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A COMPARISON OF UNIT ROOT TEST CRITERIA

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Abstract

During the past fifteen years, the ordinary least squares estimator and the corresponding pivotal statistic have been widely used for testing the unit root hypothesis in autoregressive processes. Recently, several new criteria, based on the maximum likelihood estimators and weighted symmetric estimators, have been proposed. In this article, we describe several different test criteria. Results from a Monte Carlo study that compares the power of the different criteria indicates that the new tests are more powerful against the stationary alternative. Of the procedures studied, the weighted symmetric estimator and the unconditional maximum likelihood estimator provide the most powerful tests against the stationary alternative. As an illustration, we analyze the quarterly change in business investories.

Key Words
Nonstationary Time Series; Maximum Likelihood; Weighted Symmetric; Power.

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ABSTRACT

During the past fifteen years, the ordinary least squares estimator and the corresponding pivotal statistic have been widely used for testing the unit root hypothesis in autoregressive processes. Recently, several new criteria, based on the maximum likelihood estimators and weighted symmetric estimators, have been proposed. In this article, we describe several different test criteria. Results from a Monte Carlo study that compares the power of the different criteria indicates that the new tests are more powerful against the stationary alternative. Of the procedures studied, the weighted symmetric estimator and the unconditional maximum likelihood estimator provide the most powerful tests against the stationary alternative. As an illustration, we analyze the quarterly change in business inventories.

1. INTRODUCTION

Testing for unit roots in autoregressive process has received considerable attention since the work by Fuller (1976) and Dickey and Fuller (1979). Dickey and Fuller (1979) considered tests based on the ordinary least squares estimator and the corresponding pivotal statistic. Several extensions of the procedures suggested by Dickey and Fuller (1979) exist in the literature. See Diebold and Nerlove (1990) for a survey of the unit root literature. Recently, Gonzalez–Farias (1992) and Dickey and Gonzalez–Farias (1992) considered maximum likelihood estimation of the parameters of the autoregressive process and suggested tests for unit roots based on these estimators. Elliott and Stock (1992) and Elliott, Rothenberg and Stock (1992) developed most powerful invariant tests for testing the unit root hypothesis against a particular alternative and used these tests to obtain an asymptotic power envelope. Both approaches produced tests against the alternative of a root less than one with much higher power than the test criteria based on the ordinary least squares estimators.

We summarize the new approaches, introduce a new test, and use Monte Carlo methods to compare the power of the test criteria in finite samples. In Section 2, we introduce the model and present different unit root test criteria. Extensions are given in Section 3. In Section 4, we present a Monte Carlo study that compares the power of the new approaches to that of existing methods. In Section 5, we analyze a data set to illustrate the different test criteria. We present our conclusions in Section 6.

2. TEST CRITERIA

Consider the model

$$Y_t = \mu(1-\rho) + \rho Y_{t-1} + e_t, \; t \geq 2 , \qquad (2.1)$$

where the e_t's are independent random variables with mean zero and variance σ^2. Assume that Y_1 is independent of e_t for $t \geq 2$. We are interested in testing the null hypothesis that $\rho = 1$. Different estimators and test criteria are obtained depending upon what is assumed about Y_1. Test criteria are typically constructed using likelihood procedures under the assumption that the e_t's are normally distributed. The asymptotic distributions of the test statistics are, however, valid under much weaker assumptions on the distribution of e_t. We present some different test statistics and summarize their asymptotic distributions. We refer the reader to Dickey and Fuller (1979, 1981), Elliott, Rothenberg and Stock (1992), Fuller (1992) and Gonzalez–Farias (1992) for the proofs of the asymptotic results.

2.1. Y_1 fixed

When Y_1 is considered fixed and $e_t \sim NI(0, \sigma^2)$, maximizing the log likelihood function is equivalent to minimizing

$$Q_c(\mu, \rho\sigma^2) = (n-1)\log \sigma^2 + \sigma^{-2} \sum_{t=2}^{n} [Y_t - \mu(1-\rho) - \rho Y_{t-1}]^2 . \tag{2.2}$$

In this case, the conditional maximum likelihood estimator of ρ is the same as the ordinary least squares (OLS) estimator $\hat{\rho}_{\mu,OLS}$, obtained by regressing Y_t on 1 and Y_{t-1} for $t = 2, 3, ..., n$. The OLS estimator of ρ is

$$\hat{\rho}_{\mu,OLS} = \left[\sum_{t=2}^{n} (Y_{t-1} - \bar{Y}_{(-1)})^2\right]^{-1} \sum_{t=2}^{n} (Y_{t-1} - \bar{Y}_{(-1)})Y_t .$$

There are three common approaches for constructing a test of the hypothesis that $\rho = 1$. For the first order process, one can construct a test based upon the distribution of the estimator of ρ. A second test is obtained by constructing a pivotal statistic for ρ by analogy to the usual t–test of regression analysis. This is the most used test in practice because the pivotal approach extends immediately to higher order processes. The pivotal statistic associated with the ordinary least squares estimator is

$$\hat{\tau}_{\mu,OLS} = \left[\hat{\sigma}_{OLS}^{-2} \sum_{t=2}^{n} (Y_{t-1} - \bar{Y}_{(-1)})^2\right]^{1/2} (\hat{\rho}_{\mu,OLS} - 1) , \tag{2.3}$$

where $[\bar{Y}_{(0)}, \bar{Y}_{(-1)}] = (n-1)^{-1}\Sigma_{t=2}^{n}[Y_t, Y_{t-1}]$ and

$$\hat{\sigma}_{OLS}^2 = (n-3)^{-1} \sum_{t=2}^{n} [Y_t - \bar{Y}_{(0)} - \hat{\rho}_{\mu,OLS}(Y_{t-1} - \bar{Y}_{(-1)})]^2 .$$

A third test can be constructed on the basis of the likelihood ratio. The null model with $\rho = 1$ reduces (2.1) to the random walk. The sum of squares associated with the null model is $\Sigma_{t=2}^{n}(Y_t - Y_{t-1})^2$ and a likelihood ratio type statistic for testing $\rho = 1$ is

$$\Phi_{OLS} = (2\hat{\sigma}_{OLS}^2)^{-1}\left[\sum_{t=2}^{n} (Y_t - Y_{t-1})^2 - (n-3)\hat{\sigma}_{OLS}^2\right] . \tag{2.4}$$

The limiting distributions of the statistics, derived by Dickey and Fuller (1979, 1981), are given in Table 2.1, where $\xi = 0.5[T^2 - 1]$, $T = W(1)$,

$$G = \int_0^1 W^2(t)dt , \quad H = \int_0^1 W(t)dt ,$$

and $W(t)$ is a standard Brownian motion on $[0, 1]$. Empirical percentiles for $n(\hat{\rho}_{\mu,OLS} - 1)$ and $\hat{\tau}_{\mu,OLS}$ may be found in Tables 8.5.1 and 8.5.2 of Fuller (1976). The percentiles for Φ_{OLS} are given in Dickey and Fuller (1981).

2.2. $Y_1 \sim N(\mu, \sigma^2)$

If Y_1 is assumed to be a normal random variable with mean μ and variance σ^2, maximizing the log of the likelihood function is equivalent to minimizing

$$Q_1(\mu, \rho, \sigma^2) = \log\sigma^2 + \sigma^{-2}(Y_1 - \mu)^2 + Q_c(\mu, \rho, \sigma^2), \tag{2.5}$$

where $Q_c(\mu, \rho, \sigma^2)$ is defined in (2.1). The first observation enters the quantity (2.5), but not (2.2), because Y_1 is random with variance σ^2 under the model that leads to (2.5). The maximum likelihood estimators minimize $Q_1(\mu, \rho, \sigma^2)$ and satisfy

$$\hat{\mu}_{1,ML} = \frac{Y_1 + (1 - \hat{\rho}_{1,ML}) \sum_{t=2}^{n} (Y_t - \hat{\rho}_{1,ML}Y_{t-1})}{1 + (n - 1)(1 - \hat{\rho}_{1,ML})^2}, \tag{2.6}$$

$$\hat{\rho}_{1,ML} = \frac{\sum_{t=2}^{n} (Y_t - \hat{\mu}_{1,ML})(Y_{t-1} - \hat{\mu}_{1,ML})}{\sum_{t=2}^{n} (Y_{t-1} - \hat{\mu}_{1,ML})^2}, \tag{2.7}$$

and

$$\hat{\sigma}^2_{1,ML} = n^{-1}(Y_1 - \hat{\mu}_{1,ML})^2 + n^{-1} \sum_{t=2}^{n} [Y_t - \hat{\mu}_{1,ML}(1 - \hat{\rho}_{1,ML}) - \hat{\rho}_{1,ML}Y_{t-1}]^2 . \tag{2.8}$$

Substituting (2.6) in (2.7) and simplifying, we get that $n(\hat{\rho}_{1,ML} - 1)$ is a solution to a fifth degree polynomial. Using the arguments of Gonzalez–Farias (1992), it is possible to show that the limiting distribution of $n(\hat{\rho}_{1,ML} - 1)$ is that of $G^{-1}\xi$. Recall that $G^{-1}\xi$ is also the limiting representation of $n(\hat{\rho}_{OLS} - 1)$, where $\hat{\rho}_{OLS}$ is the OLS estimator obtained by regressing Y_t on Y_{t-1} without an intercept. The percentiles of $G^{-1}\xi$ may be found in Table 8.5.1 of Fuller (1976). The pivotal and the likelihood ratio type statistics for testing $\rho = 1$ are

$$\hat{\tau}_{1,ML} = [\hat{\sigma}^{-2}_{1,ML} \sum_{t=2}^{n} (Y_{t-1} - \hat{\mu}_{1,ML})^2]^{1/2} (\hat{\rho}_{1,ML} - 1) \tag{2.9}$$

and

$$\hat{\ell}_{1,ML} = n(s^2 - \hat{\sigma}^2_{1,ML})(\hat{\sigma}^2_{1,ML})^{-1} , \tag{2.10}$$

where $s^2 = (n-1)^{-1} \Sigma^n_{t=2} (Y_t - Y_{t-1})^2$. The limiting distributions of these statistics are given in Table 2.1. We make a few remarks before considering the next case.

Remark 2.1. Assume $\rho = 1$. Let $\hat{\rho}$ be any estimator of ρ such that $\hat{\rho} = 1 + O_p(n^{-1})$. Then, $\hat{\mu}$ in (2.6) evaluated at $\hat{\rho}$ is $Y_1 + O_p(n^{-1/2})$. Likewise, if μ is fixed at $\hat{\mu}$, then the value of ρ that maximizes the likelihood is obtained by regressing $Y_t - \hat{\mu}$ on $Y_{t-1} - \hat{\mu}$. If $\hat{\mu}$ is such that $\hat{\mu} = Y_1 + O_p(n^{-1/2})$, then $n[\hat{\rho}(\hat{\mu}) - 1] \xrightarrow{\mathcal{L}} G^{-1}\xi$ and the estimator has the same large sample behavior as the estimator obtained by regressing $Y_t - Y_1$ on $Y_{t-1} - Y_1$, which is suggested in Dickey and Fuller (1979).

Remark 2.2. Based on Remark 2.1, several approximations to the maximum likelihood estimator are possible. We consider the following estimators in our study.

(a) Let $\hat{\rho}^{(0)}_{1,ML} = \hat{\rho}_{\mu,OLS}$. Compute iteratively $\hat{\mu}^{(i)}_{1,ML}$ and $\hat{\rho}^{(i)}_{1,ML}$ using (2.6) and (2.7) with $\hat{\rho}_{1,ML}$ and $\hat{\mu}_{1,ML}$ replaced by $\hat{\rho}^{(i-1)}_{1,ML}$ and $\hat{\mu}^{(i)}_{1,ML}$. This iterative procedure, if it converges, converges to the maximum likelihood estimator. In our study, we use the statistics obtained at the tenth iteration $(i = 10)$. From Remark 2.1, it follows that the statistics have the asymptotic representations given in Table 2.1. In our simulations, we call these estimators the maximum likelihood estimators and omit the superscripts.

(b) Elliott and Stock (1992) suggest using the statistic

$$\Phi_{ES} = n\hat{\sigma}_{ES}^{-2}[s^2 - \hat{\sigma}_{ES}^2] + 7 , \tag{2.11}$$

where $\hat{\mu}_7 = \hat{\mu}(\rho_7)$ is the $\hat{\mu}$ of (2.6) evaluated with ρ_7 in place of $\hat{\rho}_{1,ML}$, $\rho_7 = 1 - 7n^{-1}$, $\hat{\sigma}_{ES}^2$ is the estimator (2.8) with $\hat{\mu}_{1,ML}$ replaced with $\hat{\mu}_7$. They argue that Φ_{ES} is the most powerful invariant test for testing $\rho = 1$ against the alternative $H_a: \rho = \rho_7$. The alternative ρ_7 was selected by finding the point that is approximately tangent to the asymptotic power envelope at a power of 50%. Since $\hat{\mu}_7 = Y_1 + O_p(n^{-1/2})$, it follows that Φ_{ES} converges in distribution to $\hat{\tau}^2$.

(c) Elliott, Rothenberg and Stock (1992) consider

$$\hat{\rho}_{7,GLS} = \hat{\rho}(\hat{\mu}_7) \tag{2.12}$$

obtained by regressing $Y_t - \hat{\mu}_7$ on $Y_{t-1} - \hat{\mu}_7$, without an intercept. They call the estimator $\hat{\rho}_{7,GLS}$ the Dickey–Fuller generalized least squares estimator of ρ. Let $\hat{\tau}_{7,GLS}$ denote the corresponding pivotal statistic for testing $\rho = 1$. The limiting distribution of $\hat{\tau}_{7,GLS}$ is given in Table 2.1.

2.3. $Y_1 \sim N[\mu, (1-\rho^2)^{-1}\sigma^2]$

Suppose that Y_1 is a normal random variable with mean μ and variance $(1-\rho^2)^{-1}\sigma^2$, for $|\rho| < 1$. Then maximizing the log of the likelihood function is equivalent to minimizing

$$Q_U(\mu, \rho, \sigma^2) = \log \sigma^2 + \log(1-\rho^2)$$

$$+ (1-\rho^2)\sigma^{-2}(Y_1-\mu)^2 + Q_c(\mu, \rho, \sigma^2) . \tag{2.13}$$

Gonzalez–Farias (1992) showed that the unconditional maximum likelihood estimator (UMLE) $\hat{\rho}_{U,ML}$ of ρ is a solution to a fifth degree polynomial. She shows that the asymptotic distribution of $n[\hat{\rho}_{U,ML} - 1]$ is that of the unique negative root of

$$\sum_{i=0}^{4} b_i x^i = 0 , \tag{2.14}$$

where $b_0 = -4$, $b_4 = 2(G - H^2)$,

$$b_1 = -8(\xi - TH + H^2) + 2(T - 2H)^2 + 4 ,$$

$$b_2 = 8(G - H^2) + 8(\xi - TH + H^2) - 1 ,$$

and

$$b_3 = -2(\xi - TH + H^2) - 8(G - H^2) .$$

For a given ρ , the value of μ that minimizes (2.13) is

$$\hat{\mu}_U(\rho) = \frac{Y_1 + (1-\rho)\sum_{t=2}^{n-1} Y_t + Y_n}{2 + (n-2)(1-\rho)} . \tag{2.15}$$

Also, for a given μ , (2.13) is maximized at the ρ that is the solution to a cubic equation [see Hasza (1980)]. The equation (2.15) and the cubic equation can be solved iteratively. If the iterations converge, the estimators converge to the unconditional ML estimators of μ and ρ . Gonzalez–Farias (1992) also derives the asymptotic distribution of $\hat{\rho}_U^{(i)} =$

$\hat{\rho}_U(\hat{\mu}_U^{(i)})$, obtained at the i–th iteration, starting with an initial estimator of ρ . Even though the asymptotic distribution of $n[\hat{\rho}_U^{(i)} - 1]$ obtained for a finite i is not the same as that of $n[\hat{\rho}_{U,ML} - 1]$, the observed empirical distribution and the power are similar for the two procedures. In this paper, we consider $n[\hat{\rho}_U^{(6)} - 1]$ obtained in the 6–th iteration, starting the iterations with the simple symmetric estimator given in Section 2.4. The choice of the initial estimator and the number of iterations are the same as the ones considered by Gonzalez–Farias (1992). We shall call $\hat{\rho}_U^{(6)}$ the unconditional maximum likelihood estimator and shall omit the (6) exponent. The corresponding pivotal statistic is

$$\hat{\tau}_U = [\hat{V}\{\hat{\rho}_U\}^{-1}]^{-1/2} n[\hat{\rho}_U - 1] ,$$

where $\hat{V}\{\hat{\rho}_U\}$ is the variance of $\hat{\rho}_U$ computed from the estimated information matrix. The limiting distribution of $\hat{\tau}_U$ is given in Gonzalez–Farias (1992).

2.4. Symmetric estimators

If a normal stationary autoregressive process satisfies (2.1), it also satisfies the equation

$$Y_t = \mu(1 - \rho) + \rho Y_{t+1} + u_t ,$$

where $u_t \sim NI(0, \sigma^2)$. This symmetry leads one to consider estimators of ρ that minimize

$$Q_w(\rho) = \sum_{t=2}^{n} w_t(y_t - \rho y_{t-1})^2 + \sum_{t=1}^{n-1} (1 - w_{t+1})(y_t - \rho y_{t+1})^2 ,$$

where w_t, $t = 2, 3, ..., n$ are weights, $y_t = Y_t - \bar{y}$ and $\bar{y} = n^{-1}\Sigma_{t=1}^{n} Y_t$. Observe that the ordinary least squares estimator is a member of this class with all w_t equal to one.

Dickey, Hasza and Fuller (1984) discuss the properties of the simple symmetric estimator obtained by setting $w_t = 0.5$. In this study we consider the estimator obtained by setting $w_t = n^{-1}(t-1)$, which we call the weighted symmetric estimator. The weighted symmetric estimator is

$$\hat{\rho}_{ws} = \hat{\rho}_{ws}(\hat{\mu}) = \frac{\sum_{t=2}^{n} y_{t-1} y_t}{\sum_{t=2}^{n-1} y_t^2 + n^{-1} \sum_{t=1}^{n} y_t^2}$$

and

$$\hat{\tau}_{ws}(\hat{\mu}) = [\hat{\rho}_{ws} - 1]\left[\sum_{t=2}^{n-1} y_t^2 + n^{-1} \sum_{t=1}^{n} y_t^2\right]^{1/2} \hat{\sigma}_{ws}^{-1}$$

is the pivotal statistic, where $\hat{\sigma}_{ws}^2 = (n-2)^{-1} Q_w(\hat{\rho}_{ws})$. The limiting distributions of the statistics are given in Table 2.1. The limiting distribution of the two–step weighted symmetric estimator of ρ that replaces $\bar{y}$ with the estimator of μ, denoted by $\hat{\mu}_{ws}$, obtained by substituting $\hat{\rho}_{ws}(\bar{y})$ into (2.15) has been obtained. Our simulations indicate that $\hat{\rho}_{ws}(\hat{\mu}_{ws})$ and $\hat{\tau}_{ws}(\hat{\mu}_{ws})$ have about the same power as $\hat{\rho}_{ws}(\bar{y})$ and $\hat{\tau}_{ws}(\bar{y})$, respectively. Also, iterating the procedure did not produce any significant change in power and, hence, the iterated estimators are not included in the reported simulation results.

We included the simple symmetric estimator in our Monte Carlo simulations. Because the power of the weighted symmetric estimator always exceeded that of the simple symmetric estimator, we do not report the results for the simple symmetric estimator.

2.5. Other criteria

We considered two other estimation criteria. The test criteria associated with the $\bar{\rho}$ obtained by regressing $Y_t - \hat{\mu}_U(\hat{\rho}_{\mu,OLS})$ on $Y_{t-1} - \hat{\mu}_U(\hat{\rho}_{\mu,OLS})$, where $\hat{\mu}_U(\ )$ is defined in (2.15), had powers much smaller than those based on the maximum likelihood and weighted symmetric estimators and, hence, the powers are not reported here. Also, we investigated an estimator $\hat{\rho}_{7,ULS}$ similar to $\hat{\rho}_{7,GLS}$ that is obtained by regressing $Y_t - \hat{\mu}_U(\rho_7)$ on $Y_{t-1} - \hat{\mu}_U(\rho_7)$, where $\rho_7 = 1 - n^{-1}7$. The powers of criteria based on $\hat{\rho}_{7,ULS}$, are small compared to the powers of the weighted symmetric estimator and, hence, are not reported.

Finally, we studied a likelihood ratio test statistic based on the distribution of $X_t = Y_t - Y_{t-1}$, $t = 2, ..., n$, given that the original model is (2.1). The estimator ρ that maximizes the likelihood is a solution to a fifth degree polynomial. The empirical powers of the corresponding likelihood ratio criteria are much smaller than those of the weighted symmetric criteria and, hence, are not reported here.

3. EXTENSIONS

The test statistics presented here can be extended to the case where a time trend is included in the model and also can be extended to higher order autoregressive processes. Elliott, Rothenberg and Stock (1992) extend their procedure to the case where a linear trend is included. They also extend their procedure to include autoregressive and moving average processes. They derive the limiting distribution of $n(\hat{\rho}_{7,GLS} - 1)$, $\hat{\tau}_{7,GLS}$, and Φ_{ES} under the assumption that e_t satisfy some stationary and mixing conditions. The limiting distributions depend on a nuisance parameter ω^2, the limiting variance of $n^{-1/2}\Sigma_{t=1}^{n} e_t$. They use two approaches to estimate ω^2. One approach approximates e_t by a higher order autoregressive process, whereas the second approach uses a weighted sum of covariance type spectral estimator suggested by Phillips and Perron (1988).

Gonzalez–Farias (1992) gives an extension of the unconditional maximum likelihood estimation for higher order autoregressive processes. Under some regularity conditions on the roots, the MLE of the largest root has the same asymptotic distribution as that of $\hat{\rho}_{U,ML}$ in the first order autoregressive model. This procedure, however, requires a search for the largest root on the real line that maximizes the likelihood. Typically, this is not a severe problem in moderately large samples.

The pivotal statistic $\hat{\tau}_{ws,p}$ for testing $\theta_1 = 1$ in the weighted symmetric regression, obtained by minimizing

$$\sum_{t=p+1}^{n} (t-p)\left[y_t - \theta_1 y_{t-1} - \sum_{i=2}^{p} \theta_i Z_{t-i}\right]^2$$

$$+ \sum_{t=p+1}^{n} (n-t+p)\left[y_{t-p} - \theta_1 y_{t-p+1} + \sum_{i=2}^{p} \theta_i Z_{t-p+i}\right]^2 , \qquad (3.1)$$

where $y_t = Y_t - \bar{y}$ and $Z_t = Y_t - Y_{t-1}$, has the same asymptotic distribution as $\hat{\tau}_{ws}(\bar{y})$ for the first order process. A Monte Carlo study indicates that the weighted symmetric estimator performs well in second order processes. The weighted symmetric estimator is simple to compute and extends easily to the case where a linear trend is included in the fitted model. If $y_t = Y_t - \hat{a} - \hat{b}t$, where $\hat{a}$ and $\hat{b}$ are the ordinary least squares estimators obtained by regressing Y_t on 1 and t , is used in (3.1), then the pivotal statistic $\hat{\tau}_{ws,p}$, for testing $\theta_1 = 1$, has the limiting distribution given by

$$[G - H^2 - 3K^2]^{-1/2}[\xi - 3TK - HT - 12HK + 2H^2 - G + 12K^2] , \qquad (3.2)$$

where

$$K = 2\int_0^1 tW(t)dt - \int_0^1 W(t)dt .$$

4. POWER STUDY

In this section, we compare the empirical power of the different test criteria described in Section 2. For tests based on the estimators of ρ, the test is the test of $\rho = 1$ against the alternative that $\rho < 1$. We first present the percentiles of the test criteria and then study their empirical powers.

4.1. Empirical Percentiles

Our model is

$$Y_t = \mu(1-\rho) + \rho Y_{t-1} + e_t, \; t \geq 2 , \tag{4.1}$$

where $e_t \sim NI(0, \sigma^2)$. To construct the percentiles, we let $Y_1 = e_1$, $\mu = 0$, $\rho = 1$ and generate the e_t as independent standard normal random variables. The RANNOR function in SAS (1985) is used to generate the e_t's. For a given sample size n, we generated 20,000 samples of size n and computed the different test statistics. The empirical critical value for a 5% level test is computed for the set of 20,000 samples. The procedure was replicated three times and the average of the three critical values is reported in Table 4.1.

We consider the test criteria based on

1. OLS: $n(\hat{\rho}_{\mu,OLS} - 1)$, $\hat{\tau}_{\mu,OLS}$, $\hat{\Phi}_{OLS}$
2. MLE for $Y_1 \sim N(\mu, \sigma^2)$: $n(\hat{\rho}_{1,ML} - 1)$, $\hat{\tau}_{1,ML}$, $\hat{\Phi}_{1,ML}$

3. Elliott and Stock: $n(\hat{\rho}_{7,GLS} - 1)$, $\hat{\tau}_{7,GLS}$, $\hat{\Phi}_{ES}$

4. MLE for $Y_1 \sim N[\mu, (1-\rho^2)^{-1}\sigma^2]$: $n(\hat{\rho}_U - 1)$, $\hat{\tau}_U$

5. Weighted symmetric: $n(\hat{\rho}_{ws} - 1)$, $\hat{\tau}_{ws}$.

Except for $\hat{\Phi}_{OLS}$ and $\hat{\Phi}_{1,ML}$, we reject H_0: $\rho = 1$, for values of the statistic less than the critical values given in Table 4.1.

4.2. Empirical power

In this section, we study the empirical power of the statistics described in Section 4.1. Critical values for 5% level tests are given in Table 4.1. The percentiles for finite samples are estimated by generating e_t's from a standard normal distribution. We consider two cases: (I) $Y_1 \sim N(0, 1)$ and (II) $Y_1 \sim N[0, (1-\rho^2)^{-1}]$ and generated samples of size $n = 25, 50, 100,$ and 250, with $\rho = 0.98, 0.95, 0.93, 0.90, 0.85, 0.80,$ and 0.70. For case (I), we also include $\rho = 1$, and for case (II), we include $\rho = 0.99$. The powers are computed based on 5,000 Monte Carlo replications. In generating Tables 4.2 – 4.9, the same e_t's were used in both cases, except that $Y_1 = e_1$ in case (I), whereas $Y_1 = e_1(1-\rho^2)^{-1/2}$ in case (II). The power is higher for case (I) than for case (II) for all test statistics.

(I): $Y_1 \sim N(0, 1)$

Empirical powers for this case are summarized in Tables 4.2 – 4.5. The powers of the pivotal statistics, $\hat{\tau}_{\mu,OLS}$, $\hat{\tau}_{7,GLS}$, $\hat{\tau}_U$ and $\hat{\tau}_{ws}$ are plotted in Figure 4.1. From the tables, we observe that

1. The criteria based on ordinary least squares have low power compared to the remaining criteria.

2. Except for OLS estimation, the tests based on the pivotal statistics $(\hat{\tau})$ have higher power than those based on $n(\hat{\rho}-1)$.
3. The test criteria based on the unconditional likelihood, $\hat{\rho}_U$ and $\hat{\tau}_U$, have powers very similar to those based on the weighted symmetric estimator, $\hat{\rho}_{ws}$ and $\hat{\tau}_{ws}$ with $\hat{\tau}_U$ being somewhat superior in small samples $(n \leq 50)$.
4. For sample sizes $n = 25$ and 50, tests based on the weighted symmetric and the unconditional likelihood estimators have higher power than those based on the ML estimator $(\hat{\rho}_{1,ML}, \hat{\tau}_{1,ML})$ derived under the assumption that Y_1 has variance σ^2 .
5. For sample sizes $n = 100$ and 250 , the tests based on the true model, $\hat{\rho}_{1,ML}$ and $\hat{\tau}_{1,ML}$ have higher power than those based on the weighted symmetric and the unconditional likelihood estimators.
6. Except when $n = 250$, Φ_{ES} has higher power than $\hat{\rho}_{7,GLS}$ and $\hat{\tau}_{7,GLS}$ for values of $\rho \geq 0.9$. Also, for $\rho \geq 0.9$, these three test criteria have higher power than the remaining criteria. For values of ρ smaller than 0.9, these criteria tend to have lower power than the criteria based on the unconditional likelihood and weighted symmetric estimators.

(II). $Y_1 \sim N[0, (1-\rho^2)^{-1}]$

Empirical powers for this case are summarized in Tables 4.6 – 4.9. In Figure 4.2, we plot the powers of the pivotal statistics $\hat{\tau}_{\mu,OLS}$, $\hat{\tau}_{7,GLS}$, $\hat{\tau}_U$ and $\hat{\tau}_{ws}$. From the tables, we observe that comments 1 – 4 for Case (I) are also applicable for Case (II).

5. For all sample sizes, the criteria based on the unconditional likelihood and weighted symmetric estimators have higher power than those based on $\hat{\rho}_{1,ML}$ and $\hat{\tau}_{1,ML}$, except, perhaps, for $\rho = 0.99$ and $n = 25$.

6. The criteria based on the weighted symmetric and the unconditional likelihood have higher power than the criteria suggested by Elliott, Rothenberg and Stock (1992), except for $\rho = 0.99$ and $n = 250$.
7. The weighted symmetric criteria have slightly larger power than the criteria based on the unconditional stationary likelihood for $n = 100$ and 250. The powers of the two procedures are similar for $n = 50$ with neither estimator uniformly superior. The unconditional likelihood criteria had slightly larger power than the weighted symmetric criteria for $n = 25$.

5. EXAMPLE

In this section, we present an example to illustrate the differences among test criteria. Pankratz (1983) analyzed the quarterly seasonally adjusted change in business inventories, stated at annual rates in billions of dollars. He examined 60 observations covering the period from the first quarter of 1955 through the fourth quarter of 1969. The data are plotted in Figure 5.1. Pankratz (1983) concluded that a first order autoregressive

Table 5.1. Statistics for the change in business inventories.

Test Statistic	Est. of ρ 60 obs.	Ratio of test statistic to 5% critical value		
		60 obs.	49 obs.	48 obs.
$\hat{\tau}_{\mu,OLS}$	0.690	1.11	0.85	0.68
$\hat{\tau}_{1,ML}$	0.691	1.33	1.04	0.82
$\hat{\tau}_{7,GLS}$	0.700	1.41	1.12	0.90
$\hat{\tau}_{U}$	0.680	1.30	1.04	0.87
$\hat{\tau}_{WS}$	0.680	1.29	1.02	0.83

model fits the data adequately. The OLS estimated model is

$$Y_t = \underset{(0.73)}{1.92} + \underset{(0.096)}{0.690}\, Y_{t-1} + e_t$$

with $\hat{\sigma}^2 = 11.42$. The first column of numbers in Table 5.1 contains the estimates of ρ obtained by the alternative procedures. The values of the unconditional likelihood estimator and the weighted symmetric estimator are very similar. This will be true except for samples with a maximum likelihood estimate very close to one. The test statistics for these two procedures will also be very close for samples in which the test statistics tend to reject the null in favor of the stationary alternative.

In Table 5.1, we give the values of six pivotal test statistics for testing the null hypothesis H_0: $\rho = 1$. To facilitate the comparison, we divide the value of the test statistic by the 5% critical value. Thus, if the value in the table is greater than or equal to one, the hypothesis that $\rho = 1$ is rejected at the 5% level. If all 60 observation are used in the computations, every test statistic rejects the null hypothesis in favor of the stationary alternative.

Let $y_t = Y_t - \bar{Y}$ denote the mean adjusted series, and let $\hat{\sigma}_y^2$ denote an estimate of the "variance" of the observations. Then the estimators $\hat{\rho}_{\mu,OLS}$, $\hat{\rho}_{ws}$ and $\hat{\rho}_U$ can be written as

$$\hat{\rho} = \left[\sum_{t=2}^{n-1} y_t^2 + \hat{\sigma}_y^2\right]^{-1} \sum_{t=2}^{n} y_t y_{t-1} .$$

The values of $\hat{\sigma}_y^2$ in the denominators of $\hat{\rho}_{\mu,OLS}$ and $\hat{\rho}_{ws}$ are y_1^2 and $n^{-1}\Sigma_{t=1}^n y_t^2$, respectively. On the other hand, $\hat{\rho}_U$ uses $\hat{\sigma}_y^2 = \hat{\sigma}^2(1 - \hat{\rho}_U^2)^{-1}$ where $\hat{\sigma}^2$ is the maximum likelihood estimator of the innovation variance. Thus, it is the first and last observations that produce different results for the different estimators. To illustrate this, we reanalyze the data using the first 48 and the first 49 observations. The observation Y_{48} differs from the rest considerably whereas Y_{49} is between Y_{48} and Y_{60}. When 48 observations are used,

$$\left[(46)^{-1}\sum_{t=2}^{47} y_t^2\right]^{-1}(y_1^2, y_{48}^2) = (0.046, 7.60)\ ,$$

when 49 observations are used,

$$\left[(47)^{-1}\sum_{t=2}^{48} y_t^2\right]^{-1}(y_1^2, y_{49}^2) = (0.052, 1.78)$$

and when 60 observations are used,

$$\left[(58)^{-1}\sum_{t=2}^{59} y_t^2\right]^{-1}(y_1^2, y_{60}^2) = (0.134, 0.000)\ .$$

The values of the estimator of ρ for $n = 48$ are 0.781, 0.763, and 0.762 for $\hat{\rho}_{\mu,OLS}$, $\hat{\rho}_{ws}$ and $\hat{\rho}_U$, respectively. The corresponding values are 0.744, 0.730, and 0.730 for $n = 49$ and 0.690, 0.680, and 0.680 for $n = 60$.

Based on the first 48 observations, all pivotal statistics fail to reject $H_0: \rho = 1$ at $\alpha = 0.05$. On the other hand, when the test statistics are computed using the first 49 observations, only the OLS test criterion fails to reject $H_0: \rho = 1$ at $\alpha = 0.05$.

The test statistics $\hat{\tau}_{\mu,OLS}$, $\hat{\tau}_{1ML}$ and $\hat{\tau}_{7,GLS}$ treat the first observation differently from other observations. We can illustrate this by computing the test statistic for the data in reverse order. If this is done, the values for the 48 observation data set are −3.50, −2.46, and −1.76 for $\hat{\tau}_{\mu,OLS}$, $\hat{\tau}_{1,ML}$ and $\hat{\tau}_{7,GLS}$, respectively. The ratios of the test statistics to the 5% critical values are 1.19, 0.97, and 0.77, respectively. The corresponding ratios for 49 observations are 0.98, 1.08, and 1.01, respectively and the ratios for 60 observations are 1.10, 1.31, and 1.43, respectively. Because y_{48}^2 is large and y_1^2 is small, reversing the direction of the calculations has a large effect on the ordinary least squares statistic. The hypothesis of a unit root is rejected by $\hat{\tau}_{\mu,\mathrm{OLS}}$ when y_{48} is used as the first observation, but is accepted when y_1 is used as the first observation.

6. SUMMARY

We have discussed criteria for testing the null hypothesis of a unit root in a first order autoregressive process. Based on our simulation study, it is clear that the OLS criteria are the least powerful of the statistics studied. The criteria suggested by Elliott, Rothenberg and Stock (1992) are very powerful under certain alternatives but are not the most powerful against the stationary alternative.

The criteria based on the unconditional likelihood, and those based on the weighted symmetric estimator, are the most powerful and, hence, are the recommended test statistics.

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Table 2.1. Limiting Null Distributions of Statistics

Procedure, Statistic	Limiting Distribution
OLS	
$n(\hat{\rho}_{\mu,OLS} - 1)$	$[G - H^2]^{-1}[\xi - TH]$
$\hat{\tau}_{\mu,OLS}$	$[G - H^2]^{-1/2}[\xi - TH]$
$\hat{\Phi}_{OLS}$	$T^2 + [G - H^2]^{-1}[\xi - TH]^2$
MLE for $Y_1 \sim N(\mu, \sigma^2)$	
$n(\hat{\rho}_{1,ML} - 1)$	$G^{-1}\xi$
$\hat{\tau}_{1,ML}$	$G^{-1/2}\xi$
$\hat{\Phi}_{1,ML}$	$G^{-1}\xi^2$
Elliott and Stock	
$n(\hat{\rho}_{7,GLS} - 1)$	$G^{-1}\xi$
$\hat{\tau}_{7,GLS}$	$G^{-1/2}\xi$
$\hat{\Phi}_{ES}$	$G^{-1}\xi^2$
Weighted symmetric	
$n(\hat{\rho}_{ws} - 1)$	$[G - H^2]^{-1}[\xi - TH - G + 2H^2]$
$\hat{\tau}_{ws}$	$[G - H^2]^{-1/2}[\xi - TH - G + 2H^2]$

Table 4.1. **Empirical 5% critical values for different unit root test criteria.**

Test Statistic	n: 25	50	100	250	∞
$n[\hat{\rho}_{\mu,OLS}-1]$	–12.50	–13.30	–13.70	–14.00	–14.10
$n[\hat{\rho}_{1,ML}-1]$	–12.24	–11.93	–10.41	–8.95	–8.10
$n[\hat{\rho}_{7,GLS}-1]$	–10.95	–10.16	–9.35	–8.64	–8.10
$n[\hat{\rho}_{U}-1]$	–12.02	–12.49	–12.76	–12.95	–13.13
$n[\hat{\rho}_{ws}-1]$	–12.03	–12.48	–12.69	–12.88	–13.07
$\hat{\tau}_{\mu,OLS}$	–3.00	–2.93	–2.89	–2.88	–2.86
$\hat{\tau}_{1,ML}$	–2.75	–2.53	–2.28	–2.07	–1.95
$\hat{\tau}_{7,GLS}$	–2.56	–2.30	–2.14	–2.03	–1.95
$\hat{\tau}_{U}$	–2.75	–2.69	–2.66	–2.65	–2.64
$\hat{\tau}_{ws}$	–2.66	–2.61	–2.56	–2.54	–2.50
$\hat{\Phi}_{OLS}$	5.25	4.86	4.72	4.61	4.59
$\hat{\Phi}_{1,ML}$	6.46	5.51	4.86	4.39	4.39
$\hat{\Phi}_{ES}$	2.87	2.98	3.09	3.19	3.96

Table 4.2. **Empirical powers for 5% level test criteria ($n = 25$, $Y_1 \sim N(0, 1)$, 5000 replications)**

Statistic	ρ							
	1.00	0.98	0.95	0.93	0.90	0.85	0.80	0.70
$\hat{\rho}_{\mu,OLS}$	5.46	6.70	8.10	8.56	10.18	14.34	19.70	32.84
$\hat{\rho}_{1,ML}$	5.36	6.40	8.16	8.36	10.04	14.44	19.84	33.22
$\hat{\rho}_{7,GLS}$	5.42	6.10	8.48	9.02	11.24	16.80	23.60	37.76
$\hat{\rho}_{U}$	5.24	6.24	8.54	8.70	10.74	15.94	22.14	36.18
$\hat{\rho}_{ws}$	5.28	6.20	8.54	8.62	10.76	15.82	21.98	36.04
$\hat{\tau}_{\mu,OLS}$	5.62	6.54	6.22	6.34	6.94	8.46	12.14	20.64
$\hat{\tau}_{1,ML}$	5.26	6.14	8.50	8.58	10.26	14.64	20.52	34.42
$\hat{\tau}_{7,GLS}$	5.42	6.22	8.68	8.94	11.24	16.56	23.34	37.86
$\hat{\tau}_{U}$	5.10	6.24	8.68	9.28	11.32	17.42	25.24	39.70
$\hat{\tau}_{ws}$	5.12	6.12	8.64	9.26	11.30	17.28	25.16	39.66
$\hat{\Phi}_{OLS}$	4.52	4.48	4.28	4.16	4.16	4.78	7.70	14.08
$\hat{\Phi}_{1,ML}$	5.18	5.58	7.88	8.44	10.40	15.48	22.80	37.14
$\hat{\Phi}_{ES}$	4.78	6.30	8.84	9.96	12.28	18.20	26.64	41.54

Table 4.3. **Empirical powers for 5% level test criteria ($n = 50$, $Y_1 \sim N(0, 1)$, 5000 replications)**

Statistic	ρ							
	1.00	0.98	0.95	0.93	0.90	0.85	0.80	0.70
$\hat{\rho}_{\mu,OLS}$	5.22	7.20	10.72	13.16	19.50	30.88	47.22	80.20
$\hat{\rho}_{1,ML}$	5.02	7.06	12.10	15.28	23.24	37.58	54.98	86.00
$\hat{\rho}_{7,GLS}$	4.86	7.24	12.84	17.62	26.78	42.48	61.26	88.40
$\hat{\rho}_{U}$	5.04	7.20	12.36	15.74	24.10	38.44	56.24	86.58
$\hat{\rho}_{ws}$	4.94	7.10	12.06	15.50	23.46	38.06	55.66	86.30
$\hat{\tau}_{\mu,OLS}$	5.24	6.24	6.84	8.04	11.76	19.50	32.30	64.70
$\hat{\tau}_{1,ML}$	4.88	7.32	12.34	15.56	23.64	37.98	56.06	86.12
$\hat{\tau}_{7,GLS}$	5.00	7.40	13.06	17.74	26.62	43.12	61.50	88.60
$\hat{\tau}_{U}$	5.02	7.46	12.90	17.00	25.50	40.84	59.32	88.02
$\hat{\tau}_{ws}$	4.88	7.30	12.36	16.54	24.90	40.00	58.38	87.60
$\hat{\Phi}_{OLS}$	5.22	4.46	4.20	4.96	8.16	13.74	23.82	53.00
$\hat{\Phi}_{1,ML}$	4.90	6.50	11.48	15.90	23.80	39.60	57.86	86.52
$\hat{\Phi}_{ES}$	5.08	7.52	13.42	19.22	27.40	44.58	61.82	85.02

Table 4.4. **Empirical powers for 5% level test criteria $(n = 100, Y_1 \sim N(0, 1)$, 5000 replications)**

Statistic	ρ							
	1.00	0.98	0.95	0.93	0.90	0.85	0.80	0.70
$\hat{\rho}_{\mu,OLS}$	4.90	9.10	19.32	28.66	46.82	78.82	95.50	99.94
$\hat{\rho}_{1,ML}$	4.84	10.88	27.40	41.88	63.66	90.50	98.92	100.00
$\hat{\rho}_{7,GLS}$	5.06	10.86	29.16	43.70	66.74	90.30	97.92	99.76
$\hat{\rho}_U$	4.80	10.18	24.24	36.52	56.66	86.68	98.04	100.00
$\hat{\rho}_{ws}$	4.84	10.28	24.60	36.98	57.14	87.04	98.16	100.00
$\hat{\tau}_{\mu,OLS}$	5.46	6.68	11.70	17.80	31.06	62.10	87.70	99.70
$\hat{\tau}_{1,ML}$	4.84	10.92	27.60	42.02	64.40	90.54	98.84	100.00
$\hat{\tau}_{7,GLS}$	4.96	10.94	29.62	44.08	67.28	90.80	97.90	99.72
$\hat{\tau}_U$	4.96	10.30	25.86	38.82	59.98	88.38	98.34	100.00
$\hat{\tau}_{ws}$	4.96	10.48	26.08	39.16	60.22	88.72	98.36	100.00
$\hat{\Phi}_{OLS}$	5.10	4.40	7.98	12.10	22.00	49.76	78.82	98.98
$\hat{\Phi}_{1,ML}$	5.08	9.52	26.36	39.72	62.88	88.42	97.42	99.90
$\hat{\Phi}_{ES}$	5.10	11.02	29.76	44.66	67.44	88.56	95.28	98.76

Table 4.5. Empirical powers for 5% level test criteria ($n = 250$, $Y_1 \sim N(0, 1)$, 5000 replications)

Statistic	ρ							
	1.00	0.98	0.95	0.93	0.90	0.85	0.80	0.70
$\hat{\rho}_{\mu,OLS}$	5.08	18.76	61.32	87.08	99.34	100.00	100.00	100.00
$\hat{\rho}_{1,ML}$	5.36	30.64	85.88	97.90	99.98	100.00	100.00	100.00
$\hat{\rho}_{7,GLS}$	5.38	30.94	85.52	97.78	99.86	100.00	100.00	100.00
$\hat{\rho}_U$	5.14	24.80	74.56	94.20	99.76	100.00	100.00	100.00
$\hat{\rho}_{ws}$	5.16	25.16	75.06	94.48	99.78	100.00	100.00	100.00
$\hat{\tau}_{\mu,OLS}$	5.06	11.72	44.08	73.62	96.74	100.00	100.00	100.00
$\hat{\tau}_{1,ML}$	5.38	31.42	85.84	97.88	99.96	100.00	100.00	100.00
$\hat{\tau}_{7,GLS}$	5.20	31.60	85.66	97.52	99.88	99.98	99.98	100.00
$\hat{\tau}_U$	4.88	26.18	76.88	95.10	99.90	100.00	100.00	100.00
$\hat{\tau}_{ws}$	5.06	26.70	77.88	95.34	99.94	100.00	100.00	100.00
$\hat{\Phi}_{OLS}$	5.04	8.22	34.76	63.42	93.42	100.00	100.00	100.00
$\hat{\Phi}_{1,ML}$	5.42	27.06	81.60	96.36	99.82	100.00	100.00	100.00
$\hat{\Phi}_{ES}$	5.16	31.06	85.02	96.78	99.82	99.94	99.94	99.98

Table 4.6. **Empirical powers for 5% level test criteria (n = 25, $Y_1 \sim N[0, (1-\rho^2)^{-1}]$, 5000 replications)**

Statistic	ρ							
	0.99	0.98	0.95	0.93	0.90	0.85	0.80	0.70
$\hat{\rho}_{\mu,OLS}$	5.40	6.04	6.72	8.10	8.50	12.70	18.28	31.78
$\hat{\rho}_{1,ML}$	5.22	5.82	6.56	7.72	8.44	12.44	18.52	31.92
$\hat{\rho}_{7,GLS}$	5.32	5.72	6.74	7.54	8.72	13.88	20.00	34.28
$\hat{\rho}_U$	5.30	5.70	6.80	7.54	8.70	12.94	19.30	33.68
$\hat{\rho}_{ws}$	5.22	5.64	6.80	7.50	8.70	12.84	19.30	33.64
$\hat{\tau}_{\mu,OLS}$	5.68	6.40	6.08	6.98	7.42	9.42	13.18	21.68
$\hat{\tau}_{1,ML}$	5.18	5.74	6.66	7.70	8.70	12.84	19.20	32.90
$\hat{\tau}_{7,GLS}$	5.14	5.82	7.12	7.46	8.88	13.86	20.34	34.78
$\hat{\tau}_U$	5.00	5.78	7.08	7.84	9.14	14.10	20.98	36.28
$\hat{\tau}_{ws}$	5.02	5.78	7.08	7.78	9.10	13.86	20.80	36.10
$\hat{\Phi}_{OLS}$	4.24	4.20	4.00	4.74	4.76	5.76	8.48	14.76
$\hat{\Phi}_{1,ML}$	4.70	5.18	6.12	7.06	7.94	12.22	18.58	33.20
$\hat{\Phi}_{ES}$	4.70	5.82	6.98	7.96	9.16	14.08	20.94	35.52

Table 4.7. **Empirical powers for 5% level test criteria ($n = 50$, $Y_1 \sim N[0, (1-\rho^2)^{-1}]$, 5000 replications)**

Statistic	ρ							
	0.99	0.98	0.95	0.93	0.90	0.85	0.80	0.70
$\hat{\rho}_{\mu,OLS}$	5.22	6.52	8.70	11.98	18.04	29.98	46.62	80.22
$\hat{\rho}_{1,ML}$	4.76	6.46	9.28	13.20	19.34	32.74	50.92	84.34
$\hat{\rho}_{7,GLS}$	5.16	6.10	9.48	13.44	19.92	32.90	51.06	82.12
$\hat{\rho}_{U}$	4.92	6.44	9.60	13.62	19.46	33.22	51.80	84.88
$\hat{\rho}_{ws}$	4.86	6.26	9.32	13.32	19.14	33.08	51.08	84.38
$\hat{\tau}_{\mu,OLS}$	5.66	6.08	7.28	8.64	12.88	21.04	34.12	66.18
$\hat{\tau}_{1,ML}$	4.82	6.38	9.52	13.18	19.44	33.26	51.64	84.72
$\hat{\tau}_{7,GLS}$	5.22	6.08	9.96	13.50	19.98	33.48	51.66	82.42
$\hat{\tau}_{U}$	5.02	6.20	10.06	13.46	20.18	33.78	52.66	85.44
$\hat{\tau}_{ws}$	4.86	6.06	9.78	13.16	19.68	33.10	51.84	84.80
Φ_{OLS}	4.88	4.30	4.86	5.58	9.08	14.64	25.36	54.76
$\Phi_{1,ML}$	4.72	5.50	8.80	11.92	18.38	30.40	47.66	80.30
Φ_{ES}	5.46	6.04	10.32	13.88	20.24	32.38	48.60	74.90

Table 4.8. **Empirical powers for 5% level test criteria ($n = 100$, $Y_1 \sim N[0, (1-\rho^2)^{-1}]$, 5000 replications)**

Statistic	ρ							
	0.99	0.98	0.95	0.93	0.90	0.85	0.80	0.70
$\hat{\rho}_{\mu,OLS}$	6.28	7.64	17.14	26.52	45.90	78.64	95.46	99.94
$\hat{\rho}_{1,ML}$	6.34	8.18	19.48	30.98	52.02	82.02	95.94	99.96
$\hat{\rho}_{7,GLS}$	6.54	8.08	19.20	29.64	48.66	74.80	88.12	97.46
$\hat{\rho}_U$	6.48	8.00	18.80	29.90	51.56	83.94	97.22	100.00
$\hat{\rho}_{ws}$	6.62	8.06	19.06	30.12	52.06	84.24	97.24	100.00
$\hat{\tau}_{\mu,OLS}$	5.50	7.08	11.80	19.06	30.14	56.20	78.92	97.64
$\hat{\tau}_{1,ML}$	6.46	8.12	19.48	31.18	51.92	82.08	95.88	99.96
$\hat{\tau}_{7,GLS}$	6.36	8.10	19.50	30.20	49.80	75.32	88.36	97.60
$\hat{\tau}_U$	6.40	8.00	19.48	30.68	52.10	83.70	97.16	100.00
$\hat{\tau}_{ws}$	6.54	8.10	19.62	31.08	52.18	83.86	97.08	100.00
$\hat{\Phi}_{OLS}$	4.94	5.06	8.92	13.52	23.80	52.06	80.24	99.04
$\hat{\Phi}_{1,ML}$	5.70	6.84	17.12	26.68	45.42	73.28	89.10	99.06
$\hat{\Phi}_{ES}$	6.64	8.24	19.44	28.42	47.22	69.78	82.84	93.76

Table 4.9. **Empirical powers for 5% level test criteria ($n = 250$, $Y_1 \sim N[0, (1-\rho^2)^{-1}]$, 5000 replications)**

Statistic	ρ							
	0.99	0.98	0.95	0.93	0.90	0.85	0.80	0.70
$\hat{\rho}_{\mu,OLS}$	9.00	17.16	61.08	87.16	99.32	100.00	100.00	100.00
$\hat{\rho}_{1,ML}$	10.04	19.12	60.98	80.36	93.88	99.56	99.90	100.00
$\hat{\rho}_{7,GLS}$	9.90	18.96	58.52	76.16	89.84	96.56	98.98	99.90
$\hat{\rho}_{U}$	9.96	19.64	68.16	91.52	99.72	100.00	100.00	100.00
$\hat{\rho}_{ws}$	10.08	19.74	68.62	91.80	99.72	100.00	100.00	100.00
$\hat{\tau}_{\mu,OLS}$	7.08	12.26	46.30	75.58	97.02	100.00	100.00	100.00
$\hat{\tau}_{1,ML}$	9.98	19.54	60.82	80.64	94.00	99.58	99.96	100.00
$\hat{\tau}_{7,GLS}$	9.88	19.16	58.80	76.44	89.88	96.68	98.96	99.90
$\hat{\tau}_{U}$	9.58	19.28	67.92	91.38	99.72	100.00	100.00	100.00
$\hat{\tau}_{ws}$	9.78	19.88	68.76	91.68	99.76	100.00	100.00	100.00
$\hat{\Phi}_{OLS}$	5.28	8.86	37.36	65.72	93.98	100.00	100.00	100.00
$\hat{\Phi}_{1,ML}$	8.36	16.30	54.24	73.58	89.38	97.58	99.66	100.00
$\hat{\Phi}_{ES}$	9.74	18.78	56.26	73.32	87.06	95.28	98.26	99.80

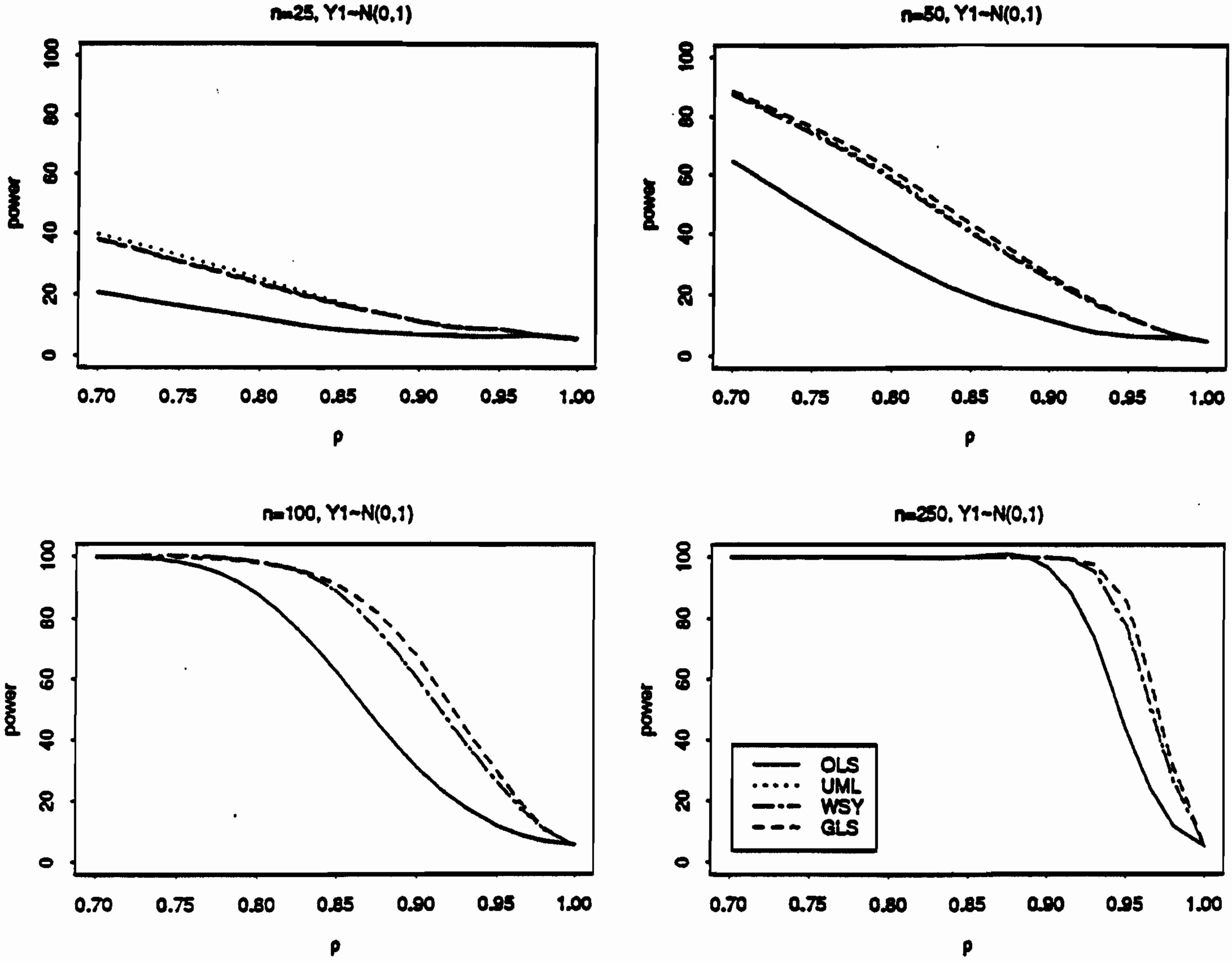

Figure 4.1. Powers of pivotal statistics when Y_1 is generated from $N(0, 1)$.

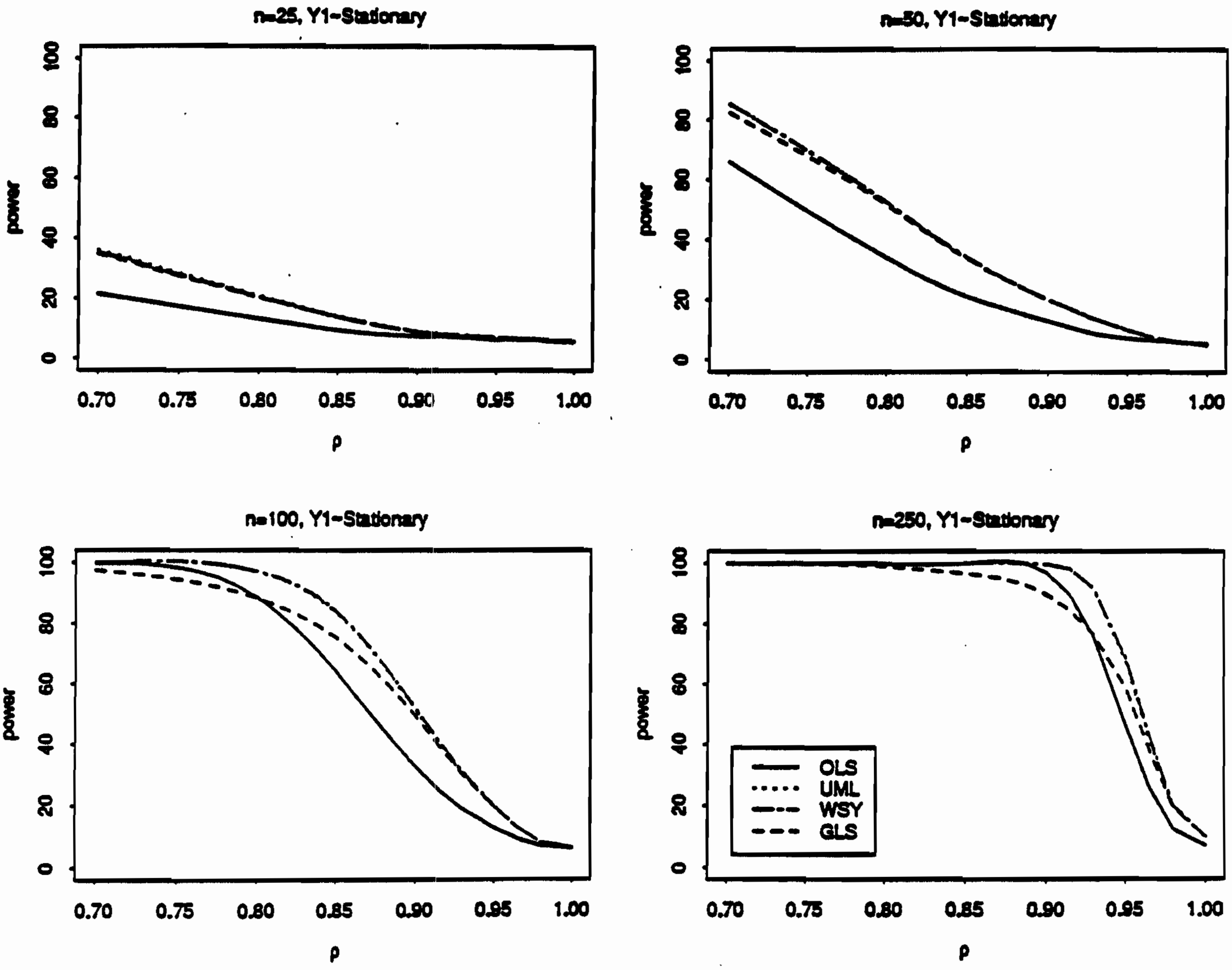

Figure 4.2. Powers of pivotal statistics when Y_1 is generated from $N[0, (1-\rho^2)^{-1}]$.

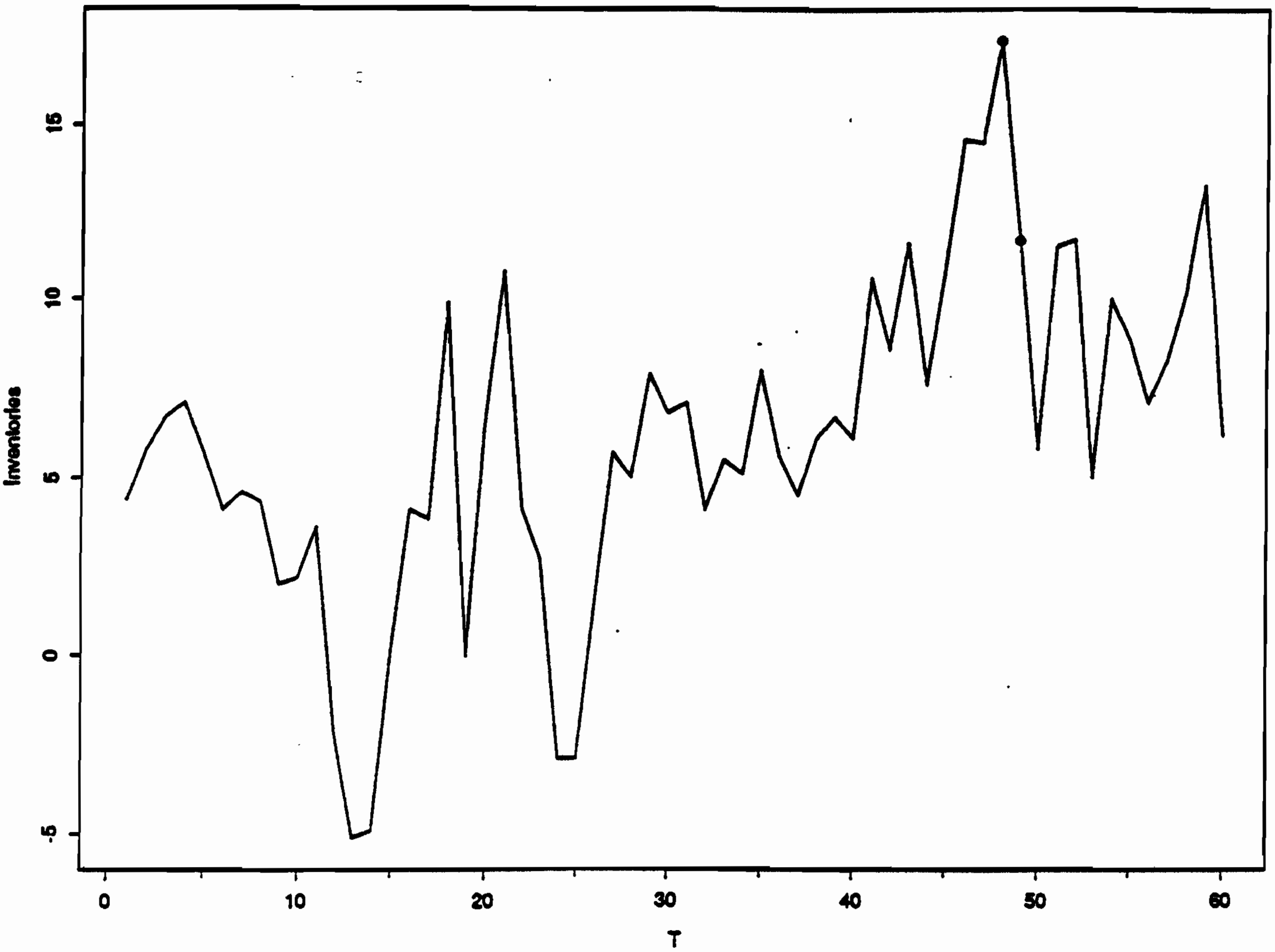

Figure 5.1. Quarterly change in business inventories. Observations 48 and 49 are highlighted.