Deaño, A.; Huertas, E. J.; Román, P. (2016). "Asymptotics of orthogonal polynomials generated by a Geronimus perturbation of the Laguerre measure". *Journal of Mathematical Analysis and Applications*, v. 433, Issue 1, January, pp. 732-746.
DOI: 10.1016/j.jmaa.2015.08.002

Proyectos:
MTM2012-36732-C03-01
MTM2012-34787
PIP 112-200801-01533

Asymptotics of orthogonal polynomials generated by a Geronimus perturbation of the Laguerre measure

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A B S T R A C T

This paper deals with monic orthogonal polynomials generated by a Geronimus canonical spectral transformation of the Laguerre classical measure, i.e.,

$$\frac{1}{x-c}x^{\alpha}e^{-x}dx + N\delta(x-c),$$

for $x \in [0, \infty)$, $\alpha > -1$, a free parameter $N \in \mathbb{R}_+$ and a shift $c < 0$. We analyze the asymptotic behavior (both strong and relative to classical Laguerre polynomials) of these orthogonal polynomials as n tends to infinity.

Keywords:
Orthogonal polynomials
Canonical spectral transformations of measures
Asymptotic analysis
Hypergeometric functions

1. Introduction

Let μ be a positive Borel measure supported on a finite or infinite interval $E = \text{supp}(\mu)$, such that the convex hull verifies $C_0(E) = [a, b] \subseteq \mathbb{R}$. In the last years several authors have studied the so-called *canonical spectral transformations* of μ, which are a way to construct new families of orthogonal polynomials from a perturbed version of μ. They have been studied from several points of view, including the corresponding Jacobi matrices (see [2,27]) or the Stieltjes functions associated with such a kind of transformations (see [28] among others).

Let us introduce the sequences of monic orthogonal polynomials (SMOP in the sequel) associated with one of the aforesaid canonical transformations, called the *Geronimus canonical transformation on the real line*. The basic *Geronimus perturbation* of μ is defined as

$$\frac{1}{x-c}d\mu(x) + N\delta(x-c),$$

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where $N \in \mathbb{R}_+$, $\delta(x-c)$ is the Dirac delta function located at $x=c$, and the shift of the perturbation verifies $c \notin E$. Observe that it is given simultaneously by a rational modification of μ by a positive linear polynomial whose real zero c is the point of transformation (also known as the *shift* of the transformation) jointly with the addition of a Dirac mass at the point of transformation as well.

This transformation was introduced by Geronimus in his pioneer works [12] and [13] on procedures of constructing new sequences of orthogonal polynomials from other known families, and it was also studied by Shohat in a different scheme involving mechanical quadratures (see [22]). Years later, Maroni (see [17]) returned to the problem and gave a first expression of the Geronimus perturbed orthogonal polynomials in terms of so-called co-recursive polynomials of the classical orthogonal polynomials. More recently, Bueno and Marcellán reinterpreted this perturbation in the framework of the discrete Darboux transformations (see [3]). In [2] the authors present a new computational algorithm for computing the Geronimus transformation with large shifts. In [1] the authors provide sharp limits (and the speed of convergence to them) of the zeros of the Geronimus perturbed SMOP, and also, when μ is semi-classical they obtain the corresponding electrostatic model for the zeros of the Geronimus perturbed SMOP, showing that they are the electrostatic equilibrium points of positive unit charges interacting according to a logarithmic potential under the action of an external field. In [16] the authors extend the standard Geronimus transformation to a cubic case. [9] provides a new revision of the Geronimus transformation in terms of symmetric bilinear forms in order to include certain Sobolev and Sobolev-type orthogonal polynomials into the scheme of Darboux transformations. Finally, [8] deals with multiple Geronimus transformations and show that they lead to discrete (non-diagonal) Sobolev type inner products, and it is shown that every discrete Sobolev inner product can be obtained as a multiple Geronimus transformation.

In view of the foregoing, this transformation has been extensively studied in the literature, mainly in analytic and algebraic frameworks. However, to the best of our knowledge, the asymptotic properties of the family of orthogonal polynomials as $n \to \infty$ have not been studied in detail, save for the particular case when $N=0$ and the perturbed measure is the Laguerre classical measure (see [10]).

2. Laguerre polynomials and functions of the second kind

The classical Laguerre polynomials $L_n^\alpha(x)$ are defined as the polynomials orthogonal with respect to the $\mathrm{L}^2([0,\infty))$ inner product

$$\langle p,q\rangle_\alpha = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx, \quad \alpha > -1, \quad p,q \in \mathbb{P},$$

see, among others [4] or [23].

In order to fix notation, we denote by $\widehat{L}_n^\alpha(x)$ the *monic* Laguerre polynomials, so $\widehat{L}_n^\alpha(x) = x^n + \dots$. These monic polynomials are connected to standard Laguerre polynomials $L_n^\alpha(x)$ by the formula

$$\widehat{L}_n^\alpha(x) = (-1)^n n! L_n^\alpha(x), \quad n \geq 0. \tag{1}$$

They satisfy a three term recurrence relation that we write in the following form

$$x\widehat{L}_n^\alpha(x) = \widehat{L}_{n+1}^\alpha(x) + \beta_n \widehat{L}_n^\alpha(x) + \gamma_n \widehat{L}_{n-1}^\alpha(x), \tag{2}$$

where

$$\beta_n = 2n+\alpha+1, \qquad \gamma_n = n(n+\alpha), \tag{3}$$

and we have initial data $\widehat{L}_0^\alpha(x) = 1$ and $\widehat{L}_1^\alpha(x) = x - \alpha - 1$. We will also make use of the $\mathrm{L}^2([0,\infty))$ norm of the monic Laguerre polynomials. Since

$$\|L_n^\alpha\|_\alpha^2 = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)},$$

we have

$$\|\widehat{L}_n^\alpha\|_\alpha^2 = \Gamma(n+\alpha+1)\Gamma(n+1). \tag{4}$$

A second (independent) solution of the recurrence relation (2) is the function of the second kind, obtained via a Stieltjes transform of the Laguerre polynomials

$$\widehat{F}_n^\alpha(z) = \int_0^\infty \frac{\widehat{L}_n^\alpha(t)}{t-z} t^\alpha e^{-t} dt, \tag{5}$$

which is an analytic function for $z \in \mathbb{C} \setminus [0,\infty)$. Here $n \geq 0$, and we define $F_{-1}^\alpha(c) = 1$, analogously to [11, §2.4.4]. Using the Rodrigues formula for Laguerre polynomials [19, §18.5(ii)] and a standard integral representation, see [19, Eq. (13.4.4)] for instance, it is possible to write $\widehat{F}_n^\alpha(z)$ in terms of the confluent hypergeometric function of the second kind, or Kummer U function:

$$\widehat{F}_n^\alpha(z) = (-1)^n n! \Gamma(n+\alpha+1) U(n+1, 1-\alpha, ze^{\pm\pi i}), \tag{6}$$

with plus sign if $-\pi < \arg z \leq 0$ and minus sign if $0 < \arg z \leq \pi$. This representation will be a key element for all the asymptotic analysis later on in this manuscript. For more information about the confluent hypergeometric functions, we refer the reader for instance to [19, Chapter 13].

Let us introduce the following inner product in the linear space $\mathbb{P}$ of polynomials with real coefficients

$$\langle p, q\rangle_{\nu_N} = \int_0^\infty p(x)q(x)\frac{x^\alpha}{x-c} e^{-x} dx + N\delta(x-c), \tag{7}$$

where $\alpha > -1$, and $N \geq 0$, and $c \in (-\infty, 0)$. Namely, we deal with a measure that consists of an absolutely continuous part, which is a rational perturbation of the Laguerre weight on $[0,+\infty)$, plus a Dirac delta located at point $x = c$:

$$d\nu_N(x) = \frac{x^\alpha e^{-x}}{x-c} dx + N\delta(x-c).$$

Equivalently, we will say that ν_N is a Geronimus perturbation of the standard Laguerre measure (see [2,1] and the references therein), and we will denote by $\widehat{Q}_n^{\alpha,c,N}(x)$ the monic orthogonal polynomials with respect to (7).

As explained in [1] (and the references therein), the *Laguerre–Geronimus monic orthogonal polynomials* can be written in terms of the monic Laguerre OPs using the following simple connection formula:

$$\widehat{Q}_n^{\alpha,c,N}(x) = \widehat{L}_n^\alpha(x) + \Lambda_n^N \widehat{L}_{n-1}^\alpha(x), \tag{8}$$

where the coefficient Λ_n^N depends on n, α, c and N. More precisely:

Proposition 1. *The connection coefficient* Λ_n^N *can be written as follows:*

$$\Lambda_n^N = -\frac{\Gamma(n+\alpha)\Gamma(n)}{\widehat{L}_{n-1}^{\alpha}(c)\widehat{F}_{n-1}^{\alpha}(c) - N\widehat{L}_{n-1}^{\alpha}(c)^2} - \pi_{n-1}(c), \tag{9}$$

where we have defined, for $c \in (-\infty, 0)$,

$$\pi_n(c) = \frac{\widehat{L}_{n+1}^{\alpha}(c)}{\widehat{L}_n^{\alpha}(c)}. \tag{10}$$

In particular, when $N = 0$, *we have* $\Lambda_n^N = -r_{n-1}(c)$.

Proof. In Remark 1 in [1], Λ_n^N is given in terms of Laguerre polynomials and functions of the second kind:

$$\Lambda_n^N = \left(\frac{1}{\pi_{n-1}(c) - r_{n-1}(c)} - N\frac{\widehat{L}_{n-1}^{\alpha}(c)^2}{\|\widehat{L}_{n-1}^{\alpha}\|^2}\right)^{-1} - \pi_{n-1}(c), \tag{11}$$

where

$$r_n(c) = \frac{\widehat{F}_{n+1}^{\alpha}(c)}{\widehat{F}_n^{\alpha}(c)}, \quad c \in (-\infty, 0). \tag{12}$$

Therefore, we can rewrite Λ_n^N as follows:

$$\Lambda_n^N = \left(\frac{\widehat{L}_{n-1}^{\alpha}(c)\widehat{F}_{n-1}^{\alpha}(c)}{\Delta_n(c)} - N\frac{\widehat{L}_{n-1}^{\alpha}(c)^2}{\|\widehat{L}_{n-1}^{\alpha}\|^2}\right)^{-1} - \pi_{n-1}(c),$$

with the Casoratian determinant

$$\Delta_n(c) = \widehat{L}_n^{\alpha}(c)\widehat{F}_{n-1}^{\alpha}(c) - \widehat{L}_{n-1}^{\alpha}(c)\widehat{F}_{n-1}^{\alpha}(c) = -\Gamma(n+\alpha)\Gamma(n), \qquad n \geq 1.$$

The last equality follows from the fact that both $\widehat{L}_n^{\alpha}(c)$ and $\widehat{F}_n^{\alpha}(c)$ are solutions of the recurrence relation (2), with $x = c$, and then it is straightforward to check that $\Delta_n(c) = \gamma_{n-1}\Delta_{n-1}(c)$, with $\Delta_1(c) = -\Gamma(\alpha+1)$. Using (4) for the norm of the polynomials and simplifying, we arrive at (9).

Finally, the case $N = 0$ follows directly from formula (11). $\square$

The aim of this paper is to obtain strong and relative asymptotics of the sequence of OPs $\widehat{Q}_n^{\alpha,c,N}(x)$, for large n. The simplicity of the connection formula (8) makes it a very attractive identity to use in conjunction with classical asymptotic approximations for Laguerre polynomials (such as Perron, Fejér or Mehler–Heine expansions, see [23, §8.22]), in order to obtain the corresponding result for the Laguerre–Geronimus OPs. The only element that is missing so far in the literature is a study of the asymptotic behavior of the coefficient Λ_n^N. We observe that because of (9), Λ_n^N depends on the ratios $\pi_n(c)$ and $r_n(c)$.

The structure of the paper is as follows: in Section 3 we obtain large n asymptotic expansions for $\pi_{n-1}(c)$ and $r_{n-1}(c)$, which lead to asymptotic approximations for Λ_n^N in Section 4. Putting together this result and the connection formula (8), we obtain the strong and relative asymptotics for $\widehat{Q}_n^{\alpha,c,N}(x)$ in Section 5.

3. Asymptotic expansions for π_{n-1} and r_{n-1}

The ratios $\pi_{n-1}(z)$ and $r_{n-1}(z)$ could in principle be studied using standard techniques for the asymptotic behavior of solutions of three-term recurrence relations, such as the Perron theorem, see for instance [14, §4.3]. However, since the recurrence coefficients in (2) satisfy $\beta_n \sim 2n$ and $\gamma_n \sim n^2$ as $n \to \infty$, the theorem is inconclusive about the existence of minimal and dominant solutions, and it does not give detailed asymptotic information about the behavior of ratios of solutions. We refer the reader to [6, Section 4] for more details.

In this paper we work with strong asymptotics of the Laguerre polynomials and functions of the second kind directly. For z away from $[0, \infty)$, the strong asymptotics for the $L_n^\alpha(z)$ can be obtained from the classical expansion due to Perron, see for instance [23, Theorem 8.22.3]:

$$L_n^\alpha(z) = \frac{1}{2\sqrt{\pi}} e^{z/2} (-z)^{-\frac{\alpha}{2}-\frac{1}{4}} n^{\frac{\alpha}{2}-\frac{1}{4}} e^{2\sqrt{-nz}} \left(1 + \mathcal{O}(n^{-1/2})\right), \tag{13}$$

which is valid for fixed $\alpha > -1$ and $z \in \mathbb{C} \setminus [0, \infty)$. The fractional powers are assumed to take their principal values, with phase between $-\pi$ and π. In [5, Theorem 3] higher terms in this asymptotic expansion have been obtained, using a related expansion for confluent hypergeometric functions due to Buchholz, see also [15] and references therein. The ratio asymptotics is given in [5] as well:

$$\begin{aligned} \frac{L_n^\alpha(z)}{L_{n-1}^\alpha(z)} &= 1 + \sqrt{-\frac{z}{n-1}} + \frac{2\alpha - 2z - 1}{4(n-1)} + \mathcal{O}(n^{-3/2}) \\ &= 1 + \sqrt{-\frac{z}{n}} + \frac{2\alpha - 2z - 1}{4n} + \mathcal{O}(n^{-3/2}), \end{aligned}$$

as $n \to \infty$, for fixed α and $z \in \mathbb{C} \setminus [0, \infty)$. Therefore, as a direct consequence, for the monic polynomials we have

$$\pi_{n-1}(z) = \frac{\widehat{L}_n^\alpha(z)}{\widehat{L}_{n-1}^\alpha(z)} = -n \frac{L_n^\alpha(z)}{L_{n-1}^\alpha(z)} = -n - \sqrt{-zn} + \frac{2z - 2\alpha + 1}{4} + \mathcal{O}(n^{-1/2}), \tag{14}$$

as $n \to \infty$.

Regarding the asymptotic behavior of the functions of the second kind, we present the following result:

Proposition 2. *Given $\alpha > -1$, the functions of the second kind $\widehat{F}_n^\alpha(z)$, defined by (5), satisfy*

$$\begin{aligned} \widehat{F}_n^\alpha(z) &= (-1)^n \sqrt{\pi} (-z)^{\frac{\alpha}{2}-\frac{1}{4}} e^{-z/2 - 2\sqrt{-zn}} \Gamma(n + \alpha + 1) n^{-\frac{\alpha}{2}-\frac{1}{4}} \\ &\times \left[e_0 + \frac{e_1}{\sqrt{-zn}} + \frac{e_2}{-zn} + \mathcal{O}(n^{-3/2}) \right], \qquad n \to \infty. \end{aligned} \tag{15}$$

The expansion is valid for bounded $z \in \mathbb{C} \setminus [0, \infty)$, with principal values of the power functions. The first few coefficients e_j are

$$\begin{aligned} e_0(\alpha, z) &= 1, \\ e_1(\alpha, z) &= \frac{12\alpha^2 - 3 - 24z(1-\alpha) - 4z^2}{48}, \\ e_2(\alpha, z) &= \frac{16z^4 + 192(1-\alpha)z^3 + 24(20\alpha^2 - 48\alpha + 13)z^2}{4608} \\ &\quad + \frac{144(\alpha - 1)(2\alpha + 1)(2\alpha + 3)z + 9(4\alpha^2 - 1)(4\alpha^2 - 9)}{4608}. \end{aligned}$$

Proof. In order to get the previous result, there are at least two possibilities: use the expression of $\widehat{F}_n^{\alpha}(c)$ in terms of Kummer functions, see (6), or deduce its asymptotic behavior from the Deift–Zhou steepest descent method applied to the corresponding Riemann–Hilbert problem, see the work of Vanlessen [26] and the monograph by Deift [7]. In the sequel, we elaborate on the first approach.

Following the ideas exposed in [24], we use the integral representation for the Kummer U-function:

$$U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} dt,$$

which holds for $\operatorname{Re} a, \operatorname{Re} z > 0$, see also [19, 13.4.4]. The transformation $t/(1+t) = e^{-\tau}$ gives

$$U(a,b,z) = \frac{e^{z/2}}{\Gamma(a)} \int_0^\infty e^{-a\tau - z/\tau} \tau^{-b} f(\tau) d\tau,$$

where

$$f(\tau) = e^{z\mu(\tau)} \left(\frac{\tau}{1-e^{-\tau}} \right)^b, \qquad \mu(\tau) = \frac{1}{\tau} - \frac{1}{e^{\tau}-1} - \frac{1}{2}.$$

The function $f(\tau)$ is analytic for $|\tau| < 2\pi$, and therefore it admits a power series expansion around the origin of the form

$$f(\tau) = \sum_{m=0}^{\infty} d_m(b,z) \tau^m. \tag{16}$$

Integration term by term, invoking the classical Watson lemma, [20,25], gives an expansion of the U-function of the form

$$U(a,b,z) = \sum_{m=0}^{M-1} d_m(b,z) \phi_m(a,b,z) + R_M(a,b,z),$$

where the asymptotic sequence is

$$\phi_m(a,b,z) = \frac{2\, e^{z/2}}{\Gamma(a)} \left(\frac{z}{a} \right)^{\frac{m+1-b}{2}} K_{m+1-b}(2(az)^{1/2}),$$

in terms of modified Bessel functions, using the fact that

$$K_\nu(2(z\zeta)^{1/2}) = \frac{1}{2} \left(\frac{\zeta}{z} \right)^{-\nu/2} \int_0^\infty e^{-z\tau - \zeta/\tau} \tau^{-\nu-1} d\tau,$$

valid for $\operatorname{Re} z, \operatorname{Re} \zeta > 0$. Also, we have $R_M(a,b,z) = \mathcal{O}(\phi_M(a,b,z))$ as $a \to \infty$, uniformly with respect to z in compact sets in $z \geq 0$ and uniformly with respect to b in compact sets of $\mathbb{R}$. We assume, following [24, §2.1] that M is large enough, in particular $M > b$.

If we replace $b = 1 - \alpha$, symbolic computation gives

$$\begin{aligned}
d_0(\alpha, z) &= 1,\\
d_1(\alpha, z) &= \frac{6(1-\alpha) - z}{12},\\
d_2(\alpha, z) &= \frac{z^2 - 12(1-\alpha)z + 12(\alpha-1)(3\alpha-2)}{288},\\
d_3(\alpha, z) &= \frac{-5z^3 - 90(1-\alpha)z^2 - 36(15\alpha^2 + 25\alpha + 8)z - 1080\alpha(\alpha-1)^2}{51\,840},
\end{aligned}$$

and so on, for the coefficients in the series expansion (16). Replacing a by $n+1$ in the asymptotic sequence, we obtain

$$\phi_m(n, \alpha, z) = \frac{2\, e^{z/2}}{n!} \left(\frac{z}{n+1}\right)^{\frac{m+\alpha}{2}} K_{m+\alpha}(2((n+1)z)^{1/2}). \tag{17}$$

As a consequence, using (6) and (17), we have

$$\begin{aligned}
\widehat{F}_n^{\alpha}(z) &= (-1)^n n!\, \Gamma(n+\alpha+1)\, U(n+1, 1-\alpha, ze^{\pm\pi i})\\
&= 2e^{-z/2}(-1)^n \Gamma(n+\alpha+1) \left[S_M(n, \alpha, ze^{\pm\pi i}) + R_M(n, \alpha, ze^{\pm\pi i})\right],
\end{aligned}$$

as $n \to \infty$, where

$$S_M(n, \alpha, ze^{\pm\pi i}) = \sum_{m=0}^{M-1} d_m(\alpha, -z) \left(\frac{ze^{\pm\pi i}}{n+1}\right)^{\frac{m+\alpha}{2}} K_{m+\alpha}(2((n+1)ze^{\pm\pi i})^{1/2}),$$

and $R_M(n, \alpha, ze^{\pm\pi i})$ is the remainder. Once again, we take plus sign if $-\pi < \arg z \leq 0$ and minus sign if $0 < \arg z \leq \pi$.

It is possible to re-expand this asymptotic series in inverse powers of n: using the asymptotics of the modified Bessel functions for large values of the argument and fixed order ν, see [19, 10.40.2], we have

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \sum_{\ell=0}^{\infty} \frac{a_\ell(\nu)}{z^\ell}, \qquad z \to \infty,$$

where $a_0(\nu) = 1$ and for $\ell \geq 1$,

$$a_\ell(\nu) = \frac{(4\nu^2 - 1)(4\nu^2 - 3^2) \cdots (4\nu^2 - (2\ell-1)^2)}{8^\ell \ell!}.$$

So, if we denote $s_n = 2((n+1)ze^{\pm\pi i})^{1/2}$, we have

$$K_{m+\alpha}(s_n) \sim \left(\frac{\pi}{2s_n}\right)^{1/2} e^{-s_n} \sum_{\ell=0}^{\infty} \frac{a_\ell(m+\alpha)}{s_n^\ell}.$$

Note that $s_n = 2\sqrt{nze^{\pm\pi i}}\,(1 + \mathcal{O}(n^{-1}))$ as $n \to \infty$. Assembling all the previous results and expanding in inverse powers of n, we arrive at Proposition 2. $\square$

Remark 1. We observe that the leading term in this expansion is consistent with the results in [10,18], bearing in mind that we are working with monic polynomials. We also note that the exponential factor is erroneously corrected in Proposition 3.2(a) in the first reference.

As a direct consequence, and using symbolic computation, we have an asymptotic expansion for the ratio of consecutive functions of the second kind:

Proposition 3. *As $n \to \infty$, the ratio asymptotics of the Laguerre functions of the second kind is given by*

$$r_{n-1}(z) = \frac{\widehat{F}_n^{\alpha}(z)}{\widehat{F}_{n-1}^{\alpha}(z)} = -n + \sqrt{-zn} + \frac{2z - 2\alpha + 1}{4} + \mathcal{O}(n^{-1/2}),$$

where $\alpha > -1$ and $z \in \mathbb{C} \setminus [0, \infty)$.

Proof. The result follows from Proposition 2, using symbolic computation to manipulate the asymptotic expansions. We remark that division by $\widehat{F}_{n-1}^{\alpha}(z)$ is allowed for z away from the positive real axis, bearing in mind (6) and the fact that the Kummer function $U(a, b, z)$ does not have zeros for $|\arg z| < \pi$ if a is positive, see [19, §13.9]. $\square$

4. Asymptotic behavior of Λ_n^N

The main result of this section is the following:

Proposition 4. *Let $c \in (-\infty, 0)$, $\alpha > -1$ and $N \geq 0$ be fixed parameters, then*

$$\Lambda_n^N = n + \sqrt{-cn} + \frac{2\alpha - 2c - 1}{4} + \mathcal{O}(n^{-1/2}), \qquad n \to \infty, \tag{18}$$

if $N > 0$, and

$$\Lambda_n^0 = -r_{n-1} = n - \sqrt{-cn} + \frac{2\alpha - 2c - 1}{4} + \mathcal{O}(n^{-1/2}), \qquad n \to \infty. \tag{19}$$

Proof. We deduce this result from the asymptotic expansions derived before: combining (13) with (15), we obtain

$$\widehat{L}_{n-1}^{\alpha}(c)\widehat{F}_{n-1}^{\alpha}(c) = \frac{1}{2\sqrt{-cn}}\Gamma(n)\Gamma(n+\alpha)\left(1 + \mathcal{O}(n^{-1})\right),$$

as $n \to \infty$. It can be checked that the term of order $\mathcal{O}(n^{-1/2})$, which is to be expected here, is actually equal to 0. Also,

$$\begin{aligned}\widehat{L}_{n-1}^{\alpha}(c)^2 &= \frac{D_{\alpha,c}}{2\sqrt{-cn}}\Gamma(n)^2 n^{\alpha} e^{4\sqrt{-cn}}\left(1 + \mathcal{O}(n^{-1/2})\right),\\ &= \frac{D_{\alpha,c}}{2\sqrt{-cn}}\Gamma(n)\Gamma(n+\alpha) e^{4\sqrt{-cn}}\left(1 + \mathcal{O}(n^{-1/2})\right),\end{aligned}$$

using the fact that $\Gamma(n+\alpha)/\Gamma(n) = n^{\alpha}(1 + \mathcal{O}(n^{-1}))$, as $n \to \infty$, and the notation

$$D_{\alpha,c} = \frac{e^c(-c)^{-\alpha}}{2\pi}.$$

Putting everything together and using the results in [5], we have

$$\begin{aligned}\Lambda_n^N &= -\frac{2\sqrt{-cn}}{1+\mathcal{O}(n^{-1}) - ND_{\alpha,c}e^{4\sqrt{-cn}}(1+\mathcal{O}(n^{-1/2}))} + n\frac{L_n^\alpha(c)}{L_{n-1}^\alpha(c)}, \\ &= -\frac{2\sqrt{-cn}}{1+\mathcal{O}(n^{-1}) - ND_{\alpha,c}e^{4\sqrt{-cn}}(1+\mathcal{O}(n^{-1/2}))} \\ &+ n + \sqrt{-cn} + \frac{2\alpha-2c-1}{4} + \mathcal{O}(n^{-1/2}).\end{aligned}$$

It is important to observe that if $N>0$, the first term in the previous sum is exponentially small in n, so it does not contribute to the final result. However, if $N=0$ the first term does contribute, since the exponential term is not present, and then we have the difference in sign in the subleading term given in Proposition 4. □

5. Asymptotics for $\widehat{Q}_n^{\alpha,c,N}(z)$

Using the estimates for Λ_n^N (18) and (19), we describe first the strong asymptotics for $\widehat{Q}_n^{\alpha,c,N}(z)$, for z away from the interval of orthogonality (outer asymptotics):

Proposition 5. *Given fixed values of $N \geq 0$, $\alpha > -1$, $c \in (-\infty, 0)$ and $z \in \mathbb{C}\setminus[0,\infty)$, as $n\to\infty$, the monic polynomials $\widehat{Q}_n^{\alpha,c,N}(z)$ verify the following strong asymptotics*

$$\widehat{Q}_n^{\alpha,c,N}(z) = \frac{(-1)^n n!}{2\sqrt{\pi}} e^{z/2+2\sqrt{-nz}}(-z)^{-\frac{\alpha}{2}-\frac{1}{4}} n^{\frac{\alpha}{2}-\frac{3}{4}} \left(\sqrt{-z}\mp\sqrt{-c}\right)\left(1+\mathcal{O}(n^{-1/2})\right), \tag{20}$$

and the following ratio asymptotics with respect to monic Laguerre polynomials:

$$\frac{\widehat{Q}_n^{\alpha,c,N}(z)}{\widehat{L}_n^\alpha(z)} = \frac{\sqrt{-z}\mp\sqrt{-c}}{\sqrt{n}} - \frac{(\sqrt{-z}\mp\sqrt{-c})^2}{2n} + \mathcal{O}(n^{-3/2}). \tag{21}$$

In both cases, the upper sign corresponds to the case $N>0$ and the lower sign to $N=0$.

Proof. We rewrite the connection formula (8) as follows:

$$\widehat{Q}_n^{\alpha,c,N}(z) = \left(1+\Lambda_n^N \frac{\widehat{L}_{n-1}^\alpha(z)}{\widehat{L}_n^\alpha(z)}\right)\widehat{L}_n^\alpha(z), \tag{22}$$

and we use the information obtained so far. Observe that division by $\widehat{L}_n^\alpha(z)$ does not cause any problem, since the zeros of this polynomial are contained in $[0,\infty)$. From (14), we deduce

$$\frac{\widehat{L}_{n-1}^\alpha(z)}{\widehat{L}_n^\alpha(z)} = -\frac{1}{n}\left[1-\sqrt{-\frac{z}{n}}+\frac{1-2z-2\alpha}{4n}+\mathcal{O}(n^{-3/2})\right], \qquad n\to\infty,$$

and combining this with the asymptotic behavior for Λ_n^N given by Proposition 4 and (14), we obtain (20). Then formula (21) for the relative asymptotic behavior is a direct consequence of (22). □

We remark that this is consistent with the result in [10, Proposition 3.4a)], taking $N=0$ and $M=1$.

Applying the connection formula again, but with the inner asymptotic expansion for Laguerre polynomials, we can obtain strong asymptotics of $\widehat{Q}_n^{\alpha,c,N}(x)$ for $x\in(0,\infty)$.

Proposition 6. *Given fixed values of $N \geq 0$, $\alpha > -1$, $c \in (-\infty, 0)$ and x in compact intervals of $(0,\infty)$, as $n \to \infty$, the monic polynomials $\widehat{Q}_n^{\alpha,c,N}(x)$ verify the following strong asymptotics*

$$\widehat{Q}_n^{\alpha,c,N}(x) = (-1)^{n+1} n! \, \frac{n^{\alpha/2-3/4} e^{x/2}}{\sqrt{\pi}\, x^{\alpha/2+1/4}} \left[\sqrt{x}\, \sin\theta_n^\alpha(x) \pm \sqrt{-c}\, \cos\theta_n^\alpha(x) + \mathcal{O}(n^{-1/2})\right], \tag{23}$$

where the phase function is

$$\theta_n^\alpha(x) = 2\sqrt{nx} - \left(\frac{\alpha}{2} + \frac{1}{4}\right)\pi, \tag{24}$$

and again the upper sign corresponds to the case $N > 0$ and the lower sign to $N = 0$.

Proof. We rewrite (8) as follows:

$$\widehat{Q}_n^{\alpha,c,N}(x) = \left(1 - \frac{\Lambda_n^N}{n}\right) \widehat{L}_n^\alpha(x) + \frac{\Lambda_n^N}{n} \widehat{L}_n^{\alpha-1}(x), \tag{25}$$

where we have used the following identity for standard Laguerre polynomials:

$$L_{n-1}^\alpha(z) = L_n^\alpha(z) - L_n^{\alpha-1}(z), \tag{26}$$

see for example [19, 18.9.13]. Then, we use the classical Fejér formula for Laguerre polynomials, see [23, Theorems 8.22.1, 8.22.2], adapted to the monic case:

$$\widehat{L}_n^\alpha(x) = (-1)^n n! \, \frac{n^{\alpha/2-1/4} e^{x/2}}{\sqrt{\pi}\, x^{\alpha/2+1/4}} \left[\cos\theta_n^\alpha(x) + \mathcal{O}(n^{-1/2})\right], \tag{27}$$

valid for x in compact intervals of $(0,\infty)$, with phase function (24).

From the asymptotic expansion of Λ_n^N we deduce that

$$1 - \frac{\Lambda_n^N}{n} = \mp\sqrt{\frac{-c}{n}} + \mathcal{O}(n^{-1}), \qquad \frac{\Lambda_n^N}{n} = 1 \pm \sqrt{\frac{-c}{n}} + \mathcal{O}(n^{-1}),$$

where the upper sign corresponds to the case $N > 0$ and the lower sign to $N = 0$. Using this information and (27), we expand as $n \to \infty$ in (25), bearing in mind that from the definition $\cos\theta_n^{\alpha-1}(x) = -\sin\theta_n^\alpha(x)$, and we arrive at the result. $\square$

Another useful asymptotic behavior that one can find in the literature is the Mehler–Heine type formulas. Is very well known that for $j \in \mathbb{N} \cup \{0\}$, the standard Laguerre polynomials verify

$$\lim_{n\to\infty} \frac{L_n^{(\alpha)}\left(z/(n+j)\right)}{n^\alpha} = z^{-\alpha/2} J_\alpha\left(2\sqrt{z}\right), \tag{28}$$

uniformly for z in compact subsets of $\mathbb{C}$, see [23, Theorem 8.1.3], where J_α is the Bessel function of the first kind, and the square root takes its principal value. Using this result, we can prove the following:

Proposition 7. *For given values of $N \geq 0$, $\alpha > -1$, $c \in (-\infty, 0)$ and z in compact subsets of $\mathbb{C}$, the polynomials $\widehat{Q}_n^{\alpha,c,N}(z)$ verify*

$$\lim_{n\to\infty} \frac{(-1)^n}{n!} \frac{\widehat{Q}_n^{\alpha,c,N}(z/n)}{n^{\alpha-1/2}} = \mp\sqrt{-c}\, J_\alpha\left(2\sqrt{z}\right), \tag{29}$$

where the upper sign corresponds to the case $N > 0$ and the lower sign to $N = 0$.

Proof. In order to prove (29), we start with the connection formula (8) for monic polynomials. In terms of standard Laguerre polynomials, recall (1), we have

$$\begin{aligned}\frac{(-1)^n}{n!}\,\widehat{Q}_n^{\alpha}(z/n) &= L_n^{\alpha}(z/n) - \frac{\Lambda_n^N}{n}L_{n-1}^{\alpha}(z/n)\\ &= \left(1-\frac{\Lambda_n^N}{n}\right)L_n^{\alpha}(z/n) + \frac{\Lambda_n^N}{n}L_n^{\alpha-1}(z/n),\end{aligned}$$

where we have used (26) again. Consequently,

$$\frac{(-1)^n}{n!}\,\frac{\widehat{Q}_n^{\alpha}(z/n)}{n^{\alpha}} = \left(1-\frac{\Lambda_n^N}{n}\right)\frac{L_n^{\alpha}(z/n)}{n^{\alpha}} + \frac{\Lambda_n^N}{n^2}\,\frac{L_n^{\alpha-1}(z/n)}{n^{\alpha-1}}. \tag{30}$$

Next, using the asymptotic expansion for Λ_n^N, we deduce that as $n\to\infty$,

$$1-\frac{\Lambda_n^N}{n} = \mp\sqrt{\frac{-c}{n}} + \mathcal{O}(n^{-1}), \qquad \frac{\Lambda_n^N}{n^2} = \mathcal{O}(n^{-1}),$$

where the upper sign corresponds to the case $N>0$ and the lower sign to $N=0$. Then multiplication by $n^{1/2}$ in (30) and the use of the Mehler–Heine asymptotics for Laguerre polynomials (28) give the result. □

Again, we note that this is consistent with [10, Proposition 3.4b)], taking $N=0$ and $M=1$.

6. Three-term recurrence relation

The monic Laguerre polynomials satisfy the three-term recurrence relation (2), and from that we can obtain an analogous recurrence for the perturbed polynomials $\widehat{Q}_n^{\alpha,c,N}(x)$. This result follows from the ones in the references [3,28], but for the benefit of the reader we include here a short proof for the case of Laguerre–Geronimus polynomials.

Proposition 8. *The polynomials $\widehat{Q}_n^{\alpha,c,N}(x)$ satisfy a three term recurrence relation*

$$\widehat{Q}_{n+1}^{\alpha,c,N}(x) = (x-\tilde{\beta}_n)\widehat{Q}_n^{\alpha,c,N}(x) - \tilde{\gamma}_n\widehat{Q}_{n-1}^{\alpha,c,N}(x), \tag{31}$$

where the coefficients are given by

$$\begin{aligned}\tilde{\beta}_n &= \beta_n + \Lambda_n^N - \Lambda_{n+1}^N,\\ \tilde{\gamma}_n &= \frac{\Lambda_n^N}{\Lambda_{n-1}^N}\gamma_{n-1},\end{aligned}$$

and β_n and γ_n are given by (3).

Proof. Let us denote by $\tilde{\beta}$ and $\tilde{\gamma}$ the coefficients of the three term recurrence relation for the polynomials $\widehat{Q}_n^{\alpha,c,N}(x)$:

$$x\widehat{Q}_n^{\alpha,c,N}(x) = \widehat{Q}_{n+1}^{\alpha,c,N}(x) + \tilde{\beta}_n\widehat{Q}_n^{\alpha,c,N}(x) + \tilde{\gamma}_n\widehat{Q}_{n-1}^{\alpha,c,N}(x).$$

Using the connection formula (8) on both sides of the previous equation we get

$$x\widehat{L}_n^{\alpha}(x) + \Lambda_n^N x\widehat{L}_{n-1}^{\alpha}(x) = \widehat{L}_{n+1}^{\alpha}(x) + (\Lambda_{n+1}^N + \tilde{\beta}_n)\widehat{L}_n^{\alpha}(x) + (\tilde{\beta}_n\Lambda_n^N + \tilde{\gamma}_n)\widehat{L}_{n-1}^{\alpha}(x) + \gamma_n\Lambda_{n-1}^N\widehat{L}_{n-2}^{\alpha}(x).$$

We use the three-term recurrence relation for the Laguerre polynomials (2) on the left hand side of the previous equation and we obtain

$$\begin{aligned}&\widehat{L}_{n+1}^{\alpha}(x)+(\beta_n+\Lambda_n^N)\widehat{L}_n^{\alpha}(x)+(\gamma_n+\Lambda_n^N\beta_{n-1})\widehat{L}_{n-1}^{\alpha}(x)+\Lambda_n^N\gamma_{n-1}\widehat{L}_{n-2}^{\alpha}(x)\\&=\widehat{L}_{n+1}^{\alpha}(x)+(\Lambda_{n+1}^N+\tilde{\beta}_n)\widehat{L}_n^{\alpha}(x)+(\tilde{\beta}_n\Lambda_n^N+\tilde{\gamma}_n)\widehat{L}_{n-1}^{\alpha}(x)+\gamma_n\Lambda_{n-1}^N\widehat{L}_{n-2}^{\alpha}(x).\end{aligned}$$

Since the Laguerre polynomials are a basis for the space of polynomials, we obtain the following equations

$$\beta_n+\Lambda_n^N=\Lambda_{n+1}^N+\tilde{\beta}_n, \tag{32}$$

$$\gamma_n+\Lambda_n^N\beta_{n-1}=\tilde{\beta}_n\Lambda_n^N+\tilde{\gamma}_n, \tag{33}$$

$$\Lambda_n^N\gamma_{n-1}=\tilde{\gamma}_n\Lambda_{n-1}^N. \tag{34}$$

The proposition follows directly from formulas (32) and (34). □

Remark 2. From (33), substituting the expressions for $\tilde{\gamma}_n$ and $\tilde{\beta}_n$, we obtain the following non-linear recursion for Λ_n^N:

$$\Lambda_{n+1}^N-\Lambda_n^N=-\frac{\gamma_n}{\Lambda_n^N}+\frac{\gamma_{n-1}}{\Lambda_{n-1}^N}+2 \tag{35}$$

It follows directly from (35) that

$$\Lambda_{n+1}^N-\Lambda_2^N=\sum_{i=3}^{n+1}(\Lambda_i^N-\Lambda_{i-1}^N)=-\frac{\gamma_n}{\Lambda_n^N}+2(n-1)+\frac{\gamma_1}{\Lambda_1^N}.$$

This gives the following recursion for Λ_n^N:

$$\Lambda_{n+1}^N=-\frac{n(n+\alpha)}{\Lambda_n^N}+2(n-1)+\frac{\gamma_1}{\Lambda_1^N}+\Lambda_2^N. \tag{36}$$

If we define $\Lambda_n^N=\varrho_n^N/\varrho_{n-1}^N$, then (36) becomes

$$\varrho_{n+1}^N-\left(2(n-1)+\frac{\gamma_1}{\Lambda_1^N}+\Lambda_2^N\right)\varrho_n^N+n(n+\alpha)\varrho_{n-1}^N=0. \tag{37}$$

From this result and the asymptotic expansion for Λ_n^N, see (18) and (19), it is straightforward to deduce the large n asymptotic behavior of the recurrence coefficients $\tilde{\beta}_n$ and $\tilde{\gamma}_n$, in terms of the recurrence coefficients for monic Laguerre polynomials, β_n and γ_n, given by (3):

Theorem 3. *As $n\to\infty$, for fixed $c\in(-\infty,0)$ and $N\geq 0$, the coefficients $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ of the three term recurrence relation for monic Laguerre–Geronimus orthogonal polynomials satisfy*

$$\begin{aligned}\tilde{\beta}_n&=\left(1-\frac{1}{2n}\mp\frac{\sqrt{-c}}{4n^{3/2}}+\mathcal{O}(n^{-2})\right)\beta_n,\\\tilde{\gamma}_n&=\left(1+\frac{1}{n}\mp\frac{\sqrt{-c}}{2n^{3/2}}+\mathcal{O}(n^{-2})\right)\gamma_{n-1},\end{aligned}$$

where the upper sign corresponds to the case $N>0$ and the lower sign to the case $N=0$.

7. Hypergeometric representation of $\widehat{Q}_n^{\alpha,c,N}(x)$

In this section we will derive a representation of the Geronimus perturbed family of orthogonal polynomials as hypergeometric functions. For this we need the connection formula (8) together with the hypergeometric representation of the monic Laguerre polynomials, that can be obtained from [19, 18.5.12]:

$$\begin{aligned}\widehat{L}_n^{\alpha}(x) &= \frac{(-1)^n \Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} {}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix};x\right)\\ &= (-1)^n (\alpha+1)_n \sum_{k=0}^{\infty} \frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}.\end{aligned} \tag{38}$$

Theorem 4. *The monic polynomials $\widehat{Q}_n^{\alpha,c,N}(x)$ have the following hypergeometric representation*

$$\widehat{Q}_n^{\alpha,c,N}(x) = C_{n,\alpha}\ {}_2F_2\left(\begin{matrix}-n,\ 1+e_n^N\\ \alpha+1,\ e_n^N\end{matrix};x\right),$$

where

$$C_{n,\alpha} = \left(1-\frac{\Lambda_n^N}{n+\alpha}\right)(-1)^n(\alpha+1)_n\,, \quad e_n^N = \frac{n(n+\alpha-\Lambda_n^N)}{\Lambda_n^N}.$$

Proof. It follows from the connection formula (8) and (38) that

$$\begin{aligned}\widehat{Q}_n^{\alpha,c,N}(x) = (-1)^n (\alpha+1)_n &\sum_{k=0}^{\infty} \frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}\\ &+ \Lambda_n^N\ (-1)^{n-1}(\alpha+1)_{n-1}\sum_{k=0}^{\infty}\frac{(-n+1)_k}{(\alpha+1)_k}\frac{x^k}{k!}.\end{aligned}$$

By a straightforward calculation, $\widehat{Q}_n^{\alpha,c,N}(x)$ can be written as

$$\widehat{Q}_n^{\alpha,c,N}(x) = (-1)^n(\alpha+1)_n\sum_{k=0}^{\infty}\left(\frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}\left[1-\frac{(n-k)\Lambda_n^N}{n(\alpha+n)}\right]\right). \tag{39}$$

Next, we rewrite the expression in square brackets as

$$1-\frac{(k-n)\Lambda_n^N}{(-n)(\alpha+n)} = \frac{\Lambda_n^N}{n(\alpha+n)}\left(k+e_n^N\right) = \frac{\Lambda_n^N}{n(\alpha+n)}\,e_n^N\,\frac{(1+e_n^N)_k}{(e_n^N)_k}, \tag{40}$$

where

$$e_n^N = \frac{n(n+\alpha-\Lambda_n^N)}{\Lambda_n^N}.$$

By replacing (40) into (39), we obtain

$$\begin{aligned}\widehat{Q}_n^{\alpha,c,N}(x) &= \frac{(-1)^n(\alpha+1)_n\,\Lambda_n^N\,e_n^N}{n(\alpha+n)}\sum_{k=0}^{\infty}\left(\frac{(-n)_k}{(\alpha+1)_k}\,\frac{(1+e_n^N)_k}{(e_n^N)_k}\,\frac{x^k}{k!}\right)\\ &= \left(1-\frac{\Lambda_n^N}{n+\alpha}\right)(-1)^n(\alpha+1)_n\,{}_2F_2\left(\begin{matrix}-n,\ 1+e_n^N\\ \alpha+1,\ e_n^N\end{matrix};x\right).\end{aligned}$$

This completes the proof of the theorem. $\square$

Remark 5. The hypergeometric functions ${}_2F_2$ are solutions to a third-order differential equation [19, 16.8.3]. Therefore, Theorem 4 implies that the perturbed polynomials $\widehat{Q}_n^{\alpha,c,N}$ are solutions to

$$x^2y''' - x(x - e_n^N - \alpha - 2))y'' - ((e_n^N - n + 2)x - (\alpha + 1)e_n^N)y' + n(e_n^N + 1)y = 0. \tag{41}$$

This differential equation can be easily obtained from the holonomic equation for the polynomials $\widehat{Q}_n^{\alpha,c,N}$ (see [1, Section 4.1]):

$$y'' + R(x)y' + S(x)y = 0, \tag{42}$$

where

$$R(x) = -\frac{\Lambda_n^N}{x\Lambda_n^N + (n - \Lambda_n^N)(n + \alpha - \Lambda_n^N)} + \frac{\alpha + 1}{x} - 1,$$
$$S(x) = \frac{x\Lambda_n^N + (n - \Lambda_n^N)(n + \alpha)}{x(x\Lambda_n^N + (n - \Lambda_n^N)(n + \alpha - \Lambda_n^N))} + \frac{n - 1}{x}.$$

Indeed, (41) is obtained by multiplying the derivative of (42) by x^2 and adding it to (42) multiplied by $-x(x - e_n^N - \alpha - 2) - x^2R(x)$.

Acknowledgments

The financial support from the project MTM2012-36732-C03-01 (A. Deaño and E. J. Huertas) and the project MTM2012-34787 (A. Deaño), both from Ministerio de Economía y Competitividad of Spain, is gratefully acknowledged.

P. Román was partially supported by CONICET grant PIP 112-200801-01533 and by SeCyT-UNC.

The authors thank Paco Marcellán (Universidad Carlos III, Madrid) for useful and stimulating discussions on the topic and scope of this paper.

The authors thank the anonymous referee of the article for his/her corrections and remarks, that have lead to an improved version of the manuscript.

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