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#Envelope theorem in dynamic economic models with recursive utility[☆]

Juan Pablo Rincón-Zapatero[a], Yanyun Zhao[b,*]

[a] *Department of Economics, Universidad Carlos III de Madrid, 28903 Getafe (Madrid), Spain*

[b] *Wenlan School of Business, Zhongnan University of Economics and Law, 430073 Wuhan, China*

Abstract

This paper investigates the first-order differentiability properties of the value function in dynamic economic models with recursive preferences where the optimal policy may lie at the boundary of the feasible set under several regular assumptions originating from the static optimization theory plus an aditional asymptotic condition.

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[*]Corresponding Author. Email: yanyunzhao@zuel.edu.cn.

1. Introduction

Most dynamic economic models presume an additive and separable functional across states of nature and the existence of interior optimal solution. This discounted, time–separable preference structure shares analytical tractability and mathematical simplification and the interiority condition is helpful in characterizing the smoothness property of the value function, as discussed in Benveniste and Scheinkman (1979).

The main purpose of this paper is to establish the first-order differentiability properties of the value function with recursive utility, departing from the time additively separable utility function and also dropping the interiority of the optimal policy. Several assumptions mainly from the static optimization theory are postulated to circumvent the interiority conditions: there exists some optimal choice in the interior of the domain and the gradient of the saturated constraints is linearly independent as well as some asymptotic condition due to the dynamic nature of the problem. Section 2 presents the aggregator approach to recursive utility. The main results on the differentiability of the value function are considered in Section 3. In Section 4 we apply the results to both the one-sector growth model and heterogenous two-sector growth model. Finally, there is an appendix that contains some auxiliary definitions and results.

2. Recursive utility defined by means of aggregators

Recursive utility could be recovered via the means of the aggregator approach as shown in Lucas and Stokey (1984). Here we consider the following abstract optimization framework.

The state space is denoted by $\mathrm{X} \subseteq \mathbb{R}^n$. An accumulation path $\boldsymbol{x} = (x_t)_{t=0}^{\infty}$ in X is said to be feasible if $x_{t+1} \in \Gamma(x_t)$ for all $t \geq 0$ starting from some initial condition $x_0 \in \mathrm{X}$, where $\Gamma : \mathrm{X} \longrightarrow 2^{\mathrm{X}}$ is the feasible correspondence. Let $\Pi(x_0)$ denotes the collection of all feasible paths from x_0 and let $\mathrm{A} = \{(x, y) : y \in \Gamma(x)\} \subset \mathrm{X} \times \mathrm{X}$ be the graph of Γ.

An aggregator is a function

$$\mathrm{W} : \mathrm{A} \times \mathbb{R} \longrightarrow \mathbb{R}.$$

$\mathrm{W}(x, y, z)$ is interpreted as the utility enjoyed from now on if the pair (x, y) is feasible, i.e., $y \in \Gamma(x)$, and if the accumulation path from tomorrow on yields $z \in \mathbb{R}$ utils as of tomorrow.

Definition 1. *Given an aggregator* W *and* $\mathcal{L}$ *be the space of sequences* $(e_0, e_1, e_2, \dots,)$ *with* $e_t \in \mathrm{X},\ t = 0, 1, 2, \dots,$ *we say that* $\mathrm{U} : \mathcal{L} \to \mathbb{R}$ *is recursive if and only if*

$$\mathrm{U}(\boldsymbol{x}) := \mathrm{U}(x_0, x_1, \cdots) = \mathrm{W}(x_0, x_1, \mathrm{U}({}_1\boldsymbol{x})) \tag{1}$$

where ${}_1\boldsymbol{x} := (x_1, x_2, \cdots) \in \Pi(x_0)$.

The usual time additively separable utility function could be nested in this formulation.

Given some initial value $x_0 \in \mathrm{X}$, the value function $\mathcal{J}(x_0) = \max_{\boldsymbol{x}\in\Pi(x_0)} \mathrm{U}(\boldsymbol{x})$. A path $\boldsymbol{x}^*$ is said to be optimal from x_0 if $\boldsymbol{x}^* \in \Pi(x_0)$ and $\mathcal{J}(x_0) = \mathrm{U}(\boldsymbol{x}^*)$. As shown in Boyd III (1990), standard arguments show that the Bellman equation is given as:

$$\mathcal{J}(x) = \max_{y\in\Gamma(x)} \{\mathrm{W}(x, y, \mathcal{J}(y))\}.$$

We will impose the following basic assumptions of continuity and convexity on the data.

(B1) : Γ is non-empty valued, compact valued and continuous, and its graph is a convex set;

(B2) : W is concave and continuous on $\mathrm{A} \times \mathbb{R}$;

(B3) : For any $(x, y) \in \mathrm{A}$, the function $z \longmapsto W(x, y, z)$ is nondecreasing.

These assumptions assure that the value function $\mathcal{J}$ is concave and continuous in X. See the Appendix. Moreover, by the Maximum Theorem of Berge, the optimal policy correspondence $\mathcal{H} : \mathrm{X} \to \mathrm{X}$ defined as $\mathcal{H}(x) = \mathrm{argmax}_{y\in\Gamma(x)}\ \mathrm{W}(x, y, \mathcal{J}(x))$ is upper hemi–continuous and compact valued.

Along the paper, a dynamic optimization problem $(\mathrm{X}, \Gamma, \mathrm{W})$ satisfies (B1)–(B3) above.

3. Differentiability of the value function

3.1. Assumptions

In order to establish differentiability of the value function we need some regularity conditions for boundary solutions. Following Rincón-Zapatero and Santos (2009), we introduce the main assumptions below:

(D1) : W is continuously differentiable in an open neighborhood of $A \times \mathbb{R}$.

(D2) : For each $x \in \mathrm{int}(\mathrm{X})$, there exists $y \in \mathcal{H}(x)$ with $y \in \mathrm{int}(\mathrm{X})$.

(D3) : There exists a finite collection of functions $g = (g^1, \cdots, g^m)$ with $\mathrm{A} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : g(x, y) \geq 0\}$. Each function $g^i : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is quasiconcave and has continuous first-order partial derivatives for $i = 1, 2, \cdots, m$. Let $s(x, y) = \{i : g^i(x, y) = 0\}$ denote the set of saturated constraints. Then for each x there exists some $y \in \mathcal{H}(x)$ with $y \in \mathrm{int}(\mathrm{X})$ such that the rank of the matrix of partial derivatives $\{D_y g^i(x, y) : i \in s(x, y)\}$ is equal to $\#s(x, y)$.

(D1) is a smoothness assumption on the aggregator. (D2) allows us to select some optimal solution in the interior of the domain. It is important not confuse this property with interiority of the optimal policy. (D3) says that the feasible correspondence is given by a finite set of quasiconcave functions.

Let us denote with $D_x g(x,y)$(*resp.* $D_y g(x,y)$) the $m \times n$ matrix of partial derivatives of g at (x,y) with respect to $x = (x_1, \dots x_n)$(*resp.* $y = (y_1, \dots y_n)$). Accordingly, the $s \times n$ matrix $D_x g_s(x,y)$ (*resp.* $D_y g_s(x,y)$) is defined similarly. The $s \times n$ generalized inverse matrix of $D_y g_s$ is defined as $D_y g_s^{+}(x,y) = (D_y g_s D_y g_s^{\top})^{-1} D_y g_s(x,y)$ at each point (x,y), where $D_y g_s^{\top}$ denotes the transpose matrix. We define an $n \times n$ matrix G as

$$G(x,y) := \begin{cases} -D_x g_s(x,y)^{\top} D_y g_s^{+}(x,y), & \text{if } s(x,y) \neq \emptyset \\ O_n, & \text{if } s(x,y) = \emptyset \end{cases}, \tag{2}$$

where O_n and I_n denotes the null matrix and the identity matrix of order $n \times n$ respectively. Here $G(x,y)$ stands for the matrix of the marginal rate of transformation between x and y at the saturated constraints. Finally, $\partial \mathcal{J}$ denotes the superdifferential of $\mathcal{J}$. Precise definitions and further results are given in the Appendix.

3.2. Main result

Now we establish an envelope theorem in the framework of concave and non–smooth optimization as follows:

Theorem 3.1. *Consider a dynamic optimization problem* $(\mathrm{X}, \Gamma, \mathrm{W})$. *Let (D1)–(D3) be satisfied. Then for any* $x_0 \in int(\mathrm{X})$ *and* $T \geq 1$, $q_0 \in \partial \mathcal{J}(x_0)$ *if and only if there exists* $q_T \in \partial \mathcal{J}(x_T)$ *such that*

$$q_0 = \sum_{t=0}^{T-1} \beta_t \mathcal{G}_t \Big\{ D_x \mathrm{W}(x_t, x_{t+1}, \mathcal{J}(x_{t+1})) + G(x_t, x_{t+1}) D_y \mathrm{W}(x_t, x_{t+1}, \mathcal{J}(x_{t+1})) \Big\} + \beta_T \mathcal{G}_T q_T \tag{3}$$

where $(x_{t+1})_{t=0}^{\infty}$ *is an optimal path from* x_0, *and for* $t = 1, 2, \dots$

$$\beta_t = \prod_{i=1}^{t} D_z \mathrm{W}(x_{i-1}, x_i, \mathcal{J}(x_i)), \quad \beta_0 = 1$$
$$\mathcal{G}_t = G(x_0, x_1) G(x_1, x_2) \cdots G(x_{t-1}, x_t), \quad \mathcal{G}_0 = I_n.$$

Expression (3) defines the superdifferential of the value function at the current state in terms of the differential of the instantaneous utility and the superdifferential of the value function evaluated at the optimal path, where β_t plays the role of a non–constant discount factor. Note that the Inada condition explored in the Appendix shows that $\partial \mathcal{J}(x_0)$ is not empty if the optimal path touches regions of the feasible set A.

If we have the following asymptotic condition, then we could obtain that $\partial \mathcal{J}(x_0)$ is a singleton and thus proving that $\mathcal{J}$ is differentiable at x_0.

(D4) : Let $\{x_{t+1}\}_{t\geq 0}$ be an optimal plan starting from $x_0 \in \text{int}(\text{X})$. For all $q_T \in \partial \mathcal{J}(x_T)$,

$$\lim_{T\to\infty} \beta_T \mathcal{G}_T q_T = 0 \tag{4}$$

Remark 1. *This asymptotic condition (D4) is automatically fulfilled in two simple cases. First, if the optimal path is not at the boundary of the feasible correspondence at every period of time, that is, if for some t, $x_{t+1} \in \text{int}(\Gamma(x_t))$, then $G(x_t, x_{t+1}) = O_n$, since $s(x_t, x_{t+1}) = \emptyset$; Second, if for some t with $s(x_t, x_{t+1}) \neq \emptyset$, then $D_x g_s(x_t, x_{t+1}) = 0$, as it happens when the saturated constraints are independent of x and then $G(x_t, x_{t+1}) = O_n$. In both cases, $\mathcal{G}_T = O_n$ for any $T \geq t$.*

Theorem 3.2. *Consider a dynamic optimization problem* $(\text{X}, \Gamma, \text{W})$. *Let $\{x_{t+1}\}_{t\geq 0}$ be an optimal plan satisfying (D1)–(D4) with $x_0 \in \text{int}(\text{X})$. Then, the value function $\mathcal{J}(\cdot)$:* $\text{int}(\text{X}) \longrightarrow \mathbb{R}$ *is differentiable at x_0 and the derivative is given by*

$$D_x \mathcal{J}(x_0) = \sum_{t=0}^{\infty} \beta_t \mathcal{G}_t \big\{ D_x \text{W}(x_t, x_{t+1}, \mathcal{J}(x_{t+1}) + G(x_t, x_{t+1}) D_y \text{W}(x_t, x_{t+1}, \mathcal{J}(x_{t+1})) \big\} \tag{5}$$

As explained above the theorem, (5) is a finite sum if either the optimal path is interior or it saturates constraints at some T which are state-independent. The case with $T = 1$ is the celebrated Mirman-Zilcha, Benveniste-Scheinkman envelope theorem for interior solutions. In both cases,

$$D_x \mathcal{J}(x_0) = D_x \text{W}(x_0, x_1, \mathcal{J}(x_1)).$$

However, Theorem 3.2 covers the general case where the optimal policy has the boundary solution.

4. Optimal growth with recursive utility

In this section we study both the one-sector growth model and the two-sector heterogeneous growth model with recursive utility.

4.1. One-sector and two-sector growth model with recursive utility

Consider an economy where there exists one non–negative capital stock k_t on hand at each period t. Let $c_t \in \mathbb{R}_+$ denote the perishable consumption good satisfying $c_{t+1} + k_{t+1} = f(k_t)$, $t = 0, 1, \ldots$, where the production function $f(\cdot)$ satisfies $f(\cdot) \geq 0$ and $f'(\cdot) \geq 0$.

Let $\mathrm{W}(\cdot,\cdot,\cdot)$ be an aggregator satisfying (B1)–(B3) and $\mathrm{W}(k,k',z) = w(f(k)-k',z)$, where $w : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuously differentiable and with the function $c \mapsto w(c,z)$ being increasing for all $z \in \mathbb{R}$.

The recursive formulation of the planner's problem is

$$\mathcal{J}(k) = \max\Big\{w(f(k)-k',\mathcal{J}(k')) : 0 \le k' \le f(k)\Big\}.$$

The constraint correspondence is $\Gamma(k) = [0, f(k)]$, whose graph is bordered by the functions $g^1(k,k') = k'$ and $g^2(k,k') = f(k) - k'$. If the Inada condition $f'(0^+) = \infty$ holds, then $k' = 0$ cannot be optimal, as demonstrated in Corollary 1.2 of the Appendix, thus (D2) is fulfilled. (D3) is always satisfied since $D_{k'}g^1(k,k') = 1$ and $D_{k'}g^2(k,k') = -1$ for every (k,k'). Notice that the optimal path cannot have $k_{t+1} = f(k_t)$ for each t, since this implies $c_t = 0$ for every t, which is not optimal because w is strictly increasing in the first variable. Thus, the optimal path becomes interior at some finite T, which implies (D4) as explained in Remark 1.

In consequence, $\mathcal{J}$ is differentiable at some initial value $k_0 > 0$. For a given $k_0 > 0$ such that the optimal $k_1, \dots, k_{T-1}$ satisfy $k_t = f(k_{t-1})$ and it is interior at some time T, that is, $0 < k_T < f(k_{T-1})$, the derivative is given by

$$\begin{aligned}\mathcal{J}'(k_0) &= \sum_{t=0}^{T-1} \beta_t \mathcal{G}_t \{D_k \mathrm{W}(k_t, k_{t+1}, \mathcal{J}(k_{t+1})) + G(k_t,k_{t+1}) D_{k'} \mathrm{W}(k_t, k_{t+1}, \mathcal{J}(k_{t+1}))\} \\ &= \beta_{T-1}\mathcal{G}_{T-1} D_k \mathrm{W}(k_{T-1}, k_T, \mathcal{J}(k_T)),\end{aligned}$$

where

$$\begin{aligned}\beta_0 &= 1, \quad \mathcal{G}_0 = 1 \\ \beta_t &= \prod_{i=1}^{t} D_z \mathrm{W}(k_{i-1}, k_i, \mathcal{J}(k_i)),\ t = 1,2,\dots,T-1 \\ \mathcal{G}_t &= \prod_{i=0}^{t-1} f'(k_i),\ t = 1,2,\dots,T.\end{aligned}$$

The summatory function above simplifies since, for $T \ge 2$, the summands from $t = 0$ to $T-2$ cancel out

$$\begin{aligned}&D_k \mathrm{W}(k_t, k_{t+1}, \mathcal{J}(k_{t+1})) + G(k_t,k_{t+1}) D_{k'} \mathrm{W}(k_t, k_{t+1}, \mathcal{J}(k_{t+1})) \\ &= f'(k_t) D_c w(c_t, \mathcal{J}(k_{t+1})) - f'(k_t) D_c w(c_t, \mathcal{J}(k_{t+1})) \\ &= 0.\end{aligned}$$

The formulation in two–sector heterogenous growth model is omitted here for brevity since it is essentially the same as that presented above.

5. Conclusion

This paper considers the problem of the smoothness of the value function in concave dynamic problems with recursive utility where the optimal solution may belong to the boundary of the feasible set. We formulate several conditions stemming from the static theory and then establish the first-order differentiable properties of the value function with recursive preference by adding an asymptotic condition. The results are then illustrated by the one-sector growth model.

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