Working Paper 95-24
Economics Series 15
July 1995

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EXTENDED PARETIAN RULES AND RELATIVE UTILITARIANISM[1]

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Abstract

This paper introduces the 'Extended Pareto´ axiom on Social welfare functions and gives a characterization of the axiom when it is assumed that the Social Welfare Functions that satisfy it in a framework of preferences over lotteries also satisfy the restrictions (on the domain and range of preferences) implied by the von-Neumann-Morgenstern axioms. With the addition of two other axioms: Anonymity and Weak IIA* it is shown that there is a unique Social Welfare Function called Relative Utilitarianism that consists of normalizing individual utilities between zero and one and then adding them.

Key Words
Group Preferences, Multi-profile.

[1]This paper is a revised version of two chapters of my thesis. I am grateful to J.F. Mertens for many extremely helpful discussions. I am also indebted to my advisor, John Hillas, and to C.d´Aspremont and P.Mongin for their comments. Part of this work was done at C.O.R.E., Belgium, and I am grateful for their hospitality

1 Introduction

Arrow [1], as far back as 1963, considered the possibility of a resolution of the social choice paradox by the use of a "broader concept of rationality," meaning thereby the use of the von-Neumann-Morgenstern axioms on preferences. In this paper I provide an axiomatization of a Social Welfare Function, in the sense of Arrow [1] called "*Relative Utilitarianism*", in a framework of preferences over lotteries and using the vN-M axioms on preferences. Relative Utilitarianism consists of normalizing individual utilities and then adding them, and was introduced separately in Mertens and Dhillon [12]. This approach is not new, indeed impossibility results have already been proved in the more general context of cardinal preferences of which v-NM axioms are a special case (see e.g. Kalai and Schmeidler [10], Sen [15]). Chichilinsky [3], studies the aggregation problem when intensities are taken into account, and the SWF is assumed to be continuous,anonymous and to respect unanimity. The result of this paper is however a positive one; I show that a SWF exists and is unique under the axioms proposed.

These axioms are: the classical *Anonymity* axiom (see May [11]), a weakened version (conceptually) of Arrow's Independence of Irrelevant Alternatives, *Weak IIA**, and *Extended Pareto.* The collective choice problem is usually viewed as a map from individual preferences to social preferences. Most voting rules, on the other hand, are in "steps", i.e. they first aggregate preferences of individuals in smaller units and then use these "group" choices to derive social choices. If one were to allow different "groups" (or coalitions) in society, what reasonable restrictions could we impose on them and what do these restrictions imply for the social rule? A requirement that arises quite naturally is the analog of Pareto for groups: this is what the Extended Pareto axiom provides. Weak IIA* may be viewed both as one way to adapt Arrow's Independence axiom to the context of preferences over lotteries, and as an axiom that leads to a formulation of the problem that is quite similar to the bargaining problem without assigning special importance to a disagreement point.

The main results include a characterization of the Extended Pareto axiom in the context of vN-M preferences and an axiomatic characterization of Relative Utilitarianism. The latter result is close to and may be considered a generalization of May's [11] Theorem (on majority rule) to bigger sets of alternatives[2] Indeed, as in May, we eschew the use of interpersonal comparisons as primitives. This paper provides an alternative axiomatization of Relative Utilitarianism avoiding the use of Continuity as in Mertens and Dhillon, an axiom that has no clear ethical interpretation, except on negative considerations, i.e. "it is only a test that some solution is unsatisfactory, but does not tell us which are the specific equity considerations that force the specific solution" (Mertens and Dhillon).

There has been, in recent years, a renewed interest in Harsanyi's [9] Utilitarianism theorems (see e.g. Weymark [17], Mongin [13] Coulhon and Mongin [4], Hammond [8]). This paper shares some of the features of the Harsanyi model. In particular, the use of vN-M utilities for individuals and society and the use of Pareto rules. While Harsanyi's theorem is a single profile one however, this paper uses the classical definition (Arrow) of the SWF. We generalize Harsanyi's single profile result, and the use of additional axioms fixes the weights for individuals to be the inverse of the range of the utility function for an individual.

The rest of the paper is organized as follows: Section 2 introduces notation, Section 3 discusses the axioms used, Section 4 gives the main results and then the proofs of these, and also provides examples to show the necessity of the axioms. Section 5 concludes.

2 Preliminaries

The set of individuals is denoted by $N = \{1, ..., n, ...\}$ and there are $\#N$ individuals in the society, with $\infty > \#N \geq 3$. I denote the set of alternatives or pure prospects by A. Following Mertens and Dhillon [12], I consider a framework of preferences over the set $\Delta(A)$ of all lotteries on A (finite), which is interpreted as some set of 'pure prospects', and assume that all such preferences have a von Neumann- Morgenstern utility representation. I denote the set of preference orderings on ΔA by $\mathcal{L}$. A *preference ordering* is a reflexive, complete and transitive binary relation on $\Delta A \times \Delta A$. The n-fold cartesian product of $\mathcal{L}$ is denoted by $\mathcal{L}^N$. We use the term *preference profile* for an element of $\mathcal{L}^N$, and denote this by $\mathcal{R}^N$. For each $\mathcal{R}^N \in \mathcal{L}^N$, the ith coordinate of $\mathcal{R}^N$ is denoted by $\mathcal{R}_i$.

[2]see the heuristic proof in Mertens and Dhillon [12] for the one dimensional case which is equivalent to having only two alternatives but which is not studied in this paper.

The set of strict subsets of N is denoted by $\Im$.

Definition 1: A social welfare function is a map $\varphi : \mathcal{L}^N \to \mathcal{L}$ that associates to any profile $\mathcal{R} \in \mathcal{L}^N$ a social preference $\mathcal{R} \in \mathcal{L}$.

Definition 2: A Group Aggregation Rule for a subgroup G is a map ψ_G : $\mathcal{L}^G \to \mathcal{L}$ where $G \in \Im$.

Definition 3: A Group Aggregation Rule satisfies Individualism iff whenever all individuals in the subgroup are completely indifferent then so is the subgroup.

For all G, we assume ψ_G satisfies Individualism. In addition, we assume:

$$\psi_G = \mathcal{R}_i \text{ whenever } G = \{i\}$$

For any preference relation $\mathcal{R}$, $\mathcal{I}$ stands for the corresponding indifference relation and $\mathcal{P}$ stands for the corresponding strict preference. Society's preference ordering is denoted by $\mathcal{R}$. For any subgroup $G_i \subset N$ the preferences $\psi_{G_i}(\mathcal{R}^{G_i})$ are represented by $\mathcal{R}_{G_i}$. S denotes the space of utility functions on A, and an element of S^N is denoted by $\vec{u}$.

3 The Axioms

Axiom 1: Extended Pareto.
For any profile of preferences $\mathcal{R}^N \in \mathcal{L}^N$ and for any 2 element partition $\{G_1, G_2\}$ of N, $\exists \psi_{G_1}, \psi_{G_2}$ such that: for any pair of lotteries p and q

$$p\mathcal{R}_{G_i}q \qquad i = 1,2$$

$$\Rightarrow p\mathcal{R}q$$

And if further, $p\mathcal{P}_{G_1}q$, then

$$p\mathcal{P}q.$$

Remark.
According to the axiom if there exist functions that aggregate preferences of individuals in (disjoint) subgroups of society (e.g. states in a country of N

individuals) then the Social Welfare Function should satisfy Pareto in terms of the "aggregate preferences" of these subgroups. There are no restrictions on the functional form of these *Group Aggregation Rules* except that they depend only on the preferences of individuals in the subgroups and they satisfy Individualism. In so far as the consequence of using this axiom with the vN-M axioms goes, it is shown that in fact the Group Aggregation Rules also satisfy the Extended Pareto axiom and are of the same functional form as the SWF, hence the axiom seems to be the logical expression of what is meant by aggregating preferences in a "consistent" way. There is an obvious difficulty in checking whether any given SWF satisfies this condition (given that there may be many such Group Aggregation Rules): hence in the specific framework of this paper Theorem 1 gives a characterization of the axiom[3]. Given the assumptions on the Group Aggregation Rules we have as a consequence of the Extended Pareto condition, a "multi-profile" interpretation of the axiom using the equivalence between the Group Aggregation Rule for a subgroup G and the SWF on the profile where $N\backslash G$ is universally indifferent[4]. Then the axiom can also be written as : For any partition of N into two subgroups G_1 and G_2, and for any three profiles: $(\mathcal{R}^{G_1}, \mathcal{I}^{G_2})$, $(\mathcal{I}^{G_1}, \mathcal{R}^{G_2})$, and $\mathcal{R}^N = (\mathcal{R}^{G_1}, \mathcal{R}^{G_2})$: If for any pair of lotteries p and q:

$$p\varphi(\mathcal{R}^{G_1}, \mathcal{I}^{G_2})q,$$

and

$$p\varphi(\mathcal{I}^{G_1}\mathcal{R}^{G_2})q,$$

$$\Rightarrow p\varphi(\mathcal{R}^N)q$$

where

$$\mathcal{R}^N = (\mathcal{R}^{G_1}, \mathcal{R}^{G_2})$$

This reconstruction of the axiom has the following interpretation: consider a partition of the set of citizens of a country into group 1 and group 2. If the social welfare function is such that it would choose lottery p over lottery q whenever group 2 was unanimously indifferent between all alternatives, and group 1 has some preferences given by $\mathcal{R}^{G_1}$, that it would choose p over q

[3]It should be noted here that there may exist SWF's that satisfy the Extended Pareto axiom but not the Continuity axiom used in Mertens and Dhillon (e.g. choose the functions $F_n(u_n)$ in Theorem 1 to be discontinuous in their sense.

[4]Proved in Lemma 1.

when the situation is reversed, i.e. group 1 was unanimously indifferent between p and q, and group 2 has preferences given by $\mathcal{R}^{G_2}$, then it must be true that then society must still prefer p to q, when preferences are given by $(\mathcal{R}^{G_1}, \mathcal{R}^{G_2})$. The restrictions it imposes on the SWF are a kind of separability in group preferences and monotonicity with respect to these preferences. In the framework of interpersonal comparibility with translation invariance (which is *not* a primitive in this paper), utilitarianism is an obvious candidate for a SWF that satisfies Extended Pareto, since it is both separable in terms of the preferences of any subgroup and monotonic with respect to them. However weighted utilitarianism where the weights depend on the whole profile would not satisfy this axiom (example given in the last section of this paper).

In the case of two individuals, the axiom is equivalent to Pareto and to a form of Monotonicity (or Positive Association) (a proof of the equivalence of a form of Monotonicity and Extended Pareto is given in the appendix).

Axiom 2 : Anonymity
Any permutation of the profile of preferences leaves the social preferences unchanged.

This axiom is standard and discussions can be found in the literature (e.g. May [11], also Sen [15]).

Axiom 3: Weak IIA*
Consider any two profiles $\mathcal{R}$ and $\mathcal{R}'$, such that they coincide on lotteries on a subset A' of A, and in addition that every lottery on $A \backslash A'$ is unanimously indifferent to some lottery on A', for each of the two profiles. Then social preferences also coincide on $\Delta A'$.

Axiom 4: Neutrality expresses that the names of the alternatives do not matter. Formally, at least when $\Delta(A)$ consists of all lotteries with finite support, any permutation π of A induces a permutation of the space of preferences: $\mathcal{R} \mapsto R_\pi$ where $p\mathcal{R}_\pi q$ iff $p \circ \pi \mathcal{R} q \circ \pi$. Then

$$\varphi[(\mathcal{R})^\pi] = (\varphi[(\mathcal{R}])_\pi$$

Remark on Weak IIA*:

This axiom is weaker (conceptually[5]) than Arrow's Independence of Irrelevant Alternatives. Formally however it is difficult to compare the two as one would need a version of IIA suitable to the framework at hand i.e. of preferences over lotteries. Since it is impossible to change preferences over a subset of lotteries on A without also changing preferences over all other lotteries when underlying preferences over A have changed, the difficulty of finding an obvious analog to IIA is clear. The axiom is in the spirit of Neutrality, but in addition it implies e.g that the problem where alternative a is unanimously indifferent to b and the one where it is unanimously indifferent to c should not have different solutions, everything else fixed. Together with Pareto Indifference (and vN-M preferences) the axiom implies that one can restrict one's attention to convex sets in utility space, quite similar to the bargaining problem. The difference between the bargaining problem and the social problem lies only in the additional datum of the disagreement point. This is proved in the form of Proposition 2 below.

A note on the dimension condition.
By Pareto, (cf.Proposition 0 Appendix), social preferences are represented by (vN-M) utility functions that satisfy:

$$U = \sum_{n \in N} \lambda_n((\vec{u})_{n \in N})u_n + \beta \tag{1}$$

where λ_n is a strictly positive real number. Let the number of alternatives be m and the number of individuals be k.

Thus if we view the social utility, U, as an $m \times 1$ vector it is equal by equation (1) to the product of a "coefficient" matrix of dimension $m \times k+1$ and the vector λ of dimension $k+1 \times 1$ then the system has a unique solution in λ iff the coefficient matrix has full rank. Thus the rank of the coefficient matrix is the number of linearly independent non-constant utility vectors in the profile. Equivalently, in case A is of infinite dimension, we look at the dimension of the smallest affine subspace containing the convex set

$$\{< u_n, p > | p \in \Delta A\} \subset \mathbb{R}^N$$

This is the dimension $d(\vec{u})$ or sometimes d referred to in the rest of the paper.

[5]Because of the additional requirement on profiles that can be compared using the axiom.

4 The Results

In this section I present the results of the paper. Proofs are presented in the next section. Proposition 0 is basically a multi-profile version of Harsanyi's Aggregation Theorem [9] wherein it was shown that vN-M preferences and Pareto Indifference imply that social utility must be a weighted sum of individual utilities. Proposition 0 simply modifies this result to the case of a SWF, the difference being only that now social utility is a weighted sum of individuals utilities, the weights being functions of the profile, and satisfying (given ordinality of the representations) suitable homogeneity properties and translation invariance. What Extended Pareto accomplishes in addition to Strong Pareto as used in Proposition 0 is to add the restriction that the weight of each person n depends only on $\mathcal{R}_n$ and not the whole profile.

Proposition 0 *(Proposition* 1*, Mertens and Dhillon [12]): The social welfare functions* φ *that satisfy the Pareto axiom are those which can be represented by a map* λ *from* S^N *to* $\mathbb{R}^N$ *such that*

1. $\lambda_n(\vec{u}) > 0, \quad \forall n, \quad \forall(\vec{u}) \in S^N.$

2. *If* $\forall n \in N, \quad u_n$ *is a representation of* $\mathcal{R}_n$*, then* $\sum_{n\in N} \lambda_n(\vec{u}).u_n$ *is a representation of* $\varphi(\mathcal{R}^N)$

3.
   - $\lambda_n(\vec{u})$ *is translation invariant, i.e., if* $v_n = u_n + \alpha_n, \quad \forall n,$ *with* $\alpha_n \in \mathbb{R}$*, then* $\lambda_n(\vec{u}) = \lambda_n(\vec{v})$
   - $\lambda_n(\vec{u})$ *is positively homogeneous of degree zero in* $u_k, \quad \forall k \neq n$ *and if* u_n *is not constant, of degree minus one in* u_n*, i.e., if* $v_n = \beta_n u_n, \quad \forall n,$ *with* $\beta_n > 0$ *then* $\lambda_n(\vec{v}) = \beta_n^{-1}\lambda_n(\vec{u})$

The first result I have is a characterization of the Extended Pareto Axiom in the framework of vN-M preferences. In the theorem below the restriction on the number of alternatives arises because of the dimension condition, the result has been proved only for profiles with dimension greater than two. If a dummy axiom is added, it would be true for all profiles, as it is trivially true if dimension equals one and all individuals have the same preference, while the case where all individuals have one preference or its exact opposite can be proved as well, using the heuristics in Mertens and Dhillon [12]. The only case that is problematic is the dimension two case.

The number of individuals is assumed to be bigger than four because in case of two individuals Extended Pareto does not give any stronger restriction than Pareto, and we need more than three individuals if anonymity is not assumed. The proof is by construction of an appropriate function.

Theorem 1:
(A)*If* $\#A \geq 4$ *and* $\#N \geq 4$ *, a SWF satisfies the Extended Pareto axiom iff it can be represented by :*

$$U = \sum_{n \in N} u_n'(\mathcal{R}_n), \quad \text{whenever } d(\vec{u}) > 2, \tag{2}$$

where U *is a vN-M utility representation of social preferences , and each* u_n' *is a (unique, upto the function* F_n*) representation of individual preferences, such that*

$$u_n'(a) = (h(u_n)(a))/F_n((h(u_n)(\cdot))), \tag{3}$$

where $h(u_n) = u_n - \min_{a \in A} u_n(a)$, *is a utility function in* $\mathbb{R}^A$, *and* $F_n : \mathbb{R}^A \rightarrow \mathbb{R}_+$ *is positively homogeneous of degree* 1 *(if* u_n *is not constant) and translation invariant*[6] *. If* u_n *is constant define* $F_n(u_n) = 1$.

(B)*There exists only one function* $F_n(u_n)$ *from the space of bounded utility functions* S *to* $\mathbb{R}_{++}$ *that yields with equation (3) above, the given SWF for profiles with* $d(\vec{u}) > 2$ *(upto multiplication by a positive constant independent of* u_n *or of the profile).*

Proposition 1 then shows that with Anonymity the functions $\lambda_n(u_n)$ are the same *functions* for all $n \in N$.

Proposition 1:
For a fixed set of alternatives A, *with* $\#A \geq 4$, *and* $\#N \geq 3, d(\vec{u}) > 2$ *the social welfare functions* φ *that satisfy the Extended Pareto axiom and the Anonymity axiom are those that satisfy equation (2) of Theorem 1 and in addition the functions* $F(n,\ u)$ *are independent of individual* n.

The third result is a characterisation of the Weak IIA* axiom with Pareto Indifference in the framework of vN-M utilities.

Proposition 2:
A map φ *satisfies the Pareto Indifference and Weak IIA* axiom iff the maps*

[6] Note that $F_n((h(u_n)(\cdot)) = F_n((u_n)(\cdot))$ by translation invariance.

λ of Proposition 0 *satisfy in addition that $\lambda(\vec{u}) = \lambda(\vec{u'})$, whenever*

$$F = \{([< u_n, p >]_{n \in N} | p \in \Delta(A))\} = \{([< u'_n, p' >]_{n \in N} | p' \in \Delta(A))\}$$

Finally I define Relative Utilitarianism, and state the main theorem which gives necessary and sufficient conditions for Relative Utilitarianism.

Definition 4: Relative Utilitarianism:
Let

$$p(u_n) = \max_{a \in A} u_n(a) - \min_{a \in A} u_n(a)$$

For $u \in S^N$,

$$U = \sum_{n: p(u) > 0} \left(\frac{u_n}{p(u_n)}\right)$$

represents a social preference over lotteries which is independent of the utility representations of individual preferences.
Theorem 2:
For a fixed set of alternatives A such that $\#A \geq 4$ and for all N such that $\#N \geq 3$ and for all profiles such that $d(\vec{u}) > 2$, a SWF $\varphi((\mathcal{R}_n)_{n \in N})$ satisfies Extended Pareto, Anonymity, Weak IIA, if and only if it is "Relative Utilitarianism".*

Remarks

1. The result may be more meaningfully viewed as a representation result than a characterisation of Utilitarianism. As this issue has been adequately addressed in the literature on Harsanyi's Theorems [4] (see, e.g. Weymark [17]), I will not comment on this here, however, it could be observed vis-a-vis Sen's [14] objection that the use of vN-M utilities is arbitrary, that any monotonic transform of individual (vN-M) utilities is compatible with the same social ordering as long as the same transform[7] is used for all individuals. Thus, in this framework, it would seem that utilities have meaning only as measures of preferences.
2. Observe that we begin with no interpersonal comparibility but end up with full comparibility. Which are the axioms therefore that give us this comparibility? The answer to this is not obvious. All the axioms together

[7]transforms different across individuals would violate the vN-M postulates for society.

imply interpersonal comparibility, but if any one of the axioms has to be isolated, it must be Anonymity, since it is this axiom that rules out the use of different scaling for different individuals.
3. Finally as remarked by Sen [14], the lack of full comparibility in the Nash solution is absorbed by the fact that origins get subtracted out while the units simply change the *scale* of the product without changing the ordering, even if the origins and units change differently for different individuals. In our solution this is done in the reverse direction, i.e., changes in origins get subtracted out, while changes in units are absorbed by compensating changes in the weights.

4.1 Proofs

Proofs are presented in this section.
Proof of Proposition 0: See Appendix.
Proof of Theorem 1
Observe that if ψ_G is taken as the restriction of the SWF to the profile on subgroup G (hence having the same representation) any SWF which has the above representation satisfies Extended Pareto. Thus we now prove the converse .

The structure of the proof is as follows: Lemma 1 proves that all ψ_G that are induced by a SWF that satisfies Extended Pareto must themselves satisfy Extended Pareto (appropriately defined for G). Lemma 2 then shows that any such ψ_G, and indeed even φ must be a weighted sum of utilities of elements of the partition of G or N. Lemma 3 provides the characterization result for a subgroup of three individuals in the case of a full dimensional profile. We know such a profile exists because of the conditions imposed. This has two corollaries, (1) the result for any full dimensional profile with $\#G$ individuals and (2) the result for a subgroup of two individuals. Next Lemma 4 proves that if the result holds for subgroups with number of individuals $\#G$ and dimension d, then it holds for subgroups with number of individuals $\#G+2$ and profiles of dimension $d-1$, whenever $\#N \geq 4$ and $d(\vec{u}) > 2$. Lemma 5 proves that if the result holds for subgroups with number of individuals $\#G$ and profiles of dimension d then it holds as well for subgroups of number $\#G-1$ and profiles of dimension $d-1$, whenever $\#G \geq 3$ and $d(\vec{u}) \geq 2$. Lemmas 4 and 5 are used to prove the result for all G using

induction.
Finally the case of an arbitrary $N \geq 4$, and profiles of dimension bigger than two is solved using the solutions for the subgroups and Lemma 4 applied to N.

The lemmas are now presented.

Lemma 1:
If $\exists$ a SWF that satisfies Extended Pareto w.r.t. any functions ψ_G then all such functions for any $G \in \Im$ must satisfy:

$$\psi_G(\mathcal{R}^G) = \varphi(\mathcal{R}^G, \mathcal{I}^{N \setminus G})$$

where $\mathcal{R}^G, \mathcal{I}^{N \setminus G}$ represents the profile $\mathcal{R}^N$ whenever there is total indifference $\forall n \in N \setminus G$.

Proof
Assume there exists a SWF that satisfies Extended Pareto w.r.t some ψ_{G_i}. Consider any partition of N into 2 subgroups $G_1, G_2 \in \Im$ and any such functions $\psi_{G_i}, i = 1, 2$. Consider the profile on N where all individuals in the subgroup G_2 are completely indifferent between all alternatives. Then by Individualism ψ_{G_2} is total indifference. The result follows from Extended Pareto.

♣

Corollary :
If $\exists$ a SWF φ that satisfies Extended Pareto, then,

1. *The functions ψ_G induced by φ satisfy Extended Pareto.*
2. *$\varphi(\mathcal{R}^N)$ satisfies Extended Pareto with respect to any such ψ_G and for any partition of N; in particular φ satisfies Pareto.*

Proof
1. Consider a partition of N into two subgroups: G_1 and G_2. Since φ satisfies Extended Pareto, by Lemma 1 one must have:

$$\psi_{G_i} = \varphi(\mathcal{R}^{G_i}, \mathcal{I}^{N \setminus G_i})$$

Now consider a further partition of G_1 into two subgroups G_{11} and G_{12}. We need to show that ψ_{G_1} satisfies Extended pareto w.r.t these two subgroups. We are given: for any $p,\ q \in \Delta A$,

$$p \quad \psi_{G_{1i}}(\mathcal{R}^{G_{1i}})q, \quad i = 1, 2$$

but we can rewrite $\psi_{G_{11}}$ (by Lemma 1) as:

$$\varphi(\mathcal{R}^{G_{11}}, \mathcal{I}^{N \setminus G_{11}})$$

and $\psi_{G_{12}}$ as:

$$\varphi(\mathcal{R}^{G_{12}}, \mathcal{I}^{N \setminus G_{12}})$$

Noting then that

$$\varphi(\mathcal{R}^{G_{12}}, \mathcal{I}^{N \setminus G_{12}}) = \varphi(\mathcal{R}^{G_{12} \cup G_2}, \mathcal{I}^{G_{11}})$$

we can use Extended Pareto for the partition $\{G_{11},\ N \setminus G_{11}\}$ to conclude that:

$$p\varphi(\mathcal{R}^{G_{11}}, \mathcal{R}^{G_{12} \cup G_2})q$$

which in turn is equivalent to:

$$p\varphi(\mathcal{R}^{G_1}, \mathcal{I}^{G_2})q$$

where $\varphi(\mathcal{R}^{G_1}, \mathcal{I}^{G_2})$ is nothing but ψ_{G_1}. One can show this for any further finite partitions of G_{11}.

2. This follows from 1.

♣

Note. Henceforth the Group Aggregation Rules referred to in the rest of the proof are the ones "induced" by a SWF satisfying Extended Pareto as shown above.

Lemma 2:
A SWF (respectively Group Aggregation Rule) satisfies Extended Pareto for any partition of N *(respectively* G*):* π *iff it can be represented as:*

$$U = \sum_{i=1,2,3\ldots} \beta_{G_i,\pi}(\vec{U}) U_{G_i} \tag{4}$$

respectively

$$U_G = \sum_{i=1,2,3\ldots} \beta_{G_i,G,\pi}(\vec{U}) U_{G_i} \tag{5}$$

where U *represents the SWF (unique upto positive monotonic transformations),* U_G *represents the preferences of the subgroup* G*,* U_{G_i} *represents the*

preferences of the subgroups $G_i, i = 1, 2, 3, 4....$
$\beta_{\pi,G_i} \in I\!R_{++}$ and $\vec{U}$ represents the "profile" of utility functions (which can be for any partitions of N).

(To simplify notation here we drop the argument π whenever the partition is the trivial one. We also drop the argument G, as in equation (4) above, whenever U is a representation of a SWF.)

<u>Proof</u>
First it is obvious that if a SWF (respectively group Aggregation Rule) can be represented by the above, then it must satisfy Extended Pareto. We now prove the converse.
Observe that by part (2) of the Corollary to Lemma 1, Extended Pareto implies Pareto. We also assume that individual and social preferences (and hence by part (1) of the corollary to Lemma 1 also group preferences) satisfy the vN-M axioms. Hence, Proposition 0 applies to both the SWF and to group preferences and both can be represented by a weighted sum of utilities of individuals in the society/group. It remains to prove that social/group preferences can be represented as in equation 4 and 5 respectively, in the specific cases where subgroups are not individuals. The proof of Proposition 0 goes through just replacing individual vN-M utilities by group vN-M utilities, and profiles of individual preferences by profiles of group preferences.
♣

Lemma 3 now proves the result for a subgroup of three individuals and a profile of preferences with dimension three.
Lemma 3:
Let $\#N \geq 4, \#A \geq 4$. For all $G \in \Im$ s.t. $\#G = 3$ and for all profiles on N such that $u_i = 0$(or constant), $\forall i \notin G$; ψ_G satisfies Extended Pareto iff $\exists F_{n,G}(u_n) \in I\!R_{++}$ such that ψ_G can be represented by $U_G : A \rightarrow I\!R$ such that:
(1)

$$U_G = \sum_{n \in G} \frac{1}{F_{n,G}(u_n)} . (u_n - \min_{a \in A} u_n(a)) \tag{6}$$

whenever $d(\vec{u}) = 3$.
(2)

$$F_{n,G}(\cdot) = F_{n,\tilde{G}}(\cdot), \forall G, \tilde{G} \in \Im,$$

s.t.

$$n \in G, n \in \tilde{G}.$$

where $u_n, U_G \in I\!R^A$, *and* $\#G = 3$.

Proof

Since it is clear that if each U_G is represented as in equation(6), it satisfies Extended Pareto in terms of any subgroups, we now prove the converse.

(1) Let the 3 individuals be $\{i,\ j,\ k\}$. Let $\vec{u}$ represent the profile of utilities of the 3 individuals, and $u,\ v,\ w$ represent utility functions for the 3 individuals $i,\ j,\ k$ respectively. $g_{ij}(u,\ v)$ denotes a function which depends on the individuals $i,\ j$ and the utility functions u and v. Note that the order of the profile is maintained in the proof, although the particular notation used is to show clearly that the first coordinate of the function refers to the utility function of the individual i who'is first in the order g_{ij}.

Claim 1.

There exists a function g_{ij} defined for all $(u,\ v) \in S^2$ which satisfy $d(u,\ v) = 2$, and for every ordered pair {i, j} of 2 different individuals such that

(a) $g_{ij}(u,\ v) = \frac{\lambda_i(\vec{u})}{\lambda_j(\vec{u})}$, $\forall\ (\vec{u}) \in S^3$ such that $\vec{u}_i = u_i, \vec{u}_j = u_j$ with 3 individuals in G s.t. i and j belong to G and $d(\vec{u}) = 3$.

(b1) $g_{ij}(u,\ v).g_{ji}(v,\ u) = 1$, whenever $d(u,\ v) = 2$.

(b2) $g_{ij}(u,\ v).g_{jk}(v,\ w).g_{ki}(w,\ u) = 1$ whenever the functions g are well-defined.

Proof:

Since $\#N \geq 3$, and $\#A \geq 4, \exists\ G' \subset G, s.t.\ G' = \{i,\ j\}$ and a profile where $N-2$ individuals are completely indifferent and $d(u,\ v) = 2$. By Lemma 1, $\psi_{G'}$ satisfies Pareto, and by Lemma 2, it is represented by :

$$U_{G'} = \sum_{n \in G'} \lambda_{n,G'}(\vec{u}) u_n. \qquad (7)$$

where $\lambda_{n,G'}$ satisfies the properties of Proposition 0. Define the function $g_{ij} = \frac{\lambda_{i,\ G'}(u,\ v)}{\lambda_{j,\ G'}(v,\ u)}$. By the above, this function is well-defined whenever $i \neq j$ and $d(u,\ v) = 2$. Now we can prove (a):

By Lemma 1 ψ_G satisfies Extended Pareto. Therefore by Lemma 2 we have, for $G_1 = \{i, j\}$, $G_2 = \{k\}$, for any ordered set $G = \{i,\ j,\ k\}$, and for the partition $\pi_1 = \{G_1, G_2\}$,

$$U_G = \alpha_{G_1,G,\pi_1}(u,\ v,\ w)(\lambda_{i,G_1}(u,v)u + \lambda_{j,G_1}(v,\ u)v) + \alpha_{k,G,\pi_1}(w,\ u,\ v)w.$$

and for $G_1 = \{i\}, G_2 = \{j\}, G_3 = \{k\}$:

$$U_G = \lambda_{i,G}(u,\ v,\ w)u + \lambda_{j,G}(v,\ u,\ w)v + \lambda_{k,G}(w,\ v,\ u)w.$$

where the coefficient vectors α, and λ are strictly positive. By the uniqueness of the coefficients in this full-dimensional case we have:

$$\frac{\lambda_{i,G'}(u,\ v)}{\lambda_{j,G'}(v,\ u)} = \frac{\lambda_{i,G}(u,\ v,\ w)}{\lambda_{j,G}(v,\ u,\ w)}. \tag{8}$$

(note the π argument is dropped as by the above argument we also show the independence of g_{ij} from the partition. The argument for the subgroup G' is also dropped from now on since g_{ij} depends on this (fixed) subgroup and on a fixed partition.)

(b1) is obvious by using the definition of the function g_{ij}.

(b2) If $d(\vec{u}) = 3$ then by part(a) the result follows.
Otherwise $\exists\ \hat{w},\ \hat{u},\ \hat{v}$ such that $d(u,\ v,\ \hat{w}) = d(v,\ w,\ \hat{u}) = d(u,\ w,\ \hat{v}) = d(\hat{u},\ \hat{v},\ \hat{w}) = d(u,\ \hat{v},\ \hat{w}) = d(\hat{u},\ v,\ \hat{w}) = d(\hat{u},\ \hat{v},\ w) = 3$ (since $\#A \geq 4$, by assumption) , such that, using Claim 1(a), equation (8):

$$g_{ij}(u,\ v) = g_{ik}(u,\ \hat{w})g_{kj}(\hat{w},\ v)$$

Similarly:

$$g_{jk}(v,\ w) = g_{ji}(v,\ \hat{u})g_{ik}(\hat{u},\ w)$$

And:

$$g_{ki}(w,\ u) = g_{kj}(w,\ \hat{v})g_{ji}(\hat{v},\ u)$$

Substituting for the functions g in equation (b2) of Lemma 3, and using successively the equivalence proved in Claim 1(a) we get:

$$g_{ij}(u,\ v)g_{jk}(v,\ w)g_{ki}(w,\ u) = g_{ik}(u,\ \hat{w})g_{kj}(\hat{w},\ \hat{v})g_{ji}(\hat{v},\ u)$$

Which is equivalent by Claim 1(a) to:

$$g_{ij}(u,\ \hat{v})g_{ji}(\hat{v},\ u)$$

And using (b1) the above expression equals 1.
<u>Claim 2.</u>

There exists a function $G_{ij}(u,\ v) : S^2 \to I\!R_+$ defined for any non- constant u, v and any 2 individuals $i,\ j$ such that
(a)

$$G_{ij}(u,\ v) = g_{ij}(u,\ v) \tag{9}$$

whenever $d(u,\ v) = 2$ and $i \neq j$
(b)

$$G_{ij}(u,\ v).G_{jk}(v,\ w).G_{ki}(w,\ u) = 1. \tag{10}$$

always, for any $i,\ j,\ k \in G$.
Proof.

First we construct the function G_{ij}.
Define $G_{ij}(u,\ v) = g_{ik}(u,\ w)g_{kj}(w,\ v)$. Observe that this function is well-defined because given any pair of individuals i and j and any pair of non-constant utility functions u, of i, and v of j we can find an individual k, and a utility function w for this individual such that $d(u,\ w) = d(v,\ w) = 2$ and hence the functions $g_{ik}(u,\ w)$, and $g_{kj}(w,\ v)$, are well-defined. It remains to show that the function $G_{ij}(u,\ v)$ is independent of the utility function w and the individual k.
Note that $\#A \geq 4$, and $\#G \geq 3$ thus for any pair $u,\ v$ of non-constant utility functions we can find $w \in I\!R^A$, $k \notin \{i,\ j\} \subset G$ such that $d(u,\ w) = d(v,\ w) = 2$.
If $i \neq j$ and if $d(u,\ v) = 2$:
then it is possible to choose w such that $d(u,\ v,\ w) = 3$, and therefore we can prove the independence using the equivalence in Claim 1(a).

If $d(u,\ v) \neq 2$, and $i \neq j$:
we only have to prove that :

$$g_{ik}(u,\ w).g_{kj}(w,\ v) = g_{ik}(u,\ \hat{w}).g_{kj}(\hat{w},\ v). \tag{11}$$

whenever $d(u,\ w) = d(w,\ v) = d(u,\ \hat{w}) = d(v,\ \hat{w}) = 2$ and whenever $k \neq i,\ k \neq j$.
This can be done by proving the equivalence of each side of equation (11) to

$$g_{ik}(u,\ \tilde{w})g_{kj}(\tilde{w},\ v)$$

It is sufficient to prove this for one side of equation (11):
Thus choose $\tilde{w}$ such that:

$$d(u,\ \tilde{w}) = d(\tilde{w},\ v) = 2$$

This is possible given domain conditions.
Choose $u_i,\ u_j,\ u_k$ to satisfy:

$$d(u,\ u_j,\ w) = d(w,\ u_i,\ v) = d(u,\ u_j,\ \tilde{w}) = d(u_i,\ \tilde{w},\ v) = 3$$

(this is possible by the domain assumption.) Thus we have:

$$g_{ik}(u,\ w)g_{kj}(w,\ v) = g_{ij}(u,\ u_j)g_{jk}(u_j,\ w)g_{ki}(w,\ u_i)g_{ij}(u_i,\ v)$$

which is equivalent by application of Claim 1 (a) and Claim 1 (b2) to:

$$g_{ik}(u,\ u_k)g_{kj}(u_k,\ v)$$

Similarly for the LHS of equation (11) and choosing $\tilde{w} = u_k$, we have proved (11).

<u>If $i = j$ and $d(u,\ v) \neq 2$:</u> We need to prove:

$$g_{ik}(u,\ w)g_{ki}(w,\ v) = g_{ij}(u,\ \hat{w})g_{ji}(\hat{w},\ v),$$

whenever the functions $g(\cdot)$ are well defined.
This is equivalent by Claim 1 (b1) to proving:

$$g_{ji}(\hat{w},\ u)g_{ik}(u,\ w) = g_{ji}(\hat{w},\ v)g_{ik}(v,\ w)$$

But this is proved already in the case $i \neq j$ with the names of the individuals permuted.
Proof of (a):
If $d(u,\ v) = 2,\ \ i \neq j$ then by definition, $G_{ij}(u,\ v) = g_{ij}(u,\ v)$.
Proof of (b)
First the equivalence is proved for $i \neq j \neq k$:
By definition:

$$G_{ij}(u,\ v) = g_{ik}(u,\ w)g_{kj}(w,\ v)$$

Hence choosing the 3 utility functions $\hat{u},\ \hat{v},\ \hat{w}$ such that:

$$d(u,\ \hat{w}) = d(\hat{w},\ v) = d(v,\ \hat{u}) = d(\hat{u},\ w) = d(w,\ \hat{v}) = d(\hat{v},\ u) = 2$$

proving equation (10) is equivalent to proving:

$$g_{ik}(u,\ \hat{w})g_{kj}(\hat{w},\ v)g_{ji}(v,\ \hat{u})g_{ik}(\hat{u},\ w)g_{kj}(w,\ \hat{v})g_{ji}(\hat{v},\ u) = 1, \qquad (12)$$

By the domain assumption ($\#A \geq 4$)we can choose $\hat{u},\ \hat{v},\ \hat{w}$ to satisfy as well:

$$d(\hat{u},\ v,\ \hat{w}) = d(\hat{u},\ \hat{v},\ w) = d(\hat{u},\ \hat{v},\ \hat{w}) = 3$$

By successive applications of Claim 1(a), (11) is equivalent to proving:

$$g_{ik}(u,\ \hat{w})g_{kj}(\hat{w},\ \hat{v})g_{ji}(\hat{v},\ u) = 1$$

Which is true by Claim 1(b2).

<u>If $i = j$:</u>

Equation (12) above becomes:

$$g_{ik}(u,\ \hat{w})g_{ki}(\hat{w},\ v)g_{ij}(v,\ \hat{v})g_{jk}(\hat{v},\ w)g_{kj}(w,\ \hat{v})g_{ji}(\hat{v},\ u) = 1, \qquad (13)$$

This is equivalent using Claim 1(a) to:

$$g_{ik}(u,\ \hat{w})g_{kj}(\hat{w},\ \hat{v})g_{ji}(\hat{v},\ u) = 1$$

Which is true by Claim 1(b2).

<u>If $i = j = k$</u>

Equation (12) becomes:

$$g_{ik}(u,\ \hat{w})g_{ki}(\hat{w},\ v)g_{ij}(v,\ \hat{v})g_{ji}(\hat{v},\ w)g_{ik}(w,\ \hat{v})g_{ki}(\hat{v},\ u) = 1, \qquad (14)$$

This is equivalent (using Claim 1(a))to:

$$g_{ik}(u,\ \hat{w})g_{kj}(\hat{w},\ \hat{v})g_{ji}(\hat{v},\ u) = 1$$

Which is true by Claim 1(b1).

<u>Claim 3</u>

There exist functions $F_G(n,\ u)$, defined on $N \times S^*$, where S^* is the set of non-constant utility functions, such that

$$G_{ij}(u,\ v) = \frac{F_G(j,\ v)}{F_G(i,\ u)}.$$

For all $n \in G$, the function $F_G(n,\ u)$ is translation invariant, and positively homogeneous of degree 1 in $u \in I\!R^A$.

<u>Proof:</u>

Observe that Claim 2(b) yields :
-first : $G_{ii}(u, u) = 1$ (the case $i = j = k$ and $u = v = w$.)
-next $G_{ij}(u, v)G_{ji}(v, u) = 1$ (the case $i = k$ and $u = w$.)
-Finally: $\forall u, v \in S$

$$G_{ij}(u, v) = G_{ik}(u, w)G_{kj}(w, v)$$

(Since $G_{ij} = \frac{1}{G_{ji}}$, using the definition of the function G_{ij} and Claim 1(b1).)
Now fix some non-indifferent individual k_0 and some $u_{k_0} \in S^*$ Then

$$G_{ij}(u, v) = \frac{F_G(j, v)}{F_G(i, u)} \tag{15}$$

where we define:

$$F_G(n, \hat{u}) = G_{k_0 n}(u_0^k, \hat{u}),$$

Translation invariance of $F_G(n, u)$ follows from the translation invariance of λ_n(Claim 1) and from Claim (2) and Proposition 0; so do the homogeneity properties.
(**Note** In the above G refers to the subgroup whenever it is a subscript and otherwise it refers to the function $G_{ij}(u, v)$.)

Claim 4. End of the proof of Lemma 3.
Note that, by Claim 2(a) we have

$$G_{ij}(u_i, u_j) = g_{ij}(u_i, u_j)$$

whenever $d(u_i, u_j) = 2$ and $i \neq j$. And by Claim 1(a):

$$g_{ij}(u_i, u_j) = \frac{\lambda_i(u_i, u_j, u_k)}{\lambda_j(u_i, u_j, u_k)}$$

whenever

$$d(u_i, u_j, u_k) = 3.$$

Thus in the full-dimensional case we must have:

$$F_G(j, u_j).\lambda_j(u_i, u_j, u_k) = F_G(i, u_i).\lambda_i(u_i, u_j, u_k)$$

-using here Proposition 0 and the definitions of g_{ij} and G_{ij} from Claims 1 and 2, — i. e. the product is independent of the individual and is only a function of the utility profile, say $\Phi(u_i, u_j, u_k)$. We can then normalise to $\Phi(u_i, u_j, u_k) = 1$ without changing social preferences(dividing the vector λ - and hence U - by Φ) . Substituting for $\lambda_n((u_n)_{n\in N})$ in equation (5) of Lemma 2,

$$U = u_i.\frac{1}{F_i(u_i)} + u_j.\frac{1}{F_j(u_j)} + u_k.\frac{1}{F_k(u_k)}.$$

Subtracting from each u_n the value $\min_{a\in A} u_n(a)$ leaves social preferences unchanged. This gives for each non-indifferent individual $n \in N$, a uniquely (upto the function F) defined map $u'_n(\mathcal{R}_n)$ from his set of possible preferences to utility representations of those, (subtracting from an arbitrary representation u_n the value $\min_{a\in A}(u_n(a))$, and dividing by $F(n, u)$ –if not 0)– such that for any {i, j, k}, and whenever $d(\vec{u}) = 3$, subgroup preferences are represented by

$$U = u'_i(\mathcal{R}_i) + u'_j(\mathcal{R}_j) + u'_k(\mathcal{R}_k)$$

as desired.

♣

The next part of the proof extends the result of Lemma 3 to all sets G.

Corollary 1 to Lemma 3:

Lemma 3 holds for all subgroups $G \in \Im$ such that $\#G \geq 3$ and $d(\vec{u}) = \#G$ on all profiles where $N - G$ individuals are completely indifferent.

Proof

Since the same proof goes through for any G as long as there are at least 3 individuals in the subgroup, it suffices to show that the function $F_G(n,\ u)$ is independent of the subgroup G.

Thus we need to prove:

$$F_G(n,\ u) = F_{G'}(n,\ u),$$

whenever $n\ \in G$ and $n\ \in\ G'$. By definition of the function $F_G(n,\ u)$ it is sufficient to prove that $G_{k_0 n}(u_{k_0},\ u)$ is independent of any $i \notin \{k_0,\ n\}$.

This is equivalent to proving:

$$g_{ik}(u,\ w)g_{kj}(w,\ v) = g_{il}(u,\ \hat{w})g_{lj}(\hat{w},\ v), \tag{16}$$

whenever the functions $g_{ij}(\cdot)$ are well defined and whenever there exists an individual $l \in G$ such that $l \notin \{i,\ j,\ k\}$. But this is equivalent to showing:

$$g_{li}(\hat{w},\ u)g_{ik}(u,\ w) = g_{lj}(\hat{w},\ v)g_{jk}(v,\ w) \tag{17}$$

choosing $w, \hat{w}$ such that :

$$d(u,\ w,\ \hat{w}) = d(v,\ w,\ \hat{w}) = 3$$

Then each side of equation (17) equals $g_{lk}(\hat{w},\ w)$, using Claim 1.

♣

Corollary 2 to Lemma 3:
Lemma 3 holds for any subgroup G with $\#G = 2,\ d(\vec{u}) \leq 2$, whenever $\#N \geq 4,\quad \#A \geq 4$.
Proof
Claim 1
Lemma 3 holds for any subgroup G with $\#G = 2$ whenever $\#N \geq 4,\quad \#A \geq 4$.
Proof: Let the subgroup $G = \{i,\ j\}$. Since $\#N \geq 4$ and $\#A \geq 4$, there exists a profile where $N-2$ individuals are totally indifferent and $d(\vec{u}) = 2$. Similarly there exists a profile where $N-3$ individuals are totally indifferent and $d(\vec{u}) = 3$. By Lemma 3 the case $G = 3,\ d(\vec{u}) = 3$ is already solved. Hence take a partition of the 3 individuals into G_1 and G_2 such that $G_1 = \{i,\ j\}$. By Lemma 2 we have, for $s = \{G_1, G_2\}$:

$$U_G = \alpha_{G_1,s}(\cdot)U_{G_1} + \beta_{G_2,s}(\cdot)U_{G_2}$$

and by Lemma 3 we have:

$$U_G = \sum_{n \in G_1 \cup G_2} u'_n(\mathcal{R}_n)$$

By the uniqueness of the coefficients for $i,\ j$ we get:

$$U_{G_1} = \alpha_{G_1,s} \sum_{n \in G_1} u'_n$$

which can be normalised to:

$$U_{G_1} = \sum_{n \in G_1} u'_n$$

as desired.

Claim 2

Lemma 3 holds for all subgroups G with $\#G = 2$ and $d(\vec{u}) \leq 1$, whenever $\#N \geq 4, \quad \#A \geq 4$.

Proof.

Observe that, by Pareto, the statement is trivially true for any such G whenever $d(\vec{u}) = 0$ or the 2 individuals in G do not have opposite preferences. Hence it is sufficient to prove the Claim for 2 individuals with opposite preferences. Let the 2 individuals be i, j. Let u be some fixed representation of $\mathcal{R}_i$. Then, by Lemma 2, social preferences are given by Mu, where $M \in I\!R$. Let $w, \tilde{w}$ represent the utility functions of 2 individuals $k, l, k \neq l$ such that $d(w, \tilde{w}) = 2$ and $d(u, -u, w, \tilde{w}) = 3$ (noting that this profile exists and two such individuals exist such that $k, l \notin \{i, j\}$).
The preferences of the group $G = \{k, l\}$ are given (Claim 1) by

$$U_G = w'(\mathcal{R}_k) + \tilde{w}'(\mathcal{R}_l).$$

Now if we partition the set N into $\{G\}$ and $\{N \backslash G\}$, we get the following:

$$\begin{aligned} U &= \alpha_s(w'(\mathcal{R}_k) + \tilde{w}'(\mathcal{R}_l)) + \beta_s(Mu) \\ &= \alpha_r(w'(\mathcal{R}_k) + u'(\mathcal{R}_i)) + \beta_r(\tilde{w}'(\mathcal{R}_l) + (-u)'(\mathcal{R}_i)) \end{aligned} \tag{18}$$

since the case $G = 2, d(\vec{u}) = 2$ has been solved already. This implies by the condition $d(u, w, w') = 3$, that

$$\alpha_r = \beta_r = \alpha_s$$

We obtain then:

$$\beta_s(Mu) = \alpha_s[u'(\mathcal{R}_i) + (-u)'\mathcal{R}_i] = \alpha_s \sum_{n \in G} u'_n(\mathcal{R}_n)$$

and normalising to $\alpha_s = 1$ gives the same social preferences as before.
♣

Lemma 4: Let the dimension of a profile be denoted by d, and $[g, d]$ represent profiles of g individuals and d dimension where $d \leq g$. Whenever $\#N \geq 4$

and $\#A \geq 4$ Theorem 1[8] *holds for any $[g, d]$, $d > 2$ whenever it holds for any profiles $[g-2, d-1]$.*

Proof

For $\#G \geq 3$, we can partition G into 2 groups: $s = \{m, l\} \subset G$ and $G' = \{G \backslash s\}$ with $m \neq l$. We have by Lemma 2 (since the case $G-2$ is fully solved by assumption and the case $G = 2$ is solved by the Corollary to Lemma 3:

$$U = \alpha_s(u'_m(\mathcal{R}_m) + u'_l(\mathcal{R}_l)) + \beta_s(\sum_{n \in G} u'_n(\mathcal{R}_n) - u'_m(\mathcal{R}_m) - u'_l(\mathcal{R}_l))$$

with both α_s and β_s strictly positive.

—*I.e.*

$$U = (\alpha_s - \beta_s)(u'_m(\mathcal{R}_m) + u'_l(\mathcal{R}_l)) + \beta_s(\sum_{n \in G} u'_n(\mathcal{R}_n))$$

We need to show that for some s,

$$\alpha_s - \beta_s = 0$$

Suppose $\forall$ s, $(\alpha_s - \beta_s) \neq 0$ Then, we have $\forall m \neq l$

$$u'_m(\mathcal{R}_m) + u'_l(\mathcal{R}_l) = \frac{U}{\alpha_s - \beta_s} + \sum_{n \in N} u'_n(\mathcal{R}_n) . \frac{\beta_s}{\alpha_s - \beta_s}$$

Adding the above equations for $\{m,\ l\} = \{i,\ j\}$ and subtracting for $\{m, l\} = \{j, k\}$ and for $\{m,\ l\} = \{i,\ k\}$ we get:

$$u'_j(\mathcal{R}_j) = \delta U + \rho \sum_{n \in G} (u'_n(\mathcal{R}_n))$$

—*i.e.* each $u'_n(\mathcal{R}_n)$ is a linear combination of ≤ 2 linearly independent vectors contradicting $d(\vec{u}) > 2$.

Thus we have , choosing s such that $\alpha_s = \beta_s$,

$$U = \beta_s \sum_{n \in G} u'_n(\mathcal{R}_n)$$

for some $\beta_s > 0$, as desired.

[8] Appropriately interpreted for subgroups.

♣

Lemma 5 Let the dimension of a profile be represented by d. Then for all profiles with $\#G \geq 3, d(\vec{u}) \geq 2$, *Theorem 1 holds for any subgroup* $[g, d]$ *if it holds for the case* $[g+1,\ d+1]$.

Proof Let the subgroup be G and the dimension of the profile be d. By hypothesis the case $G+1,\ d(\vec{u}) = d+1$ is already solved. Hence add a non-indifferent individual i to G. Consider now the following partitions of the $G+1$ individuals. If $G_1 = G$ and $G_2 = \{i\}$ denoting $G_1 \cup G_2$ by G_{12} : By Lemma 2 we have:

$$U_{G_{12}} = \alpha_s U_{G_1} + \beta_s U_{G_2}$$

and by hypothesis (Theorem 1) we have:

$$U_{G_{12}} = \sum_{n \in G_{12}} u'_n(\mathcal{R}_n)$$

By the uniqueness of the coefficient for the (non-indifferent) individual in G_2 we get:

$$U_G = \alpha_s \sum_{n \in G} u'_n$$

where $\alpha_s > 0$, as desired.

♣

Claim 5 proves the result for all subgroups G, whenever the conditions of the theorem are satisfied.

Lemma 6: For all profiles with $\#N \geq 4$ *and* $\#A \geq 4$, *the theorem holds for any* $G \subset N$, *and all profiles with* $d \geq 2$ *except the profile [N-1, 2].*

Proof Observe that the case $\#G = 3,\ d(\vec{u}) = 3$ and $\#G = 2$ are completely solved and that the full-dimensional case for all G is solved by the Corollary 1 to Lemma 3. It remains to prove therefore the cases $\#G \geq 3,\ 2 \leq d < \#G$, where d denotes the dimension of the preference profile.

We do this using the following induction steps:

Let the number of individuals in the subgroup be g and the dimension be d. By the Corollary to Lemma 3, the case $[g,\ g]$ is solved. (The second coordinate denotes the dimension). It remains to prove $[g, g-x]$, where $g - x \geq 2$.

Observe that by Lemma 4, the cases $[g,\ g]$ imply the result for all $[g+2,\ g+1]$, with starting point of $[2,\ 2]$. Hence all cases $[g,\ g-1]$, will be solved by this for all $g \geq 4$. It also implies that all cases $[g,\ g-2]$ will be solved for all G

such that $\#G \geq 6$ and that all cases $[g,\ g-3]$ will be solved for G such that $\#G \geq 8$ and so on.
In the next step observe that we can use Lemma 5 to solve the case $[3,2]$ (since we cannot use Lemma 4 to solve cases of dimension 2). Similarly we can solve $[5,3]$ using Lemma 5 on $[6,4]$ and hence $[4,2]$. One can then use Lemma 4 again to solve all cases $[G,\ G-3]$ for all G such that $\#G \geq 6$. This implies by Lemma 5 the case $[5,2]$. Again, use Lemma 4 to solve all cases $[g,\ g-4]$ for all G such that $\#G \geq 7$ and so on.
The idea is to use Lemma 4 and Lemma 5 successively, Lemma 4 can be used for solving cases of dimension greater than 2 and Lemma 5 for the cases of dimension 2.

♣

Now we can prove Theorem 1.

Proof of Theorem 1 Part (A):

Step 1

The statement is true for all profiles such that $\#N \geq 4$ and $d(\vec{u}) = \#N$.

Proof

Observe that the domain conditions of Lemma 3 are true in this case. Therefore we can use the method of Lemma 3 to construct functions $F(n,\ u)$ for any $n \in N$. For any n choose any set of 3 individuals containing individual n, and any profile $\vec{u}$ which fulfills the dimension conditions.

Claim 1 of Lemma 3 can now be read :

$$g_{ij}(u,\ v) = \frac{\lambda_i(\vec{u})}{\lambda_j(\vec{u})}$$

whenever the profile $\vec{u}$ has full dimension (i.e. $d(\vec{u}) = N$). (By the same proof).

♣

Step 2

The statement is true for any $N \geq 4$, for all profiles such that $d(\vec{u}) > 2$

Proof

It is sufficient by Step 1 to prove the result for the less than full dimensional case. For this we need Lemma 6.

For an arbitrary $N \geq 4$, $d(\vec{u}) \geq 3$, observe that the theorem holds for all G, with $d \geq 2$ except the case [N-1, 2] by Lemma 6. Use therefore Lemma

4 and cases of $\#N - 2$ individuals to solve all profiles $[\#N, \#N - x]$ where $\#N - x > 2$.

Proof of Part B. Fix a SWF that satisfies Extended Pareto. Then by Part (A) of the proof, we know that it can be represented as:

$$U = \sum_{n \in N} u'_n(\mathcal{R}_n) \tag{19}$$

for some functions $F(n,\ u)$ (see equation (2)).
It is sufficient to prove this for the profile where $G = 3$ and $d(\vec{u})$ is full, since this was the starting point of the proof of Part A (Lemma 3).
Suppose that there exists a ψ_G such that there are 2 functions $F(n, u)$ and $F'(n, u)$ such that equation (19) holds for both. Since by hypothesis the SWF, hence ψ_G, is fixed, the representation U' with the functions $F'(n, u)$ must be such that:

$$U'_G = \beta U_G + \gamma$$

with $\beta > 0$. Since we are in the full dimensional case we have by the uniqueness of the coefficients that:

$$\beta \frac{1}{F'(n,u)} = \frac{1}{F(n,u)},$$

as desired.

♣

Proof of Proposition 1:
It is clear that whatever be the map F, the above SWF satisfies our axioms. Now we prove the converse.
It is sufficient to prove that the functions $F(n, u)$ of Theorem 1 are such that $F(n, u) = F(u)$, the rest follows from the proof of Part 1.
Fix a representation of individual and social preferences. Since the SWF satisfies Extended Pareto, by Part (1) Theorem 1, the representation of the SWF is as given by equation (2). Thus for any full-dimensional case we have that whenever the preferences (utility functions) of any 2 individuals are permuted then by Anonymity the social preferences (and hence the utility function up to a positive affine transformation) must remain the same. This implies by the full-dimensionality that $F(n, u) = F(u)$.

♣

Proof of Proposition 2 is in the Appendix.

Proof of Theorem 2:

It is obvious that "Relative Utilitarianism" satisfies the axioms above. Thus we now prove the converse.

Fix an SWF satisfying Extended Pareto, Weak IIA*, Anonymity. We need to show that then it can be represented by:

$$U = \sum_{n \in N} \hat{u}(\mathcal{R}_n)$$

where $\hat{u} = \frac{u_n(a) - \min_{a \in A} u_n(a)}{p(u_n)}$

It is sufficient to show that if the SWF satisfies Weak IIA* in addition to the other axioms then $F(u) = p(u)$, where $p(u) = \max_{a \in A} u(a) - \min_{a \in A} u(a)$, for all subgroups of 2 individuals with full dimension, since this implies the result for all other profiles, if we show that this implies that the case $G = 3$ is solved[9]. This is easy to see using Extended Pareto, and different combinations of the three individuals in subgroups.

Claim 1 Let u be a utility function on A', and P a set of lotteries on A'. Then $u^P \in S$ is defined as follows: $u^P(a) = u(a)$ for all $a \in A'$, and $\forall a_0 \in A \backslash A'$, $u^P(a_0) = \langle p_{a_0},\ u \rangle$, for some $p_{a_0} \in \Delta A'$. Let $\lambda(u^P) = \frac{1}{F(u^P)}$. Then for every pair of lottery sets P and Q on A', and for every non-constant u,

$$\lambda(u^P) = \lambda(u^Q).$$

Proof

Observe that $N \geq 3$, and $\#A \geq 4$. Take any subgroup of 2 individuals, such that if u and v represent their utility functions on A', $d(u,\ v) = 2$ (this is possible since $\#A \geq 4$). For any set of lotteries P on A', let u^P and v^P represent the corresponding utilities on A. Let $t(P, u) = \lambda(u^P)$. By Weak IIA*, for every 2 sets P and Q:

$$\exists \beta > 0, \quad \gamma,$$

such that:

$\forall a \in A'$,

$$t(P,u)u(a) + t(P,v)v(a) = \beta(t(Q,u)u(a) + t(Q,v)v(a)) + \gamma$$

[9]Formally: $G = 2, d(\vec{u}) = 2 \Longrightarrow G = 3, d(\vec{u}) = 3$

By the linear independence of u and v, we get:

$$\frac{t(P,u)}{t(Q,u)} = \frac{t(P,v)}{t(Q,v)},\ \forall P,Q \tag{20}$$

whenever u and v are linearly independent. This is true as well whenever u, v are non-constant, since there exists some $w \in S$, which is linearly independent of both u and v and it is easily shown that equation (20) holds for both u and v with w.
Thus we can fix v non-constant, at $\bar{v}$ and we define a function $H(P) = t(P,\bar{v}), \quad \forall P$. Hence we have:

$$\frac{t(P,u)}{H(P)} = \frac{t(Q,u)}{H(Q)}\ \ \forall u \in S^* \text{ and } \forall P,\ Q \text{ on } A' \tag{21}$$

This ratio is therefore independent of P and we can define $G(u) = \frac{t(\bar{Q},u)}{H(\bar{Q})}$, for any fixed set $\bar{Q}$ on A'. Hence,

$$\lambda(u^P) = G(u)H(P) \quad \forall\ P\ \in\ \Delta A' \text{and } \forall\ u \in\ S^*$$

Next, we show that the function $H(P)$ is constant. From the above equations we know that $H(P) = H(Q)$ whenever the hypothesis of Weak IIA* is satisfied by two profiles u^P and u^Q. Thus, it is sufficient to show that there exist such profiles for any two sets P, Q. This is equivalent to requiring the existence of profiles of dimension at least two. This is guaranteed by our assumptions.

This means that the function $H(P)$ is constant. Thus for all non-constant u, $H(P)$ is constant and therefore $\lambda(u^P)$ is independent of P.
Claim 2

$$\lambda(u) = \frac{1}{p(u)}$$

Proof. Note that Claim 1 implies that $\lambda(u^P)$ is independent of P and hence of $u(a_0)$ for all a_0 that satisfy the condition that $\exists p_{a_0} \in \Delta A'$ such that $\langle u, p_{a_0}\rangle = u(a_0)$, i.e. that $\min_{a\in A'}(u) \leq u(a_0) \leq \max_{a\in A'}(u)$. This implies that $\lambda(u)$ depends only on $\max_{a\in A'} u(a), \min_{a\in A'} u(a)$. Translation Invariance of $\lambda(u)$ then implies the result.
♣

Axiom 4: Neutrality expresses that the names of the alternatives do not matter. Formally, at least when $\Delta(A)$ consists of all lotteries with finite support, any permutation π of A induces a permutation of the space of preferences:
$\mathcal{R} \mapsto R_\pi$ where $p\mathcal{R}_\pi q$ iff $p \circ \pi \mathcal{R} q \circ \pi$. Then

$$\varphi[(\mathcal{R})^\pi] = (\varphi[(\mathcal{R}])_\pi$$

Neutrality is obviously satisfied by "Relative Utilitarianism."

4.2 Necessity of the Axioms

The Independence Axiom

We assume the vN-M axioms are satisfied for individuals but we relax the vN-M axioms for society in particular the Independence axiom or the sure thing principle.

$$U = \sum_{n \in N} \left(\frac{u_n}{\max_{a \in A}(u_n(a)) - \min_{a \in A}(u_n(a))}\right)^2$$

where the utilities for individuals are the usual vN-M utilities. This example is due to Epstein and Segal [6].

Continuity of preferences

An example that violates the vN-M axioms of continuity of preferences is the leximin rule (Sen [15]) that lexically chooses the (normalized) utility of the worst off individual for a given alternative as the social utility.

Extended Pareto

Since Extended Pareto implies Monotonicity (Mertens and Dhillon, [12]), the e.g. used here is the same as for Monotonicity, i.e. take the gradient of the Nash product for the non-dummy players at the maximising point (in the closure of $C(u)$), when $[\min_{a \in A} u(n) | n \in N]$ is taken as the disagreement point. The weight of the dummy players is arbitrary.

Anonymity.

Otherwise use $\sum_n \lambda_n \frac{u_n}{p(u_n)}$ — with $\lambda_n > 0$ - as social utility.

Neutrality. Otherwise use $\sum_n \frac{u_n}{q(u_n)}$, where $\mu_n = \sum_{a \in A} w(a) u_n(a)$, and $q(u_n) = \sqrt{\sum_a w(a)[u_n(a) - \mu_n]^2}$ (if not zero) - with $w(a) > 0$, $\sum_a w(a) = 1$.

If one chooses all $w(a)$ equal, one obtains an example satisfying in addition neutrality, but not Weak I.I.A.*

5 Conclusion

This paper introduced the Extended Pareto axiom in a framework of preferences over lotteries. It was shown that if the von-Neumann-Morgenstern axioms on preferences are satisfied by individuals and by society then this axiom implies that the SWF is a weighted sum of utilities where the weights for each individual depend only on his utility function in the profile. The axiom thus implies additive separability in the SWF in this sense[10] . The axiom may be viewed as an analog (in the context of ordinal preferences) of the separability condition (Fleming [7], Arrow[2], and discussed by d'Aspremont [5]) which is imposed in the context of cardinal and fully comparable preferences, except that in addition it embodies Pareto. With two additional axioms, Anonymity and Weak IIA* a SWF, Relative Utilitarianism, was characterized for all profiles of preferences where the corresponding utility vectors were of dimension two at least. The Anonymity axiom is standard while Weak IIA* was motivated by Arrow's IIA applied to a framework of preferences over lotteries.

The results used quite strongly the mathematical structure imposed by the vN-M axioms. In principle, these results can be extended to the case where we do not directly use the vN-M axioms. Harsanyi's theorem e.g. has been extended in this way by Coulhon and Mongin [4], and in Mongin [13] using the more general notion of *mixture sets.* Mongin [4] has a section on Algebraic Preliminaries which would be directly relevant if we do not restrict ourselves only to lotteries over a set of A, but are concerned with (more generally) convex subsets of vector spaces, and affine functions on these.

APPENDIX :

Consider Monotonicity:

Axiom 0 (Axiom 5′ [12]. Assume $\mathcal{R}_n^*$ is total indifference, and $\mathcal{R}_k^* = \mathcal{R}_k \; \forall k \neq n$. Then

[10]Note however that it does not imply the usual form of additive separability since in principle the weights for each alternative are not separable; indeed they depend on the utility *function.*

$$\left.\begin{array}{l} p\mathcal{I}_n q \\ p\mathcal{I}^* q \end{array}\right\} \Longrightarrow p\mathcal{I}q$$

and

$$\left.\begin{array}{l} p\mathcal{P}_n q \\ p\mathcal{R}^* q \end{array}\right\} \Longrightarrow p\mathcal{P}q$$

It is easy to see that Extended Pareto implies this form of Monotonicity. In the other direction, a weak form of Extended Pareto is implied by this Monotonicity axiom where only partitions of this type are permitted, and moreover where the role of the one individual subgroup is priveleged relative to the other (see the latter part of the axiom). It is obvious that they are both equivalent to Pareto in the case of two individuals.

♣

Proposition 0. *The social welfare functions φ that satisfy the Pareto axiom are those which can be represented by a map λ from S^N to $\mathbb{R}^N$ such that*

a) $\lambda_n(\vec{u}) > 0 \;\; \forall n, \forall \vec{u} \in S^N$.

b) *If $\forall n \in N, u_n$ is a representation of $\mathcal{R}_n$, then $\sum_n \lambda_n(\vec{u}).\; u_n$ is a representation of $\varphi(\mathcal{R})$.*

c) • *$\lambda_n(\vec{u})$ is translation invariant, i.e.*

$$\text{if } v_n = u_n + \alpha_n \;\; \forall n, \text{ with } \alpha_n \in \mathbb{R}, \text{ then } \lambda_n(\vec{u}) = \lambda_n(\vec{v}).$$

• *$\lambda_n(\vec{u})$ is positively homogeneous of degree zero in u_k $\forall k \neq n$ and, if u_n is not constant, of degree minus one in u_n, i.e.*

$$\text{if } v_n = \beta_n u_n \;\; \forall n, \text{ with } \beta_n > 0, \text{ then } \lambda_n(\vec{v}) = \beta_n^{-1}\lambda_n(\vec{u}).$$

Proof. Let us first show the "single profile" result of Harsanyi. For $\vec{u} \in S^N$, U is a corrsponding social utility satisfying Pareto.
Clearly
$C = \Big\{ \big[(\langle u_n, p\rangle - \langle u_n, q\rangle)_{n\in N}, \langle U, q\rangle - \langle U, p\rangle\big] \mid p \in \Delta(A), q \in \Delta(A) \Big\}$ is symmetric (around zero) and convex, and $C \cap [\mathbb{R}^N_+ \times \mathbb{R}_+] = \{0\}$. So $F \doteq$

$\cup_{\lambda>0}(\lambda C)$ is a vector subspace, with $F \cap [\mathbb{R}_+^N \times \mathbb{R}_+] = \{0\}$. Hence, by Farkas' lemma, there exists $\lambda_n > 0$ and $\mu > 0$ such that $\sum \lambda_n x_n + \mu y \leq 0$ for $(x, y) \in F$, and hence equality since F is a vector space, i.e.,

$$\sum \lambda_n \langle u_n, p\rangle - \sum \lambda_n \langle u_n, q\rangle = \mu\langle U, p\rangle - \mu\langle U, q\rangle$$

for all $p, q \in \Delta(A)$.

Hence, dividing λ_n by μ, we obtain

$$\langle U, p\rangle = \sum_n \lambda_n \langle u_n, p\rangle + \beta \ , \ \ \forall p \in \Delta(A)$$

with $\beta = \langle U, q\rangle - \sum_n \lambda_n \langle u_n, q\rangle$ for some $q \in \Delta(A)$.
Consider now, for every $(\mathcal{R}) \in \mathcal{L}^{\mathcal{N}}$, the corresponding utility profile $(u_n)_{n \in N}$ where each u_n is normalised such as to have $\max_a u_n(a) = 1$, $\min_a u_n(a) = 0$ or $u_n(a) = 0 \ \forall a \in A$.

Let also U be a similarly normalised representation of $\varphi\,[(\mathcal{R})]$.

Let $\Lambda = \{(\lambda_n)_{n \in N} | \lambda_n > 0, U - \sum_n \lambda_n u_n$ is constant $\}$.

Let $M = \inf\{\sum_n \lambda_n | \lambda \in \Lambda\} + 1$ and $\Lambda^0 = \{\lambda \in \Lambda | \sum_n \lambda_n \leq M\}$.

Then Λ^0 is, by the above, non-empty, and is convex and bounded.

So the barycenter $\lambda^{\mathcal{R}}$ of Λ^0 exists, and belongs to Λ^0. This yields a map from preference profiles to N-tuples of positive numbers.

Define now $\lambda(\vec{u})$ for any utility profile $\vec{u}$ from its values for the normalised profiles by using (c). Then clearly (a), (b) and (c) hold.

Conversely, it is clear that any map λ satisfying (a) and (c) defines through (b) a map from preferences to preferences that satisfies Pareto.

♣

Proof of Proposition 2:

Observe that the proof of Proposition 0 does not use Strong Pareto; any form of Pareto, even Pareto Indifference is sufficient, given the vN-M axioms. The only difference due to using different forms of Pareto is in the signs of the coefficients (see e.g Mongin [13]) i.e. the vector λ. Thus in particular, Pareto Indifference with the vN-M axioms implies that social welfare can be represented as an affine function on the set F.
Second, if the sets F are equal, they will be equal even after normalising utilities, and the homogeneity properties of λ imply that if $\lambda(\vec{u}) = \lambda(\vec{u'})$ holds for the normalised problem, it holds too for the original problem. Thus we can assume the utilities are normalised.

It suffices to prove that the set C of Proposition 0 is the same in both problems, since the vector λ constructed there depends only on this set C. This will follow from the equality of the two sets F if we prove that U and U' are the same affine function on F, since by Pareto Indifference we know that they are affine functions on F. Note that the hypothesis of Weak IIA*, implies that

$$F = \{([< u, p >]_{n \in N} | p \in \Delta(A))\} = \{([< u, p' >]_{n \in N} | p' \in \Delta(A'))\}.$$

where $A' \subset A$. That the set F contains the latter set (call it F') is obvious. In the other direction, for any lottery in the set $F \backslash F'$ by the hypothesis of the axiom, there exists a lottery in F' to which all individuals are indifferent.
To prove Proposition 2, we distinguish two cases: (I) in which the extreme points of the convex hull of the set of alternatives in utility space ($\Delta A'$) are the same, and (II) in which they could be different. It is convenient and sufficient to prove this for the case $\#G = 2$ and $d(\vec{u}) = 2$, since this implies the result for the case of three individuals and a full dimensional profile.
In the first situation, we can use Weak IIA* directly to conclude that (normalised) social utilities must be the same on the set F' for the two profiles and by Pareto Indifference every lottery in $F \backslash F'$ is socially indifferent to some lottery in F'. This being true for both profiles, the set C is equal in the two cases, and hence the conclusion that $\lambda(\vec{u}) = \lambda(\vec{u'})$.
There remains to show the proof for the case II. We can show that Weak IIA* and Pareto Indifference implies Neutrality in this framework, and hence one can permute the profile to reach the situation of (I) again in a finite number of steps. Thus consider two profiles of normalized utilities $\vec{u}$ and $\vec{u'}$, which satisfy the requirement that $F = F'$. Since the number of alternatives is more than 4, there is at least one alternative p that is unanimously indifferent to some lottery on the others.Thus one can move this alternative up or down in the individual ranking without changing the vector λ as proved above. One can also use Pareto Indifference to derive the social preferences for an alternative which is unanimously indifferent to another. Consider w.l.o.g that we need to permute an alternative a to an alternative d. If these alternatives are extreme points in the convex hull F then we can construct intermediate profiles where we use Weak IIA* and Pareto Indifference to permute the two. In the first instance therefore move p to be unanimously indifferent to a without changing the vector λ. We can then use Pareto Indifference to

derive the social preferences with respect to p and in the next step move a to be unanimously indifferent to d. Next move d to the position of p without changing the vector λ and use Pareto Indifference to derive preferences for the permuted profile. Since there is a finite number of alternatives this can be done in a finite number of steps. In case either a or d or both are not extreme points of the convex hull, the proof proceeds in the same way, without using p now.
Hence the set C is equal for any two such profiles, hence also the SWF when viewed as a linear functional on F.
♣

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