold Z_f with horizontal projector $\mathbf{h}$ which is a solution of the De Donder equations (3.8) and, in addition, the connection is projectable via Leg_f to a connection in the fibration $\pi_{X\tilde{Z}} : \tilde{Z} \longrightarrow X$ along the submanifold M_f, then the horizontal projector of the projected connection $Tleg_f \circ \Gamma \circ Tleg_f^{-1}$ is a solution of the Hamilton equations (3.10), and in particular it satisfies the boundary condition. Conversely, given a connection in the fibration $\pi_{X\tilde{Z}} : \tilde{Z} \longrightarrow X$ along the submanifold M_f, with horizontal projector $\tilde{\mathbf{h}}$ which is a solution of the Hamilton equations (3.10), then every connection in the fibration $\pi_{XZ} : Z \longrightarrow X$ along the submanifold Z_f that projects onto $\tilde{\mathbf{h}}$ is a solution of the De Donder equations (3.8).

3.3 Cartan formalism in the space of Cauchy data

In many field theories, the manifold X is characterised by having a differentiated coordinate. For example, X can represent a space-time manifold, where time plays a different role from the rest of the spatial coordinates.

In this section, we introduce the analysis of such situations, starting with the study of the embeddings of submanifolds of X having codimension 1.

The analysis of such submanifolds, roughly speaking Cauchy surfaces, has been studied in the multisymplectic theoretical formalism by other authors as well, such as [9, 61, 65, 85].

3.3.1 Cauchy surfaces. Initial value problem

Definition 3.3.1. *A **Cauchy surface** is a pair (M, τ) formed by a compact oriented n-manifold M embedded in the base space X by $\tau : M \longrightarrow X$, such that $\tau(\partial M) \subseteq \partial X$, and the interior of M is included in the interior of X. Two of such Cauchy surfaces are considered the same up to an orientation and volume preserving diffeomorphism of M.*

In what follows, we shall fix M, and consider certain space $\tilde{X}$ of such embeddings. We shall rather call ***Cauchy surfaces*** *to such embeddings.*

The choice of M and $\tilde{X}$ depends on the physical theory which we aim to describe with this model.

Definition 3.3.2. *A* ***space of Cauchy data*** *is the manifold of embeddings $\gamma : M \to Z$ such that there exists a section ϕ of π_{XY} satisfying*

$$\gamma = (j^1\phi) \circ \tau$$

where $\tau := \pi_{XZ} \circ \gamma \in \tilde{X}$, and $\gamma(\partial M) \subseteq B$.
The space of such embeddings will be denoted by $\tilde{Z}$, and we shall denote by $\pi_{\tilde{X}\tilde{Z}}$ the projection $\pi_{\tilde{X}\tilde{Z}}(\gamma) = \pi_{XZ} \circ \gamma$. We shall also require this projection to be a locally trivial fibration.

Definition 3.3.3. *The* ***space of Dirichlet data*** *is the manifold $\tilde{Y}$ of all the embeddings $\delta : M \longrightarrow Y$ of the form $\delta = \pi_{YZ} \circ \gamma$ for $\gamma \in \tilde{Z}$. We also define $\pi_{\tilde{Y}\tilde{Z}} : \tilde{Z} \longrightarrow \tilde{Y}$ as $\pi_{\tilde{Y}\tilde{Z}}(\gamma) = \pi_{YZ} \circ \gamma$. We denote by $\pi_{\tilde{X}\tilde{Y}}$ the unique mapping from $\tilde{Y}$ to $\tilde{X}$ such that $\pi_{\tilde{X}\tilde{Z}} = \pi_{\tilde{X}\tilde{Y}} \circ \pi_{\tilde{Y}\tilde{Z}}$ (see Figure 3.3)*

A tangent vector v at $\gamma \in \tilde{Z}$ can be seen as a vector field along γ, that is, $v : M \longrightarrow TZ$ such that $\tau_Z \circ v = \gamma$, where $\tau_Z : TZ \longrightarrow Z$ is the canonical projection. Therefore, we identify vectors in $T_\gamma\tilde{Z}$ with vector fields on $\gamma(M)$. Thus, a vector field ξ_Z on Z induces a vector field $\xi_{\tilde{Z}}$ on $\tilde{Z}$, where for every $\gamma \in \tilde{Z}$, its representative tangent vector at $\gamma \in \tilde{Z}$ is given by

$$\xi_{\tilde{Z}}(\gamma) = \xi_Z \circ \gamma$$

And conversely, forms on Z can be considered to act upon tangent vectors of $\tilde{Z}$, for if $z = \gamma(u)$, α is a r-form on Z and $v \in T_\gamma\tilde{Z}$, then $\iota_v\alpha$ is a $(r-1)$-form on Z defined by

$$(\iota_v\alpha)_z := \iota_{v(u)}\alpha_z$$

In practice, no distinction between them will be made.

3.3.2 Integration of forms

Integration gives a standard method for obtaining k-forms on $\tilde{Z}$ from $(k+n)$-forms on Z as follows.

Definition 3.3.4. *If α is a $(k+n)$-form in Z such that $i_B^*\alpha = d\beta$, we define the k-form $\tilde{\alpha}$ on $\tilde{Z}$ by*

$$\iota_{\tilde{\zeta}_1} \dots \iota_{\tilde{\zeta}_k}\tilde{\alpha}_\gamma = \int_M \gamma^*\iota_{\zeta_1} \dots \iota_{\zeta_k}\tilde{\alpha}_\alpha - (-1)^k \int_{\partial M} \gamma^*\iota_{\zeta_1} \dots \iota_{\zeta_k}\beta \tag{3.12}$$

for $\tilde{\zeta}_1, \dots, \tilde{\zeta}_k \in T_\gamma\tilde{Z}, \gamma \in \tilde{Z}$.

In particular, the Poincaré-Cartan $(n+1)$-form Θ_L and $(n+2)$-form Ω_L also induce a 1-form $\widetilde{\Theta_L}$ and a 2-form $\widetilde{\Omega_L}$ on $\tilde{Z}$, given by:

$$(\widetilde{\Theta_L})_\gamma(\tilde{\xi}) = \int_M \gamma^*(\iota_\xi\Theta_L) + \int_{\partial M} \gamma^*(\iota_\xi\Pi)$$

and also

$$\widetilde{\Omega_L}(\tilde{\xi}_1, \tilde{\xi}_2) = \int_M \gamma^*(\iota_{\xi_2}\iota_{\xi_1}\Omega_L).$$

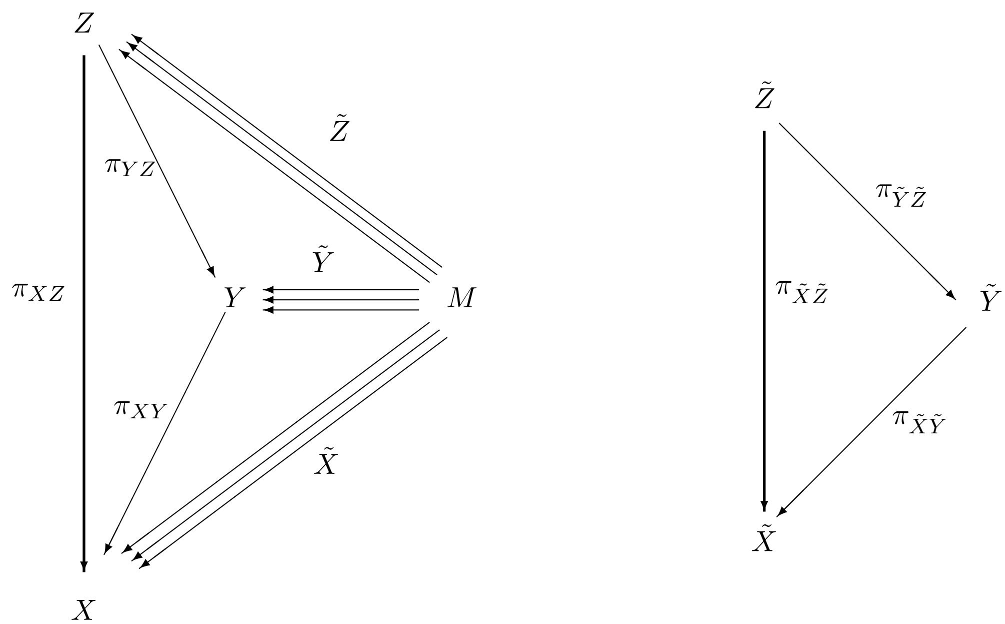

Figure 3.3: Spaces of Cauchy Data

Lemma 3.3.5. *If $\tilde{\xi}$ is a vector field on $\tilde{Z}$ defined from a vector field ξ on Z, and α is an n-form on Z such that $i_B^*\alpha = d\beta$ then*

$$d\tilde{\alpha}(\tilde{\xi})_\gamma = (\pounds_{\tilde{\xi}}\tilde{\alpha})_\gamma = \int_M \gamma^*(\pounds_\xi \alpha) - \int_{\partial M} \gamma^*(\pounds_\xi \beta)$$

Proof. First observe that $\tilde{\alpha}$ is a function. In this case, if $c_{\tilde{Z}}(t)$ is a curve such that $c_{\tilde{Z}}(0) = \gamma$ and $\dot{c}_{\tilde{Z}}(0) = \xi(\gamma)$, then

$$\begin{aligned} d\tilde{\alpha}(\tilde{\xi})_\gamma &= \tilde{\xi}_\gamma(\tilde{\alpha}) = \frac{d}{dt}(\tilde{\alpha} \circ c_{\tilde{Z}}(t))_{|t=0} = \frac{d}{dt}\left[\int_M \left(c_{\tilde{Z}}(t)^*\alpha\right) - \int_{\partial M} \left(c_{\tilde{Z}}(t)^*\beta\right)\right]_{|t=0} \\ &= \int_M \frac{d}{dt}\left(c_{\tilde{Z}}(t)^*\alpha\right)_{|t=0} - \int_{\partial M} \frac{d}{dt}\left(c_{\tilde{Z}}(t)^*\beta\right)_{|t=0} = \int_M \gamma^*(\pounds_\xi \alpha) - \int_{\partial M} \gamma^*(\pounds_\xi \beta). \end{aligned}$$ ∎

The previous result can be also extended for forms of higher degree, and for arbitrary fibrations over X.

Let ξ be a complete vector field on a fibration W over X, and let us denote by $\tilde{W}$ certain space of embeddings in W, and by $\tilde{\xi}$ the vector field defined on $\tilde{W}$ from ξ (that is, $\tilde{\xi}(\gamma) = \xi \circ \gamma$).

Fix $\gamma \in \tilde{W}$. For every $u \in M$, consider an integral curve c^u of ξ through $\gamma(u)$, that is

$$\begin{aligned} c^u(0) &= \gamma(u) \\ \dot{c}^u(0) &= \xi(\gamma(u)) \end{aligned}$$

Let us define a curve $\tilde{c}$ on $\tilde{W}$ by

$$\tilde{c}(t)(u) = c^u(t).$$

Then we have that

Proposition 3.3.6. *$\tilde{c}$ is an integral curve of $\tilde{\xi}$ through γ.*

Proof. To see this, we just have to compute

$$\tilde{c}(0)(u) = c^u(0) = \gamma(u)$$

and

$$\dot{\tilde{c}}(0)(u) = \frac{d}{dt}(\tilde{c}(t))|_{t=0}(u) = \frac{d}{dt}(\tilde{c}(t)(u))|_{t=0} = \frac{d}{dt}c^u(t)|_{t=0} = \dot{c}^u(t) = \xi(\gamma(u)) = \tilde{\xi}(\gamma)(u). \quad \blacksquare$$

$\tilde{c}$ will be said to be the associated curve to the flow given by the c^u's.

In particular, if we also have a diffeomorphism $F : W \longrightarrow W$, it is easy to see that the curve (denoted by $\widetilde{F \circ c}$) associated to the family $F \circ c^u$ (that is, $\widetilde{F \circ c}(t)(u) = (F \circ c)^u(t)$) is precisely $\tilde{F} \circ \tilde{c}$.

To see this, and using the preceding notation, note first that

$$\widetilde{F \circ c}(t)(u) = (F \circ c)^u(t) = (F \circ c^u)(t) = F(c^u(t)) = F(\tilde{c}(t)(u)) = (\tilde{F} \circ \tilde{c}(t))(u),$$

from which we deduce

Corollary 3.3.7. *If $F : W \longrightarrow W$ is a diffeomorphism, then $T\tilde{F}(\tilde{\xi}) = \widetilde{TF(\xi)}$.*

The next step is to study the pullback of forms.

Proposition 3.3.8. *If $F : W \longrightarrow W$ is a diffeomorphism, and α is a $(n+k)$-form on W, such that $i_B^*\alpha = d\beta$, then*

$$\tilde{F}^*\tilde{\alpha} = \widetilde{F^*\alpha}$$

Proof. Let $\widetilde{V_1}, \ldots, \widetilde{V_k} \in T_{\tilde{F}^{-1}(\gamma)}\tilde{W}$. We have that

$$\begin{aligned}
\iota_{\widetilde{V_1}} \cdots \iota_{\widetilde{V_k}} \tilde{F}^*\tilde{\alpha} &= \tilde{\alpha}(T\tilde{F}(\widetilde{V_1}), \ldots, T\tilde{F}(\widetilde{V_k})) = \tilde{\alpha}(\widetilde{TF(V_1)}, \ldots, \widetilde{TF(V_k)}) \\
&= \int_M \gamma^* \iota_{TF(V_1)} \cdots \iota_{TF(V_k)}\alpha - (-1)^k \int_{\partial M} \gamma^* \iota_{TF(V_1)} \cdots \iota_{TF(V_k)}\beta \\
&= \int_M (F^{-1} \circ \gamma)^* F^* \iota_{TF(V_1)} \cdots \iota_{TF(V_k)}\alpha - (-1)^k \int_{\partial M} (F^{-1} \circ \gamma)^* F^* \iota_{TF(V_1)} \cdots \iota_{TF(V_k)}\beta \\
&= \int_M (F^{-1} \circ \gamma)^* \iota_{V_1} \cdots \iota_{V_k} F^*\alpha - (-1)^k \int_{\partial M} (F^{-1} \circ \gamma)^* \iota_{V_1} \cdots \iota_{V_k} F^*\beta \\
&= \iota_{\widetilde{V_1}} \cdots \iota_{\widetilde{V_k}} \widetilde{F^*\alpha}.
\end{aligned}$$

$\blacksquare$

Finally,

Proposition 3.3.9. *If ξ is a vector field on $\tilde{W}$, then*

$$\pounds_{\tilde{\xi}}\tilde{\alpha} = \widetilde{\pounds_\xi \alpha}$$

Proof. Let $\widetilde{V_1}, \ldots, \widetilde{V_k} \in T_\gamma \tilde{W}$, and denote by ϕ_t the flow of ξ. Then we have that

$$\begin{aligned}
\iota_{\widetilde{V_1}} \ldots \iota_{\widetilde{V_k}} \pounds_{\tilde{\xi}}\tilde{\alpha} &= \iota_{\widetilde{V_1}} \ldots \iota_{\widetilde{V_k}} \frac{d}{dt}\tilde{\phi_t}^* \tilde{\alpha}|_{t=0} = \iota_{\widetilde{V_1}} \ldots \iota_{\widetilde{V_k}} \frac{d}{dt}\widetilde{\phi_t^* \alpha}|_{t=0} \\
&= \frac{d}{dt}\left(\iota_{\widetilde{V_1}} \ldots \iota_{\widetilde{V_k}} \widetilde{\phi_t^*\alpha}\right)|_{t=0} = \frac{d}{dt}\left(\int_M \iota_{V_1} \ldots \iota_{V_k} \phi_t^*\alpha - (-1)^k \int_{\partial M} \iota_{V_1} \ldots \iota_{V_k} \phi_t^* \beta\right)|_{t=0} \\
&= \int_M \iota_{V_1} \ldots \iota_{V_k} \frac{d}{dt}\left(\phi_t^*\alpha\right)|_{t=0} - (-1)^k \int_{\partial M} \iota_{V_1} \ldots \iota_{V_k} \frac{d}{dt}\left(\phi_t^*\beta\right)|_{t=0} \\
&= \int_M \iota_{V_1} \ldots \iota_{V_k} \pounds_\xi \alpha - (-1)^k \int_{\partial M} \iota_{V_1} \ldots \iota_{V_k} \pounds_\xi \beta \\
&= \iota_{\widetilde{V_1}} \ldots \iota_{\widetilde{V_k}} \widetilde{\pounds_\xi \alpha}.
\end{aligned}$$

where for the last bit just notice that $i_B^* \pounds_\xi \alpha = \pounds_\xi i_B^* \alpha = \pounds_\xi d\beta = d\pounds_\xi \beta$. ∎

Back to the fibration $Z \longrightarrow X$, the consistency of our definition of forms respect to the exterior derivative is ensured by the following proposition

Proposition 3.3.10. *If α is an n-form or an $(n+1)$-form, then*

$$\widetilde{d\alpha} = d\tilde{\alpha}$$

In particular,

$$\widetilde{\Omega_L} := -d\widetilde{\Theta_L}$$

Proof. For n-forms we use the previous lemma

$$\begin{aligned}
(d\tilde{\alpha})_\gamma(\tilde{\xi}) &= \int_M \gamma^* \pounds_\xi \alpha - \int_{\partial M} \gamma^* \pounds_\xi \beta \\
&= \int_M \gamma^* \iota_\xi d\alpha + \int_M \gamma^* d\iota_\xi \alpha - \int_{\partial M} \gamma^* (i_\xi d\beta + d i_\xi \beta) \\
&= \int_M \gamma^* \iota_\xi d\alpha = (\widetilde{d\alpha})_\gamma(\xi)
\end{aligned}$$

For $(n+1)$-forms:

$$\begin{aligned}
d\widetilde{\alpha}(\xi,\zeta)_\gamma &= \{\xi(\widetilde{\alpha}(\zeta)) - \zeta(\widetilde{\alpha}(\xi)) - \widetilde{\alpha}([\zeta,\xi])\}_\gamma \\
&= \int_M \gamma^*\{\pounds_\xi(\iota_\zeta\alpha) - \pounds_\zeta(\iota_\xi\alpha) - \iota_{[\xi,\zeta]}\alpha\} \\
&\quad + \int_{\partial M} \gamma^*\{\pounds_\xi(\iota_\zeta\beta) - \pounds_\zeta(\iota_\xi\beta) - \iota_{[\xi,\zeta]}\beta\} \\
&= \int_M \gamma^*\{\iota_\zeta\iota_\xi d\alpha - d\iota_\zeta\iota_\xi\alpha\} \\
&\quad + \int_{\partial M} \gamma^*\{\iota_\zeta\iota_\xi d\beta - d\iota_\zeta\iota_\xi\beta\} \\
&= \int_M \gamma^*(\iota_\zeta\iota_\xi d\alpha) - \int_{\partial M} \gamma^*(\iota_\zeta\iota_\xi(d\beta-\alpha)) \\
&= \int_M \gamma^*(\iota_\zeta\iota_\xi d\alpha) \\
&= \widetilde{d\alpha}(\xi,\zeta)_\gamma.
\end{aligned}$$

as we wanted to show. ∎

3.3.3 The De Donder equations in the space of Cauchy data

The De Donder equations of Field Theories have a presymplectic counterpart in the spaces of Cauchy data. The relationship between both can be found in [9] (see also [64]), and requires the definition of a slicing of the base manifold X.

Definition 3.3.11. *We say that a curve $c_{\tilde{X}}$ in $\tilde{X}$ defined on a domain $I \subseteq \mathbb{R}$* ***splits*** *X if the mapping $\Phi : I \times M \longrightarrow X$, such that $\Phi(t,u) = c_{\tilde{X}}(t)(u)$, is a diffeomorphism. In particular, the partial mapping $\Phi(t,\cdot)$ (defined by $\Phi(t,\cdot)(u) = \Phi(t,u)$) is an element of $\tilde{X}$ for all $t \in I$. In this case, $c_{\tilde{X}}$ is said to be a* ***slicing****.*

In this situation, we can rearrange coordinates in X such that if $\frac{\partial}{\partial t}$ generates the tangent space to I, then $T\Phi(\frac{\partial}{\partial t}) = \frac{\partial}{\partial x^0}$, and we consider $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ as local tangent vector fields on M or X.

Definition 3.3.12. *We can also define the concept of* ***infinitesimal slicing*** *at $\tau \in \tilde{X}$ as a tangent vector $v \in T_\tau\tilde{X}$ such that for every $u \in M$, $v(u)$ is transverse to $Im\,\tau$.*

If $c_{\tilde{Z}}$ is a curve in $\tilde{Z}$ such that its projection $c_{\tilde{X}}$ to $\tilde{X}$ splits X, then it defines a local section σ of π_{XZ} by

$$\sigma(c_{\tilde{X}}(t)(u)) = c_{\tilde{Z}}(t)(u) \tag{3.13}$$

Conversely, if σ is a section of π_{XZ}, and $c_{\tilde{X}}$ is a curve on $\tilde{X}$ (not necessarily a slicing), we define a curve $c_{\tilde{Z}}$ on Z by using (3.13). The following result relating equations in Z and $\tilde{Z}$ can be found in [9].

Theorem 3.3.13. *If σ satisfies the De Donder equations, then $c_{\tilde{Z}}$ defined as above verifies*

$$\iota_{\dot{c}_{\tilde{Z}}}\widetilde{\Omega_L} = 0 \tag{3.14}$$

Conversely, if $c_{\tilde{Z}}$ is a curve on Z satisfying (3.14)*, and its projection $c_{\tilde{X}}$ to $\tilde{X}$ splits X, then the section σ of π_{XZ} defined by* (3.13) *verifies the De Donder equations.*

Proof. Assume that σ verifies the De Donder equations. From (3.13) we obtain that $\dot{c}_{\tilde{Z}} = \sigma_* \dot{c}_{\tilde{X}}$, whence

$$c_{\tilde{Z}}(t)^*(\iota_{\dot{c}_{\tilde{Z}}} \iota_\xi \Omega_L) = c_{\tilde{X}}(t)^* \sigma^* (\iota_{\dot{c}_{\tilde{Z}}} \iota_\xi \Omega_L) = c_{\tilde{X}}(t)^*(\iota_{\dot{c}_{\tilde{X}}} \sigma^* \iota_\xi \Omega_L) = 0$$

for all ξ. Now integrate over M to obtain the desired result. For the converse, consider the integral

$$0 = \int_M c_{\tilde{X}}(t)^*(\iota_{\dot{c}_{\tilde{X}}} \sigma^* \iota_\xi \Omega_L) = 0.$$

Since this is true for every ξ, from the Fundamental Theorem of Calculus of Variations, we deduce

$$c_{\tilde{X}}(t)^*(\iota_{\dot{c}_{\tilde{X}}} \sigma^* \iota_\xi \Omega_L) = 0$$

Now if $c_{\tilde{X}}$ splits X, then $\dot{c}_{\tilde{X}}(t)$ is transverse to $c_{\tilde{X}}(t)(M)$, which implies that σ satisfies the De Donder equations. ▮

Note that, in particular, if $\mathbf{h}$ is the horizontal projector of a connection which is a solution of the De Donder equations for a connection

$$\iota_{\mathbf{h}} \Omega_L = n \Omega_L \tag{3.15}$$

and if σ is a horizontal local section of $\mathbf{h}$, the results above show that the solution to (3.14) is the horizontal lift of $\dot{c}_{\tilde{X}}$ through $\mathbf{h}$. Or more generally, the solutions are obtained as horizontal lifts of infinitesimal slicings through the connection solution to (3.15).

3.3.4 The singular case

For a singular Lagrangian, we cannot guarantee the existence of a curve $c_{\tilde{Z}}$ in $\tilde{Z}$ as a solution of the De Donder equations in $\tilde{Z}$.

Therefore, we propose an algorithm similar to that of a general presymplectic space (developed in [63, 66, 67]; see also [25, 101, 103] for the time dependent case), where to the condition that defines the manifold obtained in each step (which is the existence of a tangent vector verifying the De Donder equations), we add the fact that this tangent vector must project onto an infinitesimal slicing.

Naming $\tilde{Z}_1 := \tilde{Z}$, we define $\tilde{Z}_2$ and the subsequent subsets (requiring them to be submanifolds) as follows

$$\tilde{Z}_2 := \{\gamma \in \tilde{Z}_1 | \exists v \in T_\gamma \tilde{Z}_1 \text{ such that } T\pi_{\tilde{X}\tilde{Z}}(v) \text{ is an infinitesimal slicing and } \iota_v \widetilde{\Omega_L}|_\gamma = 0\}$$
$$\tilde{Z}_3 := \{\gamma \in \tilde{Z}_2 | \exists v \in T_\gamma \tilde{Z}_2 \text{ such that } T\pi_{\tilde{X}\tilde{Z}}(v) \text{ is an infinitesimal slicing and } \iota_v \widetilde{\Omega_L}|_\gamma = 0\}$$
$$\ldots$$

In the favourable case, the algorithm will stop at certain final non-zero dimensional constraint submanifold $\tilde{Z}_f$.

This algorithm is closely related to the algorithm in the finite dimensional spaces. We turn now to state the link between them.

Proposition 3.3.14. *Suppose that we have $v \in T_\gamma \tilde{Z}_1$ such that $T\pi_{\tilde{X}\tilde{Z}}(v)$ is an infinitesimal slicing and $\iota_v \widetilde{\Omega_L}|_\gamma = 0$. Then, for every $u \in M$ we have that*

$$H_{\gamma(u)} := T_u\gamma(T_u M) \oplus \langle v(u) \rangle$$

is a horizontal subspace of $T_{\gamma(u)}Z$ whose horizontal projector $\mathbf{h}$ *verifies the De Donder equations for connections satisfying* (3.15) *at $\gamma(u)$:*

$$\iota_{\mathbf{h}} \Omega_L|_{\gamma(u)} = n\Omega_L|_{\gamma(u)}$$

Proof. The fact that v projects onto an infinitesimal slicing guarantees that $H_{\gamma(u)}$ is indeed horizontal.

The other hypothesis states that

$$\gamma^*(\iota_\xi \iota_{v_{\gamma(u)}} \Omega_L) = 0$$

for every $\xi \in T_{\gamma(u)}Z$, that is, if $\{v_1, v_2, \ldots, v_n\}$ is a basis for $T_u M$, then

$$\iota_\xi \iota_{v_{\gamma(u)}} \Omega_L(T_u\gamma(v_1), T_u\gamma(v_2), \ldots, T_u\gamma(v_n)) = 0$$

or in other words,

$$\Omega_L(\xi, H_1, H_2, \ldots, H_{n+1}) = 0$$

for every $\xi \in T_{\gamma(u)}Z$ and every collection $H_1, H_2, \ldots, H_{n+1}$ of horizontal tangent vectors.

We want to prove that $\iota_{\mathbf{h}} \Omega_L|_{\gamma(u)} = n\Omega_L|_{\gamma(u)}$, or equivalently, $\iota_\xi \iota_{\mathbf{h}} \Omega_L|_{\gamma(u)} = n\iota_\xi \Omega_L|_{\gamma(u)}$, for every $\xi \in T_{\gamma(u)}Z$.

From the previous remarks, we see that the condition results to be true when it is evaluated on $n+1$ horizontal tangent vectors.

Suppose that V_1 is a vertical tangent vector to $\gamma(u)$. Then (as $\mathbf{h}(V_1) = 0$),

$$\iota_{\mathbf{h}} \Omega_L(\xi, V_1, H_1, \ldots, H_n) = \Omega_L(\mathbf{h}(\xi), V_1, H_1, \ldots, H_n) + n\Omega_L(\xi, V_1, H_1, \ldots, H_n)$$

where the first term vanishes due to the previous remarks. Thus, the expression holds when applied to any two tangent vector, and to any n horizontal tangent vectors.

For the next step, having two vertical vectors, remember that Ω_L is annihilated by three vertical tangent vectors (remember lemma 3.1.5). Therefore,

$$\begin{aligned} \iota_{\mathbf{h}} \Omega_L(\xi, V_1, V_2, H_1, \ldots, H_{n-1}) &= \Omega_L(\mathbf{h}(\xi), V_1, V_2, H_1, \ldots, H_{n-1}) \\ &\quad + (n-1)\Omega_L(\xi, V_1, V_2, H_1, \ldots, H_{n-1}) \\ &= \Omega_L(\xi, V_1, V_2, H_1, \ldots, H_{n-1}) + (n-1)\Omega_L(\xi, V_1, V_2, H_1, \ldots, H_{n-1}) \\ &= n\Omega_L(\xi, V_1, V_2, H_1, \ldots, H_{n-1}) \end{aligned}$$

Finally, from the mentioned properties of Ω_L, the expression also holds for a higher number of vertical tangent vectors, and so the expression holds in general. ∎

As an immediate result, we have that

Corollary 3.3.15. *If $\gamma \in \tilde{Z}_2$, then $Im\gamma \subseteq Z_2$.*

and in general,

Proposition 3.3.16. *If $\gamma \in \tilde{Z}_a$, then $Im\gamma \subseteq Z_a$.*

Proof. If $\gamma \in \tilde{Z}_a$ (which implies that there exists $v \in T\tilde{Z}_a$ such that $\iota_v \widetilde{\Omega_L}|_\gamma = 0$), then for every $u \in M$ we define $H_{\gamma(u)} := T\gamma_u(T_u M) \oplus \langle v(u) \rangle$.

We need to justify in each step that $H_{\gamma(u)} \subseteq T_{\gamma(u)} Z_a$, which amounts to prove that $T\gamma_u(T_u M) \subseteq T_{\gamma(u)} Z_a$ and $v(u) \in T_{\gamma(u)} Z_a$. The first assertion is true by construction of the subsets.

To see that $v(u) \in T_{\gamma(u)} Z_i$, we proceed inductively, starting on $a = 2$, for which the result is true because of the preceding corollary.

We assume it to be true for all the steps until the a-th, and we prove that $v(u) \in T_{\gamma(u)} Z_{a+1}$.

As $\gamma \in \tilde{Z}_{a+1}$, there exists $v \in T_\gamma \tilde{Z}_a$ such that $\iota_v \widetilde{\Omega_L} = 0$. Thus, there exists a curve $c : (-\varepsilon, \varepsilon) \longrightarrow \tilde{Z}_a$ (and thus $Im(c)(t) \subseteq Z_a$) such that $c(0) = \gamma$ and $\dot{c}(0) = v$. We deduce that $v(u) \in T_{\gamma(u)} Z_a$. ∎

Remark 3.3.17. Suppose now that $\tilde{X}$ admits an slicing. In the case in which $z \in Z_a$ is such that $\pi_{XZ}(z)$ belongs to the image of the slicing, and $\mathbf{h}_z$ is integrable, then there exists $\gamma \in \tilde{Z}_a$, and $u \in M$ such that $\gamma(u) = z$.

As before, we prove first the case $a = 2$. If σ is an horizontal local section of $\mathbf{h}$ at z, then we use the slicing to define the curve $c_{\tilde{Z}}(t)$ which verifies the De Donder equations in $\tilde{Z}$, and projects onto the slicing. Therefore we can take $\gamma = c_{\tilde{Z}}(t)$ for some t.

For the case $a > 2$, simply observe that if $H_{\gamma(u)} \subseteq T_{\gamma(u)} Z_i$, then $\dot{c}_{\tilde{Z}}(t)(u')$ must be tangent to Z_a for all $u' \in M$, and a very similar argument to that of the preceding section proves that $\gamma = c_{\tilde{Z}}(t) \in \tilde{Z}_a$.

3.3.5 Brackets

Notice that, in general, the only fact over $\widetilde{\Omega_L}$ that we can guarantee is that it is presymplectic, as we cannot guarantee nor the existence neither the uniqueness of Hamiltonian vector fields associated to functions defined on $\tilde{Z}$. For further details see [107] and [111] (and for higher order [110]).

Definition 3.3.18. *Given a function f in $\tilde{Z}$ and a vector field $\tilde{\xi}$ on $\tilde{Z}$, we shall say that f is a **Hamiltonian function**, and that $\tilde{\xi}$ is a **Hamiltonian vector field** for f if*

$$\iota_{\tilde{\xi}} \widetilde{\Omega_L} = df$$

Proposition 3.3.19. *If α is a Hamiltonian n-form in Z for Ω_L which is exact on ∂Z, say $\alpha_{|\partial Z} = d\beta$, then $\tilde{\alpha}$ is a Hamiltonian function on $\tilde{Z}$ for $\widetilde{\Omega_L}$. More precisely, if X_α is a Hamiltonian vector field for α, then $X_{\tilde{\alpha}}$ defined on $\tilde{Z}$ by*

$$[X_{\tilde{\alpha}}(\gamma)](u) = X_\alpha(\gamma(u))$$

is a Hamiltonian vector field for $\tilde{\alpha}$

Proof. Take a tangent vector $\tilde{\xi}$ to $\tilde{Z}$, then by lemma (3.3.5)

$$\begin{aligned}(d\tilde{\alpha})(\tilde{\xi})|_\gamma &= \int_M \gamma^*(\pounds_\xi \alpha) - \int_{\partial M} \gamma^*(\pounds_\xi \beta) \\ &= \int_M \gamma^* \iota_\xi d\alpha + \int_M \gamma^* d\iota_\xi \alpha - \int_{\partial M} \gamma^* \iota_\xi d\beta \\ &= \int_M \gamma^* \iota_\xi d\alpha = \int_M \gamma^* \iota_\xi \iota_{X_\alpha} \Omega_L = \iota_{\tilde{X}_\alpha} \widetilde{\Omega_L}(\tilde{\xi})|_\gamma.\end{aligned}$$

which proves that $d\tilde{\alpha} = \iota_{X_{\tilde{\alpha}}} \widetilde{\Omega_L}$. ∎

If f is a Hamiltonian function on $\tilde{Z}$, then its associated Hamiltonian vector field is defined up to an element in the kernel of $\widetilde{\Omega_L}$, therefore we can define the bracket operation for these functions as follows.

Definition 3.3.20. *If f and g are Hamiltonian functions on $\tilde{Z}$, with associated Hamiltonian vector fields X_f and X_g, then we define:*

$$\{f, g\} := \widetilde{\Omega_L}(X_f, X_g)$$

Notice that $i_B^* \Omega_L = 0$, thus if α_1 and α_2 are Hamiltonian forms which are exact on the boundary, then $i_B^*\{\alpha_1, \alpha_2\} = 0$.

Proposition 3.3.21. *If α_1 and α_2 are Hamiltonian n-forms which are exact on ∂Z, then*

$$\{\tilde{\alpha_1}, \tilde{\alpha_2}\} = \widetilde{\{\alpha_1, \alpha_2\}}$$

Proof.

$$\{\tilde{\alpha_1}, \tilde{\alpha_2}\} = \widetilde{\Omega_L}(X_{\tilde{\alpha_1}}, X_{\tilde{\alpha_2}}) = \int_M \gamma^* \iota_{X_{\alpha_2}} \iota_{X_{\alpha_1}} \Omega_L = \int_M \gamma^* \{\alpha_1, \alpha_2\} = \widetilde{\{\alpha_1, \alpha_2\}}.$$

∎

In [17, 50, 51] and [58] the authors explore the properties of a generalisation of this bracket, which satisfies the graded versions of several properties, such as skew-symmetry and Jacobi identity.

Remark 3.3.22. We could alternatively use the space of Cauchy data $\tilde{Z}^*$, defined in the obvious way. But nothing different or new would be obtained. In fact, assume for simplicity that L is hyper-regular. Then we would have a diffeomorphism $\widetilde{leg_L} : \tilde{Z} \longrightarrow \tilde{Z}^*$ defined by composition:

$$\widetilde{leg_L}(\gamma) = leg_L \circ \gamma$$

for all $\gamma \in \tilde{Z}$.

If the Lagrangian is not regular, but at least is almost regular, it is possible to develop the corresponding scheme. The only delicate point is that we have to consider the second order problem in the Lagrangian side, so that $\widetilde{leg_L} : \tilde{Z} \longrightarrow \tilde{Z}^*$ becomes a fibration.

3.4 Tulczyjew's triples in Classical Field Theory

On view of the identification of TT^*M with T^*TM introduced by Tulczyjew in [154, 155], we have developed a research work to implement the same sort of ideas and proofs to the jet manifold ambient. What follows is the result of our research paper in [116].

3.4.1 The multisymplectomorphism $\tilde{\alpha}$

Consider the vector bundle $\Lambda_2^{n+2}Z$ with generic elements of the form

$$a_i dy^i \wedge d^{n+1}x + b_i^\mu dz_\mu^i \wedge d^{n+1}x$$

This allows us to introduce local coordinates $(x^\mu, y^i, z_\mu^i, a_i, b_i^\mu)$ in the manifold $\Lambda_2^{n+2}Z$.

On the other hand, we shall denote by J^1Z^* the manifold of 1-jets of local sections of the fibred manifold $\pi_{XZ^*} : Z^* \longrightarrow X$. We have a canonical projection

$$j^1\pi_{YZ^*} : J^1Z^* \longrightarrow Z$$

Denote by $(x^\mu, y^i, p_i^\mu, y_\nu^i, p_{i\nu}^\mu)$ the induced coordinates on J^1Z^* respect to $\pi_{XZ^*} : Z^* \longrightarrow X$, such that

$$j^1\pi_{YZ^*}(x^\mu, y^i, p_i^\mu, y_\nu^i, p_{i\nu}^\mu) = (x^\mu, y^i, y_\nu^i).$$

Define a mapping

$$\alpha : J^1Z^* \longrightarrow \Lambda_2^{n+2}Z$$

by

$$\alpha(x^\mu, y^i, p_i^\mu, y_\nu^i, p_{i\nu}^\mu) = (x^\mu, y^i, y_\mu^i, \sum_\mu p_{i\mu}^\mu, p_i^\mu).$$

The mapping α is a surjective submersion, or in other words, $\alpha : J^1Z^* \longrightarrow \Lambda_2^{n+2}Z$ is a fibred manifold. In order to obtain a diffeomorphism, we need to "reduce" the manifold J^1Z^*. To do that, we introduce the following equivalence relation:

$$j_x^1\sigma_1 \equiv j_x^1\sigma_2 \text{ if and only if they have the same divergence,}$$

which in local coordinates $(x^\mu, y^i, p_i^\mu, y_\nu^i, p_{i\nu}^\mu)$ and $(x^\mu, \bar{y}^i, \bar{p}_i^\mu, \bar{y}_\nu^i, \bar{p}_{i\nu}^\mu)$ means

$$\bar{y}^i = y^i, \quad \bar{p}_i^\mu = p_i^\mu, \quad \bar{y}_\nu^i = y_\nu^i, \quad \sum_\mu \bar{p}_{i\mu}^\mu = \sum_\mu p_{i\mu}^\mu.$$

The corresponding quotient manifold will be denoted by $\widetilde{J^1Z^*}$, and we have a fibration $\tilde{pr} : J^1Z^* \longrightarrow \widetilde{J^1Z^*}$. The induced mapping

$$\tilde{\alpha} : \widetilde{J^1Z^*} \longrightarrow \Lambda_2^{n+2}Z$$

is a diffeomorphism, and we have an induced projection

$$\widetilde{j^1\pi_{YZ^*}} : \widetilde{J^1Z^*} \longrightarrow Z$$

Therefore, we can transport the canonical multisymplectic $(n+2)$-form $(\Omega_Z)_2^{n+2} = -d(\Theta_Z)_2^{n+2}$ on $\Lambda_2^{n+2}Z$ to $\widetilde{J^1Z^*}$ such that $(\widetilde{J^1Z^*}, \Omega_\alpha)$ is a multisymplectic manifold, where $\Omega_\alpha = \tilde{\alpha}^*(\Omega_Z)_2^{n+2})$.

Remark 3.4.1. Following the terminology introduced by W.M. Tulczyjew in the symplectic context, and accordingly to Definition 2.2.4, we could call $(\widetilde{J^1Z^*}, \Omega_\alpha)$ a special multisymplectic manifold, since it is multisymplectomorphic to a bundle of forms, and the multisymplectic $(n+2)$-form is $\Omega_\alpha = -d\Theta_\alpha$ (where $\Theta_\alpha = \widetilde{\alpha}^*((\Theta_Z)_2^{n+2})$). In addition, the diagram 3.4 is commutative.

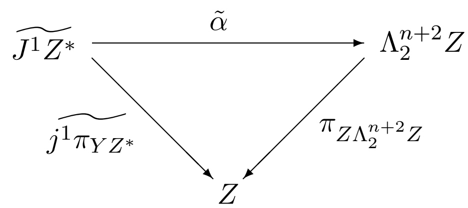

Figure 3.4: The morphism $\tilde{\alpha}$

Let $\mathbb{L} : Z \longrightarrow \Lambda^{n+1}X$ be a Lagrangian form, that is, $\mathbb{L}$ is an $(n+1)$-form on Z along the projection $\pi_{XZ} : Z \longrightarrow X$.

We put

$$\mathcal{N}_{\mathbb{L}} = \{u \in \widetilde{J^1Z^*} | \left(\widetilde{j^1\pi_{XZ^*}}\right)^* (d\mathbb{L})_u = (\Theta_\alpha)_u\}$$

Theorem 3.4.2. *$\mathcal{N}_{\mathbb{L}}$ is a (n+2)-Lagrangian submanifold of the multisymplectic manifold $(\widetilde{J^1Z^*}, \Omega_\alpha)$. In addition, the local equations defining $\mathcal{N}_{\mathbb{L}}$ are just the Euler-Lagrange equations for L, where $\mathbb{L} = L\eta$.*

Proof. From the definition it follows that

$$\tilde{\alpha}(\mathcal{N}_{\mathbb{L}}) = \operatorname{im} d\mathbb{L},$$

In addition, we have

$$\begin{aligned} (\Theta_Z)_2^{n+2} &= a_i dy^i \wedge d^{n+1}x + b_i^\mu dz_\mu^i \wedge d^{n+1}x \\ \alpha^*((\Theta_Z)_2^{n+2}) &= p_{i\mu}^\mu dy^i \wedge d^{n+1}x + p_i^\mu dy_\mu^i \wedge d^{n+1}x \\ d\mathbb{L} &= \frac{\partial L}{\partial y^i} dy^i \wedge d^{n+1}x + \frac{\partial L}{\partial z_\mu^i} dy_\mu^i \wedge d^{n+1}x. \end{aligned}$$

Since

$$(\widetilde{j^1\pi_{XZ^*}})^*(d\mathbb{L}) = \Theta_\alpha$$

if and only if

$$\tilde{pr}^*(\widetilde{j^1\pi_{XZ^*}}^*(d\mathbb{L}) - \Theta_\alpha) = 0$$

which is in turn equivalent to

$$(j^1\pi_{XZ^*})^*(d\mathbb{L}) = \alpha^*(\Theta_Z)_2^{n+2},$$

we deduce that $\mathcal{N}_{\mathbb{L}}$ is locally defined by

$$\sum_{\mu} p^{\mu}_{i\mu} = \frac{\partial L}{\partial y^i} \tag{3.16}$$

$$p^{\mu}_{i} = \frac{\partial L}{\partial z^i_{\mu}} \tag{3.17}$$

Equations (3.16) imply that $\tilde{\alpha}(\mathcal{N}_{\mathbb{L}}) = \operatorname{Im} d\mathbb{L}$, and hence $\mathcal{N}_{\mathbb{L}}$ is a $(n+2)$-Lagrangian submanifold of $(\widetilde{J^1 Z^*}, \Omega_\alpha)$.

Furthermore, we have

$$\sum_{\mu} p^{\mu}_{i\mu} = \sum_{\mu} \frac{\partial}{\partial x^{\mu}}(\frac{\partial L}{\partial z^i_{\mu}}) = \frac{\partial L}{\partial y^i}$$

which are just the Euler-Lagrange equations for L. ∎

3.4.2 The multisymplectomorphism $\tilde{\beta}$

Recall that there exists a one-to-one correspondence between connections in the fibred manifold $\pi_{XZ^*} : Z^* \longrightarrow X$ and sections of the 1-jet prolongation $\pi_{Z^* J^1 Z^*} : J^1 Z^* \longrightarrow Z^*$. (At a pointwise level we have a one-to-one correspondence between horizontal subspaces in the fibred manifold $\pi_{XZ^*} : Z^* \longrightarrow X$ and 1-jets in $J^1 Z^*$.)

Define a mapping

$$\beta : J^1 Z^* \longrightarrow \Lambda^{n+1}_2 Z^*$$

as follows: given a connection $\mathbf{h}^*$ in the fibred manifold $\pi_{XZ^*} : Z^* \longrightarrow X$, we take the $(n+2)$-form

$$\beta(\mathbf{h}^*) = i_{\mathbf{h}^*} \Omega_h - n\Omega_h.$$

An arbitrary $(n+2)$-form in $\Lambda^{n+2}_2 Z^*$ is written as

$$A_i dy^i \wedge d^{n+1}x + B^i_{\mu} dp^{\mu}_i \wedge d^{n+1}x$$

so that we can introduce local coordinates $(x^{\mu}, y^i, p^{\mu}_i, A_i, B^i_{\mu})$ on $\Lambda^{n+2}_2 Z^*$.

If we put

$$\mathbf{h}^*(\frac{\partial}{\partial x^{\mu}}) = \frac{\partial}{\partial x^{\mu}} + y^i_{\mu}\frac{\partial}{\partial y^i} + p^{\nu}_{j\mu}\frac{\partial}{\partial p^{\nu}_j}$$

or, equivalently,

$$\mathbf{h}^*(x^{\mu}, y^i, p^{\mu}_i) = (x^{\mu}, y^i, p^{\mu}_i, y^i_{\mu}, p^{\nu}_{j\mu})$$

(when $\mathbf{h}^*$ is considered as a section of $J^1 Z^* \longrightarrow Z^*$), then a straightforward computation shows that

$$\beta(x^{\mu}, y^i, p^{\mu}_i, y^i_{\mu}, p^{\nu}_{i\mu}) = (x^{\mu}, y^i, p^{\mu}_i, \sum_{\mu} p^{\mu}_{i\mu} + \frac{\partial H}{\partial y^i}, -y^i_{\mu} + \frac{\partial H}{\partial p^{\mu}_i}).$$

The mapping β is a surjective submersion. Thus, in order to have a diffeomorphism we consider the induced mapping $\tilde{\beta} : \widetilde{J^1 Z^*} \longrightarrow \Lambda^{n+2}_2 Z^*$. Therefore we obtain a commutative diagram 3.5, where $\tilde{\rho} : \widetilde{J^1 Z^*} \longrightarrow Z^*$ is the induced projection from the canonical one $\rho : J^1 Z^* \longrightarrow Z^*$.

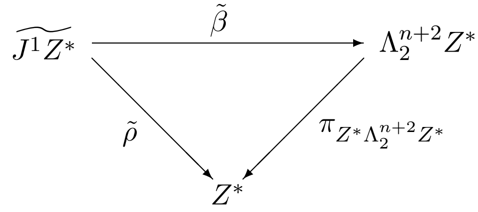

Figure 3.5: The mapping $\tilde{\beta}$

Define a $(n+2)$-form Θ_β on $\widetilde{J^1Z^*}$ as $\Theta_\beta = \tilde{\beta}^*((\Theta_{Z^*})_2^{n+2})$. Therefore, the pair $(\widetilde{J^1Z^*}, \Omega_\beta)$, $\Omega_\beta = -d\Theta_\beta$, is a multisymplectic manifold of type $(n+2, 2)$.

Remark 3.4.3. It should be noticed that the pair $(\widetilde{J^1Z^*}, \Omega_\beta)$ is a special multisymplectic manifold.

Theorem 3.4.4. *Let* $\mathbf{h}^*$ *be a solution of the de Donder equations. Then the projection* $\mathcal{N}_h$ *of the image of* $\mathbf{h}^*$ *by* $\tilde{pr}$ *is a* $(n+2)$*-Lagrangian submanifold of the multisymplectic manifold* $(\widetilde{J^1Z^*}, \Omega_\beta)$*. In addition, the local equations defining* $\mathcal{N}_h$ *are just the Hamilton equations for* h*.*

Proof.

Since

$$(\Theta_{Z^*})_2^{n+2} = A_i dy^i \wedge d^{n+1}x + B_\mu^i dp_i^\mu \wedge d^{n+1}x$$

we have

$$\beta^*((\Theta_{Z^*})_2^{n+2}) = (p_{i\mu}^\mu + \frac{\partial H}{\partial y^i})dy^i \wedge d^{n+1}x + (-y_\mu^i + \frac{\partial H}{\partial p_i^\mu})dp_i^\mu \wedge d^{n+1}x.$$

Therefore, the projection $\mathcal{N}_h$ of the image of $\mathbf{h}^*$ by $\tilde{pr}$ is just the inverse image of the zero-cross section of $\Lambda_2^{n+2}Z^*$, and hence it is a $(n+2)$-Lagrangian submanifold of $(\widetilde{J^1Z^*}, \Omega_\beta)$.

The second part of the theorem follows directly from the preceding discussion. ∎

3.4.3 Relating $\tilde{\alpha}$ and $\tilde{\beta}$

The above constructions are collected in the diagram 3.6.

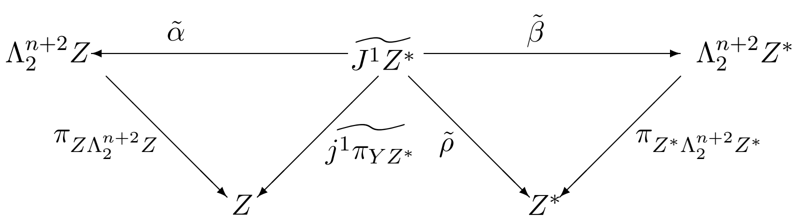

Figure 3.6: Relating $\tilde{\alpha}$ and $\tilde{\beta}$

Since

$$\begin{aligned} \tilde{pr}^*(\Theta_\alpha) &= p_{i\mu}^\mu dy^i \wedge d^{n+1}x + p_i^\mu dy_\mu^i \wedge d^{n+1}x \\ \tilde{pr}^*(\Theta_\beta) &= (p_{i\mu}^\mu + \frac{\partial H}{\partial y^i})dy^i \wedge d^{n+1}x + (-y_\mu^i + \frac{\partial H}{\partial p_i^\mu})dp_i^\mu \wedge d^{n+1}x \end{aligned}$$

we deduce that

$$\begin{aligned}\tilde{pr}^*(\Theta_\alpha - \Theta_\beta) &= dh - \left(y^i_\mu dp^\mu_i + p^\mu_i dy^i_\mu\right) \wedge d^{n+1}x \\ &= dh - d(p^\mu_i y^i_\mu) \wedge d^{n+1}x \\ &= d\left(h - (p^\mu_i y^i_\mu) \wedge d^{n+1}x\right)\end{aligned}$$

which implies that $\Omega_\alpha = \Omega_\beta$.

Theorem 3.4.5. *Let L be a regular Lagrangian, and assume that $h = leg_L \circ (Leg_L)^{-1}$. Then $N_{\mathbb{L}} = N_h$.*

CHAPTER 4

Symmetries and preserved quantities

One of the most interesting concepts of study for dynamical systems of any kind are their symmetries. This well known concept has one of its more important results in the famous theorem by **Noether**, which for dynamical systems of mechanical type (the evolution equations obtained from the minimisation of the action defined by a Lagrangian), discovers the presence of a **preserved quantity** (that is, a function which is constant along solutions) from every symmetry of the Lagrangian. Preserved quantities are a very valuable information about a system (see [97, 122, 134, 139, 143, 144]).

Furthermore, in classical mechanics, the concept of symmetry can also be used to reduce the complexity of the system, for instance by the process of symplectic reduction (as defined by Marsden and Weinstein), which leads to simplified equations defined on a lower dimension manifold, and a subsequent process of reconstruction of the original solutions from the solutions on the reduced space.

The concept of symmetry of the Lagrangian has been generalised to transformations of the phase space that preserve other geometrical objects, such as the Pointaré-Cartan 1-form (see [111, 117, 143, 144]).

In this chapter, we shall study and classify the symmetries for the field equations. Through the whole chapter we shall assume that we have a fibration $\pi : Y \longrightarrow X$, for $\dim X = n + 1$, $\dim Y = n + 1 + m$, X is oriented with volume form η, and we have a Lagrangian function $L : Z = J^1\pi \longrightarrow \mathbb{R}$.

If X has a boundary ∂X, so does Z, defining $\partial Z := \pi_{XZ}^{-1}(\partial X)$. A boundary condition is introduced as a subfibration $B \subseteq \partial Z$ of the fibration $\partial Z \longrightarrow \partial X$.

We shall be using the concepts and notations introduced in the previous chapters. The results obtained in this chapter has been published in [115].

4.1 Symmetries for the Euler-Lagrange equations

In our framework for field theory, we define a preserved quantity in the following manner:

Definition 4.1.1. *A **preserved quantity for the Euler-Lagrange equations** is an n-form α on Z such that $(j^1\phi)^*d\alpha = 0$ for every solution ϕ of the Euler-Lagrange equations. If α is a preserved quantity, then $\tilde{\alpha}$ is called its associated **momentum**.*

Notice that if α is a preserved quantity, and Λ is a closed form, then $\alpha + \Lambda$ is also a preserved quantity. Similarly, if γ is an n-form which belongs to the differential ideal $\mathcal{I}(\mathcal{C})$, then $\alpha + \gamma$ is also a preserved quantity (see [139] for a further discussion).

We turn now to obtain preserved quantities from symmetries.

4.1.1 Symmetries of the Lagrangian

We shall define the notion of symmetry based on the variation of the Poincaré-Cartan $(n+1)$-form along prolongations of vector fields. Suppose that ξ_Y is a vector field defined on Y, and abbreviate by F the function such that

$$\pounds_{\xi_Y^{(1)}}\mathcal{L} - F\eta \in \mathcal{I}(\mathcal{C})$$

having local expression

$$F = \xi_Y^{(1)}(L) + \left(\frac{\partial \xi_Y^\mu}{\partial x^\mu} + z_\nu^i \frac{\partial \xi_Y^\nu}{\partial y^i}\right) L. \tag{4.1}$$

After a lengthy computation we get that

$$\begin{aligned}\pounds_{\xi_Y^{(1)}}\Theta_L = F\eta &+ \frac{\partial F}{\partial z_\mu^i}\theta^i \wedge d^n x_\mu \\ &+ z_\nu^j \left(\frac{\partial \xi_Y^\nu}{\partial y^j}\frac{\partial L}{\partial z_\mu^i} - \frac{\partial \xi_Y^\mu}{\partial y^j}\frac{\partial L}{\partial z_\nu^i}\right)\theta^i \wedge d^n x_\mu \\ &- \frac{\partial \xi_Y^\nu}{\partial y^j}\frac{\partial L}{\partial z_\mu^i}\theta^i \wedge dy^j \wedge d^{n-1}x_{\nu\mu}\end{aligned} \tag{4.2}$$

Definition 4.1.2. *A vector field ξ_Y on Y is said to be an **infinitesimal symmetry of the Lagrangian** or a **variational symmetry** if $\pounds_{\xi_Y^{(1)}}\Theta_L \in \mathcal{I}(\mathcal{C})$ (the differential ideal generated by the contact forms), $\xi_Y^{(1)}$ is also tangent to B and verifies $\pounds_{\xi_Y^{(1)}|_B}\Pi = 0$*

We shall only deal with infinitesimal symmetries, so for brevity they will be referred simply as symmetries.

From the definition and the expression (4.2), it is obvious to see that

Proposition 4.1.3. *If a vector field ξ_Y on Y is a symmetry of the Lagrangian, then $F = 0$ (where F was defined in (4.1)).*

Remark 4.1.4. In our construction, we choose as definition of the Poincaré-Cartan $(n+1)$-form:

$$\Theta_L = \mathcal{L} + (S_\eta)^*(dL)$$

or, in fibred coordinates

$$\Theta_L = L\, d^{n+1}x + \frac{\partial L}{\partial z^i_\mu}\theta^i \wedge d^n x_\mu$$

If $n > 0$ it is possible to generalize the construction of the Poincaré-Cartan $(n+1)$-form in several different ways. The unique requirement is that the resulting π_{YZ}-semibasic $(n+1)$-form be *Lepage*-equivalent to $\mathcal{L}$, that is,

$$\Theta - \mathcal{L} \in \mathcal{I}(\mathcal{C})$$

and $i_V d\Theta \in \mathcal{I}(\mathcal{C})$ where V is an arbitrary π_{YZ}-vertical vector field. Locally,

$$\Theta = \Theta_L + \cdots \tag{4.3}$$

where the dots signify terms which are at least two-contact (see [7, 31, 62, 71, 86, 89, 90, 91]). Obviously, all them give us the same Euler-Lagrange equations.

Therefore, we may substitute in definitions 4.1.2, 4.1.8 and 4.1.13 the Poincaré-Cartan $(n+1)$-form by any $(n+1)$-form which is *Lepage*-equivalent to Θ_L. Obviously, the symmetries of the Euler-Lagrange equations are independent of the class of *Lepagean* $(n+1)$-form appearing in their definition.

We also have the following two special cases, which are easily computed from the expression of F.

Proposition 4.1.5. *If ξ_Y is a projectable symmetry of the Lagrangian ($T\pi_{XY}(\xi_Y)$ is a well defined vector field, or locally $\frac{\partial \xi^\mu_Y}{\partial y^i} = 0$), or if $\dim X = 1$ ($n = 0$), then*

$$\pounds_{\xi^{(1)}_Y}\Theta_L = 0$$

or, equivalently,

$$\pounds_{\xi^{(1)}_Y}\mathcal{L} = 0$$

Therefore,

$$\xi^{(1)}_Y(L) = -\sum_\mu \frac{d\xi^\mu_Y}{dx^\mu}L$$

Proof. Looking carefully at the last two rows of expression of (4.2), we see that if ξ_Y is projectable, then those two terms vanish since $\frac{\partial \xi^\nu_Y}{\partial y^i} = 0$, and the others because of proposition 4.1.3.

When $n = 1$, the last row does not exist, and the second row vanishes by repetition of the coordinates. The rest follows trivially from the definitions and expressions above. ∎

And as a direct consequence of Proposition 2.3.8, we have

Proposition 4.1.6. *The symmetries of the Lagrangian form a Lie subalgebra of $\mathfrak{X}(Y)$.*

Theorem 4.1.7. ***(Noether's theorem).*** *If ξ_Y is a symmetry of the Lagrangian, then $\iota_{\xi_Y^{(1)}}\Theta_L$ is a preserved quantity, which is exact on the boundary B.*

Proof. We have that

$$\pounds_{\xi_Y^{(1)}}\Theta_L = -\iota_{\xi_Y^{(1)}}\Omega_L + d\iota_{\xi_Y^{(1)}}\Theta_L$$

If ϕ is a solution of the Euler-Lagrange equations, then

$$0 = (j^1\phi)^*\pounds_{\xi_Y^{(1)}}\Theta_L = -(j^1\phi)^*\iota_{\xi_Y^{(1)}}\Omega_L + (j^1\phi)^* d\iota_{\xi_Y^{(1)}}\Theta_L,$$

where the first term vanishes by the intrinsic Euler-Lagrange equations (see Proposition 3.1.10).

Finally, to see that it is exact on the boundary, notice that from the boundary property of a symmetry of the Lagrangian we infer that $\iota_{\xi_Y^{(1)}{}_{|B}} d\Pi = -d\iota_{\xi_Y^{(1)}{}_{|B}}\Pi$, and from this we get

$$i_B^*(\iota_{\xi_Y^{(1)}}\Theta_L) = \iota_{\xi_Y^{(1)}{}_{|B}} d\Pi = -d\iota_{\xi_Y^{(1)}{}_{|B}}\Pi$$

as we wanted to prove. ∎

Observe that without the boundary condition, we obtain that $(j^1\phi)^* d\iota_{\xi_Y^{(1)}}\Theta_L = 0$, but we cannot be sure that it is exact on the boundary.

The preserved quantity can be written in local coordinates as

$$\left(\left[L - z_\mu^i \frac{\partial L}{\partial z_\mu^i}\right]\xi_X^\nu + \frac{\partial L}{\partial z_\nu^i}\xi_Y^i\right) d^n x_\nu - \frac{\partial L}{\partial z_\mu^i}\xi_X^\nu dy^i \wedge d^{n-1}x_{\mu\nu}$$

4.1.2 Noether symmetries

Definition 4.1.8. *A vector field ξ_Y on Y is said to be a* ***Noether symmetry*** *or a* ***divergence symmetry*** *if there exists an n-form on Y whose pullback α to Z (that must be exact $\alpha = d\beta$ on B) verifies $\pounds_{\xi_Y^{(1)}}\Theta_L - d\alpha \in \mathcal{I}(\mathcal{C})$, and $\xi_Y^{(1)}$ is tangent to B and verifies $\pounds_{\xi_Y^{(1)}{}_{|B}}\Pi = 0$*

The relation $dy^i = \theta^i + z_\mu^i dx^\mu$ allows us to write α locally as follows

$$\alpha = \alpha_\mu dx^0 \wedge \ldots \wedge \widehat{dx^\mu} \wedge \ldots \wedge dx^n + \theta$$

for $\theta \in \mathcal{I}(\mathcal{C})$ and

$$d\alpha - \sum_\mu \left(\frac{\partial \alpha^\mu}{\partial x^\mu} + z_\mu^i \frac{\partial \alpha^\mu}{\partial y^i}\right)\eta \in \mathcal{I}(\mathcal{C})$$

Therefore, if we define:

$$\tilde{F} = F + \sum_\mu \left(\frac{\partial \alpha^\mu}{\partial x^\mu} + z_\mu^i \frac{\partial \alpha^\mu}{\partial y^i}\right)$$

then we deduce

Proposition 4.1.9. *If a vector field ξ_Y on Y is a Noether symmetry then $\tilde{F} = 0$.*

Similarly,

Proposition 4.1.10. *(1) If ξ_Y is a $\pi_{XY}-$projectable Noether symmetry, then*

$$\pounds_{\xi_Y^{(1)}} \Theta_L = d\alpha$$

Furthermore,

$$\xi_Y^{(1)}(L) = -\sum_\mu \left(\frac{d\xi_Y^\mu}{dx^\mu} L + \frac{d\alpha^\mu}{dx^\mu} \right)$$

(2) If $\dim X = 1$ and ξ_Y is a Noether symmetry then

$$\pounds_{\xi_Y^{(1)}} \Theta_L = d\alpha$$

Proposition 4.1.11. *Noether symmetries form a Lie subalgebra of $\mathfrak{X}(Y)$, containing the Lie algebra of the symmetries of the Lagrangian.*

Proof.

$$\begin{aligned}\pounds_{[\xi_Y^{(1)},\zeta_Y^{(1)}]} \Theta_L &= \pounds_{\xi_Y^{(1)}} \pounds_{\zeta_Y^{(1)}} \Theta_L - \pounds_{\zeta_Y^{(1)}} \pounds_{\xi_Y^{(1)}} \Theta_L = \pounds_{\xi_Y^{(1)}} (d\alpha_2 + \theta_2) - \pounds_{\zeta_Y^{(1)}} (d\alpha_1 + \theta_1) \\ &= d(\pounds_{\xi_Y^{(1)}} \alpha_2 - \pounds_{\zeta_Y^{(1)}} \alpha_1) + \pounds_{\xi_Y^{(1)}} \theta_2 - \pounds_{\zeta_Y^{(1)}} \theta_1\end{aligned}$$

and $\pounds_{\xi_Y^{(1)}} \theta_2 - \pounds_{\zeta_Y^{(1)}} \theta_1 \in \mathcal{I}(\mathcal{C})$.

Finally, since $\xi_Y^{(1)}$ and $\zeta_Y^{(1)}$ are tangent to B, then $[\xi_Y^{(1)}, \zeta_Y^{(1)}]$ is also tangent to B. We also have that $\pounds_{[\xi_Y^{(1)},\zeta_Y^{(1)}]_{|B}} \Pi = \pounds_{\xi_Y^{(1)}{}_{|B}} \pounds_{\zeta_Y^{(1)}{}_{|B}} \Pi - \pounds_{\zeta_Y^{(1)}{}_{|B}} \pounds_{\xi_Y^{(1)}{}_{|B}} \Pi = 0$ on B, and that if α_1 and α_2 are exact on B, so is $\pounds_{\xi_Y^{(1)}{}_{|B}} \alpha_2 - \pounds_{\zeta_Y^{(1)}{}_{|B}} \alpha_1$. ∎

Theorem 4.1.12. ***(Noether's theorem)**. If ξ_Y is a Noether symmetry, then $\iota_{\xi_Y^{(1)}} \Theta_L - \alpha$ is a preserved quantity which is exact on the boundary.*

This Theorem is proved analogously as we did for the symmetries of the Lagrangian. We just remark a slight modification introduced to see that it is exact on the boundary:

$$i_B^*(\iota_{\xi_Y^{(1)}} \Theta_L - \alpha) = \iota_{\xi_Y^{(1)}{}_{|B}} d\Pi - d\beta = d(-\iota_{\xi_Y^{(1)}{}_{|B}} \Pi - \beta)$$

4.1.3 Cartan symmetries

Definition 4.1.13. *A vector field ξ_Z on Z is said to be a **Cartan symmetry** if its flow preserves the differential ideal $\mathcal{I}(\mathcal{C})$ (in other words, $\psi_{Z,t}^* \theta^i \in \mathcal{I}(\mathcal{C})$, or locally, $\pounds_{\xi_Z} \mathcal{I}(\mathcal{C}) \subseteq \mathcal{I}(\mathcal{C})$), and there exists an n-form α on Z (that must be exact $\alpha = d\beta$ on B) such that $\pounds_{\xi_Z} \Theta_L - d\alpha \in \mathcal{I}(\mathcal{C})$, ξ_Z is tangent to B and verifies $\pounds_{\xi_{Z|B}} \Pi = 0$.*

Proposition 4.1.14. *The Cartan symmetries form a subalgebra of $\mathfrak{X}(Z)$.*

We also have, with the same kind of proofs as above,

Theorem 4.1.15. ***(Noether's theorem)***. *If ξ_Z is a Cartan symmetry, then $\iota_{\xi_Z}\Theta_L - \alpha$ is a preserved quantity which is exact on the boundary.*

In addition we have the obvious relations between the different types of symmetries that we have exposed here. Every symmetry of the Lagrangian is a Noether symmetry. And the 1-jet prolongation of any Noether symmetry is a Cartan symmetry. Conversely, it is obvious that a projectable Cartan symmetry is the 1-jet prolongation of its projection, which is therefore a Noether symmetry.

And finally,

Proposition 4.1.16. *The flow of Cartan symmetries maps solutions of the Euler-Lagrange equations into solutions of the Euler-Lagrange equations.*

Proof. Let ψ_Z^t be the flow of a Cartan symmetry ξ_Z.

For any section $\phi \in \Gamma(\pi)$, we can locally define

$$\psi_{\phi,X}^t := \pi_{XZ} \circ \psi_Z^t \circ j^1\phi$$

$\psi_{\phi,X}^0 = Id_X$, whence for small $t's$, $\psi_{\phi,X}^t$ is a diffeomorphism. Analogously, we define

$$\psi_{\phi,Y}^t := \pi_{YZ} \circ \psi_Z^t \circ j^1\phi \circ \pi_{XY}$$

With the same argument we see that for small $t's$, $\psi_{\phi,Y}^t$ is as well a diffeomorphism.

If ϕ is a solution of the Euler-Lagrange equations, then the flow transforms ϕ into

$$\psi_{\phi,Y}^t \circ \phi \circ (\psi_{\phi,X}^t)^{-1}$$

Now, for $\theta \in \mathcal{C}$,

$$(\psi_Z^t \circ j^1\phi \circ (\psi_{\phi,X}^t)^{-1})^*\theta = ((\psi_{\phi,X}^t)^{-1})^*(j^1\phi)^*(\psi_Z^t)^*\theta = 0$$

as ξ_Z is a Cartan symmetry. This means that $\psi_Z^t \circ j^1\phi \circ (\psi_{\phi,X}^t)^{-1}$ is the 1-jet prolongation of its projection to Y,

$$\pi_{YZ} \circ \psi_Z^t \circ j^1\phi \circ (\psi_{\phi,X}^t)^{-1} = \psi_{\phi,Y}^t \circ \phi \circ (\psi_{\phi,X}^t)^{-1}$$

In other words,

$$j^1(\psi_{\phi,Y}^t \circ \phi \circ (\psi_{\phi,X}^t)^{-1}) = \psi_Z^t \circ j^1\phi \circ (\psi_{\phi,X}^t)^{-1}$$

Now we need to see that the transformed solution verifies the Euler-Lagrange equations. The preceding equation shows that, being the symmetry tangent to B, the boundary condition will be satisfied.

In addition, for every compact $(n+1)$-dimensional submanifold C, and every vertical vector field $\xi \in \mathcal{V}(\pi)$, which is annihilated at ∂C (and therefore, so does $\xi^{(1)}$),

$$\begin{aligned}
&\int_{(\psi_{\phi,X}^t)(C)} (j^1(\psi_{\phi,Y}^t \circ \phi \circ (\psi_{\phi,X}^t)^{-1}))^* \pounds_{\xi^{(1)}}\Theta_L \\
&= \int_{(\psi_{\phi,X}^t)(C)} (\psi_Z^t \circ j^1\phi \circ (\psi_{\phi,X}^t)^{-1})^* \pounds_{\xi^{(1)}}\Theta_L \\
&= \int_C (\psi_Z^t \circ j^1\phi)^* \pounds_{\xi^{(1)}}\Theta_L = \int_C (j^1\phi)^*(\psi_Z^t)^* \pounds_{\xi^{(1)}}\Theta_L
\end{aligned}$$

by means of a change of variable. The annihilation of the preceding expression is infinitesimally equivalent to the annihilation of

$$\int_C (j^1\phi)^* \pounds_{\xi_Z} \pounds_{\xi^{(1)}} \Theta_L = \int_C (j^1\phi)^* \pounds_{[\xi_Z, \xi^{(1)}]} \Theta_L - \int_C (j^1\phi)^* \pounds_{\xi^{(1)}} \pounds_{\xi_Z} \Theta_L$$

and we conclude by seeing that

$$\int_C (j^1\phi)^* \pounds_{[\xi_Z, \xi^{(1)}]} \Theta_L = -\int_C (j^1\phi)^* \iota_{[\xi_Z, \xi^{(1)}]} \Omega_L + \int_C (j^1\phi)^* d\iota_{[\xi_Z, \xi^{(1)}]} \Theta_L = 0$$

where the first term vanishes because ϕ is a solution of Euler-Lagrange equations, and second term vanishes due to the boundary condition on ξ; and

$$\begin{aligned} \int_C (j^1\phi)^* \pounds_{\xi^{(1)}} \pounds_{\xi_Z} \Theta_L &= \int_C (j^1\phi)^* \pounds_{\xi^{(1)}} (d\alpha + \theta) \\ &= \int_{\partial C} (j^1\phi)^* \pounds_{\xi^{(1)}} \alpha + \int_C (j^1\phi)^* \pounds_{\xi^{(1)}} \theta = 0 \end{aligned}$$

where the first term vanishes again by the boundary condition on ξ. ∎

4.2 Symmetries for the De Donder equations

In the discussion of the preceding section, we have used on Noether's theorem the fact that, for a solution ϕ of the Euler-Lagrange equations, we have

$$(j^1\phi)^*\theta = 0$$

for elements θ of the differential ideal generated by the contact forms. However, this result is no longer true for general solutions of the De Donder equations (more specifically, when the Lagrangian is not regular). In other words, if σ is a solution of the De Donder equations, then **not necessarily**

$$\sigma^*\theta = 0$$

for $\theta \in \mathcal{I}(\mathcal{C})$.

Therefore, our definition of symmetry must be more restrictive when we are dealing with solutions of the De Donder equations.

Definition 4.2.1. *A* ***preserved quantity for the De Donder equations*** *is a n-form α on Z such that $\sigma^* d\alpha = 0$ for every solution σ of the De Donder equations. If α is a preserved quantity, then $\tilde{\alpha}$ is called its associated* ***momentum****.*

Also note that if α is a preserved quantity and β is a closed n-form, then $\alpha+\beta$ is also a preserved quantity.

From equation (3.9) we can easily deduce the following.

Proposition 4.2.2. *Let* **h** *be a solution of the connection equation* (3.8)*. Then α is a preserved quantity for the De Donder equations if and only if $d\alpha$ is annihilated by any n horizontal tangent vectors at each point.*

Definition 4.2.3. *We have the following definitions of symmetries for the De Donder equations:*

(1) A vector field ξ_Y on Y is said to be a ***symmetry of the Lagrangian****, or a* ***variational symmetry*** *if*

$$\pounds_{\xi_Y^{(1)}}\Theta_L = 0$$

and $\xi_Y^{(1)}$ is tangent to B and verifies $\pounds_{\xi_Y^{(1)}|_B}\Pi = 0$.

(2) A vector field ξ_Y on Y is said to be a ***Noether symmetry****, or a* ***divergence symmetry*** *if*

$$\pounds_{\xi_Y^{(1)}|_B}\Theta_L = d\alpha$$

where α is the pullback to Z of a n-form on Y (that must be exact $\alpha = d\beta$ on B), $\xi_Y^{(1)}$ is tangent to B and verifies $\pounds_{\xi_Y^{(1)}{}_{|B}}\Pi = 0$.

(3) A vector field ξ_Z on Z is a ***Cartan symmetry*** *if*

$$\pounds_{\xi_Z}\Theta_L = d\alpha$$

where α is a n-form on Z (that is exact $\alpha = d\beta$ on B) (or, equivalently, if there is a n-form α' such that

$$\iota_{\xi_Z}\Omega_L = d\alpha'$$

we can put $\alpha' = \alpha + \iota_{\xi_Z}\Theta_L$), in other words, if ξ_Z is a Hamiltonian vector field, ξ_Z is tangent to B and verifies $\pounds_{\xi_Z|_B}\Pi = 0$.

There is an obvious relation between these types of symmetries, completely analogous to those between the symmetries for the Euler-Lagrange equations. Furthermore, a symmetry of the Lagrangian (resp. a Noether symmetry, Cartan symmetry) for the De Donder equations is a symmetry of the Lagrangian (resp. a Noether symmetry, Cartan symmetry) for the Euler-Lagrange equations.

Also note that a small computation shows that, in the case of a Noether symmetry, α must be necessarily the pullback of a semibasic n-form on Y, locally expressed by

$$\alpha(x, y, z) = \alpha^\mu(x, y)d^n x_\mu$$

Note from the definition of Cartan symmetry that using Cartan's formula we obtain

$$\iota_{\xi_Z}\Omega_L = d(\iota_{\xi_Z}\Theta_L + \alpha)$$

and therefore $d\iota_{\xi_Z}\Omega_L = 0$, from where

$$\pounds_{\xi_Z}\Omega_L = 0$$

Theorem 4.2.4. ***(Noether's theorem)*** *If ξ_Z is a Cartan symmetry, such that $\pounds_{\xi_Z}\Theta_L = d\alpha$, then $\iota_{\xi_Z}\Theta_L - \alpha$ is a preserved quantity which is exact on the boundary.*

For the proof, repeat that of the Noether's theorem for Euler-Lagrange equations, where

$$\pounds_{\xi_Z}\Theta_L - d\alpha$$

now vanishes by definition.

In the case of a regular Lagrangian, and $n > 0$, a computation similar to that in Proposition 3.3 for the expression $\pounds_{\xi_Z}\Omega_L = 0$ produces two terms

$$\frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu}\frac{\partial \xi^\kappa_X}{\partial y^k}\, dz^j_\nu \wedge dy^i \wedge dy^k \wedge d^{n-1}x_{\mu\kappa} = 0$$

and

$$\frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu}\frac{\partial \xi^\kappa_X}{\partial z^k_\lambda}\, dz^j_\nu \wedge dy^i \wedge dz^k_\lambda \wedge d^{n-1}x_{\mu\kappa} = 0,$$

which show that Cartan symmetries are automatically projectable. For this reason, and because projectable symmetries are typical of examples coming from Physics, we shall emphasize the role of vector fields which are projectable onto X.

Also note that the symmetries of Cartan preserve the horizontal subspaces for the connection formalism.

Proposition 4.2.5. *Assume that L is regular. If ξ_Z is a Cartan symmetry for the De Donder equations then ξ_Z preserves the horizontal distribution of any solution Γ satisfying* (3.8).

Proof. Since ξ_Z is a Cartan symmetry then $\pounds_{\xi_Z}\Omega_L = 0$. Therefore

$$\pounds_{\xi_Z} i_{\mathbf{h}}\Omega_L = 0$$

for any solution Γ of (3.8) with horizontal projector $\mathbf{h}$.

Hence,

$$\begin{aligned}
0 &= \left(\pounds_{\xi_Z} i_{\mathbf{h}}\Omega_L\right)(\xi_0, \xi_1, \ldots, \xi_n) \\
&= \xi_Z\left(i_{\mathbf{h}}\Omega_L(\xi_0, \xi_1, \ldots, \xi_n)\right) - \sum_{a=0}^{n} i_{\mathbf{h}}\Omega_L(\xi_0, \ldots, [\xi_Z, \xi_a], \ldots, \xi_n) \\
&= \sum_{b=0}^{n} \xi_Z\left(i_{\mathbf{h}(\xi_b)}\Omega_L(\xi_0, \ldots, \widehat{\xi_b}, \ldots, \xi_n)\right) \\
&\quad - \sum_{\substack{a, b = 0 \\ a \neq b}}^{n} (-1)^b i_{\mathbf{h}(\xi_b)}\Omega_L(\xi_0, \ldots, [\xi_Z, \xi_a], \ldots, \widehat{\xi_b}, \ldots, \xi_n) \\
&\quad - \sum_{b=0}^{n}(-1)^{b+1} i_{\mathbf{h}[\xi_Z, \xi_b]}\Omega_L(\xi_1, \ldots, \widehat{\xi_b}, \ldots, \xi_n) \\
&= \sum_{b=0}^{n}\left(\pounds_{\xi_Z} i_{\mathbf{h}(\xi_b)}\Omega_L\right)(\xi_0, \ldots, \widehat{\xi_b}, \ldots, \xi_n) - \sum_{b=0}^{n} i_{\mathbf{h}[\xi_Z, \xi_b]}\Omega_L(\xi_1, \ldots, \widehat{\xi_b}, \ldots, \xi_n)
\end{aligned}$$

First case $(n > 1)$**.** Since Ω_L is multisymplectic and $\pounds_{\xi_Z}\Omega_L = 0$ we deduce that

$$[\xi_Z, \mathbf{h}(\xi)] = \mathbf{h}[\xi_Z, \xi] \qquad \forall \xi \in \mathfrak{X}(Z),$$

which implies that the horizontal distribution associated to Γ is $\mathbf{h}$-invariant.

Second case ($n = 1$)**.** Taking $\xi = \frac{\partial}{\partial t}$ then $\mathbf{h}(\xi) = \xi_L$ is the Reeb vector field of the cosymplectic structure (dt, Ω_L) (being L regular). Moreover, with the notation $d_t = \frac{d}{dt}$, we have

$$\mathbf{h}[\xi_Z, \frac{\partial}{\partial t}] = -d_t\tau\xi_L, \quad dt([\xi_Z, \xi_L)] = d_t\tau$$

where $dt(\xi_Z) = \tau$. Therefore,

$$dt\left([\xi_Z, \xi_L] - \mathbf{h}[\xi_Z, \frac{\partial}{\partial t}]\right) = 0$$

Since (Ω_L, dt) is a cosymplectic structure, we deduce that

$$[\xi_Z, \xi_L] = \mathbf{h}[\xi_Z, \frac{\partial}{\partial t}] = -(d_t\tau)\xi_L, \tag{4.4}$$

which implies the invariance of the distribution $\langle\xi_L\rangle$. Observe that equation (4.4) is the classical definition of dynamical symmetry for time-dependent mechanical systems.

Moreover, the boundary conditions are fulfilled since ξ_Z preserves B. ∎

Finally, we shall justify that these symmetries are really symmetries, in the sense that they transform solutions of the De Donder equations into new solutions of the De Donder equations.

Theorem 4.2.6. *The flow of Cartan symmetries maps solutions of the De Donder equations into solutions of the De Donder equations.*

Proof. If σ is a solution of the De Donder equations, and $\xi \in \mathfrak{X}(Z)$ is a Cartan symmetry having flow ϕ_t, and we define for each t

$$\psi_t := \pi_{XZ} \circ \phi_t \circ \sigma$$

then we claim that $\phi_t \circ \sigma \circ \psi_t^{-1}$ is a solution of the De Donder equations. Being the symmetry tangent to B, the boundary condition will be automatically satisfied.

As $\psi_0 = Id$, ψ_t is a local diffeomorphism for small $t's$. Therefore, $\phi_t \circ \sigma \circ \psi_t^{-1}$ makes sense for small $t's$. In order to prove

$$(\phi_t \circ \sigma \circ \psi_t^{-1})^*(\iota_X\Omega_L) = (\psi_t^{-1})^*\sigma^*\phi_t^*(\iota_X\Omega_L) = 0$$

it suffices to see that

$$\sigma^*\phi_t^*(\iota_X\Omega_L) = 0$$

for t in a neighbourhood of 0. Now for $t = 0$, this equation reduces to the De Donder equations, therefore, it suffices to see that

$$\sigma^*(\pounds_\xi\iota_X\Omega_L) = 0$$

Using again the De Donder equations,

$$0 = \sigma^*(\iota_{[\xi,X]}\Omega_L) = \sigma^*(\pounds_\xi\iota_X\Omega_L) - \sigma^*(\iota_X\pounds_\xi\Omega_L)$$

But

$$\pounds_\xi\Omega_L = -d\pounds_\xi\Theta_L = -dd\alpha = 0$$

which completes the proof. ∎

4.3 Symmetries for singular Lagrangian systems

For the singular Lagrangian case (described in section 1.4.2), we consider diffeomorphisms $\Psi : Z \to Z$ which preserve the Poincaré-Cartan $(n+2)$-form Ω_L (i.e. $\Psi^*\Omega_L = \Omega_L$) and are π_{XZ}-projectable.

Proposition 4.3.1. *If the diffeomorphism $\Psi : Z \longrightarrow Z$ verifying $\Psi(B) \subseteq B$ preserves the $(n+2)$-form Ω_L and it is π_{XZ}-projectable, then it restricts to a diffeomorphism $\Psi_a : Z_a \longrightarrow Z_a$, where Z_a is the a-ry constraint submanifold. In particular, Ψ restricts to a diffeomorphism $\Psi_f : Z_f \longrightarrow Z_f$.*

Proof. If $z \in Z_1$ then there exists a linear mapping $\mathbf{h}_z : T_z Z \longrightarrow T_z Z$ such that $\mathbf{h}_z^2 = \mathbf{h}_z$, $\ker \mathbf{h}_z = (\mathcal{V}\pi_{XZ})_z$ and

$$i_{\mathbf{h}_z}\Omega_L(z) = n\Omega_L(z)$$

Consider the mapping

$$\mathbf{h}_{\Psi(z)} = T_z\Psi \circ \mathbf{h}_z \circ T_{\Psi(z)}\Psi^{-1}$$

It is clear that $\mathbf{h}_{\Psi(z)}$ is linear and $\mathbf{h}^2_{\Psi(z)} = \mathbf{h}_{\Psi(z)}$. Moreover, since Ψ is π_{XZ} projectable then $\ker \mathbf{h}_{\Psi(z)} = (\mathcal{V}\pi_{XZ})_{\Psi(z)}$. Finally, since $\Psi^*\Omega_L = \Omega_L$ then

$$i_{\mathbf{h}_{\Psi(z)}}\Omega_L(\Psi(z)) = n\Omega_L(\Psi(z))$$

Therefore, if $z \in Z_1$ then $\Psi(z) \in Z_1$. Thus, the proposition is true if $a = 1$. Now, suppose that the proposition is true for $a = l$ and we shall prove that it is also true for $a = l+1$.

Let z be a point in Z_{l+1} then there exists $\mathbf{h}_z : T_z Z \longrightarrow T_z Z_l$ linear such that $\mathbf{h}_z^2 = \mathbf{h}_z$, $\ker \mathbf{h}_z = (\mathcal{V}\pi_{XZ})_z$ and $i_{\mathbf{h}_z}\Omega_L(z) = n\Omega_L(z)$. Since $\Psi(Z_l) \subseteq Z_l$ and Ψ is a diffeomorphism, then $T_z\Psi(T_z Z_l) \subseteq T_{\Psi(z)} Z_l$. Thus, $\mathbf{h}_{\Psi(z)} : T_{\Psi(z)} Z \longrightarrow T_{\Psi(z)} Z_l$ and $\Psi(z) \in Z_{l+1}$. We also have that $\mathbf{h}(TB_f) \subseteq TB_f$. ∎

Corollary 4.3.2. *Let ξ_Z be a π_{XZ}-projectable vector field on X such that $\pounds_{\xi_Z}\Omega_L = 0$, then ξ_Z is tangent to Z_f*

Corollary 4.3.3. *A Cartan symmetry which is π_{XZ}-projectable is tangent to Z_f*

Proposition 4.3.1 motivates the introduction of a more general class of symmetries. If Z_f is the final constraint submanifold and $i_{f1} : Z_f \longrightarrow Z$ is the canonical immersion then we may consider the $(n+2)$-form $\Omega_{Z_f} = i^*_{f1}\Omega_L$, the $(n+1)$-form $\Theta_{Z_f} = i^*_{f1}\Theta_L$ and now analyze a new kind of symmetries.

Definition 4.3.4. *A Cartan symmetry for the system (Z_f, Ω_{Z_f}) is a vector field on Z_f tangent to $Z_f \cap B$ such that $\pounds_{\xi_{Z_f}}\Theta_{Z_f} = d\alpha_{Z_f}$, for some $\alpha_{Z_f} \in \Lambda^n Z_f$.*

If it is clear that if ξ_Z is a Cartan symmetry of the De Donder equations then using Proposition 4.3.1 we deduce that $X_{|Z_f}$ is a Cartan symmetry for the system (Z_f, Ω_{Z_f}).

4.4 Symmetries in the Hamiltonian formalism

We can define as well symmetries in the Hamiltonian formalism as we did for the De Donder equations, which are closely related by the equivalence theorem in the regular case.

Definition 4.4.1. *Given a Hamiltonian h, we have the following definitions of symmetries for the Hamilton equations:*

(1) A vector field ξ_Y on Y is said to be a ***Noether symmetry****, or a* ***divergence symmetry*** *if there exists a semibasic n-form on Y whose pullback α to $\Lambda_2^{n+1}Y$ (which is exact $\alpha = d\beta$ on B^*, and we shall use the same notation α for its pullback to Z^*) and verifies*

(a) The α-lift of ξ_Y to $\Lambda_2^{n+1}Y$ is projectable to a vector field $\xi_Y^{(1*)}$ on Z^*

(b) $\pounds_{\xi_Y^{(1*)}}\Theta_h = d\alpha$, $\xi_Y^{(1*)}$ is also tangent to B^* and verifies $\pounds_{\xi_Y^{(1*)}|_{B^*}}\Pi^* = 0$.

(2) A vector field ξ_Z on Z^* is a ***Cartan symmetry*** *if*

$$\pounds_{\xi_Z}\Theta_h = d\alpha$$

where α is an n-form on Z^* (which is exact $\alpha = d\beta$ on B^*), ξ_Z is also tangent to B^* and verifies $\pounds_{\xi_Z|_{B^*}}\Pi^* = 0$

As usual, Noether symmetries induce Cartan symmetries on Z^*.

Suppose that ξ is a vector field on Y, and α is the pullback to $\Lambda_2^{n+1}Y$ of a π_{XY}-semibasic form on Y. If the α-lift of ξ to $\Lambda_2^{n+1}Y$ projects onto a vector field on Z^* then ξ_Y is a Noether symmetry.

Theorem 4.4.2. ***(Noether's theorem)*** *If ξ_{Z^*} is a Cartan symmetry, such that $\pounds_{\xi_{Z^*}}\Theta_h = d\alpha$, then $\sigma^* d(\iota_{\xi_{Z^*}}\Theta_h - \alpha) = 0$ for every solution σ of the Hamilton equations. Furthermore, $\iota_{\xi_{Z^*}}\Theta_h - \alpha$ is exact on ∂Z^*.*

The proof of this theorem is entirely analogous to that of the Noether's theorem for De Donder equations.

Finally, we shall justify that these are real symmetries, in the sense that they transform solutions of the Hamilton equations into new solutions of the Hamilton equations.

Theorem 4.4.3. *The flow of Cartan symmetries maps solutions of the Hamilton equations into solutions of the Hamilton equations.*

The proof is identical to that given for the De Donder equations in Theorem 4.2.6.

4.5 The Legendre transformation and the symmetries

In this section we shall finally relate the symmetries of the De Donder equations to the symmetries of the Hamiltonian formalism, under the assumption of hyper-regularity. Within this section, we shall assume that L is a hyper-regular Lagrangian.

Proposition 4.5.1. *If ξ_Z is a Cartan symmetry for the De Donder equations, then $Tleg_L(\xi_Z)$ is a Cartan symmetry for the Hamilton equations. The converse is also true.*

Proof. If we just apply $(leg_L^{-1})^*$ to the Cartan condition for the De Donder equations we get the Cartan condition for the Hamilton equations:

$$0 = (leg_L^{-1})^*(\pounds_{\xi_Z}\Theta_L - d\alpha) = \pounds_{Tleg_L(\xi_Z)}(leg_L^{-1})^*\Theta_L - d\tilde{\alpha} = \pounds_{Tleg_L(\xi_Z)}\Theta_h - d\tilde{\alpha}.$$

where $leg_L^*\tilde{\alpha} = \alpha$. Boundary preservation is trivial, because of the way B^* has been defined, and the compatibility with the Legendre map. ∎

In a similar way we prove the following result

Lemma 4.5.2. *If ξ_Y is a Noether symmetry for the De Donder equation, such that $\pounds_{\xi_Y^{(1)}}\Theta_L - d\alpha$, then $TLeg_L(\xi_Y^{(1)})$ is the α-lift of ξ_Y.*

From which we can obtain

Proposition 4.5.3. *Every Noether symmetry for the De Donder equations is a Noether symmetry for the Hamilton equations. The converse is also true.*

Proof. We have that

$$Tleg_L(\xi_Y^{(1)}) = (T\mu \circ TLeg_L)(\xi_Y^{(1)})$$

therefore the α-lift of ξ_Y projects onto $Tleg_L(\xi_Y^{(1)})$ on Z^*, and as $\xi_Y^{(1)}$ is a Cartan symmetry, its image $Tleg_L(\xi_Y^{(1)})$ also verifies the Cartan condition (as $\pounds_{Tleg_L(\xi_Y^{(1)})}\Theta_h - d\tilde{\alpha} = \pounds_{Tleg_L(\xi_Y^{(1)})}(leg_L^{-1})^*\Theta_L - d(leg_L^{-1})^*\alpha = (leg_L^{-1})^*(\pounds_{\xi_Y^{(1)}}\Theta_L - d\alpha) = 0$). As usual, one can see that boundary conditions are trivially fulfilled. ∎

4.6 Symmetries in the Hamiltonian formalism for almost regular Lagrangians

On the final constraint submanifold M_f we have the following definition (see [110]):

Definition 4.6.1. *A **Cartan symmetry** for the system (M_f, Ω_{M_f}) is a vector field on M_f tangent to $M_f \cap B^*$ such that $\pounds_{\xi_{M_f}}\Theta_{M_f} = d\alpha_{M_f}$, for some $\alpha_{M_f} \in \Lambda^n M_f$.*

From where we deduce the following obvious proposition:

Proposition 4.6.2. *If ξ_{M_f} is a Cartan symmetry of (M_f, Ω_{M_f}) then any vector field ξ_{Z_f}, such that $Tleg_f(\xi_{Z_f}) = \xi_{M_f}$ is a Cartan symmetry of (Z_f, Ω_{Z_f}).*

4.7 Symmetries in the Cauchy data space

The symmetries of presymplectic systems were exhaustively studied in [107, 111] (see also [43, 69]). In [111] (Proposition 4.1 and Corollary 4.1) it was proved that for a general presymplectic system given by (M, ω, Λ), where M is a differentiable manifold, ω a closed 2-form and Λ a closed 1-form, a vector field ξ such that

$$\iota_\xi \omega = dG, \qquad \iota_\xi \Lambda = 0$$

where $G : M \to \mathbb{R}$, is a Cartan symmetry of the presymplectic system. If $\Lambda = 0$ (as it is our case), the condition reduces to $\iota_\xi \omega = dG$.

The following proposition explains the relationship between Cartan symmetries of the De Donder equations and Cartan symmetries for the presymplectic system $(\tilde{Z}, \widetilde{\Omega})$.

Proposition 4.7.1. *Let ξ_Z be a Cartan symmetry of the De Donder equations, that is, $\pounds_{\xi_Z} \Theta_L = d\alpha$. Then the induced vector field $\xi_{\tilde{Z}}$ in $\tilde{Z}$, defined by $\xi_{\tilde{Z}}(\gamma) = \xi_Z \circ \gamma$, is a Cartan symmetry of the presymplectic system $(\tilde{Z}, \widetilde{\Omega_L})$.*

Proof. If $\pounds_{\xi_Z} \Theta_L = d\alpha$, then

$$i_{\xi_Z} \Omega_L = d(\alpha - i_{\xi_Z} \Theta_L)$$

that is, ξ_Z is a Hamiltonian vector field for the n-form $\beta = \alpha - i_{\xi_Z} \Theta_L$. Then from Proposition 3.3.19 we have

$$i_{\widetilde{\xi_Z}} \widetilde{\Omega_L} = d\tilde{\beta}$$

which shows that $\widetilde{\xi_Z}$ is a Cartan symmetry for the presymplectic system $(\tilde{Z}, \widetilde{\Omega_L})$. ∎

4.8 Conservation of preserved quantities along solutions

Proposition 4.8.1. *If α is a preserved quantity, and $c_{\tilde{Z}}$ is a solution of the De Donder equations* (3.14) *such that its projection $c_{\tilde{X}}$ to $\tilde{X}$ splits X and α is exact on $B \subseteq \partial Z$ ($\alpha_{|B} = d\beta$), then $\tilde{\alpha} \circ c_{\tilde{Z}}$ is constant; in other words, the following function*

$$\int_M c_{\tilde{Z}}(t)^* \alpha - \int_{\partial M} c_{\tilde{Z}}(t)^* \beta$$

is constant with respect to t.

Proof. Pick $t_1 < t_2$ two real numbers in the domain of the solution curve, and let us denote by $M_1 = c_{\tilde{X}}(t_1)$ and $M_2 = c_{\tilde{X}}(t_2)$. As $c_{\tilde{X}}$ splits X, then we can consider the piece $U \subseteq X$ identified with $M \times [t_1, t_2]$, M_1 is identified with $M \times t_1$, M_2 is identified with $M \times t_2$, and let us denote by V the boundary piece corresponding to $\partial M \times [t_1, t_2]$. On view of (3.13), then

$$c_{\tilde{Z}}(t)^* d\alpha = 0 \quad \text{for all} \quad t$$

whence if we integrate and apply Stoke's theorem, we get

$$0 = \int_{M_2} c_{\tilde{Z}}(t)^* \alpha + \int_V c_{\tilde{Z}}(t)^* \alpha - \int_{M_1} c_{\tilde{Z}}(t)^* \alpha$$

If we put $\alpha = d\beta$ on B, then $0 = \partial\partial U = \partial M_2 + \partial V - \partial M_1$, whence applying Stoke's theorem again, we obtain

$$\int_V c_{\tilde{Z}}(t)^* \alpha = \int_{\partial V} c_{\tilde{Z}}(t)^* \beta = \int_{\partial M_1} c_{\tilde{Z}}(t)^* \beta - \int_{\partial M_2} c_{\tilde{Z}}(t)^* \beta.$$

∎

Corollary 4.8.2. *In particular, if ξ_Y is a symmetry of the Lagrangian for the De Donder equations, then the preceding formula can be applied to the preserved quantity $\iota_{\xi_Y^{(1)}} \Theta_L$ and we get that the following integral is preserved along solutions of the De Donder equations* (3.14) *such that its projection $c_{\tilde{X}}$ to $\tilde{X}$ splits X:*

$$\int_M c_{\tilde{Z}}(t)^* \iota_{\xi_Y^{(1)}} \Theta_L + \int_{\partial M} c_{\tilde{Z}}(t)^* \iota_{\xi_Y^{(1)}} \Pi$$

The preceding formula can also be found on [9].

4.9 Localizable symmetries. Second Noether's theorem

Definition 4.9.1. *A symmetry of the Lagrangian ξ_Y is said to be **localizable** when $\xi_Y^{(1)}$ vanishes on ∂Z and for every pair of open sets U and U' in X with disjoint closures, there exists another symmetry of the Lagrangian ζ_Y such that*

$$\xi_Y^{(1)} = \zeta_Y^{(1)} \qquad \text{on } \pi_{XZ}^{-1}(U)$$

and

$$\zeta_Y^{(1)} = 0 \qquad \text{on } \pi_{XZ}^{-1}(U') \cup \partial Z$$

Theorem 4.9.2. ***Second Noether Theorem.*** *If ξ_Y is a localizable symmetry, and $c_{\tilde{Z}}$ is a solution of De Donder equations* (3.14)*, then*

$$(\widetilde{\iota_{\xi_Y} \Theta_L})(c_{\tilde{Z}}(t)) = 0$$

for all t. Therefore, if $\alpha = \iota_\xi \Theta_L$ is the preserved quantity, then $\tilde{\alpha}$ is a constant of motion for the De Donder equations.

Proof. The First Noether theorem guarantees that the preceding application is constant. Pick t_0 in the domain of definition of $c_{\tilde{Z}}$, the space-time decomposition of X guarantees that, for $t \neq t_0$, we can find, using tubular neighbourhoods, two disjoint open sets U and U' with disjoint closures containing $Im(c_{\tilde{Z}}(t_0))$ and $Im(c_{\tilde{Z}}(t))$ respectively.

If ζ_Y is the Cartan symmetry whose existence guarantees the notion of localizable symmetry, respect to U and U', then

$$(\widetilde{\iota_{\xi_Y} \Theta_L})(c_{\tilde{Z}}(t_0)) = (\widetilde{\iota_{\zeta_Y} \Theta_L})(c_{\tilde{Z}}(t_0)) = (\widetilde{\iota_{\zeta_Y} \Theta_L})(c_{\tilde{Z}}(t)) = 0.$$

which completes the proof. ∎

4.10 Momentum map

In this section we are interested in considering groups of symmetries acting on the configuration space Y, which induce a lifted action into Z which preserves the Lagrangian form.

4.10.1 Action of a group

If G is a Lie group acting on Y, then the action of G on Y can be lifted to an action of G on Z, and the infinitesimal generator of the lifted action corresponds to the lift of the infinitesimal generator of the action, in other words,

$$\xi_Z = \xi_Y^{(1)}$$

Definition 4.10.1. *We shall say that a Lie group G acts as a* ***group of symmetries of the Lagrangian*** *if it defines an action on Y that projects onto a compatible action on X, which 1-jet prolongation preserves B, and if the flow ϕ_Z of ξ_Z verifies*

$$\phi_Z^* \mathcal{L} = \mathcal{L} \qquad \phi_Z^* \Pi = \Pi$$

The fact that the action is fibred implies that ξ_Y is a projectable vector field. Therefore, the condition $\phi_Z^* \mathcal{L} = \mathcal{L}$, infinitesimally expressed as

$$\pounds_{\xi_Z} \mathcal{L} = 0,$$

jointly with the following two direct consequences of the definition:

(i) ξ_Z is tangent to B

(ii) $\pounds_{(\xi_Z)_{|B}} \Pi = 0,$

states the fact that ξ_Y is a symmetry of the Lagrangian.

4.10.2 Momentum map

If we have a group of symmetries of the Lagrangian G acting on Y, we can make use of the Poincaré-Cartan $(n+1)$-form on Z to construct the analogous of the momentum map in Classical Mechanics.

Definition 4.10.2. *The* ***momentum map*** *is a mapping*

$$J : Z \longrightarrow \mathfrak{g}^* \otimes \Lambda^n Z$$

or alternatively,

$$J : Z \otimes \mathfrak{g} \longrightarrow \Lambda^n Z$$

defined by $J(z, \xi) := (\iota_{\xi_Z} \Theta_L)_z$.

Therefore, $J(\cdot, \xi)$ is a n-form, that we shall denote by J^ξ.

Remark 4.10.3. On B, since $\mathcal{L}_{(\xi_Z)_{|B}}\Pi = 0$ we have that $\iota_{(\xi_Z)_{|B}}d\Pi = -d\iota_{(\xi_Z)_{|B}}\Pi$, and therefore,

$$J(z,\xi) = (\iota_{\xi_Z}\Theta_L|_B)(z) = (\iota_{\xi_Z}d\Pi)(z) = -(d\iota_{\xi_Z}\Pi)(z)$$

Notice that J^ξ is a preserved quantity, and we called $\widetilde{J^\xi}$ its associated momentum.

Proposition 4.10.4.

$$dJ^\xi = \iota_{\xi_Z}\Omega_L$$

Proof. As ξ is projectable, $\mathcal{L}_{\xi_Z}\Theta_L = 0$ (by 4.1.5), whence

$$0 = \mathcal{L}_{\xi_Z}\Theta_L = \iota_{\xi_Z}d\Theta_L + d\iota_{\xi_Z}\Theta_L = -\iota_{\xi_Z}\Omega_L + dJ^\xi.$$

∎

In the next chapter, we shall study the case of a having fixed slicing of the manifold X.

4.10.3 Momentum map in Cauchy data spaces

If G is a Lie group acting on Y as symmetries of the Lagrangian, it induces an action on $\tilde{Z}$ defined pointwise on the image of every curve in $\tilde{Z}$.

For $\xi \in \mathfrak{g}$, the vector field $\xi_{\tilde{Z}}$ is precisely the vector on $\tilde{Z}$ induced by the vector field ξ_Z on Z. And since ξ_Z is a Cartan symmetry, so is $\xi_{\tilde{Z}}$.

In a similar manner, the presymplectic form $\widetilde{\Theta_L}$ induces a momentum map

$$\tilde{J} : \tilde{Z} \longrightarrow \mathfrak{g}^*$$

defined using its pairing (for $\xi \in \mathfrak{g}$)

$$\tilde{J}^\xi = \langle \tilde{J}, \xi \rangle : \tilde{Z} \longrightarrow \mathbb{R}$$

by

$$\tilde{J}^\xi := \iota_{\xi_{\tilde{Z}}}\widetilde{\Theta_L}$$

One immediately has that $\widetilde{J^\xi} = \tilde{J}^\xi$. As we know that ξ is a Cartan symmetry for the De Donder equations on Z, then $\tilde{\xi}$ is a Cartan symmetry for the De Donder equations on $\tilde{Z}$, thus $\tilde{J}^\xi$ is a preserved quantity for the presymplectic setting.

By repeating the arguments in proposition 4.10.4, we have:

Proposition 4.10.5.

$$d\tilde{J}^\xi = \iota_{\xi_{\tilde{Z}}}\widetilde{\Omega_L}$$

4.11 Examples

4.11.1 The Bosonic string

Let X be a 2-dimensional manifold, and (B, g) a $(d+1)$-dimensional space-time manifold endowed with a Lorentz metric g of signature $(-,+,\dots,+)$. A *bosonic string* is a map $\phi : X \longrightarrow B$ (see [6, 64]).

In the following, we shall follow the Polyakov approach to classical bosonic string theory. Let $S_2^{1,1}(X)$ be the bundle over X of symmetric covariant rank two tensors of Lorentz signature $(-,+)$ or $(1,1)$. We take the vector bundle $\pi : Y = X \times B \times S_2^{1,1}(X) \longrightarrow X$. Therefore, in this formulation, a field ψ is a section (ϕ, s) of the vector bundle $Y = X \times B \times S_2^{1,1}(X) \longrightarrow X$, where $\phi : X \longrightarrow X \times B$ is the bosonic string and s is a Lorentz metric on X.

Lagrangian description

We have that $Z = J^1(X \times B) \times_X J^1(S_2^{1,1}(X))$. Taking coordinates (x^μ), (y^i) and $(x^\mu, s_{\mu\zeta})$ on X, B and $S_2^{1,1}(X)$, then the fibred local coordinates on Z are $(x^\mu, y^i, s_{\zeta\xi}, y^i_\mu, s_{\zeta\xi\mu})$. In this system of local coordinates, the Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}\sqrt{-\det(s)}s^{\zeta\xi}g_{ij}y^i_\zeta y^j_\xi d^2x \ .$$

The Cartan 2-form is

$$\Theta_L = \sqrt{-\det(s)}\left(-s^{\mu\nu}g_{ij}y^j_\nu dy^i \wedge d^1x_\mu + \frac{1}{2}s^{\mu\nu}g_{ij}y^i_\mu y^j_\nu d^2x\right)$$

and the Cartan 3-form is

$$\begin{aligned}
\Omega_L &= dy^i \wedge d\left(-\sqrt{-\det(s)}s^{\zeta\xi}g_{ij}y^j_\xi\right) \wedge d^1x_\zeta \\
&\quad -d\left(\frac{1}{2}\sqrt{-\det(s)}s^{\zeta\xi}g_{ij}y^i_\zeta y^j_\xi\right) \wedge d^2x \\
&= -\frac{1}{2}\left(\frac{\partial\sqrt{-\det(s)}}{\partial s_{\rho\sigma}}s^{\zeta\xi}g_{ij}y^i_\zeta y^j_\xi - \sqrt{-\det(s)}s^{\zeta\rho}s^{\xi\sigma}g_{ij}y^i_\eta y^j_\xi\right) ds_{\rho\sigma} \wedge d^2x \\
&\quad -\frac{1}{2}\sqrt{-\det(s)}s^{\zeta\xi}\frac{\partial g_{ij}}{\partial y^k}y^i_\zeta y^j_\xi \, dy^k \wedge d^2x - \sqrt{-\det(s)}s^{\zeta\xi}g_{ij}y^i_\zeta \, dy^j_\xi \wedge d^2x \\
&\quad +\left(\frac{\partial\sqrt{-\det(s)}}{\partial h_{\rho\sigma}}s^{\zeta\xi}g_{ij}y^j_\xi - \sqrt{-\det(s)}s^{\zeta\rho}s^{\xi\sigma}g_{ij}y^j_\xi\right) ds_{\rho\sigma} \wedge dy^i \wedge d^1x_\zeta \\
&\quad +\sqrt{-\det(s)}s^{\zeta\xi}\frac{\partial g_{ij}}{\partial y^k}y^j_\xi \, dy^k \wedge dy^i \wedge d^1x_\zeta \\
&\quad +\sqrt{-\det(s)}s^{\zeta\xi}g_{ij} \, dy^j_\xi \wedge dy^i \wedge d^1x_\zeta .
\end{aligned}$$

If we solve the equation $i_{\mathbf{h}}\Omega_L = \Omega_L$, where

$$\mathbf{h} = dx^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \Gamma^i_\mu\frac{\partial}{\partial y^i} + \gamma_{\zeta\xi\mu}\frac{\partial}{\partial s_{\zeta\xi}} + \Gamma^i_{\zeta\mu}\frac{\partial}{\partial y^i_\zeta} + \gamma_{\zeta\xi\rho\mu}\frac{\partial}{\partial s_{\zeta\xi\rho}}\right) ,$$

we obtain that:

$$\begin{aligned}
\Gamma^i_\mu &= y^i_\mu \\
0 &= \frac{1}{2}\sqrt{-\det(s)}s^{\zeta\xi}\frac{\partial g_{ij}}{\partial y^k}y^i_\zeta y^j_\xi - \sqrt{-\det(s)}s^{\zeta\xi}\frac{\partial g_{kj}}{\partial y^i}y^i_\zeta y^j_\xi - \sqrt{-\det(s)}s^{\zeta\xi}g_{kj}\Gamma^j_{\xi\zeta} \\
&- \left(\frac{\partial\sqrt{-\det(s)}}{\partial s_{\rho\sigma}}s^{\zeta\xi}g_{kj}y^j_\xi - \sqrt{-\det(s)}s^{\zeta\rho}s^{\xi\sigma}g_{kj}y^j_\xi\right)\gamma_{\rho\sigma\zeta} \ ,
\end{aligned}$$

and the constraints given by the equations

$$\frac{\partial}{\partial s_{\rho\theta}}\left(\sqrt{-\det(s)}s^{\zeta\xi}\right)g_{ij}y^i_\zeta y^j_\xi = 0 \ .$$

The previous equation corresponds to the three following constraints

$$\begin{aligned}
\left[s^{\zeta 0}s^{\xi 0}(s^2_{01} - s_{00}s_{11}) + \frac{1}{2}s^{\zeta\xi}s_{11}\right]g_{ij}y^i_\zeta y^j_\xi &= 0 \\
\left[s^{\zeta 1}s^{\xi 1}(s^2_{01} - s_{00}s_{11}) + \frac{1}{2}s^{\zeta\xi}s_{00}\right]g_{ij}y^i_\zeta y^j_\xi &= 0 \\
\left[s^{\zeta 0}s^{\xi 1}(s^2_{01} - s_{00}s_{11}) - s^{\zeta\xi}s_{01}\right]g_{ij}y^i_\zeta y^j_\xi &= 0
\end{aligned}$$

which determine Z_2.

Hamiltonian description

The Legendre transformation is given by

$$Leg_L(x^\mu, y^i, s_{\zeta\xi}, y^i_\mu, s_{\zeta\xi\mu}) = (x^\mu, y^i, s_{\zeta\xi}, -\sqrt{-\det(s)}\, s^{\mu\zeta}g_{ij}y^j_\zeta, 0)$$

Therefore, the Lagrangian L is almost-regular and, moreover, $\tilde{M}_1 = \mathrm{Im}\ Leg_L \cong M_1 = leg_L(Z) \cong J^1(X\times B)\times_X S^{1,1}_2(X)$. Take now coordinates $(x^\mu, y^i, s_{\zeta\xi}, p^\mu_i)$ on M_1 and consider the mapping $s_1 : M_1 \to \tilde{M}_1$ given by

$$s_1(x^\mu, y^i, s_{\zeta\xi}, p^\mu_i) = (x^\mu, y^i, s_{\zeta\xi}, p = \frac{1}{2\sqrt{-\det(s)}}s_{\zeta\xi}g^{ij}p^\zeta_i p^\xi_j, p^\mu_i)$$

Then, we have

$$\Omega_{M_1} = -d\left(\frac{1}{2\sqrt{-\det(s)}}s_{\zeta\xi}g^{ij}p^\zeta_i p^\xi_j\right)\wedge d^2x + dy^i\wedge dp^\mu_i\wedge d^1x_\mu$$

and the Hamilton equations are given by $i_{\tilde{\mathbf{h}}}\Omega_{M_1} = \Omega_{M_1}$. Putting

$$\tilde{\mathbf{h}} = dx^\mu\otimes\left(\frac{\partial}{\partial x^\mu} + \tilde{\Gamma}^i_\mu\frac{\partial}{\partial y^i} + \tilde{\gamma}_{\zeta\xi\mu}\frac{\partial}{\partial s_{\zeta\xi}} + \tilde{\Gamma}^\zeta_{i\mu}\frac{\partial}{\partial p^\zeta_i}\right)$$

we obtain

$$\begin{aligned}\tilde{\Gamma}^i_\mu &= -\frac{1}{\sqrt{-\det(s)}} s_{\zeta\mu} g^{ij} p^\zeta_j \\ \tilde{\Gamma}^\mu_{i\mu} &= \frac{1}{2\sqrt{-\det(s)}} s_{\zeta\xi} \frac{\partial g^{ij}}{\partial y^k} p^i_\zeta p^j_\xi \,,\end{aligned}$$

and the secondary constraints

$$\frac{g^{ij}}{\sqrt{-\det(s)}}\left(\frac{1}{2\det(s)}\frac{\partial \det(s)}{\partial s_{\rho\sigma}} s_{\zeta\xi} p^\zeta_i p^\xi_j - p^\rho_i p^\sigma_j\right) = 0$$

determining M_2.

Symmetries

Let λ be an arbitrary function on X, and we denote also by λ its pullback to Y and Z.

Consider the following $\pi_{XY}-$projectable vector field on Y

$$\xi_Y := \lambda s_{\sigma\rho}\frac{\partial}{\partial s_{\sigma\rho}}$$

Its 1-jet prolongation is given by

$$\xi_Z := \xi_Y^{(1)} = \lambda s_{\sigma\rho}\frac{\partial}{\partial s_{\sigma\rho}} + \left(\frac{\partial\lambda}{\partial x^\mu} s_{\sigma\rho} + \lambda s_{\sigma\rho,\mu}\right)\frac{\partial}{\partial s_{\sigma\rho,\mu}}$$

We shall prove that ξ_Y is a symmetry of the Lagrangian. Note that

$$\begin{aligned}\pounds_{\xi_Z}\Theta_L = \pounds_{\xi_Y}(\sqrt{-\det(s)})&\left(-s^{\mu\nu} g_{ij} y^j_\nu dy^i \wedge d^1x_\mu + \frac{1}{2}s^{\mu\nu} g_{ij} y^i_\mu y^j_\nu d^2x\right) \\ &+ \sqrt{-\det(s)}\left(-\pounds_{\xi_Y}(s^{\mu\nu}) g_{ij} y^j_\nu dy^i \wedge d^1x_\mu + \frac{1}{2}\pounds_{\xi_Y}(s^{\mu\nu}) g_{ij} y^i_\mu y^j_\nu d^2x\right)\end{aligned}$$

And a direct computation shows that

$$\xi_Y(\sqrt{-\det(s)}) = \lambda\sqrt{-\det(s)}$$

and

$$\pounds_{\xi_Y}(s^{\mu\nu}) = -\lambda s^{\mu\nu}$$

Therefore, ξ_Y is a symmetry of the Lagrangian, and as the corresponding Cartan symmetry ξ_Z is π_{XZ}-projectable, then the symmetry projects onto the final constraint manifold.

The preserved quantity given by Noether's theorem is given by

$$J^{\xi_Y} = \sum_{\sigma,\rho,\mu} \lambda s_{\sigma\rho,\mu} s_{\sigma\rho} d^1x_\mu$$

Note that the vector field
$$\xi_Y = 2\lambda s_{\sigma\rho}\frac{\partial}{\partial s_{\sigma\rho}}$$
is the infinitesimal generator of the action of the group $N = \mathcal{C}S_2^{1,1}(X) \equiv \mathcal{F}(X, \mathbb{R}^+)$ of the conformal transformations of a metric of signature $(1,1)$ given by

$$\lambda(\phi, s) := (\phi, \lambda^2 s)$$

We have that
$$det(\lambda^2 s) = \lambda^4 det(s)$$
and
$$(\lambda^2 s)^{\mu\nu} = \lambda^{-2} s^{\mu\nu};$$
therefore, the action preserves the constraint equations.

In a similar manner, we can consider the action of $H = Diff(X)$ by

$$\eta(\phi, s) := (\phi \circ \eta^{-1}, (\eta^{-1})^* s)$$

or more generally, consider the semidirect product $G = H[N]$, where the action of elements $\eta \in H$ on elements $\lambda \in N$ is given by
$$\eta \cdot \lambda := \lambda \circ \eta^{-1}$$
The group G is a group of symmetries for Y, and the action is given by

$$(\eta, \lambda) \cdot (\phi, s) := (\phi \circ \eta^{-1}, \lambda^2 (\eta^{-1})^* s)$$

Symmetries on the Hamiltonian side

L not being regular, we cannot guarantee that ξ_Y is a Noether symmetry for the Hamiltonian side. However, an easy computation gives us that

$$\xi_Y^{(1)} = \lambda s_{\sigma\rho}\frac{\partial}{\partial s_{\sigma\rho}} - \lambda p^\mu_{\sigma\rho}\frac{\partial}{\partial p^\mu_{\sigma\rho}}$$

Thus,
$$\pounds_{\xi_Y^{(1)}}\Theta_L = \pounds_{\xi_Y^{(1)}}(p^\mu_{\sigma\rho} ds_{\sigma\rho} \wedge d^1 x_\mu) = p^\mu_{\sigma\rho} s_{\sigma\rho}\frac{\partial \lambda}{\partial x^\mu} d^2 x$$

However, note that in M_1 we have that $p^\mu_{\sigma\rho} = 0$, therefore ξ_Y restricts to a symmetry there of the form
$$\lambda s_{\sigma\rho}\frac{\partial}{\partial s_{\sigma\rho}}$$
Furthermore, this is the infinitesimal generator of the restriction of the lifted action on Z^*, and one easily deduces, on view of the form of the secondary constrain equation, that the action restricts as well to the secondary constraint submanifold.

More symmetries

In general, one can consider the invariance of the equations and the Lagrangian respect to diffeomorphisms of X. If η is one of such diffeomorphisms, then $\eta(\phi, s) = (\phi \circ \eta^{-1}, (\eta^{-1})^* s)$, having infinitesimal generator

$$-(s_{\sigma\mu}\frac{\partial \xi^\mu}{\partial x^\rho} + s_{\rho\mu}\frac{\partial \xi^\mu}{\partial x^\sigma})\frac{\partial}{\partial s_{\sigma\rho}} + \xi^\mu \frac{\partial}{\partial x^\mu}$$

where $\xi^\mu \frac{\partial}{\partial x^\mu}$ is the infinitesimal generator of η.

The most general situation arises when considering the semidirect product $H[N]$ of the group $H = Diff(X)$ and the group N of the positive real functions on X defined above, given by

$$\eta \cdot \lambda := \lambda \circ \eta^{-1}$$

The action is defined as follows

$$(\eta, \lambda)(\phi, s) = (\phi \circ \eta^{-1}, \lambda^2 (\eta^{-1})^* s),$$

and the infinitesimal generator is

$$2\lambda s_{\sigma\rho}\frac{\partial}{\partial s_{\sigma\rho}} - (s_{\sigma\mu}\frac{\partial \xi^\mu}{\partial x^\rho} + s_{\rho\mu}\frac{\partial \xi^\mu}{\partial x^\sigma})\frac{\partial}{\partial s_{\sigma\rho}} + \xi^\mu \frac{\partial}{\partial x^\mu}$$

This is proved to be a symmetry of the Lagrangian (see [64]), and the corresponding preserved quantity is

$$\frac{\partial L}{\partial y^i}(y^i_\mu \xi^\nu) + \frac{\partial L}{\partial s_{\sigma\rho}}(s_{\sigma\rho,\nu}\xi^\nu - 2\lambda s_{\sigma\rho} + s_{\sigma\nu}\frac{\partial \xi^\nu}{\partial x^\rho} + s_{\rho\nu}\frac{\partial \xi^\nu}{\partial x^\sigma}) = 0$$

for arbitrary $\lambda, \xi^\nu and \frac{\partial \xi^\nu}{\partial x^\rho}$, which gives in particular the equation $\partial L/\partial s_{\sigma\rho} = 0$, which is expanded into

$$\frac{1}{2}s^{\mu\nu} g_{ij} y^i_\mu y^j_\nu s_{\sigma\rho} = g_{ij} y^i_\sigma y^j_\rho$$

which amounts to say that h is a metric conformally equivalent to $\phi^* g$ and that the conformal factor is precisely $\frac{1}{2}s^{\mu\nu} g_{ij} y^i_\mu y^j_\nu$.

4.11.2 Klein-Gordon equation

For the Klein-Gordon equation, we set (X, g) be a Minkovski space, and $Y := X \times \mathbb{R}$, where $\pi : Y \longrightarrow X$ is the first canonical projection. A section ϕ of π can be identified with a smooth function on X, say $\varphi \in \mathcal{C}^\infty(X)$, where $y(j^1\phi(x)) = \varphi(x)$ and $z_\mu(j^1\phi(x)) = \dfrac{\partial \varphi}{\partial x^\mu}(x)$.

The chosen volume form will be $\eta := \sqrt{-det\, g}$.

Lagrangian setting

The Lagrangian function will be

$$L(x^\mu, y, z_\mu) := \frac{1}{2}\left(g^{\mu\nu} z_\mu z_\nu + m^2 y^2\right)$$

which is regular, as

$$\hat{p}^{\mu} = \frac{\partial L}{\partial z_{\mu}} = g^{\mu\nu} z_{\nu}$$

and thus the Hessian matrix is precisely $(g^{\mu\nu})$.

The Poincaré-Cartan 4-form is

$$\Theta_L = \sqrt{-det\, g} \left(g^{\mu\nu} z_{\mu} dy \wedge d^3 x_{\mu} - \frac{1}{2}(g^{\mu\nu} z_{\mu} z_{\nu} - m^2 y^2) d^4 x \right).$$

The boundary condition will be $B = 0$, that is, $\sigma(\partial X) = 0$.

And the Euler-Lagrange equations in terms of φ become

$$m^2 \varphi = g^{\mu\nu} \frac{\partial^2 \varphi}{\partial x^{\mu} \partial x^{\nu}}$$

that is, the Klein-Gordon equation.

Legendre transformation and Hamiltonian setting

We compute

$$\hat{p} = \frac{1}{2}(-g^{\mu\nu} z_{\mu} z_{\nu} + m^2 y^2) \sqrt{-det\, g}$$

Thus we can write the Hamiltonian

$$H(x^{\mu}, y, p^{\mu}) = \frac{1}{2}(g_{\mu\nu} p^{\mu} p^{\nu} + m^2 y^2),$$

and the Hamilton equation for φ corresponding to a section $\phi(x^{\mu}) = (x^{\mu}, \varphi(x^{\mu}), \varphi^{\mu}(x^{\mu}))$ become

$$\frac{\partial \varphi}{\partial x^{\mu}} = g_{\mu\nu} p^{\nu}$$
$$\sum_{\mu} \frac{\partial \varphi^{\mu}}{\partial x^{\mu}} = (\sqrt{-\det g})\, m^2 \varphi$$

Symmetries

Let ξ_X be a Killing vector field on X, with coordinates

$$\xi_X = \xi^{\mu} \frac{\partial}{\partial x^{\mu}}$$

Let us call ξ_Y the vector field ξ_X as seen in Y, that is, locally,

$$\xi_Y(x, t) := \xi^{\mu} \frac{\partial}{\partial x^{\mu}}$$

Its 1-jet prolongation ξ_Z is given by

$$\xi_Z = \xi^{\mu} \frac{\partial}{\partial x^{\mu}} - z_{\nu} \frac{d\xi^{\nu}}{dx^{\mu}} \frac{\partial}{\partial z_{\mu}}$$

These vector fields are symmetries of the Lagrangian, and the associated preserved quantity is written as

$$\left[-g^{\mu\nu} z_{\mu} \xi^{\gamma} dy \wedge d^2 x_{\nu\gamma} - \frac{\xi^{\gamma}}{2} \left(g^{\mu\nu} z_{\mu} z_{\nu} - m^2 y^2 \right) d^3 x_{\gamma} \right] \sqrt{-det\, g}$$

Cauchy surfaces

The general integral expression for the preserved quantity for an arbitrary Cauchy surface M and for sections $\phi(x^\mu) = (x^\mu, \varphi(x^\mu), \frac{\partial\varphi}{\partial x^\mu}(x^\mu))$ solutions of the Euler-Lagrange equations, and verifying the boundary condition, is given by

$$\int_M \sqrt{-det\, g}\left[g^{\mu\gamma}\frac{\partial\varphi}{\partial x^\mu}\xi^\nu\frac{\partial\varphi}{\partial x^\nu} + g^{\mu\nu}\frac{\partial\varphi}{\partial x^\mu}\xi^\gamma\frac{\partial\varphi}{\partial x^\nu} - \frac{\xi^\gamma}{2}\left(g^{\mu\nu}\frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\nu} - m^2\varphi^2\right)\right] d^3x_\gamma$$

In the particular case in which we have M to be a space-like Cauchy surface, g induces a positive definite metric g_M on M, and we have that the preserved quantity is expressed as

$$\int_M \sqrt{-det\, g}\left[\frac{\partial\varphi}{\partial x^0}\xi^\nu\frac{\partial\varphi}{\partial x^\nu} + g^{\mu\nu}\frac{\partial\varphi}{\partial x^\mu}\xi^0\frac{\partial\varphi}{\partial x^\nu} - \frac{\xi^0}{2}\left(g^{\mu\nu}\frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\nu} - m^2\varphi^2\right)\right] d^3x_0$$

Whenever ξ_X is space-like (that is, parallel to M), we obtain that the preserved quantity gets

$$\int_M \left[\frac{\partial\varphi}{\partial x^0}\frac{\partial\varphi}{\partial x^\nu}\xi^\nu\right] d^3x_0$$

which is the angular momentum whenever ξ_X is an infinitesimal rotation, and linear momentum whenever it is an infinitesimal translation.

For the contrary, if $\xi_X = \frac{\partial}{\partial x^0}$ we get

$$\frac{1}{2}\int_M \left[\frac{\partial\varphi}{\partial x^0}\frac{\partial\varphi}{\partial x^0} + g^{AB}\frac{\partial\varphi}{\partial x^A}\frac{\partial\varphi}{\partial x^B} + m^2\varphi^2\right] d^3x_0$$

which is the energy of the field φ on the Cauchy surface M.

CHAPTER 5

The theory of Cauchy surfaces

In this section we continue the development of the theory in the case in which we have a fixed slicing of X, that is, we shall assume that we have a diffeomorphism of X as $\Phi : I \times M \longrightarrow X$ that decomposes X into a temporal coordinate $I \subseteq \mathbb{R}$, where I is an interval of the real line, and a spatial component given by the compact manifold M. In what follows, X will be identified with $I \times M$ in our notation.

We shall also restrict the admissible embeddings to $\tilde{X} = \{\Phi(t, \cdot) | t \in I\}$ (where $\Phi(t, \cdot) : M \longrightarrow X$ is defined by $\Phi(t, \cdot)(u) = \Phi(t, u)$). As a consequence, we have the following identification $\tilde{X} \equiv I$, where t is identified with $\Phi(t, \cdot)$, that we shall denote by $c_{\tilde{X}}(t)$. In practice, no distinction between both manifolds will be made. Therefore, $\tilde{X}$ has a canonically defined vector field $\frac{\partial}{\partial t}$, and its corresponding dual one form dt.

We shall call ξ the vector field $\dot{c}_{\tilde{X}}(t)$ on $\tilde{X}$, seen as a vector field on X. In other words

$$\xi(t, u) = \dot{c}_{\tilde{X}}(t)(u) = \frac{d}{ds} \Phi(s, u)|_{s=t}$$

We can choose adapted coordinates at each point $(t, u) \in X$ as $(t, x_1, \ldots, x_n)$, where $t \in I$ and $(x_1, \ldots, x_n)$ are local coordinates of $u \in M$. In these adapted coordinates,

$$\left.\frac{\partial}{\partial t}\right|_{t_0} (u) = \left.\frac{\partial}{\partial t}\right|_{(t_0, u)} = \xi(t, u).$$

(remember again that a tangent vector in a point of $\tilde{X}$ is a vector field along that embedding). We shall also assume that M is compact and orientable, and has a volume form η_M for which the volume of M is 1, in other words,

$$\int_M \eta_M = 1$$

Under these circumstances, we can define a volume form in X by

$$\eta = dt \wedge \eta_M$$

This will be the volume form that we shall use as an ingredient for our approach to field theory (and thus, locally, $\eta_M = d^n x_0$).

Proposition 5.0.1. *We have that*

$$\tilde{\eta} = \pi^*_{\tilde{X}\tilde{Y}} dt \equiv dt$$

Proof. dt is characterised by the property

$$\langle dt | \frac{\partial}{\partial t} \rangle = 1$$

because $\dim \tilde{X}$=1.

First observe that $i_B^* \pi^*_{XZ} \eta = 0$, since $i^*_{\partial X} \eta = 0$ (η is an $(n+1)$-form), where $i_{\partial X} : \partial X \hookrightarrow X$ is the canonical inclusion.

Suppose that $\gamma = (j^1\phi) \circ t \in \tilde{Z}$, then we have

$$\langle \tilde{\eta} | \frac{\partial}{\partial t} \rangle = \int_M \gamma^* \iota_{\frac{\partial}{\partial t}} \eta = \int_M \gamma^* d^n x_0 = \int_M \tau^* d^n x_0 = \int_M \eta_M = 1,$$

since $\eta = dt \wedge \eta_M$. Moreover, it is clear that $\langle \tilde{\eta} | V \rangle = 0$ for all V such that $T_\gamma \pi_{\tilde{X}\tilde{Z}}(V) = 0$. ∎

This form $\tilde{\eta} = dt$ will be used as the natural volume form on $\tilde{X}$.

With this new ingredient ξ and these identifications, we shall revisit several of the concepts regarding Cauchy surfaces that we introduced, and we shall try to identify the structures of the manifolds involved and the properties of the geometrical objects involved for this particular case.

5.1 Field equations revisited

The first concept to revisit will be the field equations in the space of Cauchy data, starting with a very important property of the tangent vectors on the Cauchy data spaces.

Proposition 5.1.1. *The tangent spaces $T_\delta \tilde{Y}$ and $T_\gamma \tilde{Z}$ verify*

$$\begin{aligned} T_\delta \tilde{Y} \;\subseteq\; & \{W : M \longrightarrow TY \mid W \text{ covers } \delta \\ & \text{and for all } u \in M, T_{\delta(u)}\pi_{XY} W(u) = k\xi(\pi_{XY}(\delta(u))) \text{ with } k \text{ constant}\} \\ T_\gamma \tilde{Z} \;\subseteq\; & \{V : M \longrightarrow TZ \mid V \text{ covers } \gamma \\ & \text{and for all } u \in M, T_{\gamma(u)}\pi_{XZ} V(u) = k\xi(\pi_{XZ}(\gamma(u))) \text{ with } k \text{ constant}\} \end{aligned}$$

Proof. Let $W \in T_\delta \tilde{Y}$, and let $c : I \to \tilde{Y}$ be a smooth curve defined on some open interval about zero such that $c(0) = \delta$ and $\dot{c}(0) = W$. Moreover $\pi_{\tilde{X}\tilde{Y}} \circ c(t) \in \tilde{X}$, or in other words, $\pi_{\tilde{X}\tilde{Y}} \circ c(t)$ is identified with a curve $s : I \to \mathbb{R}$, where $t \mapsto s(t)$ is a local diffeomorphism. Taking derivatives it is easy to show that

$$T_{\delta(u)}\pi_{XY}W(u) = s'(0)\xi(s(0), u) = s'(0)\xi(\pi_{XY}(\delta(u)))$$

A similar proof is used for $\tilde{Z}$. ∎

Therefore, if $c_{\tilde{Z}}(t)$ is a curve on $\tilde{Z}$, we have that

$$T_{c_{\tilde{Z}}(t)(u)}\pi_{XZ}(\dot{c}_{\tilde{Z}}(t)(u)) = k(t)\xi_{\pi_{XZ}(c_{\tilde{Z}}(t)(u))} \; .$$

Proposition 5.1.2. *If $c_{\tilde{Z}}$ is a curve on $\tilde{Z}$ verifying*

$$\iota_{\dot{c}_{\tilde{Z}}(t)}\tilde{\eta} = 1$$

then $\dot{c}_{\tilde{Z}}$ projects in $\tilde{X}$ onto $\xi_{\pi_{XZ}(c_{\tilde{Z}}(t)(u))}$.

Proof. Suppose that $\gamma = (j^1\phi) \circ \tau$. We also know that

$$T_{c_{\tilde{Z}}(t)(u)}\pi_{XZ}(\dot{c}_{\tilde{Z}}(t)(u)) = k(t)\xi_{\pi_{XZ}(c_{\tilde{Z}}(t)(u))}$$

Now notice that

$$(j^1\phi)^*\iota_{\dot{c}_{\tilde{Z}}(t)(u)}(\pi^*_{XZ}\eta) = \iota_{T\pi\dot{c}_{\tilde{Z}}(t)(u)}\eta \; .$$

Therefore, we have that

$$\begin{aligned} 1 = \iota_{\dot{c}_{\tilde{Z}}(t)}\tilde{\eta}|_\gamma &= \int_M \gamma^*\iota_{\dot{c}_{\tilde{Z}}(t)(u)}\eta = \int_M \tau^*(j^1\phi)^*\iota_{\dot{c}_{\tilde{Z}}(t)(u)}\eta = \int_M \tau^*\iota_{T\pi\dot{c}_{\tilde{Z}}(t)(u)}\eta \\ &= \int_M \tau^*\iota_{k(t)\xi(t,u)}\eta = \int_M \tau^* k(t) d^n x_0 = k(t)\int_M \eta_M = k(t) \end{aligned}$$

and consequently, we deduce that $k(t) = 1$ for all t, as we wanted to show. ∎

Definition 5.1.3. *The* ***equations of motion*** *in $\tilde{Z}$ are defined as*

$$\iota_R\widetilde{\Omega_L} = 0, \qquad \iota_R\tilde{\eta} = 1 \tag{5.1}$$

The integral curves of such vector field define solutions of the De Donder equations in the infinite dimensional setting, which are compatible with our choice of temporal coordinate.

Notice that the such an integral curve $c(t)$ projects necessarily onto the identity curve $I(t) = t$ on $\tilde{X}$, whose derivative is precisely the slicing, and thus, these curves satisfy (3.14).

5.2 Structure of $\tilde{Z}$

5.2.1 Sections of $\tilde{\pi}$

Let $\phi \in \Gamma(\pi)$, then we can define $\varphi_\phi \in \Gamma(\tilde{\pi})$ as follows: $\varphi_\phi(t) = \phi \circ t$. Furthermore, any $\varphi \in \Gamma(\tilde{\pi})$ is of the form φ_ϕ for certain $\phi \in \Gamma(\pi)$ given by $\phi(t, u) = \varphi(t)(u)$, and in addition, $\varphi_\phi = \varphi_{\phi'}$ implies $\phi = \phi'$.

However, a section φ_ϕ of $\tilde{\pi}$ can also be seen as a curve on $\tilde{Y}$ (because $\tilde{X} \equiv I \subseteq \mathbb{R}$). By using integral curves, we can easily see that

$$\dot{\varphi}_\phi(t)(u) = T_{(t,u)}\phi(\xi)$$

Therefore, we deduce that

$$j^1\phi = j^1\phi' \Rightarrow j^1\varphi_\phi = j^1\varphi_{\phi'}$$

5.2.2 Structure of $\tilde{Z}$

In this subsection, we shall identify $\tilde{Z}$ with the first order jet bundle of the fibration $\tilde{\pi} : \tilde{Y} \longrightarrow \tilde{X}$. To do this, we define first a mapping

$$\begin{aligned} \Phi :& \tilde{Z} \longrightarrow J^1\tilde{\pi} \\ & (j^1\phi) \circ t \longmapsto j^1_t\varphi_\phi \end{aligned}$$

The function is well defined, because if $(j^1\phi) \circ t = (j^1\phi') \circ t'$ then $t = t'$, and using the results of the previous subsection, we get $j^1_t\varphi_\phi = j^1_t\varphi_{\phi'}$

We define an inverse

$$\begin{aligned} \Psi :& J^1\tilde{\pi} \longrightarrow \tilde{Z} \\ & j^1_t\varphi_\phi \longmapsto (j^1\phi) \circ t \end{aligned}$$

Both mappings are clearly inverse of each other, whence we get the desired identification.

Finally, recall that if a vector field ξ_Y on Y is tangent to the image of a section ϕ, then its 1-jet prolongation is characterised as the vector field ξ_Z on Z such that it projects onto ξ_Y and is tangent to the image of $j^1\phi$. Due to the identification above, if a vector field ξ_Y induces a vector field $\xi_{\tilde{Y}}$ on $\tilde{Y}$ then the induced vector field $\xi_{\tilde{Z}}$ by its 1-jet prolongation ξ_Z is the 1-jet prolongation of $\xi_{\tilde{Y}}$.

5.2.3 Vertical endomorphism

Recall the definition of vertical lift in the setting of jet manifolds (see [150]):

$$v : \pi^*T^*X \otimes_Y \mathcal{V}\pi \longrightarrow \mathcal{V}\pi_{YZ}$$

Given $f \in (\pi^*T^*X \otimes_Y \mathcal{V}\pi)|_{j^1\varphi}$ we consider the curve $\gamma_f : \mathbb{R} \longrightarrow \pi^{-1}_{YZ}(\pi_{YZ}(j^1_x\varphi))$ defined by

$$\gamma_f(t) = j^1_x\varphi + t\ f\ ,$$

for all $t \in \mathbb{R}$. Now put

$$f^v = \frac{d}{dt}_{|t=0}\gamma_f(t)$$

In our case, we can compute the vertical endomorphism for the 1-jet prolongation of $\tilde{\pi}$

$$S_{\tilde{\eta}}(V)_\gamma = [T_{j^1_t\varphi_\phi}\pi_{\tilde{Y}\tilde{Z}}(V) - T_t\varphi_\phi \circ T_{j^1_t\varphi_\phi}\pi_{\tilde{X}\tilde{Z}}(V)]^v$$

where $\gamma = j^1_t\varphi_\phi$.

5.3 Lagrangian formalism

In what follows and for expository simplicity, we shall assume that $\partial X = \emptyset$

A Lagrangian form in Z can be used to produce a Lagrangian form in $\tilde{Z}$. We now turn to describe it, and how structures on both settings relate to each other.

5.3.1 Lagrangian form

From Proposition 5.1, we know that $V \in T_\gamma \tilde{Z}$ is of the form

$$V(u) = \left(V^0 \frac{\partial}{\partial t} + V^i(u) \frac{\partial}{\partial y^i} + V^i_\mu(u) \frac{\partial}{\partial z^i_\mu} \right) \Big|_{\gamma(u)} .$$

The $(n+1)$-form $\mathcal{L}$ produces by integration a 1-form $\tilde{\mathcal{L}}$ on $\tilde{Z}$, using the formula (3.12)

$$\tilde{\mathcal{L}}(V)_\gamma = \int_M \gamma^* \iota_V L\eta = \int_M \gamma^* L V^0 d^n x_0 = V^0 \int_M L(\gamma(u))\eta_M$$

Defining

$$\tilde{L}(\gamma) = \int_M L(\gamma(u))\eta_M = \iota_\xi \tilde{\mathcal{L}},$$

for any ξ projecting onto $\frac{\partial}{\partial t}$, whence we have that

$$\tilde{\mathcal{L}} = \tilde{L}\tilde{\eta} = \tilde{L} dt$$

5.3.2 Poincaré-Cartan form

Lemma 5.3.1. *Let V be a tangent vector at γ of $\tilde{Z}$. In local coordinates, we have that*

(i) $\iota_{V(u)} S^*_\eta dL = \frac{\partial L}{\partial z^i_\mu}(V^i(u) - z^i_0(\gamma(u))V^0) d^n x_\mu + \mathcal{C}$

(ii) $\iota_{S_{\tilde{\eta}}(V)} d\tilde{L} = \displaystyle\int_M \frac{\partial L}{\partial z^i_0}(\gamma(u))(V^i(u) - z^i_{0|\gamma(u)} V^0(u))\eta_M$

for any ξ projecting onto $\frac{\partial}{\partial t}$, and where by $\mathcal{C}$ we denote the algebraic ideal of the contact forms (see Section 2.3.2).

Proof. First at all, note that

$$S_\eta = \theta^i \wedge d^n x_\nu \otimes \frac{\partial}{\partial z^i_\nu}$$

In particular, $\iota_{V(u)} \theta^i = V^i(u) - z^i_0(\gamma(u))V^0$. Therefore,

$$\iota_{V(u)}(S^*_\eta(dL)) = \iota_V \left(\frac{\partial L}{\partial z^i_\mu} \theta^i \wedge d^n x_\mu \right) = \frac{\partial L}{\partial z^i_\mu}(V^i(u) - z^i_0(\gamma(u))V^0) d^n x_\mu + \text{ Contact Terms}$$

which proves (i).

For (ii), recall the construction of the vertical endomorphism in subsection 5.2.3. Then, for a tangent vector V as above,

$$T_{(j^1\phi\circ t)}\pi_{\tilde{Y}\tilde{Z}}(V)(u) = T_{j^1_{(t,u)}\phi}\pi_{YZ}(V(u)) = V^0\frac{\partial}{\partial x^0} + V^i(u)\frac{\partial}{\partial y^i}$$
$$T_{(j^1\phi\circ t)}\pi_{\tilde{X}\tilde{Z}}(V)(u) = T_{j^1_{(t,u)}\phi}\pi_{XZ}(V(u)) = V^0\frac{\partial}{\partial x^0}$$
$$T_t\varphi_\phi(T_{(j^1\phi\circ t)}\pi_{\tilde{X}\tilde{Z}}(V)(u)) = T_{(t,u)}\phi(V^0\frac{\partial}{\partial x^0}) = V^0\frac{\partial}{\partial x^0} + V^0(u)z_0^i\frac{\partial}{\partial y^i}$$

Therefore,

$$S_{\tilde{\eta}}(V)(u)_{\gamma(u)} = (V^i(u) - V^0 z_0^i)\left.\frac{\partial}{\partial z_0^i}\right|_{\gamma(u)}$$

where $\gamma = j^1_t\varphi_\phi \equiv \jmath^1\phi\circ t$. Therefore we conclude (ii). ∎

So we conclude

Proposition 5.3.2. *For any tangent vector $V \in T_\gamma\tilde{Z}$, we have*

$$S^*_{\tilde{\eta}}d\tilde{L}(V)_\gamma = \int_M \gamma^*\iota_V(S^*_\eta dL)$$

Proof. First of all, remember that

$$S^*_{\tilde{\eta}}d\tilde{L}(V)_\gamma = \iota_{S_{\tilde{\eta}}(V)}d\tilde{L} = \int_M \gamma^*\iota_{S_{\tilde{\eta}}(V)}dL$$

Now since for any contact form θ^i, $\gamma^*\theta^i = t^*((j^1\phi)^*\theta^i) = 0$, integrating $\iota_V(S^*_\eta dL)$ according to the previous lemma, the only remaining term is precisely

$$\int_M \gamma^*\frac{\partial L}{\partial z_0^i}(V^i - z_0^i V^0)\eta_M$$

Therefore we obtain the result. ∎

Finally, we easily deduce that

Proposition 5.3.3.

$$\widetilde{\Theta_L} = \Theta_{\tilde{L}}$$

Proof.

$$\langle\widetilde{\Theta_L}|V\rangle_\gamma = \int_M \gamma^*\iota_V\Theta_L = \int_M \gamma^*\iota_V\mathcal{L} + \int_M \gamma^*\iota_V(S^*_\eta dL) = [\iota_V\tilde{\mathcal{L}} + \iota_V S^*_{\tilde{\eta}}d\tilde{L}]_\gamma = \langle\Theta_{\tilde{L}}|V\rangle_\gamma$$

as we wanted to prove. ∎

5.4 Compatible slicing

Definition 5.4.1. *A **compatible slicing** on Y is a complete vector field on Y which projects onto ξ on X.*

In presence of a compatible slicing, two arbitrary fibres $\tilde{\pi}^{-1}(t)$ and $\tilde{\pi}^{-1}(t')$ can be identified, where a section in one fiber is intertwined by the flows of ξ and the compatible slicing on Y to produce a section on another fiber. More precisely, the identification mapping is $\Upsilon_{t,t'} : \tilde{\pi}^{-1}(t) \longrightarrow \tilde{\pi}^{-1}(t')$, defined as $\Upsilon_{t,t'}(\phi \circ t) = \Phi^Y_{t-t'} \circ \phi \circ t'$ (where Φ^Y_t denotes the flow of the compatible slicing).

Being any two fibers diffeomorphic, we shall use Q to denote a typical fiber (which can be one of the fibers which is fixed in advance). Suppose that this fixed fiber is given by $t = t_0$.

In that case, we can identify $\tilde{Z}$ with $I \times TQ$, by a diffeomorphism that we shall call $\beta : \tilde{Z} \longrightarrow I \times TQ$, and $\widetilde{Z^*}$ with $I \times T^*Q$.

Fixed $t \in I$, if we denote by $X_t := Im(t) \equiv \{(t,u)|u \in M\}$, $Y_t := \pi^{-1}(X_t)$, $Z_t := \pi^{-1}_{XZ}(X_t)$, $p_2 : X \equiv I \times M \longrightarrow M$ the second canonical projection, $\pi_t := (\pi \circ p_2)|_{Y_t} : Y_t \longrightarrow U$, and $(\mathcal{V}\pi)_t := \mathcal{V}\pi|_{Y_t}$ the restriction of the vertical bundle to Y_t, then in [65] the authors give the following time-fixed version of the diffeomorphism above:

$$\beta_t : Z_t \longrightarrow J^1(\pi_t) \times (\mathcal{V}\pi)_t$$

given by $\beta_t(j^1\phi \circ t) = (j^1(\phi|_{X_t}), T\phi \circ \xi - \xi \circ \phi)$. In coordinates, we have that $\beta_t(j^1\phi \circ t)(u^A) = (t(u^A), \phi^i \circ t(u^A), \frac{\partial \phi^i}{\partial x^A}(t(u^A)), V = T\phi \circ \xi - \xi \circ \phi)$, from which we see the identification of $j^1\phi \circ t$ with $(t, \phi \circ t, V) \in I \times TQ$.

Thus, the space $T_{\delta_0}Q$ of tangent vectors to Q at $\delta_0 = \phi \circ t_0$, with ϕ a section of π_{XY}, consists of sections the form $V : M \longrightarrow \mathcal{V}Y$ where V covers δ_0. Thus, $V = V^i(u)\frac{\partial}{\partial y^i}$ along δ_0.

Similarly, the space $T^*_{\delta_0}Q$ is identified with sections of the form $\alpha : M \longrightarrow L(\mathcal{V}Y, \Lambda^n M)$ where α covers δ_0. Here $L(\mathcal{V}Y, \Lambda^n M)$ denotes the vector bundle over Y whose fiber at $\delta_0(u) \in Y$ is the set of linear maps from $\mathcal{V}_{\delta_0(u)}Y$ to $\Lambda^n_u M$ (see for example [151]). Therefore,

$$\alpha = p^0_i(u) dy^i \otimes \eta_M$$

The pairing of such elements $V \in T_{\delta_0}Q$, $\alpha \in T^*_{\delta_0}Q$ is defined as

$$\langle \alpha | V \rangle = \int_M \delta_0^* \iota_V(u)\alpha(u) = \int_M V^i(u) p^0_i(u) \eta_M$$

is obviously a non-degenerate pairing.

This natural pairing induces a canonical form θ_Q on T^*Q, by

$$\langle \theta_Q | W \rangle_\alpha = \langle \alpha | T\pi_Q(W) \rangle$$

and its corresponding weak symplectic form

$$\omega_Q := -d\theta_Q$$

Remark. In what follows, we shall assume that we have a compatible slicing on Y, and the decomposition of $\tilde{Y}$ as $I \times Q$. The compatible slicing corresponds with a vector field on $I \times Q$ projecting onto ξ, and that will be equally denoted by ξ when no confusion arises. In local coordinates, it corresponds to $\partial/\partial t$.

5.4.1 Action integral

Consider the compact submanifold $C = [t_0, t_1] \times M$ of X, for $[t_0, t_1] \subseteq I$. We can define, for a given section ϕ, the elements $\phi_0 := \phi \circ t_0$ and $\phi_1 := \phi \circ t_1$ of $\tilde{Y} \equiv I \times Q$, which have corresponding points (t_0, q_0) and (t_1, q_1) on $I \times Q$, respectively.

The curve $\hat{c}(t) = \phi \circ t$ on $\tilde{Y}$ is identified with a curve $t \longrightarrow (t, c(t))$ on $I \times Q$ which joins (t_0, q_0) to (t_1, q_1), which lifts to a curve $\tilde{c}(t)$ on $\tilde{Z}$.

From the identification $\tilde{Z} \equiv I \times TQ$ we define a non-autonomous lagrangian $\bar{L} : I \times TQ \longrightarrow \mathbb{R}$ and the action functional is rewritten as (using Fubini's theorem):

$$\begin{aligned} S(\phi) &= \int_{[t_0,t_1]\times M} j^1\phi^*\mathcal{L} = \int_{[t_0,t_1]\times M} \tilde{c}(t)(u)^*\mathcal{L} = \int_{[t_0,t_1]\times M} \tilde{c}(t)(u)^*(L)\eta \\ &= \int_{[t_0,t_1]\times M} L(\tilde{c}(t)(u))\eta_M \wedge dt = \int_{[t_0,t_1]} \left[\int_M L(\tilde{c}(t)(u))\eta_M\right] dt \\ &= \int_{[t_0,t_1]} \tilde{L}(\tilde{c}(t))dt = \int_{[t_0,t_1]} \tilde{c}^*\tilde{L}dt \end{aligned}$$

Since the flow of the 1-jet prolongation of a vector field on Y is the 1-jet prolongation of its flow, having a regular Lagrangian L, and using the equivalence theorem of the De Donder equations of the finite and infinite dimensional settings, we have that ϕ is a solution of the Euler-Lagrange equations if and only if the curve on TQ

$$\tilde{c}(t)(u) \equiv j^1\phi(t, u)$$

is an integral curve of the flow of an Euler-Lagrange vector field on TQ associated to $\tilde{L}$.

5.4.2 Instantaneous Poincaré-Cartan form

For a fixed $t \in I$, we denote

$$i_t : TQ \equiv \{t\} \times TQ \longrightarrow I \times TQ$$

the canonical inclusion, and we define the **instantaneous Poincaré-Cartan form** at time t as

$$\theta_{L,t} := i_t^*\Theta_{\tilde{L}}$$

From the expression of Θ_L, in local coordinates we have

$$\iota_V\theta_{L,t}(\gamma) = \int_M \gamma^*\hat{p}_i^0 V^i(u)\eta_M$$

5.5 Hamiltonian formalism

Given a Hamiltonian section $h : Z^* \longrightarrow \Lambda_2^{n+1}Y$, the form $\Theta_h = h^*\Theta$ on Z^* (where Θ is the canonical multisymplectic form on Λ_2^n), with coordinate expression

$$\Theta_h = -Hd^{n+1}x + p_i^\mu dy^i \wedge d^n x_\mu$$

induces a 1-form in $\tilde{Z}^* \equiv I \times T^*Q$, which in turn can be used to define a **Hamiltonian function** on $I \times T^*Q$ in the following manner:

$$\tilde{H}(\gamma) = -\iota_\xi(\widetilde{\Theta_h})_\gamma.$$

In local coordinates, the expression of H is

$$\begin{aligned}\tilde{H}(\gamma) &= -\int_M \gamma^* \iota_\xi \Theta_h = -\int_M \gamma^* \iota_\xi(-H d^{n+1}x + p_i^\mu dy^i \wedge d^n x_\mu) \\ &= -\int_M \gamma^*(-H\eta_M - p_i^\mu dy^i \wedge d^{n-1}x_{\mu 0})\end{aligned}$$

Therefore, if $V \in TQ$, an easy computation as above shows that

$$\begin{aligned}\iota_V \widetilde{\Theta}_{h\gamma} &= \int_M \gamma^* \iota_V \Theta_h = \int_M \gamma^* \iota_V(-H d^{n+1}x + p_i^\mu dy^i \wedge d^n x_\mu) \\ &= \int_M \gamma^*(-HV^0 d^n x_0 + p_i^\mu V^i d^n x_\mu - p_i^\mu V^0 dy^i \wedge d^{n-1}x_{\mu 0}) \\ &= -V^0 \tilde{H}(\gamma) + \int_M \gamma^* p_i^0 V^i d^n x_0,\end{aligned}$$

in other words,

$$\widetilde{\Theta_h} = -\tilde{H} dt + \theta_Q$$

The integral curves of such vector field define solutions of the Hamilton equations in the infinite dimensional setting, which are compatible with our choice of temporal coordinate.

With the same arguments as in the beginning of the chapter, the evolution field for the system $(\tilde{Z}^*, \widetilde{\Omega_h}, \tilde{H})$

$$\iota_{R'} \widetilde{\Omega_h} = 0, \; \iota_{R'} \tilde{\eta} = 1 \tag{5.2}$$

precisely coincide with the solutions of the Hamilton equations in terms of Ehresmann connections on Z^* seen in $\tilde{Z}^*$ (with a proof similar to that given in 3.3.13).

5.6 The Legendre transformation

The Legendre transformation $leg_L : Z \longrightarrow Z^*$ induces by composition a mapping $\widetilde{leg_L} : \tilde{Z} \longrightarrow \widetilde{Z^*}$ by $\widetilde{leg_L}(\gamma) = leg_L \circ \gamma$.

If we have a tangent vector $V \in T_qQ$, and $t \in I$ fixed, then to compute $\widetilde{leg_{Lt}} = \widetilde{leg_L}|_t$ we apply it to $U \in T_qQ$. If through the identification (t, q) corresponds to $\delta \in \tilde{Y}$, and (t, q, V) to γ, then a way of giving a tangent vector to $\delta(u)$ which projects onto $U(u)$, is to give a tangent vector W to V, which is such that $W(u)$ projects onto $U(u)$ for all $u \in M$. Therefore, we can compute

$$\langle \widetilde{leg_L}(\gamma) | U \rangle = \int_M \gamma^* \iota_W(u) \Theta_L = \iota_W \Theta_{\tilde{L}}$$

from where we deduce that $\widetilde{leg_L} = leg_{\tilde{L}}$.

From the fact that leg_L is fibred over Y we deduce that it is fibred over X, and hence $\widetilde{leg_L}$ is fibred over I, whence it makes sense to define an **instantaneous Legendre transformation** $\widetilde{leg_L}_t$, which from the expressions of the Poincaré-Cartan form, is identified with the fibers derivative on each fibre.

Therefore, $\widetilde{leg_L}^* \Theta_{\tilde{H}} = \Theta_{\tilde{L}}$ and if $\theta_{Q,t}$ is the pullback of the canonical form on $TQ \equiv \{t\} \times TQ$ to $I \times TQ$ through the inclusion of the fibre over t, we also have that $\widetilde{leg_L}^* \theta_{Q,t} = \theta_{L,t}$.

CHAPTER 6

Geometric numerical methods

Standard methods for simulating the motion of a dynamical system, generically called numerical integrators, usually take an initial condition and move it in the direction specified by the equation of motion or an appropriate discretisation. But these standard methods ignore all the geometric features of many dynamical systems, as for instance, for Hamiltonian systems we have preservation of the symplectic form, energy (in the autonomous case) and symmetries, if any. However, new methods have been recently developed, called geometric integrators, which are concerned with some of the extra features of geometric nature of the dynamical systems. Usually, these integrators can run in simulations, for longer times with lower spurious effects (for instance, bad energy behavior for conservative systems) than the traditional ones (the typical test example is the simulation of the solar system). Therefore, there is presently a great interest in geometric integration of differential equations as, for instance, symplectic integrators of Hamiltonian systems [14, 70, 145].

In this last chapter we explore numerical methods for computation of solutions to problems in Mechanics and Optimal Control Theory that are based on the Geometric structures that we have discussed in preceding sections. In particular, we concentrate on methods based on generating functions, compared to variational methods.

The chapter begins with a geometrical description of the non-holonomic mechanics (that is, mechanics subject to non-holonomic constraints) and the optimal control theory. It follows a section on generating functions, followed by a comparison between the discrete variational integrators, and the new integrators developed using generating functions. These later methods are extended and adapted to two examples, which are the non-holonomic mechanics and the optimal control theory. The chapter ends with some ideas about extending the methods to classical field theory.

6.1 Geometric formulation of non-holonomic systems

The theory of systems with non-holonomic constrains goes back to the XIX century. D'Alembert's or Lagrange-D'Alembert's principle of virtual work and Gauss principle of least constraint can be considered to be the first solutions to the analysis of systems with constraints, holonomic or not. After a period of decay, many authors have recently shown a new interest in that theory and also in its relation to the new developments in control theory, sub-Riemannian geometry, robotics, etc (see, for instance,[136]). The main characteristic of this period is that Geometry was used in a systematic way (see L.D. Fadeev and A.M. Vershik [156] as an advanced and fundamental reference, and also, [5, 10, 19, 20, 26, 32, 80, 81, 105, 109, 108, 121])

As it is well known, in most problems of particle mechanics, the motion of the particles is constrained in some way; this is the term used to denote the condition that some motions or configurations are not allowed. Our starting point is a configuration space Q, which is a n-dimensional differentiable manifold, with local coordinates q^i. General two-side or equality constraints are functions of the form $\phi^a(q^i, \dot{q}^i) = 0, 1 \leq a \leq m$, depending, in general, on configuration coordinates and their velocities. The various kinds of constraints we are concerned with will roughly come in two types: holonomic and non-holonomic, depending whether the constraint is derived from a constraint in the configuration space or not. Therefore, the dimension of the space of configurations is reduced by holonomic constraints but not by non-holonomic constraints. Thus, holonomic constraints permit a reduction in the number of coordinates of the configuration space needed to formulate a given problem (see [136]).

We shall restrict ourselves to the case of non-holonomic constraints, since the case of holonomic constraints, and, in particular, the construction of holonomic integrators, is well established in the existing literature. Geometrically, non-holonomic constraints are globally described by a submanifold $\tilde{M}$ of the velocity phase space TQ, the tangent bundle of the configuration space Q. If $\tilde{M}$ is a vector subbundle of TQ, we are dealing with linear constraints. We shall usually refer to $\tilde{M}$ as D and, in such case, the constraints are alternatively defined by a distribution D on the configuration space Q. If this distribution is integrable, we are precisely in the case of holonomic constraints. If $\tilde{M}$ is an affine bundle modelled on a vector bundle D, we are in the case of affine constraints. In what follows, we shall denote by D the constraint submanifold on the velocity phase space, no matter if they are determined by linear or nonlinear constraints.

Given the constraints, we need to specify the dynamical evolution of the system. The central concepts permitting the extension of mechanics from the Newtonian point of view to the Lagrangian one are the notions of virtual displacements and virtual work; these concepts were formulated in the developments of mechanics, in their application to statics. In nonholonomic dynamics, the procedure is given by Lagrange-D'Alembert's principle. We usually consider nonholonomic constraints of linear type, which are the constraints that we shall regard as natural in a mechanical sense (although the extension for general nonholonomic constraint will be straightforward).

We now come to the description of the constraint forces; for constraints of that type, Lagrange-D'Alembert's principle allows us to determine the set of possible values of the constraint forces only from the set of admissible kinematic states, that is, from the constraint manifold D determined by the vanishing of the nonholonomic constraints. Therefore, the dynamical properties of the system are mathematically described by a configuration space Q, by a Lagrangian function L and by a distribution determining the linear constraints D.

Consider a time independent Lagrangian system, with Lagrangian $L : TQ \to \mathbb{R}$, subject to non-holonomic constraints, defined by a submanifold D of the velocity phase space TQ. We shall assume that $\dim D = 2n - m$ and that D is locally described by the vanishing of m independent functions ϕ^a (the "constraint functions").

In geometrical terms, D'Alembert's principle (or Chetaev's principle for nonlinear constraints) implies that the constraint forces, regarded as 1-forms on TQ along D, take their values in the subbundle $S^*(TD^o)$ of T^*TQ, where TD^o denotes the annihilator of TD in T^*TQ. In an intrinsic way, the equations of motion can be written as (see [105, 108])

$$\begin{array}{rcl} (i_X\omega_L - dE_L)_{|D} & \in & S^*(TD^o)\ , \\ X_{|D} & \in & TD\ , \end{array}$$

where ω_L is the Poincaré-Cartan 2-form defined by L and $E_L = \Delta(L) - L$ is the energy function.

In what follows we shall also assume that the following *admissibility condition* holds

$$\dim TD^o = \dim S^*(TD^o)\,.$$

This essentially means that the matrix $(\partial\phi^a/\partial\dot{q}^i)$ has rank m everywhere.

We now turn to the Hamiltonian description of the non-holonomic system on the cotangent bundle T^*Q of Q [5, 81, 121]. The constraint functions on T^*Q become $\Psi^a = \phi^a \circ Leg^{-1}$, i.e.

$$\Psi^a(q^i, p_i) = \phi^a(q^i, \frac{\partial H}{\partial p_i})\,,$$

where the Hamiltonian $H : T^*Q \to \mathbb{R}$ is defined by $H = E_L \circ Leg^{-1}$.

The equations of motion for the non-holonomic system on T^*Q can now be written as follows

$$\left\{ \begin{array}{rcl} \dot{q}^i & = & \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i & = & -\dfrac{\partial H}{\partial q^i} - \lambda_a \dfrac{\partial \Psi^a}{\partial p_j}\mathcal{H}_{ji}\,, \end{array} \right. \tag{6.1}$$

together with the constraint equations $\Psi^a(q,p) = 0$, where $\mathcal{H}_{ij}$ are the components of the inverse of the matrix $(\mathcal{H}^{ij}) = (\partial^2 H/\partial p_i \partial p_j)$. Note that

$$(\frac{\partial \Psi^a}{\partial p_j}\mathcal{H}_{ji})(q,p) = (\frac{\partial \phi^a}{\partial \dot{q}^i} \circ Leg^{-1})(q,p).$$

The symplectic 2-form ω_L is related, via the Legendre map, with the canonical symplectic form ω_Q on T^*Q. Let M denote the image of the constraint submanifold D under the Legendre transformation, and let F be the distribution on T^*Q along M, whose annihilator is given by

$$F^o = TLeg(S^*(TD^o))\,.$$

Observe that F^o is locally generated by the m independent 1-forms

$$\mu^a = \frac{\partial \Psi^a}{\partial p_i}\mathcal{H}_{ij}dq^j \ , \ 1 \leq a \leq m\,.$$

Therefore, the "Hamilton equations" for the non-holonomic system can be rewritten in intrinsic form as

$$\begin{array}{rcl} (i_X\omega_Q - dH)_{|M} & \in & F^o \\ X_{|M} & \in & TM\,. \end{array} \tag{6.2}$$

Suppose in addition that the following *compatibility condition* $F^\perp \cap TM = \{0\}$ holds, where " $\perp$ " denotes the symplectic orthogonal with respect to ω_Q. Observe that, locally, this condition means that the matrix

$$(C^{ab}) = \left(\frac{\partial \Psi^a}{\partial p_i}\mathcal{H}_{ij}\frac{\partial \Psi^b}{\partial p_j}\right) \tag{6.3}$$

is regular. On the Lagrangian side, the compatibility condition is locally written as

$$\det(\tilde{C}^{ab}) = \det\left(\frac{\partial \phi^a}{\partial \dot{q}^i}W^{ij}\frac{\partial \phi^b}{\partial \dot{q}^j}\right) \neq 0\,, \tag{6.4}$$

where W^{ij} are the entries of the Hessian matrix $\left(\dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)_{1\leq i,j\leq n}$. The compatibility condition is not too restrictive, since, taking into account the admissibility assumption, it is trivially verified by the usual systems of mechanical type (i.e. with a Lagrangian of the form kinetic minus potential energy), where the $\mathcal{H}_{ij}$ represent the components of a Riemannian metric. The compatibility condition guarantees in particular the existence of a unique solution of the constrained equations of motion (6.2) which, henceforth, will be denoted by $X_{H,M}$ on the Hamiltonian side and $\xi_{L,D}$ on the Lagrangian side.

Moreover, if we denote by X_H the Hamiltonian vector field of H, i.e., $i_{X_H}\omega_Q = dH$ then, using the constraint functions, we may explicitely determine the Lagrange multipliers λ_a as

$$\lambda_a = -\mathcal{C}_{ab}X_H(\Psi^b)\,.$$

Next, writing the 1-form

$$\Lambda = -\mathcal{C}_{ab}X_H(\Psi^b)\frac{\partial \Psi^a}{\partial p_j}\mathcal{H}_{ji}dq^i\,,$$

the non-holonomic equations are equivalently rewritten as

$$\left\{\begin{array}{rcl} \dot{q}^i & = & \dfrac{\partial H}{\partial p_i}\,, \\ \dot{p}_i & = & -\dfrac{\partial H}{\partial q^i} - \Lambda_i\,, \end{array}\right. \tag{6.5}$$

for initial conditions $(q_0, p_0) \in M$ and $\Lambda = \Lambda_i\, dq^i$. We also denote by $\tilde{\Lambda} = Leg^*(\Lambda)$ the 1-form on TQ which represents the constraint force once the Lagrange multipliers have been determined.

Now, consider the flow $F_t : M \to M$, $t \in I \subseteq \mathbb{R}$ of the vector field $X_{H,M}$, solution of the non-holonomic problem.

Since (6.5) is geometrically rewritten as

$$i_{X_{H,M}}\omega_Q = dH + \Lambda\,,$$

($i_{\xi_{L,D}}\omega_L = dE_L + \tilde{\Lambda}$, with $\tilde{\Lambda} = Leg^*\Lambda$, on the Lagrangian side) then

$$\pounds_{X_{H,M}}\theta_Q = d(i_{X_{H,M}}\theta_Q - H) - \Lambda\,,$$

or, equivalently,

$$\pounds_{X_{H,M}}\theta_Q = d(L \circ Leg^{-1}) - \Lambda\,.$$

Now, from the dynamical definition of the Lie derivative, we have

$$F_t^*\left(\pounds_{X_{H,M}}\theta_Q\right) = \frac{d}{dt}\left(F_t^*\theta_Q\right)\,,$$

and integrating, we obtain the following expression, with some abuse of notation,

$$F_h^*\theta_Q - \theta_Q = d\left(\int_0^h L \circ \tilde{F}_t\, dt\right) - \int_0^h F_t^*\Lambda\,, \tag{6.6}$$

where $\tilde{F}_t$ is the flow of the vector field $\xi_{L,D}$.

6.2 Optimal control theory

It is well known that the dynamics of a large class of engineering and economic systems can be expressed as a set of differential equations

$$\dot{q}^A = \Gamma^A(t, q(t), u(t))\,,\ 1 \leq A \leq n\,, \tag{6.7}$$

where t is the time, q^A denote the state variables and u^a, $1 \leq a \leq m$, the control inputs to the system that must be specified. Given an initial condition of the state variables and given control inputs we completely know the trajectory of the state variables $q(t)$ (all functions are assumed to be at least C^2).

Given an initial condition, usually $q_0 = q(t_0)$, our aim is to find a C^2-piecewise smooth curve $\gamma(t) = (q(t), u(t))$, satisfying the control equations (6.7) and minimizing the functional

$$\mathcal{J}(\gamma) = \int_{t_0}^T L(t, q(t), u(t))\, dt + S(T, q(T))\,, \tag{6.8}$$

for some fixed and given final time $T \in \mathbb{R}^+$. The integral $\int_{t_0}^T L(t, q(t), u(t))\, dt$ depends on the time history (from t_0 to T) of the state variables and the control inputs, and $S(\cdot, q(\cdot))$ is a cost function based on the final time and the final state of the system.

In a global description, one assumes a fiber bundle structure $\pi : \mathbb{R} \times C \longrightarrow Q$, where Q is the configuration manifold with local coordinates (q^A) and C is the bundle of controls, with coordinates (q^A, u^a), $1 \leq A \leq n$, $1 \leq a \leq m$.

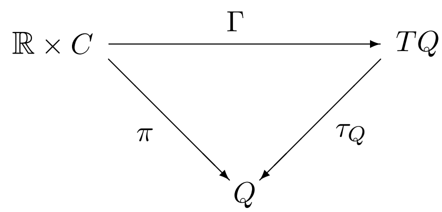

Figure 6.1: The Optimal Control equations

The time-dependent ordinary differential equations (6.7) on Q, depending on the parameters u, can be seen as a vector field Γ along the projection map π, that is, Γ is a smooth map $\Gamma : \mathbb{R} \times C \longrightarrow TQ$ such that the diagram 6.1 is commutative. This vector field is locally written as $\Gamma = \Gamma^A(t, q, u)\dfrac{\partial}{\partial q^A}$.

A necessary condition for the solutions of such problem are provided by Pontryaguin's maximum principle. If we construct the pseudo-Hamiltonian function:

$$H(t, q, p, u) = p_A\Gamma^A(t, q, u) - L(t, q, u) = p\Gamma(t, q, u) - L(t, q, u) \tag{6.9}$$

where p_A, $1 \leq A \leq n$, are now considered as Lagrange's multipliers, then a curve $\gamma : [t_0, T] \to C$, $\gamma(t) = (q(t), u(t))$ is an optimal trajectory if there exist functions $p_A(t)$, $1 \leq A \leq n$, such that they are solutions of the pseudo-Hamilton equations:

$$\begin{cases} \dot{q}^A(t) = \dfrac{\partial H}{\partial p_A}(t, q(t), p(t), u(t)) \\[2ex] \dot{p}_A(t) = -\dfrac{\partial H}{\partial q^A}(t, q(t), p(t), u(t)) \end{cases} \tag{6.10}$$

and we have

$$H(t, q(t), p(t), u(t)) = \max_v H(t, q(t), p(t), v), \quad t \in [t_0, T] \tag{6.11}$$

with transversality conditions

$$q(0) = q_0 \qquad \text{and} \qquad p_A(T) = -\frac{\partial S}{\partial q^A}(T, q(T))$$

Condition (6.11) is usually replaced by

$$\frac{\partial H}{\partial u^a} = 0, \quad 1 \leq a \leq m\ , \tag{6.12}$$

when we are looking for extremal trajectories.

It is well known that Pontryaguin's necessary conditions for extremality have a geometric interpretation in terms of presymplectic (or precosymplectic) Hamiltonian systems. The total space of the system will be $\mathbb{R} \times (T^*Q \times_Q C)$, with induced coordinates (t, q^A, p_A, u^a) .

Define Pontryaguin's Hamiltonian function $H : \mathbb{R} \times (T^*Q \times_Q C) \longrightarrow \mathbb{R}$ as follows

$$H(t, \alpha_q, u_q) = \langle \alpha_q, \Gamma(t, u_q)\rangle - L(t, u_q)$$

where $\alpha_q \in T_q^*Q$ and $(t, u_q) \in \pi^{-1}(q)$. Therefore, the coordinate expression of H is (6.9).

Let $\omega_Q = -d\theta_Q$ be the canonical symplectic form on T^*Q, where θ_Q is the Liouville form, and consider the canonical projection $\pi_1 : \mathbb{R} \times (T^*Q \times_Q C) \longrightarrow T^*Q$. Define the 2-form Ω_H on $\mathbb{R} \times (T^*Q \times_Q C)$ by $\Omega_H = \pi_1^*\omega_Q + dH \wedge dt$. Then, (Ω_H, dt) is a precosymplectic structure on $\mathbb{R} \times (T^*Q \times_Q C)$ (see [106]).

Eqs. (6.10) and (6.12) can be intrinsically written as

$$i_X\Omega_H = 0, \quad i_X dt = 1 \tag{6.13}$$

Since (Ω_H, dt) is a precosymplectic structure, in general, Eqs. (6.13) need not have a solution.

Applying the Dirac-Bergmann-Gotay-Nester algorithm [34, 63] to the precosymplectic system

$$(\mathbb{R} \times (T^*Q \times_Q C), dt, \Omega_H)$$

(see [25]) we obtain that Eqs. (6.12) correspond to the primary constraints for the precosymplectic system:

$$\phi^a = \frac{\partial H}{\partial u^a} = 0$$

Eqs. (6.13) have algebraic solution along the first constraint submanifold P_0 determined by the vanishing of the primary constraints. On the points of P_0 there is at least a pointwise solution of Eq. (6.13), but such solutions are not, in general, tangent to P_0. These points must be removed leaving a subset $P_1 \subset P_0$ (it is assumed than P_1 also is a submanifold). Thus, we have to restrict to a submanifold P_2 where the solutions of (6.13) are tangent to P_1. Proceeding further this way, we obtain a sequence of submanifolds

$$\cdots \hookrightarrow P_k \hookrightarrow \cdots \hookrightarrow P_2 \hookrightarrow P_1 \hookrightarrow P_0 \hookrightarrow \mathbb{R} \times (T^*Q \times_Q C)$$

If this algorithm stabilizes, i.e. there exists a positive integer $k \in \mathbb{N}$ such that $P_k = P_{k+1}$ and $\dim P_k \neq 0$, then we shall obtain a final submanifold $P_f = P_k$, on which a vector field X exists such that

$$(i_X\Omega_H)_{|P_f} = 0, \quad (i_X dt = 1)_{|P_f} \tag{6.14}$$

The constraints determining P_f are known, in the control literature, as **higher order conditions for optimality**.

If X is a solution of (6.14) then every arbitrary solution on P_f is of the form $X' = X + \xi$, where $\xi \in (\ker \Omega_H \cap \ker dt) \cap TP_f$.

Therefore, a necessary condition for optimality of the curve $\gamma : \mathbb{R} \to \mathbb{R} \times C$, $\gamma(t) = (t, q(t), u(t))$ is the existence of a lift $\tilde{\gamma}$ of γ to P_f such that $\tilde{\gamma}$ is an integral curve of a solution to Eqs. (6.14).

In the regular case, the final constraint manifold will be P_0 (that is, $P_0 = P_f$) and all the constraints are of the second kind following the classification of Dirac (see [106]). In such case, (P_0, η, Ω) is a cosymplectic manifold, where Ω and η denote the restrictions of Ω_H and dt, respectively, to the submanifold P_0. Denote also by ω and θ the restrictions of $\pi_1^*\omega_Q$ and $\pi_1^*\theta_Q$ to P_0.

For a singular case, see for instance [33].

The cosymplecticity of (P_0, Ω, η) is locally equivalent to the regularity of the matrix

$$\left(\frac{\partial^2 H}{\partial u^a \partial u^b}\right)_{1\leq a,b\leq m}$$

along P_0. The dynamical equations for the optimal control problem will become

$$i_X\Omega = 0, \quad i_X\eta = 1 \tag{6.15}$$

Taking coordinates (t, q^A, p_A) on P_0, then (6.15) are equivalent to:

$$\begin{cases} \dot{q}^A(t) = \dfrac{\partial H_{|P_0}}{\partial p_A}(t, q(t), p(t)) \\ \\ \dot{p}_A(t) = -\dfrac{\partial H_{|P_0}}{\partial q^A}(t, q(t), p(t)) \ , \end{cases} \tag{6.16}$$

where we have substituted in (6.10) the control variables u^a by its value $\bar{u}^a = f^a(t, q, p)$, applying the Implicit Function Theorem to the primary constraints $\phi^a = 0$. This also implies that we have a canonical projection from P_0 onto $\mathbb{R}$, say $\pi_0 : P_0 \to \mathbb{R}$.

In such case, there exists a unique solution X_{P_0} of Eq. (6.15):

$$i_{X_{P_0}}\Omega = 0, \quad i_{X_{P_0}}\eta = 1$$

and its flow preserves the cosymplectic structure given by Ω and η. That is, if we denote by F_h the flow of X_{P_0} then $F_h^*\Omega = \Omega$ and $F_h^*\eta = \eta$. In local coordinates, $F_h(t_0, q_0, p_0) = (t_0 + h, q_1, p_1)$. Denote by $F_h^{(2)}$ the mapping $F_h^{(2)}(t_0, q_0, p_0) = (q_1, p_1)$, and by $F_{t_1,t_0} : P_0^{t_0} \longrightarrow P_0^{t_1}$ the mapping defined by

$$F_{t_1,t_0}(q_0, p_0) = F_{t_1-t_0}^{(2)}(t_0, q_0, p_0) \ ,$$

where we write $P_0^t = (\pi_0)^{-1}(t)$, with $t \in \mathbb{R}$. Obviously, $F_{t_2,t_1} \circ F_{t_1,t_0} = F_{t_2,t_0}$ in their common domain.

The submanifolds P_0^t naturally inherit a symplectic structure ω_t by taking the restriction of ω to P_0^t. Similarly, denote by θ_t the restriction of θ to P_0^t, then $\omega_t = -d\theta_t$.

It is easy to deduce that, in such case, F_{t_1,t_0} is a symplectomorphism; that is, $F_{t_1,t_0}^*\omega_{t_1} = \omega_{t_0}$, noting that

$$\Omega = \omega + dH_{|P_0} \wedge \eta$$

This last remark will be interesting for constructing geometrical integrators for explicitly time-dependent optimal control systems.

6.3 Generating functions

Let (M_i, ω_i), $i = 0, 1$ be two exact symplectic manifolds (i.e. ω_i is symplectic and exact, $\omega_i = -d\theta_i$, $i = 0, 1$) and suppose that $g : M_0 \to M_1$ is a diffeomorphism. Denote by Graph(g) the graph of g,

$\text{Graph}(g) = \{(x_0, g(x_0)) \mid x_0 \in M_0\} \subseteq M_0 \times M_1$ and by $\pi_i : M_0 \times M_1 \to M_i$, $i = 0, 1$ the canonical projections. Consider the 1-form and 2-form on $M_0 \times M_1$ defined by

$$\begin{aligned} \Theta_{(1,0)} &= \pi_1^*\theta_1 - \pi_0^*\theta_0 \\ \Omega_{(1,0)} &= \pi_1^*\omega_1 - \pi_0^*\omega_0 = -d\Theta_{(1,0)} \end{aligned}$$

As it is well known, $\Omega_{(1,0)}$ is a symplectic form.

Let $i_g : \text{Graph}(g) \hookrightarrow M_0 \times M_1$ be the inclusion map, then

$$i_g^*\Omega_{(1,0)} = (\pi_{0|\text{Graph}(g)})^*(g^*\omega_1 - \omega_0)$$

Using this equality, it is clear that g is a symplectomorphism if and only if $i_g^*\Omega_{(1,0)} = 0$, that is, if $\text{Graph}(g)$ is a Lagrangian submanifold of $(M_0 \times M_1, \Omega_{(1,0)})$.

Now, if g is a symplectomorphism, we have that

$$i_g^*\Omega_{(1,0)} = -di_g^*\Theta_{(1,0)} = 0$$

and, therefore, at least locally, there exists a function $S : \text{Graph}\,(g) \to \mathbb{R}$ such that

$$i_g^*\Theta_{(1,0)} = dS \tag{6.17}$$

Let (q_0, p_0) and (q_1, p_1) be Darboux coordinates in M_0 and M_1, respectively, such that $\theta_0 = p_0\, dq_0$ and $\theta_1 = p_1\, dq_1$. Since $\text{Graph}(g)$ is diffeomorphic to M_0, we can take (q_0, p_0) as natural coordinates in $\text{Graph}(g)$. Since (q_0, p_0, q_1, p_1) are coordinates in $M_0 \times M_1$, then, along $\text{Graph}(g)$, we have $q_1 = q_1(q_0, p_0)$, $p_1 = p_1(q_0, p_0)$ and

$$p_1\, dq_1 - p_0 dq_0 = dS(q_0, p_0)$$

6.3.1 Generating functions of the first kind

Assume that, in a neighborhood of some point $x \in \text{Graph}(g)$, we can change this system of coordinates to new independent coordinates (q_0, q_1) (the local condition is that $\det(\partial q_1/\partial p_0) \neq 0$). In such case, the function S can be expressed locally as $S = S(q_0, p_0) = S_1(q_0, q_1)$.

Definition 6.3.1. *The function $S_1(q_0, q_1)$ will be called a* **generating function of the first kind** *of the symplectomorphism g.*

From (6.17) we deduce that

$$\begin{cases} p_0 = -\dfrac{\partial S_1}{\partial q_0} \\[2ex] p_1 = \dfrac{\partial S_1}{\partial q_1} \end{cases} \tag{6.18}$$

(see Fig. 6.2).

Conversely, if $S_1(q_0, q_1)$ is a function such that $\det\left(\frac{\partial^2 S_1}{\partial q_0 \partial q_1}\right) \neq 0$ then S_1 is a generating function of some symplectomorphism g implicitly determined by Eqs. (6.18), $g(q_0, p_0) = (q_1, p_1)$ (see [3]).

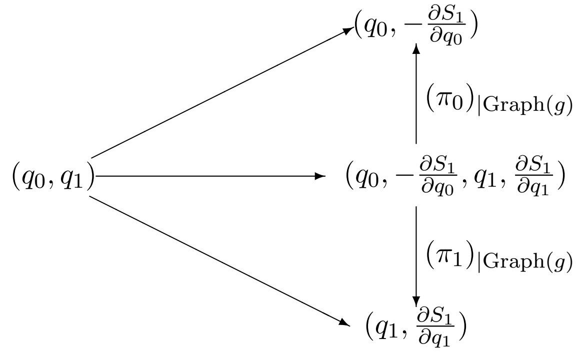

Figure 6.2: A generating function of first kind

Now suppose that M is a fiber bundle over the real line $\mathbb{R}$, $\pi : M \to \mathbb{R}$, and $M_t = \pi^{-1}(t)$ are the fibers, where each fiber M_t is equipped with a symplectic form ω_t. Let $g_{(s,t)} : M_t \to M_s$ be a two-parameter family of symplectomorphisms satisfying

$$g_{(t_2,t_1)} \circ g_{(t_1,t_0)} = g_{(t_2,t_0)}$$

Next, we shall show how this composition law can be translated in terms of their respective generating functions. Moreover, the following results will give a geometric interpretation of the Discrete Euler-Lagrange equations [129, 131]. In what follows it will be assumed that, for each interval $[s,t]$ the Lagrangian submanifold Graph $g_{(s,t)}$ has a generating function of the first kind $S_1^{(t,s)}$. The next theorem shows the relationship between these generating functions for different intervals of time.

Theorem 6.3.2. *Let $S_1^{(t_N,t_0)}$ be a function defined by*

$$S_1^{(t_N,t_0)}(q_0, q_N) = \sum_{k=0}^{N-1} S_1^{(t_{k+1},t_k)}(q_k, q_{k+1})$$

where $q_k \in M_{t_k}$, $1 \le k \le N-1$, are stationary points of the right-hand side, that is

$$0 \;=\; D_2 S_1^{(t_k,t_{k-1})}(q_{k-1}, q_k) + D_1 S_1^{(t_{k+1},t_k)}(q_k, q_{k+1}), \quad 1 \le k \le N-1.$$

If $S_1^{(t_k,t_{k-1})}$ are generating functions of the first kind of $g_{(t_k,t_{k-1})}$, then $S_1^{(t_N,t_0)}$ is a generating function of the first kind of $g_{(t_N,t_0)} : M_{t_0} \to M_{t_N}$.

Proof. Recursively, it is suffices to give the proof for $N = 2$. Consider the equation:

$$\frac{\partial}{\partial x}\left(S_1^{(t_1,t_0)}(q_0, x) + S_1^{(t_2,t_1)}(x, q_2)\right) = 0 \tag{6.19}$$

For fixed (q_0, q_2), there exist unique coordinates p_0, p_2 such that

$$(q_0, p_0, q_2, p_2) \in \text{Graph } g_{(t_2,t_0)}.$$

Therefore, from equations (6.18) we know that Equation (6.19) has at lest a solution q_1 determined by $g_{(t_1,t_0)}(q_0, p_0) = (q_1, p_1)$ or, alternatively, by $g^{-1}_{(t_2,t_1)}(q_2, p_2) = (q_1, p_1)$. Moreover, for fixed t_1, the solution of Equation (6.19) is unique, since $g_{(s,t)} : M_t \to M_s$ is a two-parameter family of symplectomorphisms.

Therefore, define

$$S_1^{(t_2,t_0)}(q_0,q_2) = S_1^{(t_1,t_0)}(q_0,q_1) + S_1^{(t_2,t_1)}(q_1,q_2)$$

Now,

$$\begin{aligned}
d(S_1^{(t_2,t_0)}(q_0,q_2)) &= d(S_1^{(t_1,t_0)}(q_0,q_1) + S_1^{(t_2,t_1)}(q_1,q_2)) \\
&= \left(\frac{\partial S_1^{(t_1,t_0)}}{\partial q_0}(q_0,q_1) + \frac{\partial S_1^{(t_1,t_0)}}{\partial q_1}(q_0,q_1)\frac{\partial q_1}{\partial q_0} + \frac{\partial S_1^{(t_2,t_1)}}{\partial q_0}(q_1,q_2)\frac{\partial q_1}{\partial q_0} \right) dq_0 \\
&+ \left(\frac{\partial S_1^{(t_2,t_1)}}{\partial q_2}(q_1,q_2) + \frac{\partial S_1^{(t_1,t_0)}}{\partial q_1}(q_0,q_1)\frac{\partial q_1}{\partial q_2} + \frac{\partial S_1^{(t_2,t_1)}}{\partial q_0}(q_1,q_2)\frac{\partial q_1}{\partial q_2} \right) dq_2
\end{aligned}$$

and applying the stationary condition and Equations (6.18) we deduce that

$$\begin{aligned}
d(S_1^{(t_2,t_0)}(q_0,q_2)) &= \frac{\partial S_1^{(t_1,t_0)}}{\partial q_0}(q_0,q_1)\, dq_0 + \frac{\partial S_1^{(t_2,t_1)}}{\partial q_2}(q_1,q_2)\, dq_2 \\
&= p_2\, dq_2 - p_0\, dq_0
\end{aligned}$$

∎

6.3.2 The action as a generating function

At this point, we are in conditions to bring this procedure to the limit when the number of subintervals increases to infinity. Consider as its continuous counterpart a cosymplectic manifold (M,ω,η), where M is still a fiber bundle over $\mathbb{R}$ ($\pi_{\mathbb{R}} : M \to \mathbb{R}$) and $\eta = \pi_{\mathbb{R}}^*(dt)$. Denote by $M_t = \pi_{\mathbb{R}}^{-1}(t)$, $t \in \mathbb{R}$. Take a Hamiltonian function $H : M \to \mathbb{R}$ and its Reeb vector field Y_H given by

$$i_{Y_H}(\omega + dH \wedge dt) = 0 \qquad \text{and} \qquad i_{Y_H}\eta = 1$$

Let $F_{(t,s)} : M_s \to M_t$ be the two-parameter family of symplectomorphisms generated by Y_H (see section 6.2) and consider as symplectic form on each fiber M_t the restriction of ω to this fiber.

We shall give a characterisation of the generating functions of the first kind associated to $F_{(t,s)}$ for t close enough to s. For doing that, consider Darboux coordinates (t, q^A, p_A) on M and assume the regularity condition $\det\left(\frac{\partial^2 H}{\partial p_A \partial p_B}\right) \neq 0$. Observe that this last condition implies that if q_1 is near to q_0 and $t_1 - t_0$ is sufficiently small then there exists a unique solution $t \to (t, q(t), p(t))$ of the Hamilton equations such that $q(t_0) = q_0$ and $q(t_1) = q_1$. Thus,

Proposition 6.3.3. *A generating function of the first kind for $F_{(t_1,t_0)}$ is given by*

$$S_1^{(t_1,t_0)}(q_0,q_1) = \int_{t_0}^{t_1} \left(p(t)\dot{q}(t) - H(t,q(t),p(t))\right) dt$$

where $t \to (t, q(t), p(t))$ is an integral curve of the Hamilton equations such that $q(t_0) = q_0$ and $q(t_1) = q_1$.

Proof. We only use Hamilton equations and integration by parts:

$$\begin{aligned}
\frac{\partial S_1^{(t_1,t_0)}}{\partial q_0}(q_0,q_1) &= \int_{t_0}^{t_1}\left(\frac{\partial p}{\partial q_0}\dot{q} + p\frac{\partial \dot{q}}{\partial q_0} - \frac{\partial H}{\partial q}\frac{\partial q}{\partial q_0} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial q_0}\right) dt \\
&= \int_{t_0}^{t_1}\left(p\frac{\partial \dot{q}}{\partial q_0} + \dot{p}\frac{\partial q}{\partial q_0}\right) dt \\
&= -p_0 + p_1\frac{\partial q_1}{\partial q_0} = -p_0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial S_1^{(t_1,t_0)}}{\partial q_1}(q_0,q_1) &= \int_{t_0}^{t_1}\left(\frac{\partial p}{\partial q_1}\dot{q} + p\frac{\partial \dot{q}}{\partial q_1} - \frac{\partial H}{\partial q}\frac{\partial q}{\partial q_1} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial q_1}\right) dt \\
&= \int_{t_0}^{t_1}\left(p\frac{\partial q}{\partial q_1} + \dot{p}\frac{\partial q}{\partial q_1}\right) dt \\
&= p_1 - p_0\frac{\partial q_0}{\partial q_1} = p_1
\end{aligned}$$

∎

6.4 Variational integrators versus methods based on Generating Function

6.4.1 Discrete Variational Integrators

Discrete variational integrators appear as a special kind of geometric integrators. These integrators have their roots in the optimal control literature in the 1960's and 1970's (Jordan and Polack [77], Cadzow [15], Maeda [119, 120]) and in 1980's by Lee [93, 94], Veselov [135, 157]. In these papers, the following concepts are introduced: the discrete action sum, discrete Euler-Lagrange equations, discrete Noether theorem... Although this kind of symplectic integrators have been considered for conservative systems [75, 78, 124, 131, 159, 160], it has been recently shown how discrete variational mechanics can include forced or dissipative systems [79, 131], holonomic constraints [59, 131], time-dependent systems [112, 131], frictional contact [140] and non-holonomic constraints (see [26, 28]). Moreover, it has been also discussed reduction theory [11, 12, 95, 126, 128], extension to field theories [76, 125] and quantum mechanics [138]. All these integrators have demonstrated exceptionally good longtime behavior and the research of this topic is interesting for numerical and geometric considerations.

At this point, we shall describe the discrete variational calculus, following the approach in [159] (see also [4, 57]). A discrete Lagrangian is a map $L_d : Q \times Q \to \mathbb{R}$ (this discrete Lagrangian may be considered as an approximation of the continuous Lagrangian $L : TQ \to \mathbb{R}$). Define the action sum $S_d : Q^{N+1} \to \mathbb{R}$ corresponding to the Lagrangian L_d by

$$S_d = \sum_{k=1}^{N} L_d(q_{k-1}, q_k) \; ,$$

where $q_k \in Q$ for $0 \leq k \leq N$. For any covector $\alpha \in T^*_{(x_1,x_2)}(Q \times Q)$, we have a decomposition $\alpha = \alpha_1 + \alpha_2$ where $\alpha_i \in T^*_{x_i}Q$. Therefore,

$$dL_d(q_0, q_1) = D_1 L_d(q_0, q_1) + D_2 L_d(q_0, q_1) \ .$$

The discrete variational principle or Cadzow's principle states that the solutions of the discrete system determined by L_d must extremise the action sum given fixed points q_0 and q_N. Extremising S_d over q_k, $1 \leq k \leq N-1$, we obtain the following system of difference equations

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0 \ .$$

These equations are usually called the **Discrete Euler-Lagrange equations**. Under some regularity hypothesis (e.g. the matrix $(D_{12}L_d(q_k, q_{k+1}))$ to be regular), this implicit system of difference equations defines a discrete flow $\Upsilon : Q \times Q \longrightarrow Q \times Q$, by $\Upsilon(q_{k-1}, q_k) = (q_k, q_{k+1})$.

The geometrical properties corresponding to this numerical method are obtained defining the discrete Legendre transformation associated to L_d by

$$\begin{array}{rccl} FL_d : & Q \times Q & \longrightarrow & T^*Q \\ & (q_0, q_1) & \longmapsto & (q_0, -D_1 L_d(q_0, q_1)) \ , \end{array}$$

and the 2-form $\omega_d = FL_d^* \omega_Q$, where ω_Q is the canonical symplectic form on T^*Q. The discrete algorithm determined by Υ preserves the symplectic form ω_d, i.e., $\Upsilon^*\omega_d = \omega_d$. Moreover, if the discrete Lagrangian is invariant under the diagonal action of a Lie group G, then the discrete momentum map $J_d : Q \times Q \to \mathfrak{g}^*$ defined by $\langle J_d(q_k, q_{k+1}), \xi \rangle = \langle D_2 L_d(q_k, q_{k+1}), \xi_Q(q_{k+1}) \rangle$ is preserved by the discrete flow. Therefore, these integrators are symplectic-momentum preserving integrators.

6.4.2 Discrete variational mechanics and generating functions

Following the notation of section 6.3.1, for this approach, we consider instead an adequate approximation $S_d^{(t,s)}$ of the action $S_1^{(t,s)}$, for instance

$$S_d^{(t,s)}(q_0, q_1) = (t-s)\mathcal{L}(\alpha q_0 + (1-\alpha) q_1, \frac{q_1 - q_0}{t-s}), \quad \alpha \in [0,1]$$

(with the supposition that Q can be regarded as a vector space). In the approximation above we see that the discrete Lagrangian can be seen as an approximation to the action (which is a generating function of this kind for the exact flow of the solution of the equations for the discrete Lagrangian).

The extremality equations

$$0 = D_2 S_d^{(t,s)}(q_{k-1}, q_k) + D_1 S_d^{(t,s)}(q_k, q_{k+1}), \quad 1 \leq k \leq N-1.$$

are precisely the Discrete Euler-Lagrange equations (see [131] and references therein).

The Legendre transformations can be seen as the projections onto T^*Q as seen in 6.2, and the symplecticity is now a direct consequence of our construction.

Notice that alternatively, we could have considered more accurate approximations. We are assuming here that $L : TQ \to \mathbb{R}$ is a Lagrangian function related via Legendre transformation with the Hamiltonian function H (see [3]), which is locally possible because of the regularity of H.

Denote by $S_1(q_0, q_1, t_0, t_1) = S_1^{(t_1,t_0)}(q_0, q_1)$. From Proposition (6.3.3), it is easy to show that:

$$\begin{aligned} D_3S_1(q_0, q_1, t_0, t_1) = D_3S_1^{(t_1,t_0)}(q_0, q_1) &= H(t_0, q_0, p_0) \\ D_4S_1(q_0, q_1, t_0, t_1) = D_4S_1^{(t_1,t_0)}(q_0, q_1) &= -H(t_1, q_1, p_1) \end{aligned}$$

(see also [131]). As a consequence

$$D_4S_1^{(t_k,t_{k-1})}(q_{k-1}, q_k) + D_3S_1^{(t_{k+1},t_k)}(q_k, q_{k+1}) = 0 \tag{6.20}$$

It should be noticed that if we take a new function $S_d^{(t_{k+1},t_k)}$ as an adequate approximation of $S_1^{(t_{k+1},t_k)}$, then solutions $\{q_0, q_1, \dots, q_N\}$ of equations

$$D_2S_d^{(t_k,t_{k-1})}(q_{k-1}, q_k) + D_1S_d^{(t_{k+1},t_k)}(q_k, q_{k+1}) = 0, \quad 1 \le k \le N-1.$$

do not satisfy (6.20) for arbitrary values of t_{k-1}, t_k, t_{k+1}. Therefore, we may write the system of difference equations

$$\left\{ \begin{array}{l} D_2S_d^{(t_k,t_{k-1})}(q_{k-1}, q_k) + D_1S_d^{(t_{k+1},t_k)}(q_k, q_{k+1}) = 0, \\ D_4S_d^{(t_k,t_{k-1})}(q_{k-1}, q_k) + D_3S_d^{(t_{k+1},t_k)}(q_k, q_{k+1}) = 0, \end{array} \right. \tag{6.21}$$

which under regularity assumptions will determine a time-dependent discrete flow

$$\Phi(q_{k-1}, q_k, t_{k-1}, t_k) = (q_k, q_{k+1}, t_k, t_{k+1})$$

with variable step size $h_k = t_{k+1} - t_k$ (see [78, 93, 94, 112, 131]).

The following sections consist of our research work based on the ideas exposed in this section. Part of this work has been published in [113] and [114].

By adapting these ideas to different situations, the analysis of the different generating functions and generalisations permits us to construct different numerical integrators with good geometrical properties for several examples, of which we shall explore two: the non-holonomic mechanics and the optimal control theory, leaving the classical field theory as a part of a future job.

6.5 Applications to non-holonomic mechanics

In a recent paper, J. Cortés and S. Martínez [28] have proposed a construction of non-holonomic integrators which is useful for numerical considerations. Their construction is based on the *Discrete Lagrange-D'Alembert's principle*. Assuming that the constraints are given by a distribution D, this principle states that

$$\left(D_1L_d(q_k, q_{k+1}) + D_2L_d(q_{k-1}, q_k)\right)_i \delta q_k^i = 0, \quad 1 \le i \le N-1 \ ,$$

where $\delta q_k \in D_{q_k}$ and, in addition $(q_k, q_{k+1}) \in D_d$. Here D_d denotes a discrete constraint space $D_d \subset Q \times Q$. This integrator has a good performance and naturally inherits some geometric properties of the continuous problem. Observe that the method is based on the discretisation of the Lagrangian and a coherent discretisation of the constraints, and both determine the discrete constraint forces.

Alternatively, we propose a non-holonomic integrator based also on the discretisation of the Lagrangian function (in a more precise sense, we discretise the action function) but now we take a coherent discretisation of the constraint forces and both determine the discrete constraint submanifold. This method gives us, in general, different integrators from those in [28]. The last considerations of the previous section will be our starting point to study non-holonomic integrators, and our equations will be conceptually equivalent to the proposed for systems with external forces (see [131]). In the particular case of mechanical systems with linear constraint in the velocities, we study a subclass of our family of non-holonomic integrators with the property of preservation of the original non-holonomic constraints.

6.5.1 Generating functions and non-holonomic mechanics

At this point, we shall follow similar arguments for the construction of generating functions for symplectic or canonical maps [3]. However, because of equation (6.6), we have that the non-holonomic flow is not a canonical transformation; i.e.

$$F_h^*\omega_Q - \omega_Q = d\left(\int_0^h F_t^*\Lambda\right) . \tag{6.22}$$

However, this description will allow us to construct a new family of non-holonomic integrators for equations (6.15). Denote by $\pi_i : T^*Q \times T^*Q \to T^*Q$, $i = 1, 2$, the canonical projections. Consider the following forms

$$\begin{aligned} \Theta &= \pi_1^*\theta_Q - \pi_0^*\theta_Q , \\ \Omega &= \pi_1^*\omega_Q - \pi_0^*\omega_Q = -d\Theta . \end{aligned}$$

Denote by $i_{F_h} : \mathrm{Graph}(F_h) \hookrightarrow T^*Q \times T^*Q$ the inclusion map and observe that $\mathrm{Graph}(F_h) \subset M \times M$. Then, from (6.22)

$$\begin{aligned} i_{F_h}^*\Omega &= (\pi_{1|\mathrm{Graph}(F_h)})^*(F_h^*\omega_Q - \omega_Q) \\ &= (\pi_{1|\mathrm{Graph}(F_h)})^*\left[d\left(\int_0^h F_t^*\Lambda\right)\right] , \end{aligned}$$

or, from (6.6),

$$i_{F_h}^*\Theta = (\pi_{1|\mathrm{Graph}(F_h)})^*\left[d\left(\int_0^h L\circ\tilde{F}_t\, dt\right) - \int_0^h F_t^*\Lambda\right] .$$

Let (q_0, p_0, q_1, p_1) be coordinates in $T^*Q \times T^*Q$ in a neighborhood of some point in $\mathrm{Graph}(F_h)$. If $(q_0, p_0, q_1, p_1) \in \mathrm{Graph}(F_h)$ then $\Psi^a(q_0, p_0) = 0$ and $\Psi^a(q_1, p_1) = 0$. Moreover, along $\mathrm{Graph}(F_h)$,

$q_1 = q_1(q_0, p_0)$ and $p_1 = p_1(q_0, p_0)$, we have

$$p_1\, dq_1 - p_0 dq_0 = d\left(\int_0^h L(q(t), \dot{q}(t))\, dt\right) - \int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t)) \;, \tag{6.23}$$

where $(q(t), \dot{q}(t)) = \tilde{F}_t(q_0, \dot{q}_0)$ with $Leg(q_0, \dot{q}_0) = (q_0, p_0)$. Here, $\tilde{F}_t$ denotes the flow of $\xi_{L,D}$. Equation (6.23) is satisfied along $\mathrm{Graph}(F_h)$.

Assume that, in a neighborhood of some point $x \in \mathrm{Graph}(F_h)$, we can change this system of coordinates to a new coordinates (q_0, q_1). Denote by

$$S^h(q_0, q_1) = \int_0^h L(q(t), \dot{q}(t))\, dt \,,$$

where $q(t)$ is a solution curve of the non-holonomic problem with $q(0) = q$ and $q(h) = q_1$. This solution always exists for adequate values of q_0 and q_1. In fact, observe that

$$q_1 = q_0 + h\frac{\partial H}{\partial p}(q_0, p_0) + O(h^2) \;,$$

hence, since $\det\left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right) \neq 0$, we locally have that $p_0 = p_0(q_0, q_1, h)$. But, in addition, $(q_0, p_0) \in M$; therefore $\varphi^a(q_0, q_1, h) = \Psi^a(q_0, p_0(q_0, q_1, h)) = 0$. Then, the curve

$$(q(t), \dot{q}(t)) = Leg^{-1}(F_t(q_0, p_0(q_0, q_1, h))) \;,$$

verifies the required assumptions if $\varphi^a(q_0, q_1, h) = 0$.

Thus, we deduce that

$$\left\{\begin{aligned} p_0 &= -\frac{\partial S^h}{\partial q_0} + \int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t))\frac{\partial q}{\partial q_0} \;, \\ p_1 &= \frac{\partial S^h}{\partial q_1} - \int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t))\frac{\partial q}{\partial q_1} \;, \end{aligned}\right. \tag{6.24}$$

where (q_0, q_1) verifies the constraint functions $\varphi^a(q_0, q_1, h) = 0$, now explicitely defined by

$$\varphi^a(q_0, q_1, h) = \Psi^a(q_0, -\frac{\partial S^h}{\partial q_0}(q_0, q_1) + \int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t))\frac{\partial q}{\partial q_0}) \;, \quad 1 \leq a \leq m \;, \tag{6.25}$$

with $q(t)$ solution of the non-holonomic problem with $q(0) = q_0$ and $q(h) = q_1$.

Next, we shall show how the group composite law of the flow F_h

$$F_{Nh} = \underbrace{F_h \circ \ldots \circ F_h}_{N}$$

is expressed in terms of the corresponding "generating functions" S^h. Notice that these functions are not proper generating functions, but we shall call them generating functions with a small abuse of language. Moreover, the following theorem will result in a construction of new numerical integrators for non-holonomic mechanics when we change the "generating function" and the constraint forces by appropriate approximations. As a generalization of theorem 6.3.2 we have the following

Theorem 6.5.1. *The function S^{Nh}, the "generating function" for F_{Nh}, is given by*

$$S^{Nh}(q_0, q_N) = \sum_{k=0}^{N-1} S^h(q_k, q_{k+1}) \ ,$$

where q_k, $1 \leq k \leq N-1$, are points verifying

$$D_2 S^h(q_{k-1}, q_k) + D_1 S^h(q_k, q_{k+1}) = \int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t)) \frac{\partial q}{\partial q_1} + \int_h^{2h} \widetilde{\Lambda}(q(t), \dot{q}(t)) \frac{\partial q}{\partial q_0} \ , \tag{6.26}$$

and $q(t)$ is a solution curve of the non-holonomic problem with $q(0) = q_{k-1}$ and $q(h) = q_k$ (respectively, $q(h) = q_k$ and $q(2h) = q_{k+1}$) for the first integral (resp., second integral) of the right-hand side.

Proof. By a recursion argument, it is suffices to prove the result for $N = 2$; that is,

$$S^{2h}(q_0, q_2) = S^h(q_0, q_1) + S^h(q_1, q_2) \ ,$$

where q_1 verifies condition (6.26).

Since

$$\begin{aligned} p_1\, dq_1 - p_0\, dq_0 &= dS^h(q_0, q_1) - \int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t)) \ , \\ p_2\, dq_2 - p_1\, dq_1 &= dS^h(q_1, q_2) - \int_h^{2h} \widetilde{\Lambda}(q(t), \dot{q}(t)) \ , \end{aligned}$$

then

$$p_2\, dq_2 - p_0\, dq_0 = d\Big(S^h(q_0, q_1) + S^h(q_1, q_2)\Big) - \int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t)) - \int_h^{2h} \widetilde{\Lambda}(q(t), \dot{q}(t)) \ .$$

Since the variables q_1 do not appear on the left-hand side term, it follows that

$$0 = D_2 S_1^h(q_0, q_1) + D_1 S_2^h(q_1, q_2) - \int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t)) \frac{\partial q}{\partial q_1} - \int_h^{2h} \widetilde{\Lambda}(q(t), \dot{q}(t)) \frac{\partial q}{\partial q_0} \ , \tag{6.27}$$

and for a choice of q_1 verifying (6.27) then

$$S^{2h}(q_0, q_2) = S^h(q_0, q_1) + S^h(q_1, q_2)$$

is a "generating function of the first kind" of F_{2h} because

$$p_2\, dq_2 - p_0\, dq_0 = dS^{2h}(q_0, q_2) - \int_0^{2h} \widetilde{\Lambda}(q(t), \dot{q}(t)) \ .$$

as we wanted to prove. ∎

Equations (6.26) determine a implicit system of difference equations which permits us to obtain q_2 from the initial data q_0 and q_1. An interesting consequence of this is that these equations preserve the constraint submanifold determined by the constraints $\varphi^a = 0$, $1 \leq a \leq m$. In fact, if $\varphi^a(q_0, q_1, h) = 0$ (that is $\Psi^a(q_0, p_0) = 0$) then

$$\varphi^a(q_1, q_2, h) = \Psi^a(q_1, \frac{\partial S^h}{\partial q_1}(q_0, q_1) - \int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t)) \frac{\partial q}{\partial q_1}) \ ,$$

and now, applying (6.24), we obtain that

$$\varphi^a(q_1, q_2, h) = \Psi^a(q_1, p_1) = 0 ,$$

since $F_h(q_0, p_0) = (q_1, p_1)$ and the flow preserves the constraints.

The next remark will be a key result for the construction of non-holonomic integrators.

Remark 6.5.2. Replace equation (6.24) by

$$\begin{cases} p_0 = -\dfrac{\partial \tilde{S}^h}{\partial q_0} + \alpha_0^h(q_0, q_1) , \\ p_1 = \dfrac{\partial \tilde{S}^h}{\partial q_1} - \alpha_1^h(q_0, q_1) , \end{cases} \tag{6.28}$$

where $\tilde{S}^h$ is a function of (q_0, q_1) coordinates and $\alpha^h = \alpha_0^h\, dq_0 + \alpha_1^h\, dq_1$, and replace the constraints functions by

$$\tilde{\varphi}^a(q_0, q_1, h) = \Psi^a(q_0, -\frac{\partial \tilde{S}^h}{\partial q^0} + \alpha_0^h(q_0, q_1)) , \tag{6.29}$$

that is,

$$p_1\, dq_1 - p_0\, dq_0 = d\tilde{S}^h - \alpha^h ,$$

along $\tilde{\varphi}^a = 0$.

Assume that

$$\det\left(\frac{\partial^2 \tilde{S}^h}{\partial q_0 \partial q_1} - \frac{\partial \alpha_0^h}{\partial q_1}\right) \neq 0 , \tag{6.30}$$

then, applying the implicit function theorem we have that, locally, $q_1 = q_1(q_0, p_0)$, and therefore, the mapping

$$G_h(q_0, p_0) = (q_1, p_1)$$

is well-defined.

Consider the mapping G_{Nh} defined by

$$G_{Nh} = \underbrace{G_h \circ \ldots \circ G_h}_{N} .$$

Following a similar argument to Theorem 6.5.1, Graph(G_{Nh}) is described by

$$\begin{cases} p_0 = -\dfrac{\partial \tilde{S}^{Nh}}{\partial q_0}(q_0, q_N) + \alpha_0^{Nh}(q_0, q_N) , \\ p_N = \dfrac{\partial \tilde{S}^{Nh}}{\partial q_N}(q_0, q_N) - \alpha_1^{Nh}(q_0, q_N) , \end{cases} \tag{6.31}$$

where $\tilde{S}^{Nh}(q_0, q_N) = \sum_{k=0}^{N-1} \tilde{S}^h(q_k, q_{k+1})$ and $\alpha^{Nh}(q_0, q_N) = \sum_{k=0}^{N-1} \alpha^h(q_k, q_{k+1})$. Here, the q_k's, $1 \leq k \leq N-1$, verify

$$D_2\tilde{S}^h(q_{k-1}, q_k) + D_1\tilde{S}^h(q_k, q_{k+1}) = \alpha_1^h(q_{k-1}, q_k) + \alpha_0^h(q_k, q_{k+1}), \quad 1 \leq k \leq N-1 . \tag{6.32}$$

Constraint error analysis

As we have seen, if our "generating function" is S^h, then we have exact preservation of the constraints φ^a. We now investigate what happens when the "generating function" is an approximation. We follow similar arguments to those in subsection 2.3 in [131].

Assume that Q, and also TQ and T^*Q, are finite-dimensional vector spaces with inner product $\langle .,.\rangle$ and corresponding norm $\| \ \|$.

Consider an "approximated generating function" $\tilde{S}^h$ and an approximated discrete constraint force $\alpha^h = \alpha_i^h\, dq^i$ for the non-holonomic problem, both of order r (all the functions are assumed to be C^2); hence, there exists an open set $U \subset D$ with compact closure and constants $c, d_i > 0$, $1 \leq i \leq n$, and $H > 0$ such that

$$\begin{aligned} \tilde{S}^h(q_0, q_1) &= S^h(q_0, q_1) + C(q_0, q_1, h)h^{r+1} && (6.33)\\ \alpha_i^h &= \int_0^h \tilde{\Lambda}_i(q(t), \dot{q}(t))\, dt + D_i(q_0, q_1, h)h^{r+1} && (6.34)\end{aligned}$$

for all solution $q(t)$ of the non-holonomic problem with $q(0) = q_0$, $q(h) = q_1$ and initial condition belonging to U and $h \leq H$. Here C and D_i, $1 \leq i \leq n$, are smooth functions such that $\|C(q_0, q_1, h)\| \leq c$ and $\|D_i(q_0, q_1, h)\| \leq d_i$ on U.

Taking derivatives we have that

$$\frac{\partial \tilde{S}^h}{\partial q_0}(q_0, q_1) = \frac{\partial S^h}{\partial q_0}(q_0, q_1) + \frac{\partial C}{\partial q_0}(q_0, q_1, h)h^{r+1}$$

and also

$$\alpha_0^h(q_0, q_1) = (\alpha_0)_i^h \frac{\partial q^i}{\partial q_0} = \int_0^h \tilde{\Lambda}_i(q(t), \dot{q}(t))\frac{\partial q^i}{\partial q_0}\, dt + \sum_{i=1}^n \frac{\partial D_i}{\partial q_0}(q_0, q_1, h)h^{r+1}$$

where now $\alpha^h = \alpha_0^h\, dq_0 + \alpha_1^h\, dq_1$

Therefore, we deduce that

$$\begin{aligned} \tilde{\varphi}^a(q_0, q_1, h) &= \Psi^a(q_0, -\frac{\partial \tilde{S}}{\partial q^0} + \alpha_0(q_0, q_1))\\ &= \Psi^a(q_0, -\frac{\partial S^h}{\partial q_0} + \int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t))\frac{\partial q}{\partial q_0}) + E^a(q_0, q_1, h)h^{r+1}\\ &= \Psi^a(q_0, p_0) + E^a(q_0, q_1, h)h^{r+1} = E^a(q_0, q_1, h)h^{r+1}\end{aligned}$$

where E^a are bounded functions. Thus, the discrete algorithm preserves the constraints up to order r.

Local error analysis.

Assuming that

$$\det\left(\frac{\partial^2 \tilde{S}^h}{\partial q_0 \partial q_1} - \frac{\partial \alpha_0^h}{\partial q_1}\right) \neq 0\ , \qquad (6.35)$$

we obtain a discrete flow $G^h : V \subseteq T^*Q \longrightarrow T^*Q$. Now, using Equations (6.24), (6.33) and (6.34) we deduce that

$$\left\{ \begin{array}{l} p_0 = -\dfrac{\partial S^h}{\partial q_0} + \displaystyle\int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t)) \dfrac{\partial q}{\partial q_0} = \dfrac{\partial \tilde{S}^h}{\partial q_0}(q_0, q_1) + \alpha_0^h(q_0, q_1) + E_0(q_0, q_1, h) h^{r+1} \,, \\ p_1 = \dfrac{\partial S^h}{\partial q_1} - \displaystyle\int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t)) \dfrac{\partial q}{\partial q_1} = \dfrac{\partial \tilde{S}^h}{\partial q_1}(q_0, q_1) - \alpha_1^h(q_0, q_1) + E_1(q_0, q_1, h) h^{r+1} \,, \end{array} \right. \tag{6.36}$$

where E_0 and E_1 are smooth and bounded functions.

Applying the implicit function theorem to (6.36), it is easy to show, from conditions (6.33) and (6.34), that G^h is an integrator of $X_{H,M}$ of order r (see for details Theorem 2.3.1 in [131]).

6.5.2 Construction of non-holonomic integrators

In what follows and for simplicity, assume that Q is a vector space. Since we have that $S^h(q_0, q_1) = \int_0^h L(q(t), \dot{q}(t))\, dt$, where $q(t)$ is a non-holonomic solution with $q(0) = q_0$ and $q(h) = q_1$, using Remark 6.5.2, we can obtain non-holonomic integrators by taking adequate approximations of the "generating function" S^h and the extra-term $\int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t))$.

Consider, for instance, the approximation

$$S^h_\alpha(q_0, q_1) = hL((1-\alpha)q_0 + \alpha q_1, \frac{q_1 - q_0}{h}) \,, \tag{6.37}$$

for some parameter $\alpha \in [0, 1]$. (In general, we shall write $S^h_\alpha(q_0, q_1) \approx S^h(q_0, q_1)$.)

A natural approximation of the constraint forces adapted to our choice of approximation for S^h are

$$\begin{array}{rcl} \displaystyle\int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t)) \frac{\partial q}{\partial q_0} & \approx & (1-\alpha) h \widetilde{\Lambda}((1-\alpha)q_0 + \alpha q_1, \dfrac{q_1 - q_0}{h}) \,, \\ \displaystyle\int_0^h \widetilde{\Lambda}(q(t), \dot{q}(t)) \frac{\partial q}{\partial q_1} & \approx & \alpha h \widetilde{\Lambda}((1-\alpha)q_0 + \alpha q_1, \dfrac{q_1 - q_0}{h}) \,. \end{array}$$

Consequently, equations (6.32) give us the following numerical method for non-holonomic systems

$$\begin{array}{l} D_2 S^h_\alpha(q_{k-1}, q_k) + D_1 S^h_\alpha(q_k, q_{k+1}) = \alpha h \widetilde{\Lambda}((1-\alpha)q_{k-1} + \alpha q_k, \dfrac{q_k - q_{k-1}}{h}) \\ \qquad + (1-\alpha) h \widetilde{\Lambda}((1-\alpha)q_k + \alpha q_{k+1}, \dfrac{q_{k+1} - q_k}{h}) \,, \quad 1 \leq k \leq N-1 \,, \end{array}$$

with initial condition satisfying

$$\tilde{\varphi}^a(q_0, q_1, h) = \Psi^a(q_0, -\frac{\partial S^h_\alpha}{\partial q_0}(q_0, q_1) + (1-\alpha) h \widetilde{\Lambda}((1-\alpha)q_0 + \alpha q_1, \frac{q_1 - q_0}{h})) = 0 \,.$$

Remark 6.5.3. Obviously, it is possible to produce a wider variety of discrete methods. For example,

$$S^h_{\mathrm{sym},\alpha} = \frac{1}{2}S^h_\alpha + \frac{1}{2}S^h_{1-\alpha}\ ,$$

gives a second-order method for any $\alpha \in [0,1]$. Also, higher-order approximations of the function S^h may be considered.

Example 6.5.4. Non-holonomic particle.

Consider the Lagrangian $L : T\mathbb{R}^3 \to \mathbb{R}$

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - (x^2 + y^2)\ ,$$

subject to the constraint

$$\phi = \dot{z} - y\dot{x} = 0\ .$$

It is easy to compute the non-holonomic differential equations

$$\begin{aligned} \ddot{x} &= -\frac{2x + y\dot{x}\dot{y}}{1+y^2} \\ \ddot{y} &= -2y \\ \ddot{z} &= \frac{-2xy + \dot{x}\dot{y}}{1+y^2}\ , \end{aligned}$$

where now the constraint 1-form is

$$\tilde{\Lambda} = \frac{2xy - \dot{x}\dot{y}}{1+y^2}(dz - ydx)\ .$$

The system being simulated here is purely conservative and so there should be no loss of energy over time.

Taking

$$\begin{aligned} S^h_{1/2}(x_0, y_0, z_0, x_1, y_1, z_1) &= \frac{h}{2}\left[\left(\frac{x_1 - x_0}{h}\right)^2 + \left(\frac{y_1 - y_0}{h}\right)^2 + \left(\frac{z_1 - z_0}{h}\right)^2\right] \\ &\quad - \left(\frac{x_0 + x_1}{2}\right)^2 - \left(\frac{y_0 + y_1}{2}\right)^2\ , \end{aligned}$$

we obtain the non-holonomic integrator

$$\begin{aligned} &\frac{x_1 - x_0}{h} - h\frac{x_1 + x_0}{2} - \frac{x_2 - x_1}{h} - h\frac{x_2 + x_1}{2} \\ &= -\frac{h}{2}\left[\frac{\frac{(x_1+x_0)(y_1+y_0)}{2} - \frac{(x_1-x_0)(y_1-y_0)}{h^2}}{1 + \left(\frac{y_1+y_0}{2}\right)^2}\cdot\frac{y_1 + y_0}{2} + \frac{\frac{(x_2+x_1)(y_2+y_1)}{2} - \frac{(x_2-x_1)(y_2-y_1)}{h^2}}{1 + \left(\frac{y_2+y_1}{2}\right)^2}\ ,\cdot\frac{y_2 + y_1}{2}\right] \\ &\frac{y_1 - y_0}{h} - h\frac{y_1 + y_0}{2} - \frac{y_2 - y_1}{h} - h\frac{y_2 + y_1}{2} = 0\ , \\ &\frac{z_1 - z_0}{h} - \frac{z_2 - z_1}{h} \\ &= \frac{h}{2}\left[\frac{\frac{(x_1+x_0)(y_1+y_0)}{2} - \frac{(x_1-x_0)(y_1-y_0)}{h^2}}{1 + \left(\frac{y_1+y_0}{2}\right)^2} + \frac{\frac{(x_2+x_1)(y_2+y_1)}{2} - \frac{(x_2-x_1)(y_2-y_1)}{h^2}}{1 + \left(\frac{y_2+y_1}{2}\right)^2}\right]\ . \end{aligned}$$

The constraint function on $\mathbb{R}^3 \times \mathbb{R}^3$ is

$$\tilde{\varphi}^a(x_0, y_0, z_0, x_1, y_1, z_1, h) = -\frac{z_1 - z_0}{h} - \frac{h}{2}\frac{\frac{(x_1+x_0)(y_1+y_0)}{2} - \frac{(x_1-x_0)(y_1-y_0)}{h^2}}{1 + \left(\frac{y_1+y_0}{2}\right)^2}$$
$$+ \quad y_0\left[\frac{x_1 - x_0}{h} + h\frac{x_1 + x_0}{2} - \frac{h}{2}\frac{\frac{(x_1+x_0)(y_1+y_0)}{2} - \frac{(x_1-x_0)(y_1-y_0)}{h^2}}{1 + \left(\frac{y_1+y_0}{2}\right)^2} \cdot \frac{y_1 + y_0}{2}\right].$$

The following figures show the preservation of energy as a key point of comparison of computational implementations of the method exposed above to other methods.

Figure 6.3 compares the method introduced here to the traditional Runge-Kutta method of fourth order, showing an improvement in several orders of magnitude. Observe that, in this scale, the value of the energy in each step of our algorithm is practically undistinguishable from the initial value of the energy, therefore our method does not artificially dissipate energy

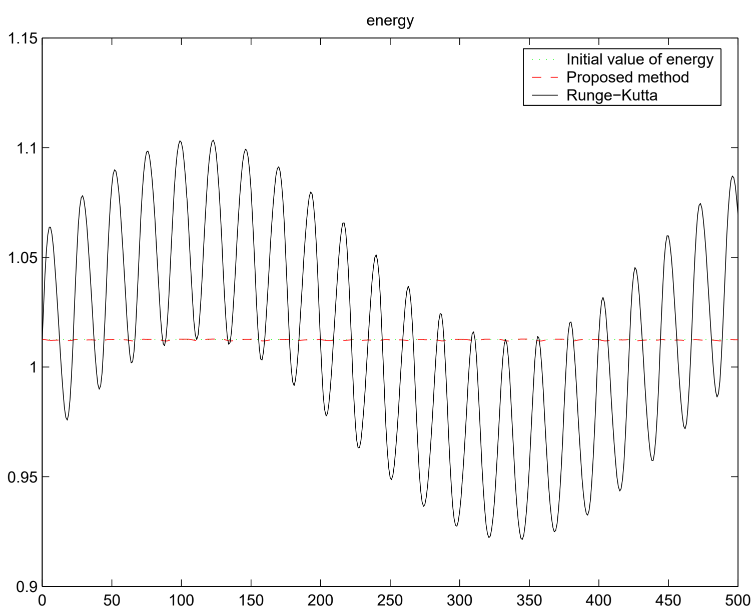

Figure 6.3: Runge-Kutta method versus our method

Figure 6.4 is a comparison between our method and the one appeared in [26, 28]. A similar behaviour is observed. Nevertheless, a slightly better behaviour can also be appreciated, where the proposed algorithm shows on average a better preservation of the original energy.

For the same initial conditions and data, figure 6.5 shows a very good behaviour of the constraint function evolution with time (notice the small scale).

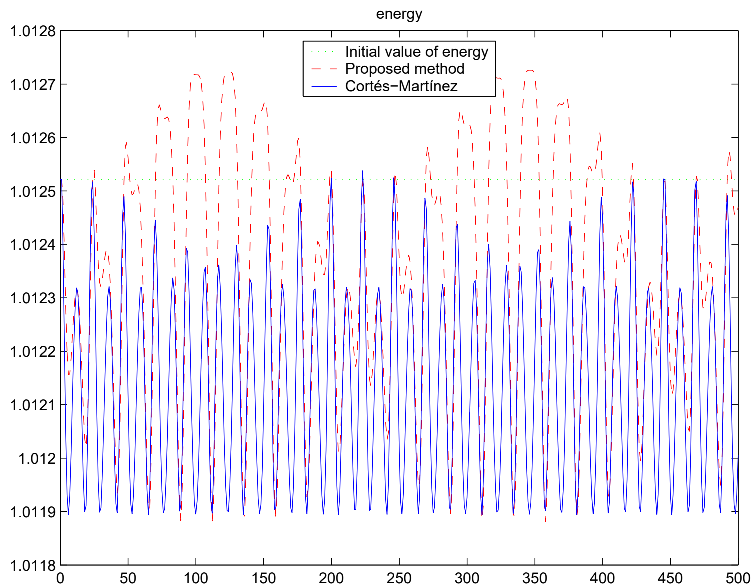

Figure 6.4: Cortés-Martínez method versus our method

Remark 6.5.5. Generating function approach and Discrete Lagrange-d'Alembert principle: a discussion

As is well known, a remarkable feature of symplectic transformations is that they can be expressed in terms of a single real-valued function S, the generating function of the canonical transformation. Therefore, any symplectic integrator has an associated generating function. Now, taking adequate approximations of the generating function associated to the exact flow of a Hamiltonian system, we generate symplectic integrators (see, for instance, symplectic and symplectic partitioned Runge-Kutta methods in [70]) by using generating functions of the first kind, second kind, etc. It is also possible to construct symplectic numerical methods of higher order considering better approximations of the generating function in the Hamilton-Jacobi equation (see [24] and section 6.7.5).

As we have seen in previous sections, the discrete variational approach and the generating function approach are in fact the same on generating function of the first kind. That is, considering the action integral as a function of (q_0, q_1), for the solution $q(t)$ of the Euler-Lagrange equations we have that

$$S^h(q_0, q_1) = \int_0^h L(q(t), \dot{q}(t))\, dt$$

(this is precisely the exact discrete Lagrangian following the notation in Marsden and West [131]). Therefore, a discrete Lagrangian L_d is a discrete approximation of the above action integral or, in other words, an approximated generating function for S^h.

In Section 3.2 of [131], the authors discuss Discrete variational mechanics with forces and using the so-called *Discrete Lagrange-d'Alembert principle*, they simulate a given forced Lagrangian or Hamiltonian system choosing discrete Lagrangians and discrete forces to approximate the exact quantities.

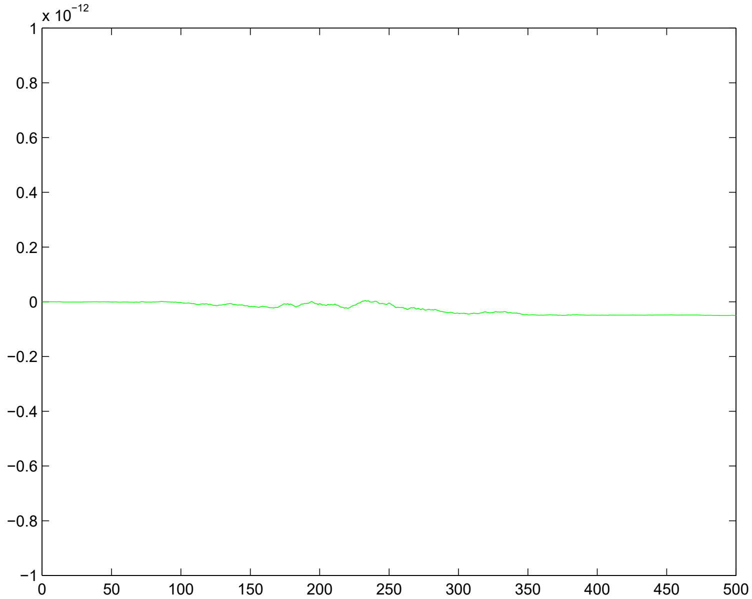

Figure 6.5: Constraint preservation

Formally, the approach followed in the previous sections is equivalent to the approach given in [131] considering a non-holonomic system as a Lagrangian system with forces determined with the constraint equations. However, a new insight is gained from the generating function approach. First, our theory is ready for the constructions of new numerical integrators for non-holonomic systems using Hamilton-Jacobi theory for non-holonomic systems (see [47]) or even "generating functions" for non-holonomic system of different kinds. Observe that, for instance, symplectic Runge-Kutta methods were generated using generating functions of the third kind (see [145]). Second, since symplectic integrators based on generating functions are strongly established in numerical analysis research, we think that our presentation will be clearer than the usual one of discrete Lagrangians frequently used by the geometrical mechanics researchers.

6.6 Mechanical systems with linear constraints. Geometric numerical methods preserving constraints

In the previous section we have constructed a family of numerical integrators for non-holonomic mechanics; these integrators do not preserve the constraint but we show that the violation of the constraint is very small. This answer is not completely satisfactory for a numerical method for a non-holonomic system (as, for instance, rolling constraints in wheeled vehicles), therefore we impose the preservation of the non-holonomic constraints obtaining a subfamily of the above numerical integrators in this section. As we shall show more insight is performed by restricting ourselves to the particular (but general in a mechanical sense) case of Lagrangians of mechanical type ($L = T - V$) and constraints linear on velocities.

Therefore, suppose that the mechanical system, given by the Lagrangian $L : TQ \to \mathbb{R}$

$$L(v_q) = \frac{1}{2} g(v_q, v_q) - V(q)$$

is subjected to non-holonomic constraints $\phi^a : TQ \longrightarrow \mathbb{R}$, $1 \leq a \leq m$. Since the non-holonomic constraints usually found in mechanics are linear in the velocities, we shall assume that

$$\phi^a(q, \dot{q}) = \mu_i^a(q)\dot{q}^i, \; 1 \leq a \leq m \, .$$

From a geometric point of view, these linear constraints are determined by prescribing a distribution $\mathcal{D}$ on Q of dimension $n - m$ such that the annihilator of $\mathcal{D}$ is locally given by

$$\mathcal{D}^o = \langle \mu^a = \mu_i^a dq^i \; ; 1 \leq a \leq m \rangle \, .$$

In this manner, the solutions of the non-holonomic Lagrangian system satisfy

$$\nabla_{\dot{c}(t)}\dot{c}(t) = -\text{grad}\; V(c(t)) + \lambda(\dot{c}(t)), \quad \dot{c}(t) \in \mathcal{D}_{c(t)} \, , \tag{6.38}$$

where λ is a section of $\mathcal{D}^\perp$ along c, and $\mathcal{D}^\perp$ stands for the orthogonal complement of $\mathcal{D}$ with respect to the metric g.

Since g is a Riemannian metric, the $m \times m$ matrix $(C^{ab}) = (\mu_i^a g^{ij} \mu_j^b)$ is symmetric and regular. Therefore, we can explicitly determine

$$\lambda(q^i(t), \dot{q}^i(t)) = C_{ab} \left((-\Gamma^i_{jk}\dot{q}^j\dot{q}^k - g^{ij}\frac{\partial V}{\partial q^j})\mu_i^a + \dot{q}^i\dot{q}^j \frac{\partial \mu_i^a}{\partial q^j} \right) Z^b \tag{6.39}$$

where (C_{ab}) is the inverse matrix of (C^{ab}), Γ^i_{jk} are the Christoffel components and the vector field Z^a is defined by

$$g(Z^a, Y) = \mu^a(Y), \quad \text{for all vector field } Y, \; 1 \leq a \leq m \, ,$$

that is, Z^a is the gradient of the 1-form μ^a. Thus, $\mathcal{D}^\perp = \langle Z^a \rangle$, $1 \leq a \leq m$. In local coordinates, we have

$$Z^a = g^{ij}\mu_i^a \frac{\partial}{\partial q^j} \, .$$

By using the metric g and the distribution $\mathcal{D}$ we can obtain two complementary projectors

$$\begin{aligned} \mathcal{P} : TQ &\to \mathcal{D} \, , \\ \mathcal{Q} : TQ &\to \mathcal{D}^\perp \, , \end{aligned}$$

with respect to g. The projector $\mathcal{Q}$ is locally described by

$$\mathcal{Q} = C_{ab} Z^a \otimes \mu^b \, .$$

Using these projectors we can obtain the equations of motion as follows. A curve $c(t)$ is a motion for the non-holonomic system if it satisfies the constraints, say, $\phi^a(\dot{c}(t)) = 0$, for all a, and, in addition, the "projected equation of motion"

$$\mathcal{P}(\nabla_{\dot{c}(t)} \; \dot{c}(t)) = -\mathcal{P}(\text{grad}\; V(c(t))) \tag{6.40}$$

is fulfilled. But these conditions are equivalent to

$$\dot c(t) \in \mathcal{D}_{c(t)} \ , \ \bar\nabla_{\dot c(t)}\dot c(t) = -\mathcal{P}(\text{grad } V(c(t))) \ ,$$

where $\bar\nabla$ is the modified linear connection defined by

$$\bar\nabla_X Y = \nabla_X Y + (\nabla_X \mathcal{Q})(Y)$$

for all vector fields X and Y on Q.

Since the constraints are linear then, from (6.25)

$$-\mu_i^a(q_0)g^{ij}(q_0)\frac{\partial S^h}{\partial q_0^j}(q_0,q_1) + \mu_i^a(q_0)g^{ij}(q_0)\int_0^h \widetilde\Lambda(q(t),\dot q(t))\frac{\partial q}{\partial q_0^j} = 0 \ , \quad 1\leq a\leq m \ , \tag{6.41}$$

or, in terms of projectors,

$$\mathcal{Q}_{|q_0}\left(D_1 S^h(q_0,q_1)\right)) = \mathcal{Q}_{|q_0}\left(D_1\int_0^h \widetilde\Lambda(q(t),\dot q(t))\right) \tag{6.42}$$

Moreover, the dynamics preserves the constraints Ψ^a which implies that

$$\Psi^a(q_1, \frac{\partial S^h}{\partial q_1}(q_0,q_1) - \int_0^h \widetilde\Lambda(q(t),\dot q(t))\frac{\partial q}{\partial q_1}) = 0 \ ,$$

or, in other words,

$$\mathcal{Q}_{|q_1}\left(D_2 S^h(q_0,q_1)\right) = \mathcal{Q}_{|q_1}\left(D_2\int_0^h \widetilde\Lambda(q(t),\dot q(t)\right) \tag{6.43}$$

Therefore, equations (6.42) and (6.43) show that the preservation of the exact constraints is equivalent to give a prescription about the relationship between the "generating function" and the constraint forces.

Thus, equations (6.26)

$$D_2S^h(q_{k-1},q_k) + D_1S^h(q_k,q_{k+1}) = \int_0^h \widetilde\Lambda(q(t),\dot q(t))\frac{\partial q}{\partial q_1} + \int_h^{2h} \widetilde\Lambda(q(t),\dot q(t))\frac{\partial q}{\partial q_0} \ ,$$

can be rewritten using expression (6.43) as follows

$$\mathcal{P}_{|q_k}\left(D_2S^h(q_{k-1},q_k)\right) + D_1S^h(q_k,q_{k+1}) = \mathcal{P}_{|q_k}\left(\int_0^h \widetilde\Lambda(q(t),\dot q(t))\frac{\partial q}{\partial q_1}\right) + \int_h^{2h} \widetilde\Lambda(q(t),\dot q(t))\frac{\partial q}{\partial q_0} \ , \tag{6.44}$$

Now, considering an approximated generating function $\tilde S^h$ and an approximate constraint force $\alpha^h = \alpha_0^h(q_0,q_1)\,dq_0 + \alpha_1^h(q_0,q_1)\,dq_1$, as in Remark 6.5.2, from the previous discussion, we now substitute the approximated constraint force by:

$$\begin{aligned}\tilde\alpha^h &= \alpha_0^h(q_0,q_1)\,dq_0 \\ &\quad + \mathcal{P}_{|q_1}(\alpha_1^h(q_0,q_1)\,dq_1) + \mathcal{Q}_{|q_1}\left(D_2\tilde S^h(q_0,q_1)\right)\end{aligned}$$

Therefore for $\tilde S^h$ and $\tilde\alpha^h$ equations (6.32) are rewritten as

$$\mathcal{P}_{|q_k}\left(D_2\tilde S^h(q_{k-1},q_k)\right) + D_1\tilde S^h(q_k,q_{k+1}) = \mathcal{P}_{|q_k}\left(\alpha_1^h(q_{k-1},q_k)\right) + \alpha_0^h(q_k,q_{k+1}), \tag{6.45}$$

for $1 \leq k \leq N-1$. The importance of equations (6.45) is that they generate an algorithm which automatically preserves the exact constraint functions Φ^a. In fact, if we apply the projector $\mathcal{Q}$ to Equations (6.45) we obtain:

$$\mathcal{Q}_{|q_k}\left(D_1\tilde{S}^h(q_k,q_{k+1})\right) = \mathcal{Q}_{|q_k}\left(\alpha_0^h(q_k,q_{k+1}\right) \tag{6.46}$$

or

$$\tilde{\varphi}^a(q_k,q_{k+1},h) = \Psi^a(q_k, -\frac{\partial\tilde{S}^h}{\partial q_0}(q_k,q_{k+1}) + \alpha_0^h(q_k,q_{k+1})) = 0$$

that is, the constraints are satisfied.

Therefore the geometric algorithm that we have obtained work as follows:

$$\mathcal{P}_{|q_k}\left(D_2\tilde{S}^h(q_{k-1},q_k)\right) + D_1\tilde{S}^h(q_k,q_{k+1}) = \mathcal{P}_{|q_k}\left(\alpha_1^h(q_{k-1},q_k)\right) + \alpha_0^h(q_k,q_{k+1}),$$

with initial condition satisfying:

$$\tilde{\varphi}^a(q_0,q_1,h) = 0$$

Choosing α_0^h and α_1^h in $\mathcal{D}^0$, we obtain equations for non-holonomic integrators with more geometric flavour:

Geometric non-holonomic integrator

$$\mathcal{P}_{|q_k}\left(D_2\tilde{S}^h(q_{k-1},q_k) + D_1\tilde{S}^h(q_k,q_{k+1})\right) = 0 \tag{6.47}$$

which is interpreted as a discretisation of Equations: (6.40)

$$\bar{\nabla}_{\dot{c}(t)}\dot{c}(t) = -\mathcal{P}(\text{grad}\,(V(c(t)))$$

6.6.1 Non-holonomic integrators preserving constraints

For the class of integrators introduced in Section 6.5.2, we find the following family of non-holonomic integrators preserving constraints:

$$\mathcal{P}_{|q_k}\left(D_2S_\alpha^h(q_{k-1},q_k)\right) + D_1S_\alpha^h(q_k,q_{k+1}) = \alpha h\mathcal{P}_{|q_k}\left(\widetilde{\Lambda}((1-\alpha)q_{k-1}+\alpha q_k, \frac{q_k-q_{k-1}}{h})\right)$$
$$+(1-\alpha)h\widetilde{\Lambda}((1-\alpha)q_k+\alpha q_{k+1}, \frac{q_{k+1}-q_k}{h})\,, \quad 1 \leq k \leq N-1\,,$$

with initial condition satisfying

$$-\mu_i^a(q_0)g^{ij}(q_0)\frac{\partial S_\alpha^h}{\partial q_0^j}(q_0,q_1) + (1-\alpha)h\mu_i^a(q_0)g^{ij}(q_0)\widetilde{\Lambda}_j((1-\alpha)q_0+\alpha q_1, \frac{q_1-q_0}{h})) = 0\,.$$

Example 6.6.1 (The non-holonomic particle revisited). Constructing the previous algorithm for the non-holonomic particle, we obtain the following preserving constraint integrator:

$$
\begin{aligned}
&\frac{1}{1+y_1^2}\left(\frac{x_1-x_0}{h}-h\frac{x_1+x_0}{2}\right)-\frac{x_2-x_1}{h}-h\frac{x_2+x_1}{2}+\frac{y_1}{1+y_1^2}\left(\frac{z_1-z_0}{h}\right)\\
&\quad=-\frac{h}{2}\left[\frac{\frac{1}{1+y_1^2}\cdot\frac{(x_1+x_0)(y_1+y_0)}{2}-\frac{(x_1-x_0)(y_1-y_0)}{h^2}}{1+\left(\frac{y_1+y_0}{2}\right)^2}\cdot\frac{y_1+y_0}{2}+\frac{\frac{(x_2+x_1)(y_2+y_1)}{2}-\frac{(x_2-x_1)(y_2-y_1)}{h^2}}{1+\left(\frac{y_2+y_1}{2}\right)^2}\cdot\frac{y_2+y_1}{2}\right.\\
&\qquad\left.-\frac{y_1}{1+y_1^2}\frac{\frac{(x_1+x_0)(y_1+y_0)}{2}-\frac{(x_1-x_0)(y_1-y_0)}{h^2}}{1+\left(\frac{y_1+y_0}{2}\right)^2}\right]\\
&\frac{y_1-y_0}{h}-h\frac{y_1+y_0}{2}-\frac{y_2-y_1}{h}-h\frac{y_2+y_1}{2}=0\,,\\
&\frac{y_1^2}{1+y_1^2}\left(\frac{z_1-z_0}{h}\right)-\frac{z_2-z_1}{h}+\frac{y_1}{1+y_1^2}\left(\frac{x_1-x_0}{h}-h\frac{x_1+x_0}{2}\right)\\
&\quad=\frac{h}{2}\left[\frac{y_1^2}{1+y_1^2}\frac{\frac{(x_1+x_0)(y_1+y_0)}{2}-\frac{(x_1-x_0)(y_1-y_0)}{h^2}}{1+\left(\frac{y_1+y_0}{2}\right)^2}+\frac{\frac{(x_2+x_1)(y_2+y_1)}{2}-\frac{(x_2-x_1)(y_2-y_1)}{h^2}}{1+\left(\frac{y_2+y_1}{2}\right)^2}\right.\\
&\qquad\left.-\frac{y_1}{1+y_1^2}\frac{\frac{(x_1+x_0)(y_1+y_0)}{2}-\frac{(x_1-x_0)(y_1-y_0)}{h^2}}{1+\left(\frac{y_1+y_0}{2}\right)^2}\cdot\frac{y_1+y_0}{2}\right]\,.
\end{aligned}
$$

with initial condition satisfying

$$
\begin{aligned}
\tilde{\varphi}^a(x_0,y_0,z_0,x_1,y_1,z_1,h)&=-\frac{z_1-z_0}{h}-\frac{h}{2}\frac{\frac{(x_1+x_0)(y_1+y_0)}{2}-\frac{(x_1-x_0)(y_1-y_0)}{h^2}}{1+\left(\frac{y_1+y_0}{2}\right)^2}\\
+\quad y_0&\left[\frac{x_1-x_0}{h}+h\frac{x_1+x_0}{2}-\frac{h}{2}\frac{\frac{(x_1+x_0)(y_1+y_0)}{2}-\frac{(x_1-x_0)(y_1-y_0)}{h^2}}{1+\left(\frac{y_1+y_0}{2}\right)^2}\cdot\frac{y_1+y_0}{2}\right]\,.
\end{aligned}
$$

For the same initial conditions and data, figure 6.6 shows the exact preservation of the constraint function evolution with time of our algorithm.

Remark 6.6.2. In numerical analysis, an approach to the numerical solution of differential equations is by projecting into a subset of invariants. These projection techniques do not deteriorate the convergence order of the method but they can, in some cases, destroy the good long-time behaviour of the solution. However a different behaviour arises in the projection techniques that we have constructed in this section.

Observe that a remarkable feature of non-holonomic systems is the non preservation of the symplectic form, therefore its flow does not act as symplectic transformations; in a geometric way

$$\pounds_{X_{H,M}}\omega_Q=d\Lambda\neq 0$$

In section 6.5.2, we have generated integrators verifying a discrete version of the previous equation. The use of projection techniques are adequate in this case, since we *recover* the geometrical properties of the non-holonomic system.

In the continuous setting it is well known, for instance in [108], how to obtain the solution of non-holonomic systems from the free dynamics using projection techniques. Also, projection

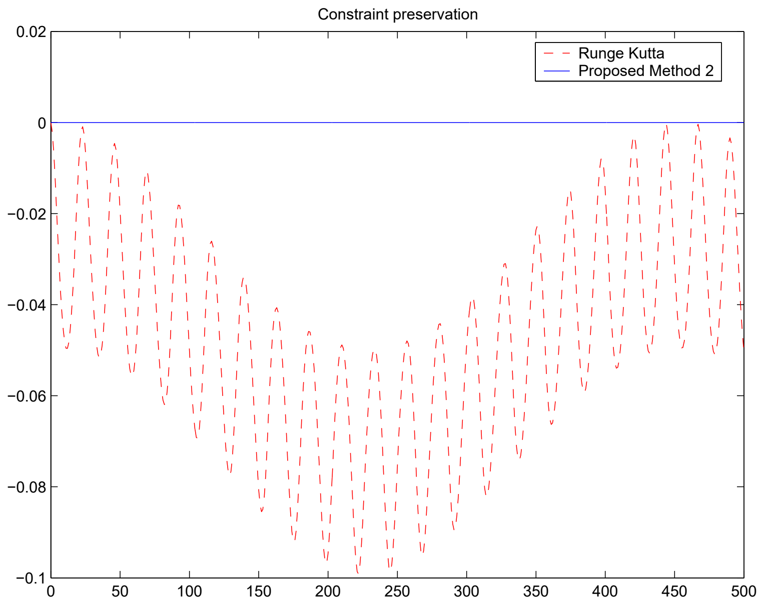

Figure 6.6: Constraint preservation of the new method versus Runge-Kutta

techniques are used in a Riemannian setting, modifying the Levi-Civita connection to obtain an affine connection which gives as the correct dynamics (see [27] and references therein). This is the main idea of the preceding section, where we show that projection techniques are useful for non-holonomic systems. Observe, for instance, the integrator proposed in (6.47). We take a variational integrator and then symplectic (a bad property for a non-holonomic integrator), and projecting orthogonally the discrete Euler-Lagrange equations, we obtain a non-holonomic integrator.

6.7 Applications to optimal control theory

The same kind of arguments applied for generating functions of different kinds can be applied to other problems, such as the optimal control, as we shall see in this section.

Here we give the general solution of an optimization problem for discrete systems and analyze its geometric behaviour, in particular, the symplecticity.

Suppose that the discrete state equations are given by the dynamical equation

$$q^A_{k+1} = f^A(k, q_k, u_k), \quad k = 0, 1, \ldots, N-1, \quad A = 1, \ldots, n \tag{6.48}$$

or, shortly, $q_{k+1} = f(k, q_k, u_k)$, where q_0 is initially given.

The associate performance index or objective function is:

$$J = \bar{S}(N, q(N)) + \sum_{k=0}^{N-1} \bar{L}(k, q_k, u_k) \tag{6.49}$$

where $\bar{S}$ is a function of the final time and state at the final time N, and $\bar{L}$ is time-varying function of the state and control input at each intermediate discrete time k.

The optimal control problem is solved finding controls u_k^*, $k = 0, 1, \dots N-1$, that drive the system along a trajectory q_k^*, $k = 0, 1, \dots, N$, verifying the state equations such that the performance index is minimized.

6.7.1 Problem solution of the discrete optimal control problem

Let us now solve the optimal control problem for the discrete optimal problem determined by (6.48) and (6.49) using the Lagrange multiplier approach. Considering the state Eqs. (6.48) as constraint equations, then we have $N \cdot n$ constraints, and we associate a Lagrange multiplier to each constraint. Next, we construct the augmented performance index J' by

$$J' = \sum_{k=0}^{N-1} \left[p_{k+1}(f(k, q_k, u_k) - q_{k+1}) - \bar{L}(k, q_k, u_k) \right] - \bar{S}(N, q(N)) \tag{6.50}$$

where $p_{k+1} = ((p_{k+1})_A)$ are considered as Lagrange multipliers with $A = 1, \dots, n$ and $k = 0, \dots, N-1$.

Taking the Hamiltonian function

$$\bar{H}(k, q_k, p_{k+1}, u_k) = p_{k+1} f(k, q_k, u_k) - \bar{L}(k, q_k, u_k)$$

we deduce that the necessary conditions for a constrained minimum are thus given by:

$$q_{k+1} = \frac{\partial \bar{H}}{\partial p}(k, q_k, p_{k+1}, u_k) = f(k, q_k, u_k) \tag{6.51}$$

$$p_k = \frac{\partial \bar{H}}{\partial q}(k, q_k, p_{k+1}, u_k) = p_{k+1} \frac{\partial f}{\partial q}(k, q_k, u_k) - \frac{\partial \bar{L}}{\partial q}(k, q_k, u_k) \tag{6.52}$$

$$0 = \frac{\partial \bar{H}}{\partial u}(k, q_k, p_{k+1}, u_k) = p_{k+1} \frac{\partial f}{\partial u}(k, q_k, u_k) - \frac{\partial \bar{L}}{\partial u}(k, q_k, u_k) \tag{6.53}$$

where $0 \le k \le N-1$, and the transversality conditions

$$p_N = -\frac{\partial \bar{S}}{\partial q}(N, q_N) \quad \text{and} \quad q_0 \quad \text{fixed.}$$

Observe that the recursion for the state q_k develops forward in time, but the co-state variable p_k develops backwards in time. Therefore, the required boundary conditions for finding a solution are the initial state q_0 and the final co-state p_N.

Assume that

$$\det \left(\frac{\partial^2 \bar{H}}{\partial u^a \partial u^b} \right) \neq 0$$

then, locally, $u_k^* = h(k, q_k, p_{k+1})$. If we denote, by

$$\tilde{H}(k, q_k, p_{k+1}) = \bar{H}(k, q_k, p_{k+1}, u_k^*)$$

then Eqs. (6.51), (6.52) are rewritten as

$$q_{k+1} = \frac{\partial \tilde{H}}{\partial p}(k, q_k, p_{k+1}) \tag{6.54}$$

$$p_k = \frac{\partial \tilde{H}}{\partial q}(k, q_k, p_{k+1}) \tag{6.55}$$

with $0 \leq k \leq N-1$.

Consider the function

$$G_k(q_k, q_{k+1}, p_{k+1}) \quad = \quad \tilde{H}(k, q_k, p_{k+1}) - p_{k+1}q_{k+1}, \quad 0 \leq k \leq N-1.$$

Then, for a fixed k:

$$dG_k \quad = \quad \frac{\partial \tilde{H}}{\partial q_k}(k, q_k, p_{k+1})\, dq_k + \frac{\partial \tilde{H}}{\partial p_{k+1}}(k, q_k, p_{k+1})\, dp_{k+1} - p_{k+1}\, dq_{k+1} - q_{k+1}\, dp_{k+1} \; .$$

Along solutions of Eqs. (6.51), (6.52) and (6.53) we have:

$$dG_{k|\mathrm{Sol}} \quad = \quad p_k\, dq_k - p_{k+1}\, dq_{k+1} \; ,$$

which implies

$$dp_k \wedge dq_k = dp_{k+1} \wedge dq_{k+1} \; . \tag{6.56}$$

along the solution of (6.51)-(6.53).

In a later subsection, we shall analyze the geometric meaning of (6.56), which it is obviously interpreted as symplecticity of discrete optimal control problems in terms of a natural symplectic form.

6.7.2 Generating functions of the second kind

The construction of more general generating functions will be useful in next sections. For instance, suppose that (q_0, p_1) are independent local coordinates on Graph(g). Then the function S is written as $S = S(q_0, p_1)$.

We have

$$p_1\, dq_1 - p_0\, dq_0 = -q_1\, dp_1 + d(q_1 p_1) - p_0\, dq_0 = dS.$$

If we define

$$S_2(q_0, p_1) = q_1 p_1 - S(q_0, p_1),$$

where q_1 is expressed in terms of q_0 and p_1, then we deduce that

$$q_1\, dp_1 + p_0 dq_0 = dS_2(q_0, p_1)$$

Using the same definition that in [3] we have the following

Definition 6.7.1. *The function $S_2(q_0, p_1)$ will be called a* **generating function of the second kind** *of the symplectomorphism g.*

We have that

$$\begin{cases} p_0 = \dfrac{\partial S_2}{\partial q_0} \\[2ex] q_1 = \dfrac{\partial S_2}{\partial p_1} \end{cases} \tag{6.57}$$

Conversely, if $S_2(q_0,p_1)$ is a generating function such that $\det\left(\dfrac{\partial^2 S_2}{\partial q_0 \partial p_1}\right) \neq 0$ then S_2 is a generating function of some local symplectomorphism determined by Eqs. (6.57) (see [3]).

Denote by $F_{(t,s)} : M_s \to M_t$ the two-parametric group of canonical transformations generated by the Hamiltonian vector field X_H, as in the preliminaries to Proposition 6.3.3. Assume also that for each admissible interval $[s,t]$ the Lagrangian submanifold Graph $F_{(t,s)}$ has a generating function of the second kind. We have the following

Theorem 6.7.2. *Let* $S_2^{(t_{k+1},t_k)}$, $k = 0,\ldots,N-1$, *be generating functions of the second kind. The function defined as*

$$S_2^{(t_N,t_0)}(q_0,p_N) = \sum_{k=0}^{N-1} S_2^{(t_{k+1},t_k)}(q_k,p_{k+1}) - \sum_{k=1}^{N-1} q_k p_k \tag{6.58}$$

where q_k, p_k, $1 \le k \le N-1$, *are stationary points of the right hand side, that is,*

$$q_k = \frac{\partial S_2^{(t_k,t_{k-1})}}{\partial p}(q_{k-1},p_k), \quad 1 \le k \le N-1, \tag{6.59}$$

$$p_k = \frac{\partial S_2^{(t_{k+1},t_k)}}{\partial q}(q_k,p_{k+1}), \quad 1 \le k \le N-1, \tag{6.60}$$

is a generating function of the second kind of $F_{(t_N,t_0)} : M_{t_0} \to M_{t_N}$.

Proof. It follows as in Theorem 6.3.2. ∎

As a consequence, we have that

$$S^{(t_N,t_0)}(q_0,p_N) = q_N p_N - S_2^{(t_N,t_0)}(q_0,p_N) = \sum_{k=0}^{N-1}\left[q_{k+1}p_{k+1} - S_2^{(t_{k+1},t_k)}(q_k,p_{k+1})\right] \tag{6.61}$$

Proposition 6.7.3. *A generating function of the second kind for* $F_{(t_1,t_0)}$ *is given by*

$$S_2^{(t_1,t_0)}(q_0,p_1) = p_1 q_1 - \int_{t_0}^{t_1} \left(p(t)\dot q(t) - H(t,q(t),p(t))\right) dt$$

where $t \to (q(t),p(t))$ *is an integral curve of the Hamilton equations such that* $q(t_0) = q_0$ *and* $p(t_1) = p_1$, *with* t_1 *close enough to* t_0

Proof. It is proved in a similar way to Proposition 6.3.3. ∎

Denote by $S_2(t,q_0,p_1) = S_2^{(t,0)}(q_0,p_1)$ then it is easy to show that (see, for instance [70])

Theorem 6.7.4 (Hamilton-Jacobi equation for S_2). *If* $S_2(t,q_0,p_1)$ *is a solution of the partial differential equation*

$$\frac{\partial S_2}{\partial t}(t,q_0,p_1) = H(t, \frac{\partial S_2}{\partial p_1}(t,q_0,p_1),p_1), \qquad S_2(0,q_0,p_1) = q_0 p_1 \tag{6.62}$$

with $\det\left(\dfrac{\partial^2 S_2}{\partial q_0 \partial p_1}\right) \neq 0$, *then the mapping* $(q_0,p_0) \longrightarrow (q_1,p_1) = (q(t),p(t))$ *defined by Eqs. (6.57) is the exact flow of the Hamiltonian system determined by* H.

$$\begin{aligned} p_0 &= \frac{\partial S_2}{\partial q_0}(t, q_0, p_1) \\ q_1 &= \frac{\partial S_2}{\partial p_1}(t, q_0, p_1) \end{aligned}$$

Differentiating the first equation with respect to t, we obtain

$$0 = \frac{\partial^2 S_2}{\partial t \partial q_0} + \frac{\partial^2 S_2}{\partial q_0 \partial p_1}\dot{p}(t)$$

and using Hamilton-Jacobi equation

$$0 = \frac{\partial H}{\partial q}_{|(t,q(t),p(t))} \frac{\partial^2 S_2}{\partial q_0 \partial p_1} + \frac{\partial^2 S_2}{\partial q_0 \partial p_1}\dot{p}(t)\,,$$

that is, $\dot{p} = -\dfrac{\partial H}{\partial q}$.

Differentiating the second equation of (6.57) and using Hamilton-Jacobi equation, we deduce that:

$$\dot{q}(t) = \frac{\partial^2 S_2}{\partial t \partial p_1} + p(t)\frac{\partial^2 S_2}{\partial p_1^2} = \frac{\partial H}{\partial q}_{|(t,q(t),p(t))} \frac{\partial^2 S_2}{\partial p_1^2} + \frac{\partial H}{\partial p}_{|(t,q(t),p(t))} + \dot{p}(t)\frac{\partial^2 S_2}{\partial p_1^2}\,.$$

Therefore $\dot{q} = \dfrac{\partial H}{\partial p}$. ∎

6.7.3 Generating functions of the second kind and discrete optimal control problems

From Proposition 6.7.3 the following function is a generating function of the second kind for the cosymplectic Hamiltonian system (P_0, η, Ω), which determines the dynamics of the optimal control problem given by (6.7) and (6.8) (see Section 6.2):

$$S_2^{(t_1,t_0)}(q_0, p_1) = p_1 q_1 - \int_{t_0}^{t_1} \big(p(t)\dot{q}(t) - H_{|P_0}(t, q(t), p(t))\big)\, dt\,, \tag{6.63}$$

where $t \to (t, q(t), p(t))$ is the integral curve on P_0 of the vector field X_{P_0}. Here X_{P_0} is the unique solution of equation

$$i_{X_{P_0}}\Omega = dH_{|P_0} \qquad i_{X_{P_0}}\eta = 1$$

with $(q(t_0), p(t_0)) = (q_0, p_0)$ and $(q(t_1), p(t_1)) = (q_1, p_1)$.

We now focus on the construction of a numerical integrator for the Hamiltonian system (P_0, η, Ω) by using an approximation of the generating function. As we shall show, the obtained method also realize the integration steps by symplectomorphism transformations; thus, it is a symplectic integrator.

First take a fixed time interval $h = t_{k+1} - t_k$, $k = 0, \dots, N-1$.

Assume that we are working on vector spaces, and consider the following natural approximation:

$$\begin{aligned}\tilde{S}_2^h(k, q_k, p_{k+1}) &= p_{k+1}q_{k+1} - hp_{k+1}\left(\frac{q_{k+1}-q_k}{h}\right) - h\tilde{L}(k, q_k, p_{k+1}) \\ &\quad + hp_{k+1}\tilde{\Gamma}(k, q_k, p_{k+1})\end{aligned}$$

where, for instance, $\tilde{L}(k, q_k, p_{k+1}) = hL_{|P_0}(t_0+kh, q_k, p_{k+1})$ and $\tilde{\Gamma}(k, q_k, p_{k+1}) = \Gamma_{|P_0}(t_0+kh, q_k, p_{k+1})$.

If we denote by $\tilde{f}(k, q_k, p_{k+1})$ the function

$$\tilde{f}(k, q_k, p_{k+1}) = h\tilde{\Gamma}(k, q_k, p_{k+1}) + q_k \tag{6.64}$$

then,

$$\tilde{S}_2^h(k, q_k, p_{k+1}) = p_{k+1}\tilde{f}(k, q_k, p_{k+1}) - \tilde{L}(k, q_k, p_{k+1}) = \tilde{H}(k, q_k, p_{k+1})\ .$$

Thus, equations

$$\left\{\begin{aligned} p_k &= \frac{\partial \tilde{S}_2^h}{\partial q^k}(k, q_k, p_{k+1}) = \frac{\partial \tilde{H}}{\partial q^k}(k, q_k, p_{k+1}) \\ q_{k+1} &= \frac{\partial \tilde{S}_2^h}{\partial p_{k+1}}(k, q_k, p_{k+1}) = \frac{\partial \tilde{H}}{\partial p_{k+1}}(k, q_k, p_{k+1}) \end{aligned}\right. \tag{6.65}$$

are exactly (6.54) and (6.55) and the symplecticity condition (6.56) for discrete optimal control problems is now a trivial consequence of the generating function construction.

Remark 6.7.5. It is also possible to construct symplectic numerical methods of higher order; for instance, considering better approximations of the Hamilton Jacobi equation (6.62) (see [24] and references therein). Assume for simplicity that the Hamiltonian is autonomous, that is, $H \equiv H(q, p)$. Now, first expand the generating function $S_2(t, q_0, p_1)$ as:

$$S_2(t, q_0, p_1) = q_0 p_1 + \sum_{i=1}^{\infty} t^i G_i(q_0, p_1),$$

apply this expression into Hamilton-Jacobi equation (6.62), and compare equal powers of t. This yields

$$\begin{aligned} G_1(q_0, p_1) &= H(q_0, p_1) \\ G_2(q_0, p_1) &= \frac{1}{2}\left(\frac{\partial H}{\partial q_0^A}\frac{\partial H}{\partial p_{1A}}\right) \\ G_3(q_0, p_1) &= \frac{1}{6}\left(\frac{\partial^2 H}{\partial p_{1A}\partial p_{1B}}\frac{\partial H}{\partial q_0^A}\frac{\partial H}{\partial q_0^B} + \frac{\partial^2 H}{\partial p_{1A}\partial q_0^B}\frac{\partial H}{\partial q_0^A}\frac{\partial H}{\partial p_{1B}} + \frac{\partial^2 H}{\partial q_0^A\partial q_0^B}\frac{\partial H}{\partial p_{1A}}\frac{\partial H}{\partial p_{1B}}\right) \\ \cdots &= \cdots \end{aligned}$$

Using the truncated series, we obtain an approximated generating function:

$$S_2^h(q_k, p_{k+1}) = q_k \cdot p_{k+1} + \sum_{i=1}^{r} h^r G_i(q_k, p_{k+1})$$

which defines a symplectic method of order r.

Other approaches are also admissible without using higher derivatives of the Hamiltonian H, for instance, symplectic or symplectic partitioned Runge-Kutta methods (see [70, 145]).

6.8 Discrete Hamiltonian systems

In [48] Erbe and Yan have considered discrete linear Hamiltonian systems of the form:

$$\begin{aligned}\Delta y(t) &= B(t)y(t+1) + C(t)z(t)\\ \Delta z(t) &= -A(t)y(t+1) - B^T(t)z(t)\end{aligned}$$

where A, C are symmetric and $I-B$ is invertible. Here $\Delta y(t) = y(t+1)-y(t)$, $\Delta z(t) = z(t+1)-z(t)$ and $y, z \in \mathbb{R}^d$.

This problem is a particular case of a discrete Hamiltonian systems of the form

$$\Delta y(t) = H_z(t, y(t+1), z(t)) \tag{6.66}$$

$$\Delta z(t) = -H_y(t, y(t+1), z(t)) \tag{6.67}$$

where $H(t, y, z) = \frac{1}{2}(y^T, z^T)\begin{pmatrix} A(t) & B^T(t) \\ B(t) & C(t)\end{pmatrix}\begin{pmatrix} y \\ z\end{pmatrix}$. The symplecticity of the discrete linear Hamiltonian system was fully studied (see [48], for instance, and references therein). The existence of a corresponding symplectic structure for discrete nonlinear Hamiltonian systems given by (6.66) and (6.67) was proposed by Ahlbrandt as an open problem ([2] and also [152]).

From the point of view of section 6.3, this problem is easily solved considering as generating function of the second kind the following one:

$$S_2^{(t+1,t)}(y(t+1), z(t)) = z(t)y(t+1) - H(t, y(t+1), z(t)) \,.$$

Then Eqs. (6.66) and (6.67) are precisely

$$\left\{\begin{aligned} y(t) &= \frac{\partial S^{(t+1,t)}}{\partial z}(y(t+1), z(t))\\ z(t+1) &= \frac{\partial S^{(t+1,t)}}{\partial y}(y(t+1), z(t)) \,,\end{aligned}\right.$$

which guarantees the symplecticity of the discrete Hamiltonian system. In order to find the canonical transformation associated to this generating map it is only necessary to impose the local condition (see [3]):

$$\det\left(\frac{\partial^2 S_2^{(t+1,t)}}{\partial y \partial z}(y(t+1), z(t))\right) \neq 0$$

Then, in a neighbourhood of a point satisfying the above condition, there exists a symplectomorphism defined by Eqs. (6.66) and (6.67).

CHAPTER 7

Conclusion. Future work

With the ideas exposed in this thesis, I have given some steps towards an ambitious program aimed to develop the multisymplectic geometry as a natural ground for the classical field theory equations, as much as symplectic geometry was for the classical mechanics. The concepts of Darboux coordinates for multisymplectic geometries, and of symmetries, preserved quantities and Tulczyjew's triples have been introduced and described. Finally, a detailed study of the geometry of the spaces of Cauchy data has also been developed.

I have as well described how to exploit the geometric properties of these equations in order to produce better numerical methods, in particular, of the concept of generating functions. To illustrate this, two cases (nonholonomic mechanics and optimal control theory) have been analised and compared to other classical methods.

I shall finish the work by mentioning open and unknown areas of study that are left for future works on this area. Some of the concepts here are presently being developed by our own group or other researchers.

On a first group I can mention the extension of several well-known concepts and geometrical objects for the symplectic geometry in terms of multisymplectic geometry whenever possible, such as the Hamilton-Jacobi theorem, the generating functions, Skinner and Rusk formalism, optimal control theory, or the reduction and reconstruction of the dynamics, and their relation to the infinite-dimensional precosymplectic counterparts in the space of Cauchy data.

Secondly, the detailed analysis of the higher order jet manifolds, and field theories, with the study of the symmetries and preserved quantities, and the extension of the Cauchy surfaces.

Also the detailed case study of some of the classical field theories, adapted to their particularities, such as the fluid equations and continuum media, or the general relativity.

Finally, regarding the numerical methods, the development of a new series of numerical methods for the classical field theories based on generating functions, inspired on those presented in the last chapter here, and to be compared with classical methods. This topic in particular is the object of the sketch provided in the following section. Also a detailed study of the numerical properties of the numerical methods that we have introduced.

7.1 Numerical methods in classical field theories

The object of study consists of applying the methods based on generating functions described in the last chapter to the notion of generating function for the presymplectic form $\tilde{\Omega}_h$ in $\tilde{Z}^*$. A first step is to verify that the ideas provided in the section 6.3 hold for the infinite dimensional setting.

Suppose that $C = [t_0, t_1] \times M$ is a compact on X. With the theory of Cauchy surfaces developed in chapter 4, elements of $\tilde{X} \equiv I$ are identified. Therefore, we can define, for a given section ϕ, the mappings $\phi_0 := \phi \circ t_0$ and $\phi_1 := \phi \circ t_1$ of $\tilde{Y} \equiv I \times Q$, which project onto points q_0 and q_1 on Q, respectively.

The curve $\hat{c}(t) = \phi \circ t$ on $\tilde{Y}$ projects onto a curve $c(t)$ on Q which joins q_0 to q_1. In particular, we have that $\phi(t, u) = c(t)(u)$.

But we also know that the flow of the 1-jet prolongation of a vector field on Y is the 1-jet prolongation of its flow, so $c(t)$ lifts to a curve $\tilde{c}(t)$ on TQ, having a regular Lagrangian $\tilde{L}$, and using the equivalence theorem of the De Donder equations of the finite and infinite dimensional settings, we have that ϕ is a solution of the Euler-Lagrange equations if and only if the curve on TQ

$$\tilde{c}(t)(u) = j^1\phi(t, u)$$

is the flow of a solution for the evolution equation on $I \times TQ$ associated to $\tilde{L}$.

Therefore, the action can be rewritten as follows, using Fubini's theorem

$$\begin{aligned}
S(\phi) &= \int_{[t_0,t_1]\times M} j^1\phi^*\mathcal{L} = \int_{[t_0,t_1]\times M} \tilde{c}(t)^*\mathcal{L} = \int_{[t_0,t_1]\times M} \tilde{c}(t)^*(L)\eta \\
&= \int_{[t_0,t_1]\times M} L(\tilde{c}(t)(u))\eta_M \wedge dt = \int_{[t_0,t_1]} \left[\int_M L(\tilde{c}(t)(u))\eta_M\right] dt \\
&= \int_{[t_0,t_1]} \tilde{L}(\tilde{c}(t))dt = S_1^{t_1-t_0}(q_0, q_1)
\end{aligned}$$

7.1.1 Geometric numerical methods based on generating functions

In order to implement symplectic numerical methods for field theory, we just have to start with an appropriate (in some sense) discretisation of the action integral.

To start with, consider a grid $\{u_1, \ldots, u_K\}$ of M, and we shall use the notation, for $q^i \in Q$, $q^i_l = q^i(u_l)$.

With that grid over M we define a suitable approximation of the action integral

$$S_1(q_0, q_1) = S(\phi) = \int_{[t_0,t_1]\times M} j^1\phi^*\mathcal{L}$$

for example (remember that $vol(M) = 1$, and denote $k = t_1 - t_0$),

$$S_d(q_0, q_1) = kL(t_0, q_0 = (q_0^1, \dots, q_0^K), \frac{q_1 - q_0}{k})$$

(where we assume Q to be a vector space). The condition of extremality on q_i on Theorem 6.3.2, in which

$$0 = D_2 S_1^t(q_{k-1}, q_k) + D_1 S_1^t(q_k, q_{k+1}), \quad 1 \leq k \leq N-1.$$

can then be read in terms of extremality of the image points $(q_1^i, \dots, q_K^i)$ on the action integral. The resulting equations above are known as the **Discrete Euler-Lagrange equations** for field theory (see [131]).

7.1.2 Examples

In [131], the authors propose the following discretisation for the case of a 1-dimensional manifold M, which is therefore a closed interval. In this case, $X = \mathbb{R} \times M$. We consider a grid $\{u_1, \dots, u_K\}$ of the interval M.

First, fix a temporal step k, and divide the compact rectangle $[t_0, t_1] \times M$ into smaller rectangles $(u_i^0, u_{i+1}^0, u_i^1, u_{i+1}^1)$, and each of these rectangles into two triangles, $(u_i^0, u_i^1, u_{i+1}^1)$ and $(u_i^0, u_{i+1}^0, u_{i+1}^1)$.

Let U be the set of all these triangles, that cover the rectangle $[t_0, t_1] \times M$ (in such a way that two triangles have in common at most one vertex or one side).

Second, propose a discretisation of the Lagrangian function $L : Z \longrightarrow \mathbb{R}$, given by a function $L_d : X^3 \times Q^3 \longrightarrow \mathbb{R}$, $L_d(x^1, x^2, x^3, q^1, q^2, q^3) \in \mathbb{R}$.

Finally, the proposed approximation to the action is given by

$$S_d(\phi) = \sum_{\Delta \in U} L_d(\Delta, \phi(\Delta))$$

where if $\Delta = (x_1, x_2, x_3)$, $\phi(\Delta)$ stands for $(\phi(x_1), \phi(x_2), \phi(x_3))$.

The resulting extremality equations are the Euler-Lagrange equations proposed by the authors.

Bibliography

[1] R. Abraham, J. E. Marsden: *Foundations of Mechanics.* Addison-Wesley, Reading, Massachusetts (USA), 1978.

[2] C. D. Ahlbrandt: Equivalence of Discrete Euler Equations and Discrete Hamiltonian Systems *J. Math. Anal. Appl.* **180** (1993), 498-478

[3] V. I. Arnold: *Mathematical Methods of Classical Mechanics* Graduate Text in Mathematics 60, Springer-Verlag New York, 1978.

[4] J. C. Baez, J. W. Gilliam: An algebraic approach to discrete mechanics *Lett. Math. Phys.* **31** (1994), 205–212

[5] L. Bates, J. Śniatycki: Nonholonomic reduction *Rep. Math. Phys.* **32** (1) (1992), 99-115

[6] C. Batle, J. Gomis, J.M. Pons: Hamiltonian and Lagrangian constraints of the bosonic string *Physical Review D* **34** (8) (1986), 2430–2432.

[7] D.E. Betounes: Extension of the Classical Cartan form *Phys. Rev. D*, **29** (4) (1984), 599–606.

[8] E. Binz, M. de León, D. Martín de Diego, D. Socolescu: *Rep. Math. Phys.* **49** (2002) 151–166

[9] E. Binz, J. Sniatycki, H. Fischer: Nonholonomic constraints in classical field theories *Geometry of classical fields.* North-Holland, Amsterdam, 1988.

[10] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, R. M. Murray: Nonholonomic mechanical systems with symmetry *Arch. Rational Mech. Anal.* **136** (1996), 21–99

[11] A. I. Bobenko, Y. B. Suris: Discrete Lagrangian reduction, discrete Euler-Poincar equations, and semidirect products *Lett. Math. Phys.* **49** (1999), 79-93

[12] A. I. Bobenko, Y. B. Suris: Discrete time Lagrangian mechanics on Lie groups, with an application to the Lagrange top *Comm. Math. Phys.* **204** (1999), 147-188

[13] R.L. Bryant, S.S. Chern, R.B. Gardner, H.L. Goldschmidt, P.A. Griffiths : *Exterior differential systems*. Mathematical Sciences Research Institute Publications, 18. Springer-Verlag, New York, 1991.

[14] C. J. Budd, M. D. Piggott: Geometric integration and its applications *Handb. Numer. Anal.* **XI** (2003), 35–139

[15] J. A. Cadzow: Discrete calculus of variations *Intern. J. Control.* **11** (1970), 393-407

[16] F. Cantrijn, A. Ibort, M. de León: On the geometry of multisymplectic manifolds *J. Austral. Math. Soc. (Series A)* **66** (1999), 303–330

[17] F. Cantrijn, A. Ibort, M. de León: Hamiltonian structures on multisymplectic manifolds *Rend. Sem. Mat. Univ. Politec. Torino* **54** (3) (1996), 225–236

[18] F. Cantrijn, M. de León, E. A. Lacomba: Gradient vector fields on cosymplectic manifolds *J. Phys. A: Math. Gen.* **25** (1992), 175–188

[19] F. Cantrijn, M. de León, J. C. Marrero, D. Martín de Diego: Reduction of constrained systems with symmetries *J. Math. Phys.* **40**, N. 2 (1999), 795–820

[20] F. Cantrijn, M. de León, J. C. Marrero, D. Martín de Diego: Reduction of nonholonomic mechanical systems with symmetries *Rep. Math. Phys.* **42**, N. 1/2 (1998), 25–45

[21] J. F. Cariñena, M. Crampin, L.A. Ibort: On the multisymplectic formalism for first order field theories *J. Geom. Phys.* **3** (1986), 353–400.

[22] M. Castrillón López, P. L. García Pérez, T. Ratiu: Euler-Poincar reduction on principal bundles. *Lett. Math. Phys.* **58** (2001), no. 2, 167–180.

[23] M. Castrillón López, T. Ratiu, S. Shkoller: Reduction in principal fiber bundles: covariant Euler-Poincar equations. *Proc. Amer. Math. Soc.* **247** (7) (2000), 2155–2164.

[24] P. J. Channell, C. Scovel: Symplectic integration of Hamiltonian Systems *Nonlinearity* **3** (1990), 231-259

[25] D. Chinea, M. de León, J.C. Marrero: The constraint algorithm for time-dependent Lagrangians *J. Math. Phys.* **35** (7) (1994), 3410–3447.

[26] J. Cortés: *Geometric, control and numerical aspects of nonholonomic systems* Lecture Notes in Mathematics, vol. 1793, Springer-Verlag, 2002.

[27] J. Cortés: Energy conserving nonholonomic integrators, *Discrete and Continuous Dynamical Systems*, added volume (2003), 189–199

[28] J. Cortés, S. Martínez: Nonholonomic integrators *Nonlinearity* **14** (2001), 1365-1392

[29] M. Crampin: The second variation formula in Lagrange and Finsler geometry *Houston J. Math.* **26** (2) (2000), 255–275

[30] M. Crampin, F. Cantrijn, W. Sarlet: Lifting geometric objects to a cotangent bundle, and the geometry of the cotangent bundle of a tangent bundle *J. Geom. Phys.* **4** (1987), **4**, 469–492

[31] M. Crampin, D. Saunders: The Hilbert-Carathodory form for parametric multiple integral problems in the calculus of variations *Acta Appl. Math.* **76** (1) (2003), 37–55.

[32] R. Cushman, D. Kemppainen, J. Śniatycki, L. Bates: Geometry of nonholonomic constraints *Rep. Math. Phys.* **36** (2/3) (1995), 275–286

[33] M. Delgado Téllez de Cepeda: *Métodos geométricos en problemas de control óptimo singulares: fundamentos y aplicaciones.* Tesis Doctoral, Universidad Carlos III (Madrid), 2004

[34] P. A. M. Dirac *Lecture on Quantum Mechanics* Belfer Graduate School of Science, Yeshiva University, New York, 1964.

[35] M. Doupovec, A. Vondra: Some natural operations between connections on fibered manifolds *Rend. Circ. Mat. Palermo (2)* **39** (1996) 73–84

[36] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy: A geometrical analysis of the field equations in field theory *Int. J. Maths. and Math. Sciences*, **29** (12) (2002) 687–700

[37] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy: Geometry of Lagrangian First-order Classical Field Theories *Fortschr. Phys.* **44** (1996) 3, 235–280.

[38] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy: Geometry of multisymplectic Hamiltonian first-order field theories *J. Math. Phys.* **41** (11) (2000), 7402–7444.

[39] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy: Multivector field formulation of Hamiltonian field theories: equations and symmetries *J. Phys. A* **32** (48) (1999), 8461–8484.

[40] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy: Multivector fields and connections. Applications to field theories *RACSAM Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Mat.* **96** (1) (2002), 83–93.

[41] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy: Multivector fields and connections: setting Lagrangian equations in field theories *J. Math. Phys.* **39** (9) (1998), 4578–4603.

[42] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy: On the multimomentum bundles and the Legendre maps in field theories *Rep. Math. Phys.* **45** (1) (2000), 85–105.

[43] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy: Reduction of presymplectic manifolds with symmetry *Rev. Math. Phys.* **11** (10) (1999), 1209–1247.

[44] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, A. Ibort: Invariant forms and automorphisms of multisymplectic manifolds *arXiv:math.DG/9805040 v2.*

[45] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, J. Marín-Solano: Sections along maps in field theories: the covariant field operators *arXiv:math-ph/0103019 v2*

[46] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, C. López, J. Marín-Solano: Lagrangian-Hamiltonian unified formalism for field theory *J. Math. Phys.* **45** (1) (2004), 360–380.

[47] R. J. Eden: The Hamiltonian dynamics of non-holonomic systems *Proc. Roy. Soc. London, Ser.* A 205, (1951) 564–583

[48] L. H. Erbe, P. Yan: Disconjugancy for linear Hamiltonian difference systems *J. Math. Anal. Appl.* **167** (1992), 355-367

[49] A. Fernández, P. L. García-Pérez, C. Rodrigo: Lagrangian reduction and constrained variational calculus on: Proceedings of the IX Fall Workshop on Geometry and Physics, Vilanova i la Geltrú, 2000 *Publicaciones de la RSME* **3** (2002) 53–64

[50] M. Forger, C. Paufler, H. Römer: A general construction of Poisson brackets on exact multisymplectic manifolds *Rep. Math. Phys.* **51** (2-3) (2003), 187–195.

[51] M. Forger, C. Paufler, H. Römer: The Poisson bracket for Poisson forms in multisymplectic field theory *Rev. Math. Phys.* **15** (2003), no. 7, 705–743

[52] P. L. García-Pérez: The Poincaré-Cartan invariant in the calculus of variations *Symp. Math.* **XIV** (1974), 219–246.

[53] P.L. García-Pérez, A. Pérez-Rendón: Symplectic approach to the theory of quantized fields, I *Comm. Math. Phys.* **13** (1969), 24–44.

[54] P.L. García-Pérez, A. Pérez-Rendón: Symplectic approach to the theory of quantized fields, II *Archive Rat. Mech. Anal.* **43** (1971), 101–124.

[55] G. Giachetta, L. Mangiarotti, G Sardanishvily: Covariant Hamilton equations for field theory *J. Phys. A* **32** (38) (1999), 6629–6642.

[56] G. Giachetta, L. Mangiarotti, G. Sardanashvily: *New Lagrangian and Hamiltonian methods in field theory.* World Scientific Publishing Co., Inc., River Edge, NJ, 1997.

[57] J. W. Gilliam: *Lagrangian and symplectic techniques in discrete mechanics* PhD thesis, University of California Riverside, Department of Mathematics, 1996.

[58] H. Goldschmidt, S. Sternberg: The Hamilton-Cartan formalism in the calculus of variations *Ann. Inst. Fourier (Grenoble)* **23** (1975), 203–267.

[59] O. González: Mechanical systems subject to holonomic constraints: Differential-algebraic formulations and conservative integration *Physica D* **132** (1999), 165-174

[60] M. J. Gotay: A multisymplectic framework for classical field theory and the calculus of variations I: Covariant Hamiltonian formalism: Mechanics, analysis and geometry. In: 200 years after Lagrange, *North-Holland Delta Ser.*, North-Holland, Amsterdam (1991), 160–188.

[61] M. J. Gotay: A multisymplectic framework for classical field theory and the calculus of variations II: Space+Time decomposition *Differential Geom. Appl.* **1** (1997), 375–390.

[62] M. J. Gotay: An exterior differential systems approach to the Cartan form. Symplectic geometry and mathematical physics (Aix-en-Provence, 1990), 160–188, *Progr. Math.*, **99**, Birkhäuser Boston, Boston, MA, USA, 1991.

[63] M. J. Gotay: *Presymplectic Manifolds, Geometric Constraint Theory and the Dirac-Bergmann Theory of Constraints*, Dissertation, Center for Theoretical Physics, University of Maryland, 1979.

[64] M.J. Gotay, J. Isenberg, J.E. Marsden: *Momentun maps and classical relativistic fields, Part I: Covariant field theory.* MSRI preprint (1997).

[65] M.J. Gotay, J. Isenberg, J.E. Marsden: *Momentun maps and classical relativistic fields, Part II: Canonical analysis of field theories.* MSRI preprint (1999).

[66] M.J. Gotay, J. Nester, Presymplectic Lagrangian systems I: the constraint algorithm and the equivalence theorem *Ann. Inst. Henri Poincaré* **A 30** (1978), 129–142.

[67] M.J. Gotay, J. Nester, Presymplectic Lagrangian systems II: the second order differential equation problem *Ann. Inst. Henri Poincaré* **A 32** (1980), 1–13.

[68] M.J. Gotay, J. Nester, G. Hinds: Presymplectic manifolds and the Dirac-Bergmann theory of constraints. *J. Math. Phys.* **19** (1978), no. 11, 2388–2399.

[69] X. Gràcia, J.M. Pons: Symmetries and infinitesimal symmetries of singular differential equations *J. Phys. A: Math. Gen.* **35** (2002), 5059–5077.

[70] E. Hairer, C. Lubich, G. Wanner: *Geometric Numerical Integration, Structure-Preserving Algorithms for Ordinary Differential Equations* Springer Series in Computational Mathematics **31**, Springer-Verlag Berlin Heidelberg, 2002.

[71] A. Haková, O. Krupková: Varitational first-order partial differential equations *J. Differential Equations* **191** (2003) 67–89.

[72] F. Hélein: Hamiltonian formalisms for multidimensional calculus of variations and perturbation theory. *arXiv:math-ph/0212036 v1.*

[73] S. Hrabak: On a Multisymplectic Formulation of the Classical BRST symmetry for First Order Field Theories Part I: Algebraic Structures *math-ph/9901012.*

[74] S. Hrabak: On a Multisymplectic Formulation of the Classical BRST Symmetry for First Order Field Theories Part II: Geometric Structures *math-ph/9901013.*

[75] G. Jaroszkiewicz, K. Norton: Principes of discrete time mechanics I: Particle systems *J. Phys. A* **30** (1997), 3115-3144

[76] G. Jaroszkiewicz, K. Norton: Principes of discrete time mechanics II: Classical field theory *J. Phys. A* **30** (1997), 3145-3163

[77] B. W. Jordan, E. Polak: Theory of a class of discrete optimal control systems *J. Electron. Control* **17** (1964), 697-711

[78] C. Kane, J. E. Marsden, M. Ortiz: Symplectic-energy-momentum integrators *J. Math. Phys.* **40**, (1999) 3353-3371

[79] C. Kane, J. E. Marsden, M. Ortiz, M. West: Variational integrators and the Newmark algorithm for conservative and dissipative mechanical systems *Internat. J. Numer. Math. Eng.* **49** (2000), 1295-1325

[80] J. Koiller: Reduction of some classical non-holonomic systems with symmetry *Arch. Rational Mech. Anal.* **118** (1992), 113-148

[81] W. S. Koon, J. E. Marsden: The Hamiltonian and Lagrangian Approaches to the Dynamics of Nonholonomic Systems *Rep. on Math. Phys.* **40** (1) (1997), 21-62

[82] I.V. Kanatchikov: Canonical structure of classical field theory in the polymomentum phase space *Rep. Math. Phys.* **41** (1) (1998), 49–90.

[83] J. Kijowski: A finite-dimensional canonical formalism in the classical field theory. *Comm. Math. Phys.* **30** (1973), 99-128.

[84] J. Kijowski, W. Szczyrba: Multisymplectic manifolds and the geometric construction of the Poisson bracket in field theory. In: *Géométrie Symplectique et Physique Mathématique*, CNRS, Paris, 1975, pp. 347-378.

[85] J. Kijowski, W. Tulczyjew: *A symplectic framework for field theories*, Lecture Notes in Physics, **107**, Springer-Verlag, New York, 1979.

[86] D. Krupka: *Some geometric aspects of variational problems in fibered manifolds.* Folia Fac. Sci. Nat. UJEP Brunensis **14** (1973), 1–65.

[87] D. Krupka: The contact ideal *Differential Geom. Appl.* **5** 3 (1995), 257–276.

[88] O. Krupková, Hamiltonian field theory *J. Geom. Phys.* **43** (2002), 93-132.

[89] O. Krupková, *The Geometry of Ordinary Variational Equations*, Lecture Notes in Mathematics 1678, Springer, Berlin, 1997.

[90] O. Krupková, D. Smetanová: On regularization of variational problems in first-order field theory *Rend. Circ. Mat. Palermo (2)* **66** (2001) 133–140

[91] O. Krupková, D. Smetanová: Legendre transformation for Regularizable Lagrangians in Field Theory *Lett. Math. Phys.* **58** (2001) 189–204

[92] J.K. Lawson: A frame bundle generalization of multisymplectic geometries *Rep. Math. Phys.* **45** (2000), no. 2, 183–205.

[93] T. D. Lee: Can time be a discrete dynamical variable? *Phys. Lett.*, **122B** (1983), 217-220

[94] T. D. Lee: Difference equations and conservation laws *J. Statis. Phys.*, **46** (1987), 843-860

[95] M. Leok: *Foundations of Computational Geometric Mechanics.* Tesis Doctoral, CALTECH (Pasadena, California, USA), 2004

[96] M. de León: Media with microstructure: constitutive theory and dynamics. In: *Proceedings of the IX Fall Workshop on Geometry and Physics*, Real Sociedad Matemática Española, Madrid, (2001) 107–118.

[97] M. de León, J. Cortés, D. Martín de Diego, S. Martínez: An introduction to mechanics with symmetry *Recent advances in Lie theory* (Vigo, 2000), 305–332, In *Res. Exp. Math.*, 25, Heldermann, Lemgo, 2002.

[98] M. de León, M. McLean, L.K. Norris, A.M. Rey, M. Salgado: Geometric Structures in Field Theory. *arXiv:math-ph/0208036.*

[99] M. de León, J. Marín-Solano, J.C. Marrero: A geometrical approach to Classical Field Theories: a constraint algorithm for singular theories. In: *New Developments in Differential Geometry (Debrecen, 1994), Math. Appl.*, **350**, Kluwer Acad. Publ., Dordrecht, 1996, pp. 291–312.

[100] M. de León, J. Marín-Solano, J.C. Marrero: Ehresmann connections in Classical Field Theories. In: *Proceedings of the III Fall Workshop: Differential Geometry and its Applications*, Granada, September 19-20, (1994), Anales de Física, Monografías, **2**, (1995). pp. 73–89.

[101] M. de León, J. Marín-Solano, J.C. Marrero: The constraint algorithm in the jet formalism *Differential Geom. Appl.* **6** (1996) 275–300.

[102] M. de León, J. Marín-Solano, J.C. Marrero, M. Muñoz-Lecanda, N. Román-Roy: Singular Lagrangian systems on jet bundles *Forschritte für Physik* **50** (2) (2002), 105–169.

[103] M. de León, J.C. Marrero: Constrained time-dependent Lagrangian systems and Lagrangian submanifolds *J. Math. Phys.* **34** (2) (1993), 622–644.

[104] M. de León M, J. C. Marrero, D. Martín de Diego: A new geometric setting for classical field theories, on *Classical and quantum integrability (Warsaw, 2001)* 189–209, Banach Center Publ., **59**, Polish Acad. Sci., Warsaw, 2003.

[105] M. de León M, J. C. Marrero, D. Martín de Diego: Mechanical systems with non-linear constraints *Int. J. Theor. Phys.* **36** (4) (1997), 973-989

[106] M. de León, J. C. Marrero, D. Martín de Diego: Time-dependent constrained Hamiltonian systems and Dirac brackets *J. Phys. A : Math. Gen.* (1996) **29** 6843-6859

[107] M. de León, D. Martín de Diego: Almost product structures and Poisson reduction of presymplectic systems *Extracta Mathematicae* **10** (1) (1995), 37–45.

[108] M. de León, D. Martín de Diego: On the geometry of non-holonomic Lagrangian systems *J. Math. Phys.* **37** (7) (1996), 3389-3414

[109] M. de León, D. Martín de Diego: Solving non-holonomic Lagrangian dynamics in terms of almost product structures *Extracta Mathematicae*, **11** (2) (1996) 325-347

[110] M. de León, D. Martín de Diego: Symmetries and Constants of the Motion for higher-order Lagrangian Systems *J. Math. Phys.* **36** (8) (1995), 4138–4161.

[111] M. de León, D. Martín de Diego: Symmetries and Constants of the Motion for Singular Lagrangian Systems *Int. J. Theor. Phys.* **35** (5) (1996), 975–1011.

[112] M. de León, D. Martín de Diego: Variational integrators and time-dependent Lagrangian systems *Rep. on Math. Phys* **49** 2/3 (2002), 183-192

[113] M. de León, D. Martín de Diego, A. Santamaría-Merino: Discrete optimal control theory and symplectic integrators *To Appear in Adv. Comput. Math.*

[114] M. de León, D. Martín de Diego, A. Santamaría-Merino: Geometric numerical integration of nonholonomic systems and optimal control problems *2nd IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control, Seville* (2003), 163-168

[115] M. de León, D. Martín de Diego, A. Santamaría-Merino: Symmetries in Classical Field Theory *To Appear in Int. J. Geom. Methods Mod. Phys.*

[116] M. de León, D. Martín de Diego, Aitor Santamaría–Merino: Tulczyjew' s triples and Lagrangian submanifolds in classical field theory. In: *Applied Differential Geometry and Mechanics*, W. Sarlet and F. Cantrijn (eds). Universiteit Gent, 2003. pp. 21-47.

[117] M. de León, P. R. Rodrigues: *Generalized Classical Mechanics and Field Theory.* North-Holland Math. Studies, 112, Amsterdam, l985.

[118] M. de León, P. R. Rodrigues: *Methods of Differential Geometry in Analytical Mechanics* North-Holland Math. Ser. 152 Amsterdam: North-Holland, 1989.

[119] S. Maeda: Canonical structure and symmetries for discrete systems *Math. Japonica* **25** (1980), 405-420

[120] S. Maeda: Extension of discrete Noether theorem *Math. Japonica* **26** (1981), 85-90

[121] Ch. M. Marle: Reduction of constrained mechanical systems and stability of relative equilibria *Commun. Math. Phys.* **174** (1995), 295-318

[122] G. Marmo, E. J. Saletan, A. Simoni, B. Vitale: *Dynamical systems: A differential geometric approach to Symmetry and Reduction* John Wiley & Sons, Great Britain, 1985.

[123] G. Marmo, W. M. Tulczyjew, P. Urbański: Dynamics of autonomous systems with external forces *Acta Phys. Polon. B* **33** (2002), no. 5, 1181–1240.

[124] J. E. Marsden *Park City Lectures on Mechanics, Dynamics and Symmetry, in Symplectic Geometry and Topology* Y. Eliashberg and L. Traynor, eds, American Mathematical Society, Providence, R1, Vol. 7 of IAS/Park City Math. Ser. (1999), 335-430

[125] J. E. Marsden, G. W. Patrick, S. Shkoller: Multisymplectic geometry, variational integrators, and nonlinear PDEs *Comm. Math. Phys.* **199** (1998), 351- 395

[126] J. E. Marsden, S. Pekarsky, S. Shkoller: Discrete Euler-Poincaré and Lie-Poisson equations *Nonlinearity* **12** (1999), 1647-1662.

[127] J.E. Marsden, S. Pekarsky, S. Shkoller: Multisymplectic Geometry, Variational Integrators and Nonlinear PDEs *Comm. Math. Phys.* **199** (1998), 351–391

[128] J. E. Marsden, S. Pekarsky, S. Shkoller: Symmetry reduction of discrete Lagrangian mechanics on Lie groups *J. Geom. Phys.* **36** (1999), 140-151

[129] J. E. Marsden, S. Pekarsky, S. Shkoller, M. West: Variational methods, multisymplectic geometry and continuum mechanics *J. Geom. Phys.* **38** (2001), 253–284

[130] J. E. Marsden, T. S. Ratiu: *Introduction to Mechanics and Symmetry*, Texts in Applied Mathematics **17** Springer-Verlag, 1994; Second Edition, 1999.

[131] J. E. Marsden, M. West: Discrete mechanics and variational integrators *Acta Numerica* (2001), 357-514

[132] G. Martin: A Darboux theorem for multi-symplectic manifolds *Letters in Math. Phys.* **16** (1988), 133–138.

[133] G. Martin: Dynamical structures for k-vector fields *Int. J. Theor. Phys.* **27**, 5 (1988), 571–585.

[134] D. Martín de Diego: *Simetrías y Constantes del movimiento de sistemas lagrangianos. Sistemas lagrangianos con ligaduras no holonómicas.* Tesis Doctoral, Universidad Complutense (Madrid), 1995.

[135] J. Moser, A. P. Veselov: Discrete versions of some classical integrable systems and factorization of matrix polynomials *Comm. Math. Phys.* **139** (1991), 217-243

[136] J. Neimark, N. Fufaev: *Dynamics of Nonholonomic Systems* Translations of Mathematical Monographs Vol. 33 Providence: Am. Math. Soc., 1972.

[137] Jet Nestruev: *Smooth manifolds and observables.* (Joint work of A.M. Astashov, A.B. Bocharov, S.V. Duzhin, A.B. Sossinsky, A.M. Vinogradov and M.M.Vinogradov). Graduate Texts in Mathematics, 220. Springer-Verlag, New York, 2003.

[138] K. Norton, G. Jaroszkiewicz: Principes of discrete time mechanics, III: Quantum field theory *J. Phys. A* **31** (1998), 977-1000

[139] P.J. Olver: *Applications of Lie Groups to Differential Equations.* Springer-Verlag, New York, 1986.

[140] A. Pandolfi, C. Kane, J. E. Marsden, M. Ortiz: Time-discretized variational formulation of nonsmooth frictional contact *Int. J. Num. Methods in Engineering* **53** (2002), 1801-1829

[141] C. Paufler, H. Römer: De Donder-Weyl Equations and Multisymplectic Geometry *Reports on Mathematical Physics* **47** 1 (2001), 101–119.

[142] C. Paufler, H. Römer; Geometry of Hamiltonean n-vectorfields in Multisymplectic Field Theory *J. Geom. Phys.* **44** 1 (2002), 52–69.

[143] G. Prince: Toward a classification of dynamical symmetries in classical mechanics *Bull. Austral. Math. Soc.* **27** (1983), no. 1, 53–71

[144] G. Prince: A complete classification of dynamical symmetries in classical mechanics *Bull. Austral. Math. Soc.* **32** (1985), no. 2, 299–308

[145] J. M. Sanz-Serna, M. P. Calvo *Numerical Hamiltonian Problems* Chapman& Hall, London, 1994.

[146] G. Sardanashvily: *Gauge theory in jet manifolds.* Hadronic Press Monographs in Applied Mathematics. Hadronic Press, Inc., Palm Harbor, FL, 1993.

[147] G. Sardanashvily: *Generalized Hamiltonian formalism for field theory. Constraint systems.* World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[148] G. Sardanashvily, O. A. Zakharov: On applications of the Hamiltonian formalism in fibered manifolds to field theory *Diff. Geom. Appl.* **3** (1993), 245-263.

[149] D. J. Saunders: An alternative approach to the Cartan form in Lagrangian field theories *J. Phys. A: Math. Gen.* **20** (1987), 339-349.

[150] D. J. Saunders: *The Geometry of Jet Bundles.* Cambridge University Press, Cambridge (UK), 1989.

[151] R. Schmid *Infinite dimensional Hamiltonian systems.* Monographs and textbooks in physical science, Bibliopolis, Nápoles (Italia), 1987.

[152] Y. Shi: Symplectic structure of Discrete Hamiltonian Systems *J. Math. Anal. Appl.* **266** (2002), 472-478

[153] J. Śniatycki: On the geometric structure of classical field theory in Lagrangian formulation *Proc. Camb. Phil. Soc.* **68** (1970), 475-483.

[154] W. M. Tulczyjew: Les sous-variétés lagrangiennes et la dynamique hamiltonienne *C. R. Acad. Sc. Paris* **283**, Série A (1976), 15–18;

[155] W. M. Tulczyjew: Les sous-variétés lagrangiennes et la dynamique lagrangienne *C. R. Acad. Sc. Paris* **283**, Série A (1976), 675–678.

[156] A. M. Vershik, L. D. Faddeev: Differential geometry and Lagrangian mechanics with constraints *Sov. Phys. Dokl.* **17** (1) (1972), 34-36

[157] A. P. Veselov: Integrable discrete-time systems and difference operators *Funct. Anal. Appl.* **22** (1988), 83-93

[158] A. Vondra: *Towards a geometry of higher order partial differential equations represented by connections on fibered manifolds.* Tesis doctoral, Military Academy en Brno, 1995.

[159] J. M. Wendlandt, J. E. Marsden Mechanical integrators derived from a discrete variational principle *Physica D* **106** (1997), 223-246

[160] J. M. Wendlandt, J. E. Marsden *Mechanical systems with symmetry, variational principies and integration algorithms* Current and Future Directions in Applied Mathematics, M. Alber, B. Hu and J. Rosenthal. In: Birkhauser (1997), 219-261.