Sobolev orthogonal polynomials in the complex plane

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Abstract

Sobolev orthogonal polynomials with respect to measures supported on compact subsets of the complex plane are considered. For a wide class of such Sobolev orthogonal polynomials, it is proved that their zeros are contained in a compact subset of the complex plane and their asymptotic zero distribution is studied. We also find the nth root asymptotic behavior of the corresponding sequence of Sobolev orthogonal polynomials.

§1. Introduction

1. Let $\{\mu_k\}_{k=0}^m$ be a set of $m+1$ finite positive Borel measures. For each $k = 0, \ldots, m$ the support $S(\mu_k)$ of μ_k is a compact subset of the complex plane $\mathbb{C}$. We will assume that $S(\mu_0)$ contains infinitely many points. If p, q are polynomials, we define

$$\langle p, q \rangle_S = \sum_{k=0}^m \int p^{(k)}(x) \overline{q^{(k)}(x)} d\mu_k(x) = \sum_{k=0}^m \langle p^{(k)}, q^{(k)} \rangle_{L_2(\mu_k)} . \tag{1}$$

As usual, $f^{(k)}$ denotes the kth derivative of a function f and the bar complex conjugation. Obviously, (1) defines an inner product on the linear space of all polynomials. Therefore, a unique sequence of monic orthogonal polynomials is associated to it with a representative of each degree. By Q_n, we will denote the corresponding monic orthogonal polynomial of degree n. The sequence $\{Q_n\}$ is called the sequence of general monic Sobolev orthogonal polynomials relative to (1).

Sobolev orthogonal polynomials have attracted considerable attention in the past decade, but only recently there has been a breakthrough in the study of their asymptotic properties for sufficiently general classes of defining measures. In this connection, we call attention to the papers [4], [5], and [7], in which the first results of general character

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where obtained regarding nth root, ratio, and strong asymptotics, respectively, of Sobolev orthogonal polynomials with respect to measures supported on the real line. The corresponding problems for the case when the measures are supported on arbitrary compact subsets of the complex plane has not yet been studied.

In this paper, we deal with the nth root asymptotic behavior; therefore, we will only comment on [4]. In that paper, for measures supported on the real line and with $m = 1$, the authors assume that $\mu_0, \mu_1 \in \mathbf{Reg}$ (in the sense defined in [10]) and that their supports are regular sets with respect to the solution of the Dirichlet problem. Under these conditions, they find the asymptotic zero distribution of the zeros of the derivatives of the Sobolev orthogonal polynomials and of the proper sequence of Sobolev orthogonal polynomials when additionally it is assumed that $S(\mu_0) \supset S(\mu_1)$. In [6], these questions were considered for arbitrary m and additional information was obtained on the location of the zeros which allowed to derive the nth root asymptotic behavior of the Sobolev orthogonal polynomials outside a certain compact set.

The object of the present paper is to extend the results of [6] to the case when the measures involved in the inner product are supported on compact subsets of the complex plane. Under a certain domination assumption on the measures involved in the Sobolev inner product, we prove in section 2 that the zeros of general Sobolev orthogonal polynomials are contained in a compact subset of the complex plane. For Sobolev inner products on the real line, we also study in section 2 the case when the supports of the measures are mutually disjoint and give a sufficient condition for the boundedness of the zeros of the Sobolev orthogonal polynomials. Section 3 is dedicated to the study of the asymptotic zero distribution and nth root asymptotic behavior of general Sobolev orthogonal polynomials. For this purpose, methods of potential theory are employed.

2. In order to state the corresponding results, let us fix some assumptions and additional notation. As above, (1) defines an inner product on the space P of all polynomials. The norm of $p \in$ P is

$$\|p\|_S = \left(\sum_{k=0}^m \int |p^{(k)}(x)|^2 \, d\mu_k(x)\right)^{1/2} = \left(\sum_{k=0}^m \|p^{(k)}\|_{L_2(\mu_k)}^2\right)^{1/2}. \tag{2}$$

We say that the Sobolev inner product (1) is **sequentially dominated** if

$$S(\mu_k) \subset S(\mu_{k-1}), \qquad k = 1, \ldots, m,$$

and

$$d\mu_k = f_{k-1} d\mu_{k-1}, \qquad f_{k-1} \in L_\infty(\mu_{k-1}), \qquad k = 1, \ldots, m.$$

For example, if all the measures in the inner product are equal, then it is sequentially dominated. The concept of sequentially dominated Sobolev inner product was introduced in [6] for the real case (when the supports of the measures are contained in the real line).

Theorem 1 *Assume that the Sobolev inner product* (1) *is sequentially dominated, then for each $p \in P$ we have that*

$$\|xp\|_S \leq C\|p\|_S \tag{3}$$

where

$$C \leq (2[C_1^2 + (m+1)^2 C_2])^{1/2}, \tag{4}$$

and

$$C_1 = \max_{x \in S(\mu_0)} |x| \,, \qquad C_2 = \max_{k=0,\cdots,m-1} \|f_k\|_{L_\infty(\mu_k)} \,.$$

As usual, two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a given normed space E are said to be equivalent if there exist positive constants c_1, c_2 such that

$$c_1 \|x\| \le \|x\| \le c_2 \|x\| \,, \qquad x \in E.$$

If a Sobolev inner product defines a norm on P which is equivalent to that defined by a sequentially dominated Sobolev inner product, we say that the Sobolev inner product is **essentially sequentially dominated**. It is immediate from the previous Theorem that a Sobolev inner product which is essentially sequentially dominated also satisfies (3) (in general, with a constant C different from (4)). Whenever (3) takes place, we say that the multiplication operator is bounded on the space of all polynomials. This property implies in turn the uniform boundedness of the zeros of Sobolev orthogonal polynomials.

Theorem 2 *Assume that for some positive constant C, we have that $\|xp\|_S \le C\|p\|_S, p \in P$. Then, all the zeros of the Sobolev orthogonal polynomials are contained in the disk $\{z : |z| \le C\}$. In particular. this is true if the Sobolev inner product is essentially sequentially dominated.*

In a recent paper, see Theorem 4.1 in [9] (for related questions see also [1]), the author proves for a large class of Sobolev inner products supported on the real line that the boundedness of the multiplication operator implies that the corresponding Sobolev norm is essentially sequentially dominated. Therefore, in terms of the boundedness of the multiplication operator on the space of polynomials, we cannot obtain more information on the uniform boundedness of the zeros of the Sobolev orthogonal polynomials than that expressed in the theorem above. It is well known that in the case of usual orthogonality the uniform boundedness of the zeros implies that the multiplication operator is bounded. In general, this is not the case for Sobolev inner products as the following result illustrates. In the sequel, $Co(K)$ denotes the convex hull of a compact set K.

Theorem 3 *For $m = 1$, assume that $S(\mu_0)$ and $S(\mu_1)$ are contained in the real line and*

$$\mathrm{Co}(S(\mu_0)) \cap \mathrm{Co}(S(\mu_1)) = \emptyset.$$

Then, for all $n \ge 1$ the zeros of Q'_n are simple, contained in the interior of $\mathrm{Co}(S(\mu_0) \cup S(\mu_1))$, and the zeros of the Sobolev orthogonal polynomials lie in the disk centered at the extreme point of $\mathrm{Co}(S(\mu_1))$ further away from $S(\mu_0)$ and radius equal to the diameter of $\mathrm{Co}(S(\mu_0) \cup S(\mu_1))$.

The statements of this theorem will be complemented below. As Theorem 3 clearly indicates, Theorem 2 is far from giving an answer to the question of uniform boundedness of the zeros of Sobolev orthogonal polynomials. The main question remains; that is, prove or disprove that for any Sobolev inner product compactly supported the zeros of the corresponding Sobolev orthogonal polynomials are uniformly bounded. This question is of vital importance in the study of the asymptotic behaviour of Sobolev orthogonal polynomials.

3. We mention some concepts needed to state the result on the nth root asymptotic behaviour of Sobolev orthogonal polynomials. For any polynomial q of exact degree n, we denote

$$\nu(q) := \frac{1}{n} \sum_{j=1}^{n} \delta_{z_j} \, ,$$

where $z_1, \ldots, z_n$ are the zeros of q repeated according to their multiplicity, and δ_{z_j} is the Dirac measure with mass one at the point z_j. This is the so called normalized zero counting measure associated with q. In [10], the authors introduce a class **Reg** of regular measures. For measures supported on a compact set of the complex plane, they prove that (see Theorem 3.1.1) $\mu \in$ **Reg** if and only if

$$\lim_{n \to \infty} \|Q_n\|_{L_2(\mu)}^{1/n} = cap(S(\mu))$$

where Q_n denotes the nth monic orthogonal polynomials (in the usual sense) with respect to μ and $cap(S(\mu))$ denotes the logarithmic capacity of $S(\mu)$. In case that $S(\mu)$ is a regular compact set with respect to the solution of the Dirichlet problem on the unbounded connected component of the complement of $S(\mu)$ in the extended complex plane, the measure μ belongs to **Reg** (see Theorem 3.2.3 in [10]) if and only if

$$\lim_{n \to \infty} \left(\frac{\|p_n\|_{S(\mu)}}{\|p_n\|_{L_2(\mu)}} \right)^{1/n} = 1 \tag{5}$$

for every sequence of polynomials $\{p_n\}, \deg p_n \leq n, p_n \not\equiv 0$. Here and in the following, $\| \cdot \|_{S(\mu)}$ denotes the supremum norm on $S(\mu)$.

Set

$$\Delta = \cup_{k=0}^{m} S(\mu_k) \, .$$

We call this set the support of the Sobolev inner product. Denote by $g_\Omega(z, \infty)$ the Green's function of the region Ω with singularity at infinity where Ω is the unbounded connected component of the complement of Δ in the extended complex plane. When Δ is regular then the Green's function is continuous up to the boundary and we extend it continuously to all of $\mathbb{C}$ assigning it the value zero on the complement of Ω. By ω_Δ we denote the equilibrium measure of Δ. Assume that there exists $l \in \{0, \ldots, m\}$ such that $\cup_{k=0}^{l} S(\mu_k) = \Delta$, where $S(\mu_k)$ is regular, and $\mu_k \in$ **Reg** for $k = 0, \ldots, l$. Under these assumptions, we say that the Sobolev inner product (1) is l-**regular**.

The next result is inspired in Theorem 1 and Corollary 3 of [4].

Theorem 4 *Let the Sobolev inner product* (1) *be l-regular. Then for each fixed $k = 0, \ldots, l$ and for all $j \geq k$*

$$\overline{\lim_{n \to \infty}} \|Q_n^{(j)}\|_{S(\mu_k)}^{1/n} \leq cap(\Delta) \, . \tag{6}$$

For all $j \geq l$

$$\lim_{n \to \infty} \|Q_n^{(j)}\|_{\Delta}^{1/n} = cap(\Delta). \tag{7}$$

Furthermore, if the interior of Δ is empty and its complement connected, then for all $j \geq l$

$$\lim_{n \to \infty} \nu(Q_n^{(j)}) = \omega_\Delta \tag{8}$$

in the weak star topology of measures.

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The following example illustrates that (8) is not a direct consequence of (7). On the unit circle, take $\mu_j, j = 0, \cdots, m$, equal to the Lebesgue measure. This Sobolev inner product is 0-regular and thus (7) takes place for all $j \geq 0$. Obviously, $\{z^n\}$ is the corresponding sequence of monic Sobolev orthogonal polynomials whose sequence of normalized zero counting measures converges in the weak star topology to the Dirac measure with mass one at zero. Under (7), (8) takes place if it is known that $\lim_{n \to \infty} \nu(Q_n^{(j)})(A) = 0$ for every compact set A contained in the union of the bounded components of $\mathbb{C} \setminus S(\omega_\Delta)$ (see Theorem 2.1 in [2]). But finding general conditions on the measures involved in the inner product which would guarantee this property is, in general, an open problem already in the case of usual orthogonality.

If the inner product is sequentially dominated, then $S(\mu_0) = \Delta$; therefore, if $S(\mu_0)$ and μ_0 are regular the corresponding inner product is 0-regular. In the sequel, $\mathbb{Z}_+ = \{0, 1, \ldots\}$. An immediate consequence of Theorems 2 and 4 is the following.

Theorem 5 *Assume that for some positive constant C, we have that $\|xp\|_S \leq C\|p\|_S, p \in P$ and that the Sobolev inner product is l-regular. Then, for all $j \geq l$*

$$\overline{\lim_{n \to \infty}} |Q_n^{(j)}(z)|^{1/n} \leq \mathrm{cap}(\Delta)e^{g_\Omega(z;\infty)}, \qquad z \in \mathbb{C}. \tag{9}$$

Furthermore,

$$\lim_{n \to \infty} |Q_n^{(j)}(z)|^{1/n} = \mathrm{cap}(\Delta)e^{g_\Omega(z;\infty)}, \tag{10}$$

uniformly on each compact subset of $\{z : |z| \leq C\} \cap \Omega$. Finally, if the interior of Δ is empty and its complement connected we have equality in (9) for all $z \in \mathbb{C}$ except for a set of capacity zero, $S(\omega_\Delta) \subset \{z : |z| \leq C\}$, and

$$\lim_{n \to \infty} \frac{Q_n^{(j+1)}(z)}{nQ_n^{(j)}(z)} = \int \frac{d\omega_\Delta(x)}{z - x}, \tag{11}$$

uniformly on compact subsets of $\{z : |z| \leq C\}$.

These results will be complemented in the sections below. In the rest of the paper, we maintain the notations and definitions introduced above.

§2. Zero location

We fix an inner product of the form (1). For simplicity in the notation, we write

$$\langle \cdot, \cdot \rangle_{L_2(\mu_k)} = \langle \cdot, \cdot \rangle_k , \qquad \| \cdot \|_{L_2(\mu_k)} = \| \cdot \|_k.$$

Proof of Theorem 1. Take C_1 and C_2 as in the statement of this theorem. Straightforward calculations lead to the estimates

$$\|xp\|_S^2 = \sum_{k=0}^m \|(xp)^{(k)}\|_k^2 = \sum_{k=0}^m \|xp^{(k)} + kp^{(k-1)}\|_k^2 \leq 2\sum_{k=0}^m (\|xp^{(k)}\|_k^2 + k^2\|p^{(k-1)}\|_k^2) \leq$$

$$2\sum_{k=0}^m (C_1^2\|p^{(k)}\|_k^2 + k^2C_2\|p^{(k-1)}\|_{k-1}^2) \leq 2[C_1^2 + (m+1)^2C_2]\sum_{k=0}^m \|p^{(k)}\|_k^2 = C^2\|p\|_S^2 ,$$

which imply (3) with C as indicated in (4). ∎

Proof of Theorem 2. Let Q_n denote the nth Sobolev orthogonal polynomial. Since it cannot be orthogonal to itself it is of degree n. Let x_0 denote one of its zeros. It is obvious that there exists a polynomial q of degree $n-1$ such that $xq = x_0 q + Q_n$. Since Q_n is orthogonal to q, and using the boundednes of the multiplication operator, we obtain

$$|x_0| \|q\|_S = \|x_0 q\|_S \leq \|xq\|_S \leq C \|q\|_S.$$

Simplifying $\|q\|_S (\neq 0)$ in the inequality above, we obtain the bound claimed on $|x_0|$ independent of n. The rest of the statements follow from Theorem 1.
∎

Now, let us consider the special case refered to in Theorem 3. For the proof of the corresponding result we need some auxiliary lemmas. Let I be a given interval of the real line (open or closed) and q a polynomial. By $c(q; I)$ and $\kappa(q; I)$ we denote the number of zeros and the number of changes of sign respectively that the polynomial q has on the interval I.

Lemma 1 *Let I be an interval of the real line and q a polynomial such that $\deg q = l \geq 1$. We have that*
$$c(q; I) + c(q'; \mathbb{C} \setminus I) \leq l \,.$$

Proof. By Rolle's Theorem, it follows that

$$c(q; I) \leq c(q'; I) + 1.$$

Therefore,

$$c(q; I) + c(q'; \mathbb{C} \setminus I) \leq c(q'; I) + 1 + c(q'; \mathbb{C} \setminus I) = c(q'; \mathbb{C}) + 1 = l$$

as we wanted to proof. ∎

As above, let Q_n denote the nth monic Sobolev orthogonal polynomial with respect to (1) where all the measures are supported on the real line.

Lemma 2 *Assume that $n \geq 1$. Then*

$$\kappa(Q_n; (\mathrm{Co}(S(\mu_0)))^o) \geq 1.$$

Proof. If, to the contrary, Q_n does not change sign on the indicated set, we immediately obtain a contradiction from the fact that Q_n is orthogonal to 1 since then

$$0 = \langle Q_n, 1 \rangle_S = \int Q_n(x) \, d\mu_0(x) \neq 0 \,.$$

∎

Unless otherwise stated, in the rest of this section we restrict our attention to the case presented in Theorem 3. That is, $m = 1$, the supports of μ_0 and μ_1 are contained in the real line and their convex hulls do not intersect. By $(Co(S(\mu_k)))^o$ we denote the interior of the convex hull of $S(\mu_k)$ with the usual euclidean topology on $\mathbb{R}$.

Lemma 3 *Under the hypothesis of Theorem 3, for $n \geq 1$, we have that*

$$\kappa(Q_n; (\mathrm{Co}(S(\mu_0)))^o) + \kappa(Q'_n; (\mathrm{Co}(S(\mu_1)))^o) \geq n - 1 . \tag{12}$$

Proof. For $n = 1, 2$ the statement follows from Lemma 2. Let $n \geq 3$ and assume that (12) does not hold. That is

$$\kappa(Q_n; (Co(S(\mu_0)))^o) + \kappa(Q'_n; (Co(S(\mu_1)))^o) = l \leq n - 2 . \tag{13}$$

Without loss of generality, we can assume that

$$Co(S(\mu_0)) = [a, b], \qquad Co(S(\mu_1)) = [c, d], \qquad b < c.$$

This reduction is always possible by means of a linear change of variables.

Let x_0 be the point in (a, b) closest to $[c, d]$ where Q_n changes sign. This point exists due to Lemma 2. There are two possibilities, either

$$Q'_n(x_0 + \epsilon) \cdot Q'_n(c + \epsilon) > 0 \tag{14}$$

for all sufficiently small $\epsilon > 0$, or

$$Q'_n(x_0 + \epsilon) \cdot Q'_n(c + \epsilon) < 0 \tag{15}$$

for all sufficiently small $\epsilon > 0$. Let us consider separately each case.

Assume that (14) holds. Let q be a polynomial of degree $\leq l$ with real coefficients, not identically equal to zero, which has a zero at each point of (a, b) where Q_n changes sign and whose derivative has a zero at each point of (c, d) where Q'_n changes sign. The existence of such a polynomial q reduces to solving a system of l equations on $l + 1$ unknowns (the coefficients of q). Thus a non trivial solution always exists. Notice that

$$l \leq c(q; (a, b)) + c(q'; (c, d))$$

with strict inequality if either q (resp. q') has on (a, b) (resp. (c, d)) zeros of multiplicity greater than one or distinct from those assigned by construction. On the other hand, because of Lemma 2 the degree of q is at least 1; therefore, using Lemma 1, we have that

$$c(q; (a, b)) + c(q'; (c, d)) \leq \deg q \leq l.$$

The last two inequalities imply that

$$l = c(q; (a, b)) + c(q'; (c, d)) = \deg q .$$

Hence, qQ_n and $q'Q'_n$ have constant sign on $[a, b]$ and $[c, d]$ respectively. We can choose q in such a way that $qQ_n \geq 0$ on $[a, b]$ (if this was not so replace q by $-q$). With this selection, for all sufficiently small $\epsilon > 0$, we have that $q'(x_0 + \epsilon)Q'_n(x_0 + \epsilon) > 0$. All the zeros of q' are contained in $(a, x_0) \cup (c, d)$, so q' preserves its sign all along the interval $(x_0, c + \epsilon)$, for all sufficiently small $\epsilon > 0$. On the other hand, we are in case (14) where Q'_n has the same sign to the right of x_0 and of c. Therefore, $q'Q'_n \geq 0$ on $[c, d]$. Since $\deg q \leq n - 2$, using orthogonality we obtain the contradiction

$$0 = \int q(x)Q_n(x)d\mu_0(x) + \int q'(x)Q'_n(x)d\mu_1(x) > 0.$$

So, (14) cannot take place if (13) is true.

Let us assume that we are in situation (15). The difference consists in that to the right of x_0 and c the polynomial Q'_n has different signs. Notice (see (13)) that we have at least one degree of freedom left to use orthogonality. Here, we construcrt q of degree $\leq l+1$ with real coefficients and not identically equal to zero with the same interpolation conditions as above plus $q'(c) = 0$. Following the same line of reasoning as above we have that qQ_n and $q'Q'_n$ preserve their sign on $[a, b]$ and $[c, d]$ respectively. Taking q so that $qQ_n \geq 0$ on $[a, b]$ one can see that also $q'Q'_n \geq 0$ on $[c, d]$. Since $\deg q = l + 1 \leq n - 1$, using orthogonality we obtain that (15) is not possible under (13). But either (14) or (15) must hold, thus (12) takes place. ■

Corollary 1 *Set* $I = \text{Co}(S(\mu_0) \cup S(\mu_1)) \setminus (\text{Co}(S(\mu_0)) \cup \text{Co}(S(\mu_1)))$. *Under the conditions of Theorem 3, we have that*
$$c(Q_n; I) + c(Q'_n; I) \leq 1.$$

Proof. It is an immediate consequence of Lemmas 1 and 3 applied to Q_n. ■

Proof of Theorem 3. We will employ the notation introduced for the proof of Lemma 3. According to Lemmas 1 and 3
$$n - 1 \leq l = \kappa(Q_n; (a, b)) + \kappa(Q'_n; (c, d)) \leq n .$$

If $l = n$, then by Rolle's Theorem we have that all the zeros of Q'_n are simple and contained in $(a, b) \cup (c, d)$ which implies our first statement.

Suppose that $l = n - 1$. We consider the same two cases (14) and (15) analyzed in the proof of Lemma 3. Following the arguments used in the proof of Lemma 3 it is easy to see that (14) is not possible with $l = n - 1$. If (15) takes place, then Q'_n has an extra zero in the interval $[x_0, c]$ and again by use of Rolle's Theorem we have that all the zeros of Q'_n are simple and contained in (a, d).

In order to prove the second part of Theorem 3 we use the following remarkable result known as Grace's Apollarity Theorem (we wish to thank T. Erdelyi for drawing our attention to this simple proof of the second statement). Let q be a polynomial of degree greater or equal to two. Take any two zeros of q in the complex plane and draw the straight line which cuts perpendicularly the segment joining the two zeros at its middle point. Then q' has at least one zero in each of the closed half planes in which the line divides the complex plane. For the proof of this result see Theorem 1.4.7 in [8] (see also pp. 23-24 of [3]).

For $n = 1$ the second statement is certainly true because from Lemma 1 we know that for all $n \geq 1, Q_n$ has a zero on (a, b). Let $n \geq 2$. If Q_n had a zero outside the circle with center at d and radius equal to $|a - d|$, from Grace's Apollarity Theorem Q'_n would have a zero outside the segment (a, d) which contradicts the first statement of the theorem. Therefore all the zeros of Q_n lie in the indicated set. ■

Remark 1. The arguments used in the proof of Lemma 3 allow to deduce some other interesting properties which resemble those satisfied by usual orthogonal polynomials. For example, the interval joining any two consecutive zeros of Q_n on (a, b) intersects $S(\mu_0)$. Analogously, the interval joining any two consecutive zeros of Q'_n on (c, d) intersects $S(\mu_1)$. In order to prove this notice that if any one of these statements were not true then in

the construction of the polynomial q in Lemma 3 we can disregard the corresponding zeros which gives us some extra degrees of freedom to use orthogonality and arrive to a contradiction as was done there. From the proof of Theorem 3 it is also clear that the zeros of Q_n in (a, b) are simple and interlace the zeros of Q'_n on that set.

Remark 2. The key to the proof of Theorem 3 is Lemma 1. It's role is to guarantee that in the construction of q in Lemma 3 no extra zeros of q or q' fall on (a, b) or (c, d) respectively. Lemma 1 can be used in order to cover more general Sobolev inner products supported on the real line as long as the supports of the measures appear in a certain order. To be more precise, following essentially the same ideas we can prove the following result.

Consider a Sobolev inner product (1) supported on the real line such that for each $k = 0, \ldots, m - 1$

$$Co(\cup_{j=0}^k S(\mu_j)) \cap S(\mu_{k+1}) = \emptyset.$$

Then for all $n \geq m$ the zeros of $Q_n^{(m)}$ are simple and they are contained in the interior of $Co(\cup_{j=0}^m S(\mu_j))$. The zeros of $Q_n^{(j)}, j = 0, \ldots, m - 1$ lie in the disk centered at z_0 and radius equal to $3^{m-j}r$, where z_0 is the center of the interval $Co(\cup_{j=0}^m S(\mu_j))$ and r is equal to the half the length of that interval.

For $m = 1$ this statement is weaker than that contained in Theorem 3 regarding the location of the zeros of the Q_n because in the present conditions we allow that the support of $S(\mu_1)$ have points on both sides of $Co(S(\mu_0))$.

§3. Regular asymptotic zero distribution

For the proof of Theorem 4, we need the following lemma which is proved in [6] and is easy to verify.

Lemma 4 *Let E be a compact regular subset of the complex plane and $\{P_n\}$ a sequence of polynomials such that $\deg P_n \leq n$ and $P_n \not\equiv 0$. Then, for all $k \in \mathbb{Z}_+$,*

$$\varlimsup_{n \to \infty} \left(\frac{\|P_n^{(k)}\|_E}{\|P_n\|_E} \right)^{1/n} \leq 1 . \tag{16}$$

The guidelines for the proof of the next result are essentially contained in [4] (for m=1) and [6].

Proof of Theorem 4. We start out showing that

$$\varlimsup_{n \to \infty} \|Q_n\|_S^{1/n} \leq cap(\Delta) . \tag{17}$$

Since each of the sets $S(\mu_k), k = 0, \ldots, l$ is regular, so is the support Δ of the Sobolev inner product. Let T_n denote the monic Chebyshev polynomial of degree n for the set Δ. It is well known that $\lim_{n \to \infty} \|T_n\|_\Delta^{1/n} = cap(\Delta)$. Then, by Lemma 4, for all $j \in \mathbb{Z}_+$

$$\varlimsup_{n \to \infty} \|T_n^{(j)}\|_\Delta^{1/n} \leq cap(\Delta) . \tag{18}$$

Therefore, by the minimizing property of the Sobolev norm of the polynomial Q_n, we have

$$\|Q_n\|_S^2 \leq \|T_n\|_S^2 = \sum_{k=0}^m \|T_n^{(k)}\|_k^2 \leq \sum_{k=0}^m \mu_k(S(\mu_k))\|T_n^{(k)}\|_\Delta^2 \ .$$

This estimate, together with (18), gives (17).

From the regularity of the measure μ_k and of its support (see (5)), we know that for each $k = 0, \ldots, l$

$$\lim_{n \to \infty} \left(\frac{\|Q_n^{(k)}\|_{S(\mu_k)}}{\|Q_n^{(k)}\|_k} \right)^{1/n} = 1 \ . \tag{19}$$

Since

$$\|Q_n^{(k)}\|_k \leq \|Q_n\|_S \ ,$$

(17) and (19) imply

$$\varlimsup_{n \to \infty} \|Q_n^{(k)}\|_{S(\mu_k)}^{1/n} \leq cap(\Delta) \ . \tag{20}$$

Taking into consideration Lemma 4, relation (6) follows from (20).

If $j \geq l$, (6) takes place for each $k = 0, \ldots, l$. Since

$$\|Q_n^{(j)}\|_\Delta = \max_{k=0,\ldots,l} \|Q_n^{(j)}\|_{S(\mu_k)} \ ,$$

using (6), we obtain

$$\varlimsup_{n \to \infty} \|Q_n^{(j)}\|_\Delta^{1/n} \leq cap(\Delta) \ .$$

On the other hand

$$\varliminf_{n \to \infty} \|Q_n^{(j)}\|_\Delta^{1/n} \geq cap(\Delta)$$

is true for any sequence $\{Q_n\}$ of monic polynomials. Hence (7) follows.

If the compact set Δ has empty interior and connected complement, it is well known (see Theorem 2.1 in [2]) that (7) implies (8). ∎

Remark 3. We wish to point out that in Theorem 4 eventually some of the measures $\mu_k, k = 2, \ldots, m-1$, may be the null measure in which case μ_k and $S(\mu_k) = \emptyset$ are considered to be regular and $\|Q_n^{(j)}\|_\emptyset = 0$. With these conventions Theorem 4 remains in force.

The so called discrete Sobolev orthogonal polynomials have attracted particular attention in the past years. They are of the form

$$\langle f, g \rangle_S = \int f(x)\overline{g(x)} \, d\mu_0(x) + \sum_{i=1}^m \sum_{j=0}^{N_i} A_{i,j} f^{(j)}(c_i)\overline{g^{(j)}(c_i)} \ . \tag{21}$$

where $A_{i,j} \geq 0, A_{i,N_i} > 0$. If any of the points c_i lie in the complement of the support $S(\mu_0)$ of μ_0, the corresponding Sobolev inner product cannot be l-regular. Nevertheless, a simple modification of the proof of Theorem 4 allows to consider this case.

Theorem 6 *Let the discrete Sobolev inner product (21) be such that $S(\mu_0)$ is regular, and $\mu_0 \in$ **Reg**. Then, (7) takes place for all $j \geq 0$, with $\Delta = S(\mu_0)$ and so does (8) under the additional assumption that $S(\mu_0)$ has empty interior and connected complement.*

Proof. Let T_n denote the nth monic Chebyshev polynomial with respect to $S(\mu_0)$. Set

$$w(z) = \prod_{i=1}^{m} (z - c_i)^{N_i+1} .$$

Let $N = \deg w$, and take $n \geq N$. Then,

$$\|Q_n\|_0^2 \leq \|Q_n\|_S^2 \leq \|wT_{n-N}\|_S^2 = \int |wT_{n-N}|^2 \, d\mu_0 \leq \mu_0(S(\mu_0))\|w\|_{S(\mu_0)}^2 \|T_{n-N}\|_{S(\mu_0)}^2 .$$

Since $\mu_0(S(\mu_0))\|w\|_{S(\mu_0)}^2 > 0$ does not depend on n, we find that

$$\varlimsup_{n\to\infty} \|Q_n\|_0^{1/n} \leq C(S(\mu_0)) .$$

From the regularity of the measure μ_0, it follows that

$$\varlimsup_{n\to\infty} \|Q_n\|_{\Delta_0}^{1/n} \leq C(S(\mu_0)) .$$

Using the regularity of the compact set $S(\mu_0)$ and Lemma 4 (for $E = S(\mu_0)$), we obtain that

$$\varlimsup_{n\to\infty} \|Q_n^{(j)}\|_{\Delta_0}^{1/n} \leq C(S(\mu_0)) ,$$

for all $j \geq 0$. This inequality is necessary and sufficient in order that (7) takes place (with $\Delta = S(\mu_0)$), which in turn implies (8). ∎

Proof of Theorem 5. Fix $j \in \mathbb{Z}_+$. Set

$$v_n(z) = \frac{1}{n} \log \frac{|Q_n^{(j)}(z)|}{\|Q_n^{(j)}\|_\Delta} - g_\Omega(z; \infty).$$

Let us show that

$$v_n(z) \leq 0, \qquad z \in \mathbb{C} \cup \{\infty\} \tag{22}$$

This function is subharmonic in the Ω and on the boundary of Ω it is ≤ 0. By the maximum principle for subharmonic functions it is ≤ 0 on all Ω. On the complement of Ω we also have that $v_n(z) \leq 0$ because by definition (and the regularity of Δ) Green's function is identically equal to zero on this set and the other term which defines v_n is obviously at most zero using the maximum principle of analytic functions. These remarks imply (9) taking upper limit in (22) and using (7) (for this inequality no use is made of the boundedness of the multiplication operator on P).

From Theorem 2, we have that for all $n \in \mathbb{Z}_+$, the zeros of the Sobolev orthogonal polynomials are contained in $\{z : |z| \leq C\}$. It is well known that the zeros of the derivative of a polynomial lie in the convex hull of the set of zeros of the polynomial itself. Therefore, the zeros of $Q_n^{(j)}$ for all $n \in \mathbb{Z}_+$ lie in $\{z : |z| \leq C\}$. Using this, we have that $\{v_n\}$ forms a family of uniformly bounded harmonic functions on each compact subset of $\{z : |z| > C\} \cap \Omega$ (including infinity). Take a sequence of indexes Λ such that $\{v_n\}_{n\in\Lambda}$ converges uniformly on each compact subset of $\{z : |z| > C\} \cap \Omega$. Let v_Λ denote its limit. Obviously, v_Λ is harmonic and ≤ 0 in $\{z : |z| > C\} \cap \Omega$ and because of (7) $v_\Lambda(\infty) = 0$. Therefore, $v_\Lambda \equiv 0$ in $\{z : |z| > C\} \cap \Omega$. Since this is true for every convergent subsequence

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of $\{v_n\}$, we get that the whole sequence converges uniformly on each compact subset of $\{z : |z| > C\} \cap \Omega$ to zero. This is equivalent to (10)

If additionally the interior of Δ is empty and its complement connected, we can use (8). Since the measures $\nu_{n,j} = \nu(Q_n^{(j)}), n \in \mathbb{Z}_+$, and ω_Δ have their support contained in a compact subset of $\mathbb{C}$. Using this and (8), from the Lower Envelope Theorem (see page 223 in [10]), we obtain

$$\varliminf_{n \to \infty} \int \log \frac{1}{|z - x|} \, d\nu_{n,j}(x) = \int \log \frac{1}{|z - x|} \, d\omega_\Delta(x) \ ,$$

for all $z \in \mathbb{C}$ except for a set of zero capacity. This is equivalent to having equality in (9) except for a set of capacity zero because (see page 7 in [10])

$$g_\Omega(z; \infty) = \log \frac{1}{cap(\Delta)} - \int \log \frac{1}{|z - x|} \, d\omega_\Delta(x) \ .$$

Let $x_{n,i}^j, i = 1, \ldots, n - j$, denote the $n - j$ zeros of $Q_n^{(j)}$. As mentioned above, all these zeros are contained in $\{z : |z| \leq C\}$. From (8), each point of $S(\omega_\Delta$ must be a limit point of zeros of $\{Q_n^{(j)}\}$; therefore, $S(\omega_\Delta \subset \{z : |z| \leq C\}$ Decomposing in simple fractions and using the definition of $\nu_{n,j}$, we obtain

$$\frac{Q_n^{(j+1)}(z)}{nQ_n^{(j)}(z)} = \frac{1}{n} \sum_{i=1}^{n-j} \frac{1}{z - x_{n,i}^j} = \frac{n-j}{n} \int \frac{d\nu_{n,j}(x)}{z - x} \ . \tag{23}$$

Therefore, for each fixed $j \in \mathbb{Z}_+$, the family of functions

$$\left\{ \frac{Q_n^{(j+1)}(z)}{nQ_n^{(j)}(z)} \right\} \ , \quad n \in \mathbb{Z}_+ \ , \tag{24}$$

is uniformly bounded on each compact subset of $\{z : |z| > C\}$.

On the other hand, all the measures $\nu_{n,j}, n \in \mathbb{Z}_+$, are supported in $\{z : |z| \leq C\}$ and for $z, |z| > C$, fixed, the function $(z - x)^{-1}$ is continuous on $\{x : |x| \leq C\}$ with respect to x. Therefore, from (8) and (23), we find that any subsequence of (24) which converges uniformly on compact subsets of $\{z : |z| > C\}$ converges pointwise to $\int (z - x)^{-1} \, d\omega_\Delta(x)$. Thus, the whole sequence converges uniformly on compact subsets of $\{z : |z| > C\}$ to this function as stated in (11). ∎

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