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# Measurable diagonalization of positive definite matrices

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## Abstract

In this paper we show that any positive definite matrix  $V$  with measurable entries can be written as  $V = U \Lambda U^*$ , where the matrix  $\Lambda$  is diagonal, the matrix  $U$  is unitary, and the entries of  $U$  and  $\Lambda$  are measurable functions ( $U^*$  denotes the transpose conjugate of  $U$ ).

This result allows to obtain results about the zero location and asymptotic behavior of extremal polynomials with respect to a generalized non-diagonal Sobolev norm in which products of derivatives of different order appear. The orthogonal polynomials with respect to this Sobolev norm are a particular case of those extremal polynomials.

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## 1. Introduction

One of the central problems in the theory of Sobolev orthogonal polynomials is to determine its asymptotic behavior (cf. [7–11] and references therein). It is possible that the first result in the

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literature about asymptotic properties for orthogonal polynomials with respect to a non-discrete Sobolev type inner product associated to general measures is contained in [8], where the authors show how to obtain the  $n$ th root asymptotic of Sobolev orthogonal polynomials if the zeros of these polynomials are contained in a compact set of the complex plane. Although the uniform bound of the zeros of orthogonal polynomials holds for every measure with compact support in the case without derivatives, it is an open problem to bound the zeros of Sobolev orthogonal polynomials. The boundedness of the zeros is a consequence of the boundedness of the multiplication operator  $Mf(z) = z f(z)$ ; in fact, the zeros of the Sobolev orthogonal polynomials are contained in the disk  $\{z : |z| \leq 2\|M\|\}$  (see [8, Theorem 2] or [9, Theorem 1.2]).

In [1,3,13–17] there are some answers to the question stated in [8] about some conditions for  $M$  to be bounded: the more general result on this topic is [1, Theorem 8.1] which characterizes in a simple way (in terms of equivalent norms in Sobolev spaces) the boundedness of  $M$  for the classical diagonal case

$$\|q\|_{W^{k,p}(\mu_0,\mu_1,\dots,\mu_N)} := \left( \sum_{k=0}^N \|q^{(k)}\|_{L^p(\mu_k)}^p \right)^{1/p}.$$

The rest of the papers mention several conditions which guarantee the equivalence of norms in Sobolev spaces, and consequently, the boundedness of  $M$ .

In [7], the authors study the asymptotic behavior of extremal polynomials with respect to the following non-diagonal Sobolev norms. Given a finite Borel measure  $\mu$  with compact support  $S(\mu)$  consisting of infinitely many points in the complex plane, let us consider the diagonal matrix  $A := \text{diag}(\lambda_j)$ ,  $0 \leq j \leq N$ , with  $\lambda_j$  positive  $\mu$ -almost everywhere measurable functions, and  $U := (u_{jk})$ ,  $0 \leq j, k \leq N$ , a matrix of measurable functions such that the matrix  $U(x) = (u_{jk}(x))$ ,  $0 \leq j, k \leq N$ , is unitary  $\mu$ -almost everywhere. If  $V := UAU^*$ , where  $U^*$  denotes the transpose conjugate of  $U$  (note that then  $V$  is a positive definite matrix), and  $1 \leq p < \infty$ , we define the Sobolev norm

$$\begin{aligned} \|q\|_{W^{N,p}(V\mu)} &:= \left( \int [(q, q', \dots, q^{(N)})V^{2/p}(q, q', \dots, q^{(N)})^*]^{p/2} d\mu \right)^{1/p} \\ &= \left( \int [(q, q', \dots, q^{(N)})U\Lambda^{2/p}U^*(q, q', \dots, q^{(N)})^*]^{p/2} d\mu \right)^{1/p}. \end{aligned} \quad (1.1)$$

It is not difficult to verify that under the assumptions imposed,  $\|\cdot\|_{W^{N,p}(V\mu)}$  defines a norm on the space of polynomials  $\mathbb{P}$ . If  $U$  is not the identity matrix  $\mu$ -almost everywhere, then (1.1) defines a generalized non-diagonal Sobolev norm in which the product of derivatives of different order appears.

We say that  $q_n(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0$  is an  $n$ th monic extremal polynomial with respect to the norm (1.1) if

$$\|q_n\|_{W^{N,p}(V\mu)} = \inf\{\|q\|_{W^{N,p}(V\mu)} : q(z) = z^n + b_{n-1}z^{n-1} + \dots + b_1z + b_0, b_j \in \mathbb{C}\}.$$

It is clear that there exists at least an  $n$ th monic extremal polynomial. Furthermore, it is unique if  $1 < p < \infty$  since (1.1) is strictly convex (see e.g. [2, pp. 22–23]). If  $p = 2$ , then the  $n$ th monic extremal polynomial is precisely the  $n$ th monic Sobolev orthogonal polynomial with respect to the inner product corresponding to (1.1).

The results of [7, Theorem 1] allow to locate the zeros of the extremal polynomials with respect to (1.1). This result is interesting since it allows the authors to obtain the asymptotic behavior of extremal polynomials (see [7, Theorems 2 and 6]).

The hypothesis  $V = U\Lambda U^*$  is essential in the proofs in [7], as well as in other proofs concerning some important results in the theory of orthogonal matrix polynomials (see, for instance, [5] and the references therein). There are non-trivial conditions in [4, Lemma 11, p. 1341] which allow to guarantee this factorization for some positive definite matrices  $V$ , although it is an open problem to know either of this factorization is possible or not for every positive definite matrix  $V$ .

In this paper we solve this problem by showing that any positive definite matrix  $V$  with measurable entries can be written as  $V = U\Lambda U^*$ , where the matrix  $\Lambda$  is diagonal, the matrix  $U$  is unitary and the entries of  $U$  and  $\Lambda$  are measurable functions (see [Theorem 2.1](#) below). Hence, we enlarge the scope of application of the results in [7].

[Theorem 2.1](#) gives an affirmative answer in its context to the following interesting question: if an algorithm provides a result from some data and we allow the data to depend on a parameter and this dependence verifies some good property, does the result verify also this property?

## 2. The main result

In what follows we will consider only column vectors.

**Theorem 2.1.** *Given a measurable space  $X$  and a function  $V$  defined on  $X$  and with values on the set of positive definite matrices, with measurable entries, there exist a diagonal matrix  $\Lambda$  and a unitary matrix  $U$  defined on  $X$  with measurable entries such that  $V = U\Lambda U^*$ .*

**Proof.** We prove the result by induction on the order  $n$  of the matrix  $V$ .

Assume first that  $n = 2$  and consider the characteristic polynomial

$$P(\lambda) := \det(V - \lambda I) = \lambda^2 + a_1\lambda + a_0,$$

where  $I$  denotes the identity matrix. Note that the coefficients  $a_0 = a_0(x)$ ,  $a_1 = a_1(x)$  are measurable functions since they are obtained by multiplying and adding the entries of  $V$ .

Define

$$\lambda_1 := \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2}, \quad \lambda_2 := \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2}.$$

These are the zeros of  $P(\lambda)$  and satisfy  $\lambda_1 \geq \lambda_2 > 0$ ; besides, they are measurable functions. Therefore,  $G := \{x \in X \mid \lambda_1(x) = \lambda_2(x)\}$  is a measurable set.

Denote by  $I_A$  the characteristic function of the set  $A$ :

$$I_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

If  $x \in G$ , then

$$V = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} I_G.$$

Consider now  $x \notin G$  and write

$$V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}.$$

The eigenspace of  $V$  corresponding to the eigenvalue  $\lambda_1$  is the set of vectors  $(u_1, u_2)^T$  (the superindex  $T$  denotes the transpose) satisfying the equations

$$\begin{cases} (v_{11} - \lambda_1)u_1 + v_{12}u_2 = 0, \\ v_{21}u_1 + (v_{22} - \lambda_1)u_2 = 0. \end{cases}$$

We also have that

$$\begin{aligned} G_{11} &:= \{x \in X \mid v_{11}(x) - \lambda_1(x) = 0\}, \\ G_{12} &:= \{x \in X \mid v_{12}(x) = 0\}, \\ G_{21} &:= \{x \in X \mid v_{21}(x) = 0\}, \\ G_{22} &:= \{x \in X \mid v_{22}(x) - \lambda_1(x) = 0\} \end{aligned}$$

are measurable sets.

Denote by  $A^c := X \setminus A$  the complement of the subset  $A \subseteq X$ . Then

$$\begin{aligned} u^1 &:= \left\{ (-v_{12}/(v_{11} - \lambda_1), 1)^T I_{G_{11}^c} + (1, -(v_{11} - \lambda_1)/v_{12})^T I_{G_{11} \cap G_{12}^c} \right. \\ &\quad \left. + (-v_{22} - \lambda_1)/v_{21}, 1)^T I_{G_{11} \cap G_{12} \cap G_{21}^c} \right. \\ &\quad \left. + (1, -v_{21}/(v_{22} - \lambda_1))^T I_{G_{11} \cap G_{12} \cap G_{21} \cap G_{22}^c} \right\} I_{G^c} \end{aligned}$$

is an eigenvector of  $V$  corresponding to the eigenvalue  $\lambda_1$  for every  $x \notin G$ , since

$$G^c = G_{11}^c \cup (G_{11} \cap G_{12}^c) \cup (G_{11} \cap G_{12} \cap G_{21}^c) \cup (G_{11} \cap G_{12} \cap G_{21} \cap G_{22}^c).$$

Besides,  $u^1$  has measurable entries.

The eigenspace of  $V$  corresponding to the eigenvalue  $\lambda_2$  is the set of vectors  $(u_1, u_2)^T$  satisfying the equations

$$\begin{cases} (v_{11} - \lambda_2)u_1 + v_{12}u_2 = 0, \\ v_{21}u_1 + (v_{22} - \lambda_2)u_2 = 0. \end{cases}$$

We also have that

$$\begin{aligned} \Gamma_{11} &:= \{x \in X \mid v_{11}(x) - \lambda_2(x) = 0\}, \\ \Gamma_{22} &:= \{x \in X \mid v_{22}(x) - \lambda_2(x) = 0\} \end{aligned}$$

are measurable sets. Then

$$\begin{aligned} u^2 &:= \left\{ (-v_{12}/(v_{11} - \lambda_2), 1)^T I_{\Gamma_{11}^c} + (1, -(v_{11} - \lambda_2)/v_{12})^T I_{\Gamma_{11} \cap \Gamma_{12}^c} \right. \\ &\quad \left. + (-v_{22} - \lambda_2)/v_{21}, 1)^T I_{\Gamma_{11} \cap \Gamma_{12} \cap \Gamma_{21}^c} \right. \\ &\quad \left. + (1, -v_{21}/(v_{22} - \lambda_2))^T I_{\Gamma_{11} \cap \Gamma_{12} \cap \Gamma_{21} \cap \Gamma_{22}^c} \right\} I_{G^c} \end{aligned}$$

is an eigenvector of  $V$  corresponding to the eigenvalue  $\lambda_2$  for every  $x \notin G$ , since

$$G^c = \Gamma_{11}^c \cup (\Gamma_{11} \cap \Gamma_{12}^c) \cup (\Gamma_{11} \cap \Gamma_{12} \cap \Gamma_{21}^c) \cup (\Gamma_{11} \cap \Gamma_{12} \cap \Gamma_{21} \cap \Gamma_{22}^c).$$

Besides,  $u^2$  has measurable entries.

Define also the normalized vector

$$N(u) := \begin{cases} \frac{u}{\|u\|}, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

Then

$$U := I_G + (N(u^1), N(u^2)) I_{G^c}, \quad \Lambda := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

are, respectively, an orthogonal and a diagonal matrix with measurable entries, and they verify  $V = U \Lambda U^*$ .

Assume that the result holds for matrices of order  $n - 1$  and fix a positive definite matrix  $V$  with measurable entries and order  $n$ .

Consider the characteristic polynomial

$$P(\lambda) := \det(\lambda I - V) = (-1)^n \det(V - \lambda I) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

Note that the coefficients  $a_j = a_j(x)$  ( $0 \leq j \leq n - 1$ ) are measurable functions since they are obtained by multiplying and adding the entries of  $V$ .

Let  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n > 0$  be the zeros of  $P(\lambda)$ .

Consider the  $n$  linearly independent solutions  $y_k^{(1)}, \dots, y_k^{(n)}$  of the difference equation

$$y_{n+k} + a_{n-1}y_{n+k-1} + \cdots + a_1y_{k+1} + a_0y_k = 0,$$

satisfying the initial values  $y_k^{(j)} = \delta_{j,k}$  for  $1 \leq j, k \leq n$ . Since  $a_1, \dots, a_{n-1}$  are measurable functions, these solutions  $y_k^{(j)}$  are measurable functions for every  $1 \leq j \leq n$  and  $k \geq 1$ , and one can check that

$$\lim_{k \rightarrow \infty} \frac{y_{k+1}^{(j)}}{y_k^{(j)}} \in \{\lambda_1, \dots, \lambda_n\}$$

for every  $1 \leq j \leq n$ . Therefore,

$$\lambda_1 = \max_{1 \leq j \leq n} \lim_{k \rightarrow \infty} \frac{y_{k+1}^{(j)}}{y_k^{(j)}},$$

and  $\lambda_1$  (the maximum eigenvalue of  $V$ ) is also a measurable function. (This is a “measurable version” of the known results about analytic eigenvalues when the entries of the matrix are analytic functions, without the hypothesis of different eigenvalues that is needed in the analytic case, see e.g. [6, Chapter 2.1, p. 62], [12, Chapter XII.1, p. 1] or [18, Theorems 9.6.6 and 9.6.8].)

Let  $\{e_1, \dots, e_n\}$  be the canonical basis of  $\mathbb{R}^n$ . We are going to prove that for any  $u \in \mathbb{R}^n$ , there exists the limit

$$u^* := \lim_{k \rightarrow \infty} \lambda_1^{-k} V^k u$$

and it belongs to  $\ker(V - \lambda_1 I)$ .

Fix any  $u \in \mathbb{R}^n$ . For each  $x \in X$ , let us choose a basis  $\{f_1, \dots, f_n\}$  of  $\mathbb{R}^n$  with  $Vf_j = \lambda_j f_j$  for every  $1 \leq j \leq n$  (this basis perhaps does not have measurable entries, but it does not matter). We can write  $u = \sum_{j=1}^n u_j f_j$  for some scalar functions  $u_1, \dots, u_n$ , and therefore

$$Vu = \sum_{j=1}^n \lambda_j u_j f_j, \quad V^k u = \sum_{j=1}^n \lambda_j^k u_j f_j.$$

Let  $r$  be the multiplicity of the eigenvalue  $\lambda_1$ ; then we have that there exists the limit

$$u^* = \lim_{k \rightarrow \infty} \lambda_1^{-k} V^k u = \lim_{k \rightarrow \infty} \sum_{j=1}^n \left(\frac{\lambda_j}{\lambda_1}\right)^k u_j f_j = \sum_{j=1}^r u_j f_j,$$

and  $u^* \in \ker(V - \lambda_1 I)$  since  $\{f_1, \dots, f_r\} \subset \ker(V - \lambda_1 I)$ . If  $u_1 = \cdots = u_r = 0$ , then  $u^* = 0$ ; otherwise,  $u^* \neq 0$ . Therefore, there exists some  $1 \leq j \leq n$  ( $j = j(x)$ ) with  $e_j^* \neq 0$ .

Since  $\lambda_1$  is a measurable function,  $u^*$  has measurable entries for every  $u \in \mathbb{R}^n$  (recall that the definition of  $u^*$  does not depend on the choice of the basis  $\{f_1, \dots, f_n\}$ ); then  $e_1^*, \dots, e_n^*$  have measurable entries. Define the measurable sets  $F_j := (e_j^*)^{-1}(\{0\})$  for  $1 \leq j \leq n$ .

Since there exists some  $1 \leq j \leq n$  with  $e_j^* \neq 0$ ,  $F_1 \cap F_2 \cap \dots \cap F_{n-1} \cap F_n = \emptyset$ ; then the vector

$$E_1 := N(e_1^*)I_{F_1^c} + N(e_2^*)I_{F_1 \cap F_2^c} + N(e_3^*)I_{F_1 \cap F_2 \cap F_3^c} + \dots + N(e_n^*)I_{F_1 \cap \dots \cap F_{n-1} \cap F_n^c}$$

is unitary and has measurable entries. It is clear that  $E_1$  is an eigenvector of  $V$  corresponding to  $\lambda_1$ .

Given any  $u \in \mathbb{R}^n$  and  $1 \leq j \leq n$ , let  $P_j u$  be the  $j$ th coordinate of the vector  $u$  in the canonical basis  $\{e_1, \dots, e_n\}$ . Then  $P_j E_1$  is a measurable function and  $Q_j := (P_j E_1)^{-1}(\{0\})$  is a measurable set for each  $1 \leq j \leq n$ . Let us define the vector with measurable entries

$$E'_1 := e_1 I_{Q_1^c} + e_2 I_{Q_1 \cap Q_2^c} + e_3 I_{Q_1 \cap Q_2 \cap Q_3^c} + \dots + e_n I_{Q_1 \cap \dots \cap Q_{n-1} \cap Q_n^c}.$$

Since  $E_1$  is a unitary vector for every  $x \in X$ ,  $Q_1 \cap \dots \cap Q_{n-1} \cap Q_n = \emptyset$  and  $E'_1 \in \{e_1, \dots, e_n\}$  for every  $x \in X$ . Note that if  $E'_1 = e_j$ , then the  $j$ th coordinate of  $E_1$  is not 0; hence  $\{E_1, e_1, \dots, e_n\} \setminus E'_1$  is a basis of  $\mathbb{R}^n$ .

If we apply the Gram–Schmidt process to  $\{E_1, e_1, \dots, e_n\} \setminus E'_1$ , then we obtain a new orthonormal basis  $\{E_1, E_2, \dots, E_n\}$  with measurable entries.

If we define the unitary matrix  $U_1 := (E_1, E_2, \dots, E_n)$ , then one can check that the positive definite matrix  $V_1 := U_1^* V U_1$  has the following properties:

- $\lambda_1$  is the entry  $(1, 1)$  of  $V_1$ ,
- the other entries in the first row and the first column of  $V_1$  are 0,
- the matrix  $V_2$  with order  $n - 1$  obtained from  $V_1$  by deleting the first row and the first column is positive definite and has measurable entries.

Then

$$V_1 = \left( \begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & V_2 \end{array} \right).$$

Applying the induction hypothesis to  $V_2$  we obtain a diagonal matrix  $A_2$  and a unitary matrix  $U_2$  with order  $n - 1$  and measurable entries such that  $V_2 = U_2 A_2 U_2^*$ . Let  $A_3$  be the diagonal matrix with order  $n$  obtained from  $A_2$  by adding  $\lambda_1$  as the entry  $(1, 1)$ , i.e.,

$$A_3 = \left( \begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & A_2 \end{array} \right).$$

Let  $U_3$  be the unitary matrix with order  $n$  obtained from  $U_2$  by adding a first row and a first column such that 1 is the entry  $(1, 1)$  of  $U_3$ , i.e.,

$$U_3 = \left( \begin{array}{c|c} 1 & 0 \\ \hline 0 & U_2 \end{array} \right).$$

It is clear that the entries of  $A_3$  and  $U_3$  are measurable and that  $U_1^* V U_1 = V_1 = U_3 A_3 U_3^*$ .

Then  $V = U_1 U_3 A_3 U_3^* U_1^*$ , and we have  $V = U A U^*$  with  $A := A_3$  and  $U := U_1 U_3$ ; furthermore, the entries of  $A$  and  $U$  are measurable functions.  $\square$

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