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QUANTUM ALGEBRAS $SU_Q(2)$ AND $SU_Q(1,1)$ ASSOCIATED WITH CERTAIN Q-HAHN POLYNOMIALS: A REVISITED APPROACH*

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Abstract. This contribution deals with the connection of q-Clebsch-Gordan coefficients (q-CGC) of the Wigner-Racah algebra for the quantum groups $SU_q(2)$ and $SU_q(1,1)$ with certain q-Hahn polynomials. A comparative analysis of the properties of these polynomials and $su_q(2)$ and $su_q(1,1)$ Clebsch-Gordan coefficients shows that each relation for q-Hahn polynomials has the corresponding partner among the properties of q-CGC and vice versa. Consequently, special emphasis is given to the calculations carried out in the linear space of polynomials, i.e., to the main characteristics and properties for the new q-Hahn polynomials obtained here by using the Nikiforov-Uvarov approach [29, 30] on the non-uniform lattice $x(s)=\frac{q^s-1}{q-1}$. These characteristics and properties will be important to extend the q-Hahn polynomials to the multiple case [7]. On the other hand, the aforementioned lattice allows to recover the linear one $x(s)=s$ as a limiting case, which doesn't happen in other investigated cases [14, 16], for example in $x(s)=q^{2s}$. This fact suggests that the q-analogues presented here (both from the point of view of quantum group theory and special function theory) are *'good'* ones since all characteristics and properties, and consequently, all matrix element relations will converge to the standard ones when q tends to 1.

Key words. Clebsch-Gordan coefficients, discrete orthogonal polynomials (q-discrete orthogonal polynomials), Nikiforov-Uvarov approach, quantum groups and algebras

AMS subject classifications. 81R50, 33D45, 33C80

The author dedicates this contribution to the Russian (Soviet) physicists and mathematicians A. F. Nikiforov and Yu. F. Smirnov for their scientific works in this area. Both are prize winners for research works. The first one was awarded with the Lenin Prize in 1962 while the second one was recently awarded (in 2002) with the Lomonosov Prize.

1. Introduction.

The theory of quantum groups is a very fascinating subject at the border of many areas, mainly group theory, differential and difference equations and special function among others. Consequently, many techniques have been developed and are available; therefore there is some productive competition between various approaches to the subject. The crucial role that the group representation theory concepts play as effective tool to unify independent areas of Physics and Mathematics such as the Quantum Theory and the Special Function Theory (Orthogonal Polynomials) is magnificently presented in [44, 45]. This role has inspired the main goal of this contribution, i.e., the study of the interrelation between these two areas by constructing a q-analogue of the Hahn polynomials using the Nikiforov-Uvarov approach. In such a way, a useful and fruitful parallelism for the study of q-Clebsch-Gordan coefficients and q-Hahn polynomials is established (see Section 4). Moreover, the construction presented here for the q-Hahn polynomials is a very simple one and it corresponds to the standard theory of orthogonal polynomials i.e., the orthogonality conditions are considered with respect to a regular linear functional [15]. Indeed, the regularity condition for certain modifications of the linear functionals [5] is the corner-stone that will allow the extension of the quantum algebras studied here.

During the last years the study of q-discrete analogues of the classical orthogonal polynomials and the connection with the representation theory of quantum algebras in relation with several applications has received an increasing interest (see for instance [6, 9, 17, 23, 27] as well as [39]-[40]). The notion of quantum groups and algebras (Hopf algebras [18]) appears as a consecuence of the study of solutions of the Yang-Baxter equation [26] and the development of the quantum inverse problem method [19]. Their applications in quantum and

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statistical physics were practically immediate [43], especially for the study of the q-deformed oscillator [9, 27], description of the rotational and vibrational spectra of deformed nuclei [10, 12, 32] and diatomic molecules [3, 11, 13]. As a consequence, a big amount of material can be gathered under the title 'quantum algebras', and therefore, any communication on this topic that serves both as a survey and research paper -like this communication- must necessarily be summarized. This partially answers to the question about the dedicatory of this contribution, since here are summarizing some ideas of both Researches. Despite the contributions of other authors we have focused the attention of the reader in the techniques and methods developed by A. F. Nikiforov and Yu. F. Smirnov. However, a suitable list of references is included.

The knowledge of CGC, Racah coefficients ($6j$ symbols) and $9j$ symbols is essential for understanding the corresponding quantum physical problem since all matrix elements of the physical quantities are proportional to them (see [36, 37]). Based on this well known fact we investigate the relation between the Clebsch-Gordan coefficients –also known as $3j$ symbols– [42] for the quantum algebras $su_q(2)$ and $su_q(1,1)$ with q-analogues of the Hahn polynomials on the non-uniform lattice $x(s) = \frac{q^s-1}{q-1}$. Although several authors have investigated the connection between different constructions of the Wigner-Racah algebras for the q-groups and q-algebras and the polynomials orthogonal with respect to discrete measures (see [4, 16, 22, 25, 28, 33, 34, 45]), the resulting properties for such polynomials could not have in return classical analogues. For instance, the connection of CGC with some q-analogue of the Hahn polynomials for $x(s) = q^{2s}$ was studied in [16], however to recover the parameters involved in different characterizations for these polynomials as well as their connection with standard group representation is not possible as a simple limiting case ($q \to 1$) since many of these parameters under this limiting operation disappear (go to zero); consequently the corresponding physical relations lose sense (see Section 5). The same could happen with some q-analogues of the Kravchuk and Meixner polynomials for $x(s) = q^{2s}$ in connection with the Wigner D-functions and Bargmann D-functions for $su_q(2)$ and $su_q(1,1)$ algebras that was established in [14]. These questions should be study in forthcoming publications.

Therefore a natural question appears: What q-analogues in connection with physical problems should be constructed? Recently, in [2] based on the Charlier case the authors show that the lattice $x(s) = q^s$ is not a proper choice in the sense that to recover the linear ones $x(s) = s$ is not possible by taking limits as we have already mentioned. Instead, a good choice is $x(s) = \frac{q^s-1}{q-1}$, such that when q tends to one we recover the classical linear lattice, and the constructed polynomials are 'good' q-analogues of the classical ones. Moreover, all characteristics related to such q-polynomials will tend to the classical ones for the aforementioned limit case ($q \to 1$). This fact, is crucial when we are dealing with physical concepts instead of mathematical ones.

The present communication is written in a narrative way without loss of mathematical rigor. We have stressed simplicity of ideas at the expense of a formal mathematical presentation. Consequently, the customary sequences of definitions, lemmas and theorems will be omitted; nevertheless a mathematical rigor in all the arguments has been kept. Thus, the presentation of the results is self-contained and mostly elementary. As a consequence the only prerequisites are a good understanding of the fundamentals of group theory and orthogonal polynomials. The structure of the paper is as follows. In Section 2 we summarize some necessary formulas and relations concerning $su_q(2)$ quantum algebras and q-analogues of the CGC. In Section 3 we quickly sketch the more important aspects of the Nikiforov-Uvarov approach. This will facilitate the willpower of the reader for understanding the next calculations carried out in Section 4 in order to obtain the main data of the q-Hahn polynomials. It is important to remark that the aforementioned approach is connected with the solution of the eigenvalue problem for the second order finite difference equation on a quite general lattice. The solutions for such an equation have several properties similar to the solutions of the Schrödinger equation. Thus, this approach is close to the standard quantum mechanical methods since it allows to use rather fruitfully the physical intuition in the analysis of appeared problems. The Section 4 contains a lot of calculations so the author kindly ask the reader to be patient during the reading of this section. In fact, a lack in [29] will be covered since in most of the papers related to the Nikiforov-Uvarov approach the authors did not study concrete families (up to the papers [2, 4, 33]). Actually, a non-included in [29] expression involving the difference derivatives of the q-Hahn polynomials with the polynomials itself (difference recurrence relation) is obtained. Furthermore, a comparative analysis of the properties of the q-Hahn polynomials and q-CGC is also given. Hence, the interpretation of the representation theory for $su_q(2)$ and $su_q(1,1)$ in terms of the q-Hahn polynomials is established. The Section 5 is devoted to a brief presentation of the q-Hahn polynomials in the exponential lattice $x(s) = q^s$ in order to emphasize the 'improper' choice of such a lattice for the construction of physical q-analogues. Finally, the Section 6 includes some conclusions and remarks.

2. q-Clebsch-Gordan coefficients for the $su_q(2)$ quantum algebra. The aim of this section is to prepare all the necessary results concerning the q-Clebsch-Gordan coefficients [41] for the comparison with the properties of q-Hahn polynomials given in the Section 4 below.

In [35, 36] the authors define the quantum algebra $su_q(2)$ in the standard way using the three generators $\mathcal{J}_0$, $\mathcal{J}_+$ and $\mathcal{J}_-$ with the usual properties

$$[\mathcal{J}_0, \mathcal{J}_\pm] = \pm\mathcal{J}_\pm, \quad [\mathcal{J}_+, \mathcal{J}_-] = [2\mathcal{J}_0]_q,$$
$$\mathcal{J}_0^\dagger = \mathcal{J}_0, \quad \mathcal{J}_\pm^\dagger = \mathcal{J}_\mp,$$

where

$$[\mathcal{J}_\nu, \mathcal{J}_\upsilon] = \mathcal{J}_\nu\mathcal{J}_\upsilon - \mathcal{J}_\upsilon\mathcal{J}_\nu, \quad \nu,\upsilon = \{+,-,\pm,\mp,0\}, \quad [n]_q = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{\kappa_q}, \quad q = e^{\frac{h}{2}},$$

and

$$\kappa_q = q^{\frac{1}{2}} - q^{-\frac{1}{2}}. \tag{2.1}$$

The above expression $[2\mathcal{J}_0]_q$ means the corresponding infinite formal series.

For $su_q(2)$ quantum algebra the irreducible representations D^j with the highest weight $j = 0, \frac{1}{2}, 1, \ldots$ are determined by the highest weight vector $|jj\ \rangle_q$, such that

$$\mathcal{J}_+\, |jj\ \rangle_q = 0, \quad \mathcal{J}_0\, |jj\ \rangle_q = j\, |jj\ \rangle_q\,, \quad \langle jj|jj\rangle_q = 1,$$

and

$$|jm\ \rangle_q = \sqrt{\frac{[j+m]_q!}{[2j]_q![j-m]_q!}}\, (\mathcal{J}_-)^{j-m}\, |jj\ \rangle_q\,, \quad \text{with} \quad -j \le m \le j.$$

Here we have used the symmetric q-factorial symbol $[n]_q!$ which can be expressed through the symmetric q-Gamma function $\tilde{\Gamma}_q(s)$ defined in [29] and that is connected with the classical q-Gamma function $\Gamma_q(s)$ [20, 29]. Indeed, by [29]

$$\tilde{\Gamma}_q(s) = q^{-\frac{(s-1)(s-2)}{4}}\Gamma_q(s),$$

and

$$\Gamma_q(s) = \begin{cases} f(s;q) = (1-q)^{1-s} \dfrac{\prod_{k\geq 0}(1-q^{k+1})}{\prod_{k\geq 0}(1-q^{s+k})}, & 0<q<1, \\ q^{\frac{(s-1)(s-2)}{2}} f(s;q^{-1}), & q>1. \end{cases} \tag{2.2}$$

Notice that $\tilde{\Gamma}_q(s+1) = [s]_q \tilde{\Gamma}_q(s)$ and $[n]_q! = \tilde{\Gamma}_q(n+1)$ for nonnegative integer n.

On the other hand, if $\mathcal{D}^{j_1}$ and $\mathcal{D}^{j_2}$ denote the two irreducible representation of the quantum algebra $su_q(2)$, then the tensor product can be decomposed into the direct sum

$$\mathcal{D}^{j_1} \otimes \mathcal{D}^{j_2} = \sum_{j=|j_1-j_2|}^{j_1+j_2} \oplus D^j,$$

usually known as Clebsch-Gordan series. Its generators (co-products) are given by the expressions

$$\begin{gathered} \mathcal{J}_0(1,2) = \mathcal{J}_0(1) + \mathcal{J}_0(2), \\ \mathcal{J}_\pm(1,2) = q^{\frac{1}{2}\mathcal{J}_0(2)} \mathcal{J}_\pm(1) + q^{-\frac{1}{2}\mathcal{J}_0(1)} \mathcal{J}_\pm(2). \end{gathered}$$

Taking into account the explicit form of the irreducible representation

$$\begin{aligned} \langle jm'|\mathcal{J}_0|jm\rangle_q &= \delta_{m,m'}, \\ \langle jm'|\mathcal{J}_\pm|jm\rangle_q &= \sqrt{[j\mp m]_q[j\pm m+1]_q}\,\delta_{m',m+1}, \end{aligned} \tag{2.3}$$

as well as the Casimir operator

$$\begin{gathered} \mathcal{C}_2 = \mathcal{J}_-\mathcal{J}_+ + [\mathcal{J}_0 + \tfrac{1}{2}]_q^2, \\ \mathcal{C}_2 \left|jm\ \right\rangle_q = [j+\tfrac{1}{2}]_q^2 \left|jm\ \right\rangle_q, \end{gathered} \tag{2.4}$$

one can define the q-CGC in a similar way to the classical case. Thus, for the basis vectors of the irreducible representations $\mathcal{D}^j$ we have

$$\begin{gathered} \left|j_1j_2, jm\ \right\rangle_q = \sum_{m_1,m_2} \langle j_1m_1j_2m_2, jm\rangle_q \left|j_1m_1\ \right\rangle_q \left|j_2m_2\ \right\rangle_q, \\ \mathcal{C}_2(12)\left|j_1j_2, jm\ \right\rangle_q = [j+\tfrac{1}{2}]_q^2 \left|j_1j_2, jm\ \right\rangle_q, \end{gathered} \tag{2.5}$$

where $\langle j_1m_1j_2m_2, jm\rangle_q$ denotes the q-Clebsch-Gordan coefficients. Notice that for these coefficients the following orthogonality conditions

$$\sum_{m_1,m_2} \langle j_1m_1j_2m_2, jm\rangle_q \langle j_1m_1j_2m_2, j'm'\rangle_q = \delta_{j,j'}\delta_{m,m'},$$

$$\sum_{j,m} \langle j_1m_1j_2m_2, jm\rangle_q \langle j_1m_1'j_2m_2', jm\rangle_q = \delta_{m_1,m_1'}\delta_{m_2,m_2'}.$$

hold. Here $\delta_{k,l}$ represents the Kronecker delta symbol.

Using the Casimir operator (2.4) as well as the expression (2.5) the matrix elements $\langle j_1m_1j_2m_2|\mathcal{C}_2(1,2)|j_1j_2jm\rangle_q$ can be computed. From these matrix elements one deduces

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the following three-term recurrence relation in m_1, m_2 for the q-Clebsch-Gordan coefficients [33, 35]

$$
\begin{aligned}
&q^{-1}\sqrt{[m_2-j_2-1]_q[j_2+m_2]_q[m_1-j_1]_q[j_1+m_1+1]_q}\,\langle j_1 m_1+1 j_2 m_2-1|jm\rangle_q \\
&+\sqrt{[m_2-j_2]_q[j_2+m_2+1]_q[j_1+m_1]_q[m_1-j_1-1]_q}\,\langle j_1 m_1-1 j_2 m_2+1|jm\rangle_q \\
&+\left(q^{-m_1}[j_2+m_2+1]_q[j_2-m_2]_q+q^{m_2}[j_1+m_1+1]_q[j_1-m_1]_q\right. \\
&\left.+[j+\tfrac{1}{2}]_q^2-[m+\tfrac{1}{2}]_q^2\right)q^{-\frac{1}{2}(m_2-m_1+1)}\langle j_1 m_1 j_2 m_2|jm\rangle_q=0.
\end{aligned}
\tag{2.6}
$$

Based on the symmetry property for the q-Clebsch-Gordan coefficients

$$
\langle j_1 m_1 j_2 m_2|jm\rangle_q=(-1)^{j_1+j_2-j}\langle j_2 m_2 j_1 m_1|jm\rangle_{q^{-1}}, \tag{2.7}
$$

the expression (2.6) is invariant with respect to the change j_1 by j_2 whereas q is replaced by q^{-1}.

Following the same ideas used for the computation of (2.6), but in this case for the matrix element $\langle j_1 m_1 j_2 m_2|\mathcal{J}_0(2)|j_1 j_2, jm\rangle_q$ one gets a similar three-term recurrence relation with respect to the variable j [38]

$$
\begin{aligned}
&\sqrt{\frac{[j-m]_q[j+m]_q[j_1+j_2+j+1]_q[j_2-j_1+j]_q[j-j_2+j_1]_q[j_1+j_2-j+1]_q}{[2j+1]_q[2j-1]_q[2j]_q^2}}\,\langle j_1 m_1 j_2 m_2|j-1m\rangle_q \\
&+\sqrt{\frac{[j-m+1]_q[j+m+1]_q[j_1+j_2+j+2]_q[j_2-j_1+j+1]_q[j-j_2+j_1+1]_q}{[2j+3]_q[2j]_q[2j+2]_q^2[j_1+j_2-j]_q^{-1}}}\,\langle j_1 m_1 j_2 m_2|j+1m\rangle_q \\
&+\left(\frac{[2j]_q[2j_1+2]_q-[2]_q[j_1+j_2-j+1]_q[j-j_1+j_2]_q}{[2]_q[2j]_q[2j+2]_q}\left(q^{-\frac{1}{2}(j+1)}[j+m]_q-q^{\frac{1}{2}(j+1)}[j-m]_q\right)\right. \\
&\left.-\frac{q^{-\frac{m_2}{2}}}{[2]_q}\left(q^{-\frac{1}{2}(j_1+1)}[j_1+m_1]_q-q^{\frac{1}{2}(j_1+1)}[j_1-m_1]_q\right)\right)\langle j_1 m_1 j_2 m_2|jm\rangle_q=0.
\end{aligned}
$$

Finally, from (2.3) a straightforward calculation leads to

$$
\langle j_1 m_1 j_2 m_2|\mathcal{J}_\pm|j_1 j_2, jm\rangle_q=\sqrt{[j\mp m]_q[j\pm m+1]_q}\,\langle j_1 m_1 j_2 m_2|jm\rangle_q. \tag{2.8}
$$

3. Nikiforov-Uvarov approach. A polynomial sequence $\{P_n(x)\}_{n\geq 0}$ orthogonal with respect to a positive measure μ on the real line is such that P_n has degree n and satisfies the conditions

$$
\int_\Omega P_n(x)x^k\,d\mu(x)=0,\quad k=0,1,\ldots,n-1,\quad \Omega\subset\Re.
$$

This defines the polynomial up to a multiplicative factor. In the case of discrete orthogonal polynomials, we have a discrete measure μ (with finite moments)

$$
\mu=\sum_{k=0}^{N}\rho_k\delta_{x_k},\quad \rho_k>0,\ x_k\in\Re \text{ and } N\in\{1,2,\ldots\}\cup\{+\infty\},
$$

which is a linear combination of Dirac measures on the $N+1$ points $x_0,\ldots,x_N$. The orthogonality conditions of a discrete orthogonal polynomial P_n on the lattice $\{x(s)\mapsto\Re^+:$

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$s = 0, 1, \ldots, N\}$ are usually written as

$$\sum_{s=0}^{N} P_n(x(s)) x^k(s)\, \rho(s) = 0, \quad k = 0, 1, \ldots, n-1.$$

There exist several mathematical methods and approaches to the study of classical polynomials orthogonal with respect to a discrete measure, usually known as discrete orthogonal polynomials [29]. Nevertheless, for physicists perhaps the more easy method to get in touch with these kind of mathematical objects (discrete orthogonal polynomials) is via the discretization of the first and second derivatives $y'(x)$ and $y''(x)$ involved in the hypergeometric differential equation [29]

$$\tilde{\sigma}(x) y''(x) + \tilde{\tau}(x) + \lambda y(x) = 0,$$

where $\deg \tilde{\sigma} \leq 2$, $\deg \tilde{\tau} = 1$ and $\lambda = const$. This method is usually known as Nikiforov-Uvarov approach.

Thus, the corresponding hypergeometric-type difference equation is [29]

$$\begin{gathered} \sigma(s) \frac{\triangle}{\triangle x(s-\frac{1}{2})} \frac{\nabla y(s)}{\nabla x(s)} + \tau(s) \frac{\triangle y(s)}{\triangle x(s)} + \lambda y(s) = 0, \\ \sigma(s) = \tilde{\sigma}(x(s)) - \tfrac{1}{2}\tilde{\tau}(x(s)) \triangle x(s-\tfrac{1}{2}), \quad \tau(s) = \tilde{\tau}(x(s)), \end{gathered} \tag{3.1}$$

where $\nabla y(s) = y(s) - y(s-1)$ and $\triangle y(s) = y(s+1) - y(s)$ denote the backward and forward finite difference, respectively.

Of course, any partition $x(s)$ can not guarantee the existence of polynomial solutions of (3.1). In [8, 29] it is shown that $x(s)$ should be of the form

$$x(s) = c_1 q^s + c_2 q^{-s} + c_3, \ (q \in \Re^+ \backslash \{1\}) \quad \text{or} \quad x(s) = as^2 + bs + c,$$

where c_1, c_2, c_3, a, b and c are constants. In fact, the lattice $x(s) = \frac{q^s - 1}{q-1}$ belongs to this class.

The polynomial solutions of (3.1) can be orthogonalized constructing a Sturm-Liouville problem. First, the equation (3.1) is written in the self-adjoint form

$$\frac{\triangle}{\triangle x(s-\frac{1}{2})} \left[\sigma(s)\rho(s) \frac{\nabla y(s)}{\nabla x(s)} \right] + \lambda \rho(s) y(s) = 0, \tag{3.2}$$

for two different polynomial solution of degree n and m, respectively. The function $\rho(s)$ (the so-called symmetrization factor of (3.1)) is the solution of the Pearson-type difference equation [29]

$$\frac{\triangle}{\triangle x(s-\frac{1}{2})} [\sigma(s)\rho(s)] = \tau(s)\rho(s) \text{ where } \rho(s) \triangle x(s-\tfrac{1}{2}) > 0,\ a \leq s \leq b-1. \tag{3.3}$$

Second, the equation (3.2) for polynomial solution of degree n is multiplied by the other polynomial solution of degree m of the same equation and vice versa, and then one subtracts the resulting equations (one from the other). Finally, we sum over $a \leq s \leq b-1$ for which we obtain the orthogonality property (see [29, pages 70-72])

$$\sum_{s=a}^{b-1} P_n(x(s)) P_m(x(s)) \rho(s) \triangle x(s-\tfrac{1}{2}) = \delta_{n,m} ||P_n||^2, \tag{3.4}$$

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under additional (boundary) conditions

$$\sigma(s)\rho(s)x^k(s-\tfrac{1}{2})\big|_{s=a,b}=0, \quad k=0,1,\dots$$

Since we are interested in polynomial solutions the same procedure could be carried out for the k-order finite difference of a solution of (3.1), defined by

$$y_k(x(s))=\frac{\triangle}{\triangle x_{k-1}(s)}\frac{\triangle}{\triangle x_{k-2}(s)}\cdots\frac{\triangle}{\triangle x(s)}y(x(s))\equiv\triangle^{(k)}y(s),$$

where $x_k(s)=x(s+\frac{k}{2})$, because this k-th finite difference verifies a difference equation of the same type (hypergeometric type) [29]

$$\sigma(s)\frac{\triangle}{\triangle x_k(s-\frac{1}{2})}\left[\frac{\nabla y_k(s)_q}{\nabla x_k(s)}\right]+\tau_k(s)\frac{\triangle y_k(s)_q}{\triangle x_k(s)}+\mu_k y_k(s)_q=0, \tag{3.5}$$

being

$$\tau_k(s)=\frac{\sigma(s+k)-\sigma(s)+\tau(s+k)\triangle x(s+k-\frac{1}{2})}{\triangle x_{k-1}(s)} \text{ and } \mu_k=\lambda_n+\sum_{m=0}^{k-1}\frac{\triangle\tau_m(s)}{\triangle x_m(s)}.$$

For (3.5) the symmetrization factor is

$$\rho_k(s)=\rho(s+k)\prod_{i=1}^{k}\sigma(s+i). \tag{3.6}$$

As a consequence of (3.4) the polynomial solutions of (3.1) and (3.5) verify several crucial relations that will help us to find the connection between the q-Clebsch-Gordan coefficients for the quantum algebras and the q-Hahn polynomials. Now we will write some of them (see [29] for more details).

It is well known that the polynomial solutions $P_n(x(s))$ of (3.1) are determined up to a normalizing factor B_n, by means of the discrete analog of the Rodrigues formula [29, Eq. 3.2.19, page 66]

$$P_n(x(s))=\frac{B_n}{\rho(s)}\nabla_0^{(n)}[\rho_n(s)], \quad \nabla_k^{(n)}\equiv\frac{\nabla}{\nabla x_{k+1}(s)}\frac{\nabla}{\nabla x_{k+2}(s)}\cdots\frac{\nabla}{\nabla x_n(s)}, \tag{3.7}$$

where $\rho_n(s)$ is given in (3.6). Notice that these polynomial solutions correspond to certain eigenvalues λ_n (see (3.1)) which can be computed by simple substitution of $P_n(x(s))$ into the equation (3.1) and comparing the coefficients for the powers of $x(s)$, i.e.,

$$\lambda_n=-[n]_q\left(\frac{1}{2}\left(q^{n-1}+q^{-n+1}\right)\tilde{\tau}'+[n-1]_q\frac{\tilde{\sigma}''}{2}\right), \tag{3.8}$$

where (see (3.1)) $\tilde{\sigma}(s)=\frac{\tilde{\sigma}''}{2}x(s)^2+\tilde{\sigma}'(0)x(s)+\tilde{\sigma}(0)$ and $\tilde{\tau}(s)=\tau' x(s)+\tau(0)$.

Similarly, for the k-th finite difference the following Rodrigues-type formula is also valid

$$y_{k,n}(x(s))=\triangle^{(k)}P_n(x(s))=\frac{A_{n,k}B_n}{\rho_k(s)}\nabla_k^{(n)}[\rho_n(s)] \text{ where } B_n=\frac{\triangle^{(n)}P_n}{A_{n,n}}, \tag{3.9}$$

and

$$A_{n,k}=\frac{[n]_q!}{[n-k]_q!}\prod_{m=0}^{k-1}\left(\frac{q^{\frac{1}{2}(n+m-1)}+q^{-\frac{1}{2}(n+m-1)}}{2}\tilde{\tau}'+[n+m-1]_q\frac{\tilde{\sigma}''}{2}\right). \tag{3.10}$$

From (3.7) follows an explicit expression for the polynomials P_n [2]

$$P_n(x(s)) = \frac{B_n q^{-ns+\frac{n}{4}(n+1)}}{c_1^n(q-1)^n}\sum_{m=0}^{n}\frac{[n]_q! q^{-\frac{m}{2}(n-1)}(-1)^{m+n}}{[m]_q![n-m]_q!} \times \prod_{l=0}^{n-m-1}[\sigma(s-l)]\prod_{l=0}^{m-1}[\sigma(s+l)+\tau(s+l)\bigtriangleup x(s+l-\tfrac{1}{2})], \tag{3.11}$$

with the assumption $\prod_{l=0}^{-1} f(l) \equiv 1$. Notice that this expression depends only on the coefficients of σ and τ given in (3.1).

4. q-Hahn polynomials in the non-uniform lattice $x(s) = \frac{q^s-1}{q-1}$ and q-Clebsch-Gordan coefficients. It is well known [29] the relation between Clebsch-Gordan coefficient and Hahn polynomials in the uniform lattice $x(s) = s$. Analogously, in [33] is used a relation (see (4.24) below) for the q-case in the non-uniform lattice $x(s) = q^{2s}$. In fact, we will deduce a similar relation but in the lattice $x(s) = \frac{q^s-1}{q-1}$. In such a way an useful and fruitful parallelism between q-Clebsch-Gordan coefficients for $su_q(2)$ and q-Hahn polynomials will be constructed. Therefore, studying in details the relations and properties satisfied by the q-Hahn polynomials we have the corresponding partner among the relations and properties of $su_q(2)$ Clebsch-Gordan coefficients and vice versa. For such a purpose let first to determine the q-Hahn polynomials as well as their main characteristics based on the aforementioned Nikiforov-Uvarov approach. Second, selecting a special choice of s, N, α, β, and n we will establish the connection between both mathematical objects.

In [30] has been proved that the most general orthogonal polynomial solution of (3.1) in the lattice $x(s) = c_1q^s + c_3$ corresponds to the choice

$$\begin{aligned} \sigma(s) &= \bar{A}(q^{s-\varsigma_1}-1)(q^{s-\varsigma_2}-1), \\ \sigma(s)+\tau(s)\bigtriangleup x(s-\tfrac{1}{2}) &= \bar{A}(q^{s-\bar{\varsigma}_1}-1)(q^{s-\bar{\varsigma}_2}-1). \end{aligned} \tag{4.1}$$

From (3.8) considering (4.1) the following expression for the eigenvalues of (3.1)

$$\lambda_n = -\frac{\bar{A}}{c_1^2}q^{-\frac{1}{2}(\varsigma_1+\varsigma_2+\bar{\varsigma}_1+\bar{\varsigma}_2)}[n]_q[\varsigma_1+\varsigma_2-\bar{\varsigma}_1-\bar{\varsigma}_2+n-1]_q, \tag{4.2}$$

yields. The corresponding weight function is, in this case,

$$\rho(s) = \frac{\Gamma_q(s-\bar{\varsigma}_1)\Gamma_q(s-\bar{\varsigma}_2)}{\Gamma_q(s-\varsigma_1+1)\Gamma_q(s-\varsigma_2+1)}. \tag{4.3}$$

Here $\Gamma_q(s)$ functions are those defined in (2.2).

In this case the corresponding polynomial solution of (3.1) can be represented in terms of the basic hypergeometric series [2, 31]

$$\begin{aligned} P_n(x(s)) &= B_n\left(\frac{\bar{A}}{c_1(q^{\frac{1}{2}}-q^{-\frac{1}{2}})}\right)^n q^{-\frac{n(n-1)}{4}-n\varsigma_1}(q^{\varsigma_1-\bar{\varsigma}_1};q)_n(q^{\varsigma_1-\bar{\varsigma}_2};q)_n \\ &\times {}_3\varphi_2\left(\begin{matrix} q^{-n}, q^{\varsigma_1+\varsigma_2-\bar{\varsigma}_1-\bar{\varsigma}_2+n-1}, q^{\varsigma_1-s} \\ q^{\varsigma_1-\bar{\varsigma}_1}, q^{\varsigma_1-\bar{\varsigma}_2} \end{matrix}; q\,,\, q^{s-\varsigma_2+1}\right), \end{aligned} \tag{4.4}$$

$$\begin{aligned} P_n(x(s)) &= B_n\left(-\frac{\bar{A}}{c_1(q^{\frac{1}{2}}-q^{-\frac{1}{2}})}\right)^n q^{-\frac{3n(n-1)}{4}-n(\varsigma_1+\varsigma_2-\bar{\varsigma}_1)}(q^{\varsigma_1-\bar{\varsigma}_1};q)_n \\ &\times (q^{\varsigma_2-\bar{\varsigma}_1};q)_n\,{}_3\varphi_2\left(\begin{matrix} q^{-n}, q^{\varsigma_1+\varsigma_2-\bar{\varsigma}_1-\bar{\varsigma}_2+n-1}, q^{s-\bar{\varsigma}_1} \\ q^{\varsigma_1-\bar{\varsigma}_1}, q^{\varsigma_2-\bar{\varsigma}_1} \end{matrix}; q\,,\, q\right), \end{aligned} \tag{4.5}$$

where

$$ {}_r\varphi_p\left(\begin{array}{c} a_1, a_2, ..., a_r \\ b_1, b_2, ..., b_p \end{array} ; q\,,\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k \cdots (a_r;q)_k}{(b_1;q)_k \cdots (b_p;q)_k} \frac{z^k}{(q;q)_k} \left[(-1)^k q^{\frac{k}{2}(k-1)}\right]^{p-r+1}, $$

and

$$ (a;q)_k = \prod_{m=0}^{k-1} (1 - aq^m). $$

Let the end points of the orthogonality interval $a = 0$ and $b = N$. Moreover, $c_1 = -c_3 = \frac{1}{q-1}$, i.e., $x(s) = \frac{q^s-1}{q-1}$, and

$$ A_1 = -\frac{q^{\frac{N+\alpha}{2}}}{q-1},\; B_n = \frac{(-1)^{-n}}{[n]_q!},\; \varsigma_1 = 0,\; \varsigma_2 = N+\alpha,\; \bar{\varsigma}_1 = -\beta - 1,\; \bar{\varsigma}_2 = N - 1, \tag{4.6} $$

with the restrictions $\alpha, \beta > -1$ and $n < N$. Taking into account this choice of parameters (4.6) the functions $\sigma(s)$ and $\tau(s)$ take the form

$$ \sigma(s) = -q^{s-1}[s]_q[N+\alpha-s]_q, $$

$$ \sigma(s) + \tau(s) \bigtriangleup x(s - \tfrac{1}{2}) = -q^{s+\frac{\alpha+\beta}{2}}[s+\beta+1]_q[s-N+1]_q. $$

Consequently, from (3.8) or (4.2) the eigenvalues of (3.1) are given by the expression

$$ \lambda_n = q^{\frac{\beta+2-N}{2}}[n]_q[n+\alpha+\beta+1]_q. $$

The solution $\rho(s)$ of the Pearson-type equation (3.3) is

$$ \rho(s) = q^{(\frac{\alpha+\beta}{2})s} \frac{\tilde{\Gamma}_q(s+\beta+1)\tilde{\Gamma}_q(N+\alpha-s)}{\tilde{\Gamma}_q(s+1)\tilde{\Gamma}_q(N-s)}. \tag{4.7} $$

Notice that the same result can be found if we use (4.3) and the relation $\tilde{\Gamma}_q(s) = q^{-\frac{(s-1)(s-2)}{4}}\Gamma_q(s)$.

Considering (4.5) the hypergeometric representation for the q-Hahn polynomials

$$ h_n^{\alpha,\beta}(s,N;q) = \frac{(q^{\beta+1};q)_n (q^{N+\alpha+\beta+1};q)_n}{q^{\frac{n}{2}(n+\alpha+2\beta+N)}(q;q)_n}\, {}_3\varphi_2\left(\begin{array}{c} q^{-n}, q^{n+\alpha+\beta+1}, q^{s+\beta+1} \\ q^{\beta+1}, q^{N+\alpha+\beta+1} \end{array} ; q\,,\, q\right), \tag{4.8} $$

yields. From here follows other useful characteristic of the q-Hahn polynomials, i.e., the value of this polynomials at the end points of the orthogonality interval $[0, N-1]$. Indeed,

$$ \begin{aligned} h_n^{\alpha\beta}(0,N;q) &= (-1)^n \frac{[N-1]_q!\tilde{\Gamma}_q[\beta+n+1]}{[n]_q!\tilde{\Gamma}_q[\beta+1][N-n-1]_q!} q^{\frac{1}{2}n(2\alpha+\beta+N+\frac{1}{2}(n-1))}, \\ h_n^{\alpha\beta}(N-1,N;q) &= \frac{[N-1]_q!\tilde{\Gamma}_q[\alpha+n+1]}{[n]_q![N-n-1]_q!\tilde{\Gamma}_q[\alpha+1]} q^{\frac{1}{2}n(\alpha+N-\frac{1}{2}(n-1))}. \end{aligned} $$

By $||P_n||^2$ we will denote the square norm of P_n. In [29, Chapter 3, Section 3.7.2, page 104] the authors show a convenient way to compute it. In fact,

$$ ||P_n||^2 = (-1)^n A_{n,n} B_n^2 \sum_{s=a}^{b-n-1} \rho_n(s) \bigtriangleup x_n(s - \tfrac{1}{2}). \tag{4.9} $$

TABLE 4.1
Main data for the q-Hahn polynomials in the lattice $x(s)=\frac{q^s-1}{q-1}$.

$P_n(x(s))$	$h_n^{\alpha,\beta}(s,N;q)$
Interval	$[0,N-1]$
$\rho(s)$	$q^{(\frac{\alpha+\beta}{2})s}\dfrac{\tilde\Gamma_q(s+\beta+1)\tilde\Gamma_q(N+\alpha-s)}{\tilde\Gamma_q(s+1)\tilde\Gamma_q(N-s)}$
$\sigma(s)$	$-q^{-\frac{N+\alpha}{2}}x(s)^2+q^{-\frac12}[N+\alpha]_q x(s)$
$\tau(s)$	$-q^{\frac{\beta+2-N}{2}}[\alpha+\beta+2]_q x(s)+q^{\alpha+\beta+1}[\beta+1]_q[N-1]_q$
$\tau_n(s)$	$-q^{\frac{\beta+2-N}{2}}x_n(s)-q^{\frac{\beta+2-N}{2}}[2n+\alpha+\beta+2]_q$
λ_n	$q^{\frac{\beta+2-N}{2}}[n]_q[n+\alpha+\beta+1]_q$
B_n	$\dfrac{(-1)^{-n}}{[n]_q!}$
$\vert\vert P_n\vert\vert^2$	$\dfrac{q^{N(\frac{\alpha+\beta}{2}+N)-\frac{\alpha(\alpha-3)+2}{4}}\tilde\Gamma_q(n+\alpha+1)\tilde\Gamma_q(n+\beta+1)\tilde\Gamma_q(n+\alpha+\beta+N+1)}{q^{n(\frac{\alpha-\beta}{2}+N+1)}[n]_q![N-n-1]_q!\tilde\Gamma_q(n+\alpha+\beta+1)\tilde\Gamma_q(2n+\alpha+\beta+2)}$
$\rho_n(s)$	$q^{s(\frac{\alpha+\beta}{2}+n-1)+\frac{n}{2}(\alpha+\beta+n-1)+1}\dfrac{\tilde\Gamma_q(s+n+\beta+1)\tilde\Gamma_q(N+\alpha-s)}{\tilde\Gamma_q(s+1)\tilde\Gamma_q(N-s-n)}$

Based on (4.9) after some cumbersome calculation we arrive to the expression,

$$\left\|h_n^{\alpha,\beta}(s,N;q)\right\|^2 = \frac{q^{N\left(\frac{\alpha+\beta}{2}+N\right)-\frac12}\tilde\Gamma_q(\alpha+n+1)}{q^{n\left(\frac{\alpha-\beta}{2}+N+1\right)+\frac{\alpha(\alpha-3)}{4}}[n]_q![N-n-1]_q!} \times\frac{\tilde\Gamma_q(\beta+n+1)\tilde\Gamma_q(\alpha+\beta+N+n+1)}{\tilde\Gamma_q(\alpha+\beta+n+1)\tilde\Gamma_q(\alpha+\beta+2n+2)}. \tag{4.10}$$

Finally, after the calculations of the very basic characteristics of the q-Hahn polynomials like $\rho(s)$, $\sigma(s)$, $\tau(s)$, λ_n and $\left\|h_n^{\alpha,\beta}(s,N;q)\right\|^2$ we summarize in Table 4.1 the results of the remaining calculations by using the Nikiforov-Uvarov approach.

4.1. Three-term recurrence relation. A simple consequence of (3.4) is [29]

$$x(s)P_n(x(s))=\alpha_nP_{n+1}(x(s))+\beta_nP_n(x(s))+\gamma_nP_{n-1}(x(s)),\quad n\ge 0, \tag{4.11}$$

with $P_{-1}(x(s))=0$ and $P_0(x(s))=1$.

Denoting by a_n and b_n the coefficients in the expansion $P_n(x(s))=a_nx^n(s)+b_nx^{n-1}(s)+\cdots$ we have the following explicit expressions for the recurrence coefficients [2, 29]

$$\alpha_n=\frac{a_n}{a_{n+1}},\qquad \beta_n=\frac{b_n}{a_n}-\frac{b_{n+1}}{a_{n+1}},\qquad \gamma_n=\frac{a_{n-1}}{a_n}\frac{||P_n||^2}{||P_{n-1}||^2}. \tag{4.12}$$

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TABLE 4.2
Three-term recurrence coefficients and leading coefficient for q-Hahn polynomials in the lattice $x(s)=\frac{q^s-1}{q-1}$

Coefficient	Explicit expression
α_n	$q^{-\frac{\beta+2-N}{2}}\dfrac{[n+1]_q[n+\alpha+\beta+1]_q}{[2n+\alpha+\beta+2]_q[2n+\alpha+\beta+1]_q}$
β_n	$q^{\alpha+N+\frac{n}{2}-1}\dfrac{[n+\alpha+\beta+1]_q[n+\beta+1]_q[N-n-2]_q}{[2n+\alpha+\beta+2]_q[2n+\alpha+\beta+1]_q} + \dfrac{q^{-\frac{1}{2}\{2N+\beta+n+3(\alpha+1)\}}[n+\alpha]_q[n+\alpha+\beta+N]_q[N-n]_q[n]_q}{[2n+\alpha+\beta+1]_q[2n+\alpha+\beta]_q^2[2n+\alpha+\beta-1]_q[N-n-1]_q}$
γ_n	$q^{-\frac{N+\alpha}{2}-2}\dfrac{[n+\alpha]_q[n+\beta]_q[n+\alpha+\beta+N]_q[N-n]_q}{[2n+\alpha+\beta+1]_q[2n+\alpha+\beta]_q^2[2n+\alpha+\beta-1]_q}$
a_n	$q^{n\frac{\beta+2-N}{2}}\dfrac{\tilde{\Gamma}_q(2n+\alpha+\beta+1)}{[n]_q\tilde{\Gamma}_q(n+\alpha+\beta+1)}$

To find the leading coefficient a_n it is enough to note that

$$\frac{\triangle x^n(s)}{\triangle x(s)}=[n]_q x^{n-1}(s+\tfrac{1}{2})+\cdots,\quad \text{so}\quad \left(\frac{\triangle}{\triangle x(s)}\right)^n x^n(s)=[n]_q!\,.$$

On the other hand, $\left(\frac{\triangle}{\triangle x(s)}\right)^n P_n(x(s))=[n]_q!a_n$, then using (3.9), $[n]_q!a_n=B_nA_{n,n}$, and taking into account (3.10)

$$a_n=B_n\prod_{k=0}^{n-1}\left(\frac{q^{\frac{1}{2}(n+k-1)}+q^{-\frac{1}{2}(n+k-1)}}{2}\tilde{\tau}'+[n+k-1]_q\frac{\tilde{\sigma}''}{2}\right).$$

Frequently, the computation of b_n as well as β_n is very tedious, then if we know α_n and γ_n, and $P_n(a)\neq 0$ for all n, we can use (4.11). Indeed,

$$\beta_n=\frac{x(a)P_n(a)-\alpha_nP_{n+1}(a)-\gamma_nP_{n-1}(a)}{P_n(a)}.$$

See Table 4.2 to find the result of computations for the coefficients (4.12)

4.2. Other recurrence relations. Using the Rodrigues-type formula (3.9) we find [29]

$$\sigma(s)\frac{\nabla P_n(x(s))}{\nabla x(s)}=\frac{\lambda_n}{[n]_q\tau_n'}\left[\tau_n(s)P_n(x(s))-\frac{B_n}{B_{n+1}}P_{n+1}(x(s))\right]. \tag{4.13}$$

Despite the fact the results shown here are given in [2] we sketch below most of the steps to obtain the so-called structure relations for the corresponding polynomial solutions of (3.1) in the non-uniform lattice $x(s)=c_1q^s+c_3$. Thus, the reader could follow easily all the calculations. For such a purpose we will substitute $\tau_n(s)$ as follows

$$\tau_n(s)=\tau_n'x_n(s)+\tau_n(0)=\tau_n'q^{\frac{n}{2}}x(s)+\tau_n(0)-\tau_n'c_3(q^{\frac{n}{2}}-1),$$

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in (4.13). Then, based on the three-term recurrence relation (4.11) we obtain the *first structure relation*

$$\sigma(s)\frac{\nabla P_n(x(s))}{\nabla x(s)} = \tilde{S}_n P_{n+1}(x(s)) + \tilde{T}_n P_n(x(s)) + \tilde{R}_n P_{n-1}(x(s)), \tag{4.14}$$

where

$$\tilde{S}_n = \frac{\lambda_n}{[n]_q}\left[q^{\frac{n}{2}}\alpha_n - \frac{B_n}{\tau_n' B_{n+1}}\right], \quad \tilde{T}_n = \frac{\lambda_n}{[n]_q}\left[q^{\frac{n}{2}}\beta_n + \frac{\tau_n(0)}{\tau_n'} - c_3(q^{\frac{n}{2}}-1)\right], \tag{4.15}$$
$$\tilde{R}_n = \frac{\lambda_n q^{\frac{n}{2}}\gamma_n}{[n]_q}.$$

To deduce the *second structure relation* we will transform (4.14) with the help of the identity

$$\triangle\frac{\nabla P_n(x(s))}{\nabla x(s)} = \frac{\Delta P_n(x(s))}{\Delta x(s)} - \frac{\nabla P_n(x(s))}{\nabla x(s)}. \tag{4.16}$$

Thus, using $\Delta x(s-\frac{1}{2}) = \kappa_q x(s) - c_3\kappa_q$, as well as (3.1) and (4.11), one gets

$$[\sigma(s) + \tau(s)\triangle x(s-\tfrac{1}{2})]\frac{\Delta P_n(x(s))}{\Delta x(s)} = S_n P_{n+1}(x(s)) + T_n P_n(x(s)) + R_n P_{n-1}(x(s)),$$

where

$$S_n = \tilde{S}_n - \alpha_n\lambda_n\kappa_q, \quad T_n = \tilde{T}_n - \beta_n\lambda_n\kappa_q + c_3\lambda_n\kappa_q, \quad R_n = \tilde{R}_n - \gamma_n\lambda_n\kappa_q,$$

or equivalently, using (4.15),

$$S_n = \frac{\lambda_n}{[n]_q}\left[q^{-\frac{n}{2}}\alpha_n - \frac{B_n}{\tau_n' B_{n+1}}\right], \quad T_n = \frac{\lambda_n}{[n]_q}\left[q^{-\frac{n}{2}}\beta_n + \frac{\tau_n(0)}{\tau_n'} - c_3(q^{-\frac{n}{2}}-1)\right],$$
$$R_n = \frac{\lambda_n q^{-\frac{n}{2}}\gamma_n}{[n]_q}.$$

Now we will find a *difference-recurrence relation* of the form [2]

$$P_n(x(s)) = L_n\frac{\Delta P_{n+1}(x(s))}{\Delta x(s)} + M_n\frac{\Delta P_n(x(s))}{\Delta x(s)} + N_n\frac{\Delta P_{n-1}(x(s))}{\Delta x(s)}, \tag{4.17}$$

being L_n, M_n y N_n constants. Notice that this relation is not include in [29].

Let describe briefly how to prove (4.17). Applying the operator $\frac{\triangle}{\triangle x(s)}$ on both sides of (4.14), and using (3.1) as well as $q^{-\frac{1}{2}}\triangle x(s) = \Delta x(s-\frac{1}{2})$ one obtains

$$\begin{aligned}\left[q^{\frac{1}{2}}\frac{\Delta\sigma(s)}{\Delta x(s)} - \tau(s)\right]\frac{\Delta P_n(x(s))}{\Delta x(s)} - \lambda_n P_n(x(s)) = \\ = q^{\frac{1}{2}}\tilde{S}_n\frac{\Delta P_{n+1}(x(s))}{\Delta x(s)} + q^{\frac{1}{2}}\tilde{T}_n\frac{\Delta P_n(x(s))}{\Delta x(s)} + q^{\frac{1}{2}}\tilde{R}_n\frac{\Delta P_{n-1}(x(s))}{\Delta x(s)}.\end{aligned} \tag{4.18}$$

Since

$$\sigma(s) = \frac{\sigma''}{2}x^2(s) + \sigma'(0)x(s) + \sigma(0), \quad \text{and} \quad \tau(s) = \tau' x(s) + \tau(0),$$

and $x(s+1)=q\,x(s)-c_3(q-1)$ the expression $\frac{\triangle\sigma(s)}{\triangle x(s)}$ is a polynomials $x(s)$ of degree one. Hence,

$$\left[q^{\frac{1}{2}}\frac{\triangle\sigma(s)}{\triangle x(s)}-\tau(s)\right]=Ax(s)+B,$$

where

$$A=\frac{\sigma''}{2}(1+q)q^{\frac{1}{2}}-\tau', \quad \text{and} \quad B=q^{\frac{1}{2}}\sigma'(0)-\frac{\sigma''}{2}c_3q^{\frac{1}{2}}(q-1)-\tau(0).$$

Thus, (4.18) becomes

$$\begin{aligned} &q^{-\frac{1}{2}}\left(Ax(s)\frac{\triangle P_n(x(s))}{\triangle x(s)}-\lambda_n P_n(x(s))\right)= \\ &\tilde{S}_n\frac{\triangle P_{n+1}(x(s))}{\triangle x(s)}+\tilde{T}_n\frac{\triangle P_n(x(s))}{\triangle x(s)}+\tilde{R}_n\frac{\triangle P_{n-1}(x(s))}{\triangle x(s)}-\frac{B}{q^{\frac{1}{2}}}\frac{\triangle P_n(x(s))}{\triangle x(s)}. \end{aligned} \tag{4.19}$$

Now, if we apply the operator $\frac{\triangle}{\triangle x(s)}$ on both sides of (4.11) we can eliminate the term $x(s)\frac{\triangle P_n(x(s))}{\triangle x(s)}$ in (4.19) by using the identity $x(s+1)=q\,x(s)-c_3(q-1)$, i.e.,

$$\begin{aligned} qx(s)\frac{\triangle P_n(x(s))}{\triangle x(s)} &=\alpha_n\frac{\triangle P_{n+1}(x(s))}{\triangle x(s)}+[\beta_n+c_3(q-1)]\frac{\triangle P_n(x(s))}{\triangle x(s)} \\ &+\gamma_n\frac{\triangle P_{n-1}(x(s))}{\triangle x(s)}-P_n(x(s)). \end{aligned}$$

Finally, multiplying (4.19) by q and using the above equation one gets

$$\begin{aligned} P_n(x(s)) &=\frac{\left(\alpha_n A-q^{\frac{3}{2}}\tilde{S}_n\right)}{(A+q\lambda_n)}\frac{\triangle P_{n+1}(x(s))}{\triangle x(s)}+\frac{\left(\gamma_n A-q^{\frac{3}{2}}\tilde{R}_n\right)}{(A+q\lambda_n)}\frac{\triangle P_{n-1}(x(s))}{\triangle x(s)} \\ &+\frac{\left(\beta_n A+c_3A(q-1)+qB-q^{\frac{3}{2}}\tilde{T}_n\right)}{(A+q\lambda_n)}\frac{\triangle P_n(x(s))}{\triangle x(s)}, \end{aligned}$$

which is of the form (4.17) if $A+q\lambda_n\neq 0$. In particular, we have

$$A=q^{\frac{\beta-N+4}{2}}[\alpha+\beta]_q, \quad B=[N+\alpha]_q-q^{-\frac{\alpha+N-1}{2}}-q^{\alpha+\beta+1}[\beta+1]_q[N-1]_q,$$

consequently,

$$A+q\lambda_n=q^{\frac{\beta-N+4}{2}}\left([\alpha+\beta]_q+[n]_q[n+\alpha+\beta+1]_q\right).$$

See Tables 4.3 and 4.4 to find all the recurrence coefficients described in this subsection.

4.3. q-Clebsch-Gordan coefficients. It is relatively easy to verify that the hypergeometric-type difference equation (3.1) for the q-Hahn polynomials is equivalent to the recurrence relation (2.6) for the q-Clebsch-Gordan coefficients with respect to the projection m_1 or m_2 of the angular momentum j_1 or j_2, respectively. For such a purpose let to rewrite (3.1) as a recurrence relation in s, i.e.,

$$\xi(s)y(s+1)+\eta(s)y(s)+\zeta(s)y(s-1)=0, \tag{4.20}$$

TABLE 4.3
Recurrence coefficients for q-Hahn polynomials in the lattice $x(s) = \frac{q^s-1}{q-1}$

Coefficient	Explicit expression
$\tilde{S}_n$	$-q^{-\frac{(n+\alpha+\beta+1)}{2}} \dfrac{[n+1]_q[n]_q[n+\alpha+\beta+1]_q}{[2n+\alpha+\beta+2]_q[2n+\alpha+\beta+1]_q}$
$\tilde{T}_n$	$q^{\frac{\beta+2-N}{2}}[n+\alpha+\beta+1]_q\Big\{\dfrac{[n+\alpha+\beta+1]_q[n+\beta+1]_q[N-n-2]_q}{q^{1-n-\alpha-N}[2n+\alpha+\beta+2]_q[2n+\alpha+\beta+1]_q}$ $+\dfrac{q^{-\frac{1}{2}\{2N+\beta+3(\alpha+1)\}}[n+\alpha]_q[n+\alpha+\beta+N]_q[N-n]_q[n]_q}{[2n+\alpha+\beta+1]_q[2n+\alpha+\beta]_q^2[2n+\alpha+\beta-1]_q[N-n-1]_q}+q^{\frac{n-2}{2}}[\frac{n}{2}]_q$ $+q^{\frac{N-n-\beta-3}{2}}\dfrac{\left(q^{n+1+\frac{\alpha+\beta}{2}}[\frac{n}{2}+\beta+1]_q[\frac{n}{2}-N+1]_q-[\frac{n}{2}]_q[\frac{n}{2}+N+\alpha]_q\right)}{[2n+\alpha+\beta+2]_q}\Big\}$
$\tilde{R}_n$	$q^{\frac{(n+\beta-\alpha-2N-2)}{2}}\dfrac{[n+\alpha]_q[n+\beta]_q[n+\alpha+\beta+1]_q[n+\alpha+\beta+N]_q[N-n]_q}{[2n+\alpha+\beta+1]_q[2n+\alpha+\beta]_q^2[2n+\alpha+\beta-1]_q}$
S_n	$-q^{\frac{n+\alpha+\beta+1}{2}}\dfrac{[n+1]_q[n]_q[n+\alpha+\beta+1]_q}{[2n+\alpha+\beta+2]_q[2n+\alpha+\beta+1]_q}$
T_n	$q^{\frac{\beta+2-N}{2}}[n+\alpha+\beta+1]_q\Big\{\dfrac{[n+\alpha+\beta+1]_q[n+\beta+1]_q[N-n-2]_q}{q^{1-\alpha-N}[2n+\alpha+\beta+2]_q[2n+\alpha+\beta+1]_q}$ $+\dfrac{q^{-\frac{1}{2}\{2N+\beta+2n+3(\alpha+1)\}}[n+\alpha]_q[n+\alpha+\beta+N]_q[N-n]_q[n]_q}{[2n+\alpha+\beta+1]_q[2n+\alpha+\beta]_q^2[2n+\alpha+\beta-1]_q[N-n-1]_q}-q^{-\frac{n+2}{4}}$ $+q^{\frac{N-n-\beta-3}{2}}\dfrac{\left(q^{n+1+\frac{\alpha+\beta}{2}}[\frac{n}{2}+\beta+1]_q[\frac{n}{2}-N+1]_q-[\frac{n}{2}]_q[\frac{n}{2}+N+\alpha]_q\right)}{[2n+\alpha+\beta+2]_q}\Big\}$
R_n	$q^{\frac{\beta-\alpha-n}{2}-N-1}\dfrac{[n+\alpha]_q[n+\beta]_q[n+\alpha+\beta+1]_q[n+\alpha+\beta+N]_q[N-n]_q}{[2n+\alpha+\beta+1]_q[n+\alpha+\beta]_q^2[2n+\alpha+\beta-1]_q}$

TABLE 4.4
Recurrence coefficients for q-Hahn polynomials in the lattice $x(s) = \frac{q^s-1}{q-1}$

Coefficient	Explicit expression
L_n	$\dfrac{q^{\frac{N-n-\beta-2}{2}}[n+1]_q[n+\alpha+\beta+1]_q[n+\alpha+\beta]_q}{[2n+\alpha+\beta+2]_q[2n+\alpha+\beta+1]_q\left([\alpha+\beta]_q+[n]_q[n+\alpha+\beta+1]_q\right)}$
M_n	$\dfrac{1}{([\alpha+\beta]_q+[n]_q[n+\alpha+\beta+1]_q)}\Bigg\{\dfrac{-[n+1]_q}{[2n+\alpha+\beta+1]_q}\Bigg[\dfrac{[n+\alpha+\beta+1]_q}{q^{\frac{1-\alpha-\beta}{2}-\alpha-N-n}}$ $\times\dfrac{[n+\beta+1]_q[N-n-2]_q}{[2n+\alpha+\beta+2]_q}+\dfrac{q^{-N-\alpha-1}[n+\alpha]_q[n+\alpha+\beta+N]_q[N-n]_q[n]_q}{[2n+\alpha+\beta]_q^2[2n+\alpha+\beta-1]_q[N-n-1]_q}\Bigg]$ $+\dfrac{[N+\alpha]_q}{q^{\frac{\beta+2-N}{2}}}+\dfrac{1}{q^{\frac{\alpha+\beta+3}{2}}}-\dfrac{[\beta+1]_q[N-1]_q}{q^{-\alpha-\frac{N+\beta}{2}}}-\dfrac{[\alpha+\beta+2]_q}{q}-\dfrac{[n+\alpha+\beta+1]_q[\frac{n}{2}]_q}{q^{-\frac{n}{4}}}$ $-\dfrac{[n+\alpha+\beta+1]_q\left(q^{n+1+\frac{\alpha+\beta}{2}}[\frac{n}{2}+\beta+1]_q[\frac{n}{2}-N+1]_q-[\frac{n}{2}]_q[\frac{n}{2}+N+\alpha]_q\right)}{q^{\frac{n-N+\beta+2}{2}}[2n+\alpha+\beta+2]_q}\Bigg\}$
N_n	$\dfrac{-q^{\frac{n-N+\beta-3}{2}}[n+1]_q[n+\alpha]_q[n+\beta]_q[n+\alpha+\beta]_q^{-1}[N-n]_q}{[2n+\alpha+\beta+1]_q[2n+\alpha+\beta-1]_q\left([\alpha+\beta]_q+[n]_q[n+\alpha+\beta+1]_q\right)}$

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where

$$\xi(s)=\frac{\sigma(s)+\tau(s)\bigtriangleup x(s-\frac{1}{2})}{\bigtriangleup x(s-\frac{1}{2})\bigtriangledown x(s)},\quad \zeta(s)=\frac{\sigma(s)}{\bigtriangleup x(s-\frac{1}{2})\bigtriangledown x(s)}, \\ \eta(s)=\lambda_n-\zeta(s)-\xi(s). \tag{4.21}$$

Let

$$s=j_1-m_1,\quad N=j_1+j_2-m+1,\quad \alpha=m+j_1-j_2, \\ \beta=m-j_1+j_2,\quad n=j-m. \tag{4.22}$$

Therefore, substituting in the above equations (4.21) the choice (4.22) we have

$$\begin{aligned}
\xi(j_1-m_1) &= -q^{2m_1+m_2-j_1+\frac{1}{2}}[j_2+m_2+1]_q[m_2-j_2]_q,\\
\eta(j_1-m_1) &= q^{m_1-j_1+\frac{1}{2}}\left(q^{m_2}[j-m]_q[j+m+1]_q+[j_1-m_1]_q[j_1+m_1+1]_q\right.\\
&\quad \left.+q^{m_1+m_2}[m_2+j_2+1]_q[m_2-j_2]_q\right),\\
\zeta(j_1-m_1) &= q^{m_1-j_1+\frac{1}{2}}[j_1-m_1]_q[m_1+j_1+1]_q.
\end{aligned}$$

Consequently, the equation (4.20) for the q-Hahn polynomials becomes

$$\begin{gathered}
\xi(j_1-m_1)h_{j-m}^{m+j_1-j_2,m-j_1+j_2}(j_1-m_1+1,j_1+j_2-m+1;q)\\
+\eta(j_1-m_1)h_{j-m}^{m+j_1-j_2,m-j_1+j_2}(j_1-m_1,j_1+j_2-m+1;q)\\
+\zeta(j_1-m_1)h_{j-m}^{m+j_1-j_2,m-j_1+j_2}(j_1-m_1-1,j_1+j_2-m+1;q)=0.
\end{gathered} \tag{4.23}$$

Comparing the recurrence relations (2.6) and (4.23) one deduces

$$\langle j_1m_1j_2m_2|jm\rangle_{q^{-1}}=(-1)^{n+s}\sqrt{\rho(s)\bigtriangleup x(s-\tfrac{1}{2})}\,h_n^{\alpha,\beta}(s,N;q), \tag{4.24}$$

where $h_n^{\alpha,\beta}(s,N;q)$ denotes the orthonormal q-Hahn polynomials and s, N, α, β, n coincide with (4.22). To efficiently carry out a comparison between these two recurrence relations we should consider (4.10) as well as the following useful expressions

$$\begin{aligned}
\frac{\rho(j_1-m_1)}{\rho(j_1-m_1+1)}&=q^{-m}\frac{[j_1+m_1]_q[j_1-m_1+1]_q}{[j_2-m_2]_q[j_2+m_2+1]_q},\quad \frac{\bigtriangleup x(s-\frac{1}{2})}{\bigtriangleup x(s-\frac{3}{2})}=q\\
\frac{\rho(j_1-m_1)}{\rho(j_1-m_1-1)}&=q^{m}\frac{[j_2+m_2]_q[j_2-m_2+1]_q}{[j_1-m_1]_q[j_1+m_1+1]_q},\quad \frac{\bigtriangleup x(s-\frac{1}{2})}{\bigtriangleup x(s+\frac{1}{2})}=q^{-1}.
\end{aligned}$$

For the ratio of weights we have used (4.7).

Notice that (4.24) constitutes an extension –in this case a q-analog– of the well known relation between the classical Hahn polynomials and Clebsch-Gordan coefficients. Furthermore, this expression is in accordance with the results obtained in [33] (see also [1] for more details).

From (3.4) as well as from (4.8) the symmetry property for the q-Hahn polynomials

$$h_n^{\alpha,\beta}(s,N;q)=(-1)^n q^{n(\alpha+\beta+N)}h_n^{\beta,\alpha}(N-s-1,N;q^{-1}),$$

holds. It leads as a consequence the corresponding symmetry property for the q-Clebsch-Gordan coefficients given in (2.7) by a simple substitution of the parameters (4.22). Now, the substitution of the parameters (4.22) into the expression (3.11) with the help of (4.24) gives the q-analog of the Racah formula for $su_q(2)$ Clebsch-Gordan coefficients

$$\begin{aligned}
\langle j_1m_1j_2m_2|jm\rangle_q&=(-1)^{j_1-m_1}\sqrt{\frac{\frac{[2j+1]_q[j-m]_q![j+m]_q![j_1-m_1]_q![j_2-m_2]_q![j_1+j_2-j]_q!}{[j_1+j_2+j+1]_q![j_1-j_2+j]_q![j_2-j_1+j]_q![j_1+m_1]_q![j_2+m_2]_q!}}{q^{-m_1(m+1)+\frac{1}{2}(j(j+1)+j_1(j_1+1)-j_2(j_2+1))}}}\\
&\quad\times\sum_k(-1)^k\frac{q^{\frac{1}{2}k(j+m+1)}[j_1-m_1+k]_q![j+j_2-m_1-k]_q!}{[k]_q![j-m-k]_q![j_1-m_1-k]_q![j_2-j+m_1-k]_q!}.
\end{aligned} \tag{4.25}$$

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Two particular cases are immediately derived from (4.25):

i) First, when $j_1 = m_1$

$$\langle j_1 j_1 j_2 m_2 | jm \rangle_q = \sqrt{\frac{\frac{[2j+1]_q [j+m]_q! [2j_1]_q! [j_2-m_2]_q! [j-j_1+j_2]_q!}{[j_1+j_2+j+1]_q! [j_1+j_2-j]_q! [j_1-j_2+j]_q! [j_2+m_2]_q! [j-m]_q}}{q^{j_1(j-m)-\frac{1}{2}(j_1+j_2-j)(j-j_1+j_2-1)}}},$$

which corresponds to the choice $s=0$ for the q-Hahn polynomials.

ii) Second,

$$\frac{\langle j_1 m_1 j_2 m_2 | jj \rangle_q}{(-1)^{j_1-m_1}} = \sqrt{\frac{\frac{[2j+1]_q! [j_1+m_1]_q! [j_2+m_2]_q! [j_1+j_2-j]_q!}{[j_1+j_2+j+1]_q! [j_1-j_2+j]_q! [j_2-j_1+j]_q! [j_1-m_1]_q! [j_2-m_2]_q!}}{q^{(j+1)(j_1-m_1))-\frac{1}{2}(j_1+j_2-j)(j-j_1+j_2+1))}}},$$

which coincides with the expression (4.24) for $m=j$

$$\frac{\langle j_1 m_1 j_2 m_2 | jj \rangle_q}{h_0^{m+j_1-j_2, m-j_1+j_2}(j_1-m_1, j_1+j_2-m+1; q)} = (-1)^s \sqrt{\rho(s) \bigtriangleup x(s-\tfrac{1}{2})}.$$

From the Rodrigues-type formula (3.7) and the identity (4.16) it is easy to obtain

$$h_{n+1}^{\alpha-1,\beta-1}(s+1,N;q) - h_{n+1}^{\alpha-1,\beta-1}(s,N;q) = q^{s+1+\frac{\alpha+\beta-n}{2}} h_{n+1}^{\alpha,\beta}(s,N;q).$$

Now, if we put in this expression the above selection of parameters (4.22) and consider (4.23) we deduce an interesting relation for the q-Clebsch-Gordan coefficients. Indeed,

$$\begin{array}{c} \sqrt{(j_1 \pm m_1)(j_1 \mp m_1+1)} q^{\frac{m_2}{2}} \langle j_1 m_1 \mp 1 j_2 m_2 | jm \rangle_q \\ + \sqrt{(j_2 \pm m_2)(j_2 \mp m_2+1)} q^{-\frac{m_1}{2}} \langle j_1 m_1 j_2 m_2 \mp | jm \rangle_q = \\ \sqrt{(j \mp m)(j \pm m+1)} \langle j_1 m_1 j_2 m_2 | jm+1 \rangle_q \end{array}$$

which can be also obtained from (2.8).

4.4. Relation between the Clebsch-Gordan coefficients for the quantum algebras $su_q(2)$ and $su_q(1,1)$. It is well known [21] that the quantum algebra $su_q(1,1)$ can be generated from the operators

$$[\mathcal{Q}_0, \mathcal{Q}_\pm] = \pm \mathcal{Q}_\pm, \quad [\mathcal{Q}_+, \mathcal{Q}_-] = -[2\mathcal{Q}_0]_q,$$

$$\mathcal{Q}_0^\dagger = \mathcal{Q}_0, \quad \mathcal{Q}_\pm^\dagger = \mathcal{Q}_\mp,$$

where $[\cdot,\cdot]$ represents as in the Section 2 the commutator product.

The description of this subsection is quite similar to the already presented in the Section 2 as well as [1]. Here, to avoid confusion with the Section 2 we use the notation '*prime*' over the basic vectors $|j'm'\ \rangle_q$, $m' = j'+1, j'+2, \ldots$ These basic vectors of the irreducible representations of the discrete positive series $D^{j'+}$ are determined by $|j'j'+1\ \rangle_q$, such that

$$\mathcal{Q}_- \,|j'j'+1\ \rangle_q = 0,\ \mathcal{Q}_0 \,|j'j'+1\ \rangle_q = (j'+1)\,|j'j'+1\ \rangle_q\,,\ \langle j'j'+1|j'j'+1\rangle_q = 1,$$

and

$$|j'm'\ \rangle_q = \sqrt{\frac{[2j'+1]_q!}{[j'+m']_q! [m'-j'-1]_q!}}\, (\mathcal{Q}_+)^{m'-j'-1} \,|j'j'+1\ \rangle_q\,.$$

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Analogously, the tensor product of the irreducible representation $\mathcal{D}^{j_1'+}$ and $\mathcal{D}^{j_2'+}$ of the quantum algebra $su_q(1,1)$ can be decomposed into the direct sum

$$\mathcal{D}^{j_1'+} \otimes \mathcal{D}^{j_2'+} = \sum_{j'=j_1'+j_2'+1}^{\infty} \oplus D^{j'+}.$$

Its generators (co-products) are given by the expressions

$$\mathcal{Q}_0(1,2) = \mathcal{Q}_0(1) + \mathcal{Q}_0(2),$$
$$\mathcal{Q}_\pm(1,2) = q^{\frac{1}{2}\mathcal{Q}_0(2)}\mathcal{Q}_\pm(1) + q^{-\frac{1}{2}\mathcal{Q}_0(1)}\mathcal{Q}_\pm(2).$$

The explicit form of the irreducible representation

$$\left\langle j'\tilde{m'}|\mathcal{Q}_0|j'm'\right\rangle_q = \delta_{\tilde{m'},m'},$$
$$\left\langle j'\tilde{m'}|\mathcal{Q}_\pm|j'm'\right\rangle_q = \sqrt{[m' \mp j']_q[m' \pm j' \pm 1]_q}\,\delta_{\tilde{m'},m'\pm 1},$$

as well as the Casimir operator

$$\begin{aligned} \mathcal{C}_2 &= -\mathcal{Q}_+\mathcal{Q}_- + [\mathcal{Q}_0]_q[\mathcal{Q}_0 - 1]_q, \\ \mathcal{C}_2\,|j'm'\ \rangle_q &= [j']_q[j'+1]_q\,|j'm'\ \rangle_q\,, \end{aligned} \tag{4.26}$$

are given in a quite similar form to those presented in Section 2. Analogously, the definition of the q-Clebsch-Gordan coefficients is

$$|j_1'j_2',j'm'\ \rangle_q = \sum_{m_1',m_2'} \langle j_1'm_1'j_2'm_2',j'm'\rangle_q\,|j_1'm_1'\ \rangle_q\,|j_2'm_2'\ \rangle_q\,,$$

$$\mathcal{C}_2(12)\,|j_1'j_2',j'm'\ \rangle_q = [j']_q[j'+1]_q\,|j_1'j_2',j'm'\ \rangle_q\,. \tag{4.27}$$

Now, using the Casimir operator (4.26) as well as the expression (4.27) the matrix elements $\langle j_1'm_1'j_2'm_2'|\mathcal{C}_2(1,2)|j_1'j_2',j'm'\rangle_q$ can be found. From these matrix elements (see [16]) the following three-term recurrence relation in m_1', m_2' for the q-Clebsch-Gordan coefficients

$$\begin{aligned} &\sqrt{[m_2'-j_2'-1]_q[j_2'+m_2']_q[m_1'-j_1']_q[j_1'+m_1'+1]_q}\,\langle j_1'm_1'+1j_2'm_2'-1|j'm'\rangle_q \\ &+q\sqrt{[m_2'-j_2']_q[j_2'+m_2'+1]_q[j_1'+m_1']_q[m_1'-j_1'-1]_q}\,\langle j_1'm_1'-1j_2'm_2'+1|j'm'\rangle_q \\ &+\Big(q^{-m_1'}[j_2'+m_2'+1]_q[m_2'-j_2']_q + q^{m_2'}[j_1'+m_1'+1]_q[m_1'-j_1']_q \\ &+[j'+\tfrac{1}{2}]_q^2 - [m'+\tfrac{1}{2}]_q^2\Big)\,q^{-\frac{1}{2}(m_2'-m_1'-1)}\,\langle j_1'm_1'j_2'm_2'|j'm'\rangle_q = 0, \end{aligned} \tag{4.28}$$

yields.

Finally, to conclude this section let establish a useful relation between the q-Clebsch-Gordan coefficients for the $su_q(2)$ and $su_q(1,1)$ algebras, i.e.,

$$\langle j_1m_1j_2m_2|jm\rangle_{su_q(2)} = \langle j_1'm_1'j_2'm_2'|j'm'\rangle_{su_q(1,1)}\,, \tag{4.29}$$

where

$$j_1 = \tfrac{m'+j_1'-j_2'-1}{2}, \quad m_1 = \tfrac{m_1'-m_2'+j_1'+j_2'+1}{2}, \quad j = j'$$
$$j_2 = \tfrac{m'-j_1'+j_2'-1}{2}, \quad m_2 = \tfrac{m_2'-m_1'+j_1'+j_2'+1}{2}, \quad m = j_1'+j_2'+1.$$

The relation (4.29) can be deduced by a mere comparison between the relations (2.6) and (4.28). This result is in accordance with [4, 16].

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TABLE 5.1
Three-term recurrence coefficients for q-Hahn polynomials in the lattice $x(s) = q^s$ [1]

Coefficient	Explicit expression
α_n	$\dfrac{\kappa_q q^{-\frac{1}{2}(\alpha+\beta+1)}[n+1]_q[\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+2]_q[\alpha+\beta+2n+1]_q}$
β_n	$\dfrac{q^{\frac{\alpha-\beta+N-1}{2}}}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+2]_q}\Big\{ ([N-n]_q[n]_q[\alpha+\beta+2n+2] - [N-n-1]_q[n+1]_q[\alpha+\beta+2n]_q) + q^{\frac{\beta-\alpha}{2}}\left(\dfrac{[\alpha+\beta+N+n+1]_q[n+1]_q}{[\alpha+\beta+2n]_q^{-1}} - \dfrac{[\alpha+\beta+N+n]_q[n]_q}{[\alpha+\beta+2n+2]_q^{-1}}\right)\Big\}$
γ_n	$\dfrac{\kappa_q q^{\alpha+N-\frac{1}{2}}[\alpha+n]_q[\beta+n]_q[\alpha+\beta+N+n]_q[N-n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}$

5. Remark on the q-Hahn polynomials for $x(s) = q^s$. Here we will present a brief description on some basic characteristics of these polynomials from the point of view of the Nikiforov-Uvarov approach [29], specially, the three-term recurrence relation. The importance of this relation in connection with the q-CGC was already discussed in the above section. We focus our attention in this relation since when $q \to 1$ the parameters α_n and γ_n clearly go to zero as a consequence of the multiplicative factor κ_q (see the equation (2.1) as well as Table 5.1). Thus, this relation loses sense in this limiting case. Therefore, this comparative analysis reveals that the three-term recurrence relation –for instance– is not a q-analogue relation because it has not the corresponding partner property for the classical Hahn polynomials, which does not happen in the investigated case $x(s) = \frac{q^s-1}{q-1}$ (see Section 4).

The q-Hahn polynomials in the lattice $x(s) = q^s$ have been studied in [1, 2, 33]. Despite the fact that we can use the same scheme presented in Section 4 to obtain them, we will omit it. Below we will summarize few results. Let chose $x(s) = q^s$, i.e., $c_1 = 1$ and $c_3 = 0$. In this case, with the parameters

$$B_n = \frac{(-1)^n}{q^{\frac{n}{2}}\kappa_q^n[n]_q!},$$
$$\varsigma_1 = 0, \quad \varsigma_2 = N+\alpha, \quad \bar{\varsigma}_1 = -\beta-1, \quad \bar{\varsigma}_2 = N-1, \quad A = -q^{\frac{1}{2}(N+\alpha)},$$

we obtain the q-Hahn polynomials [24, 33]. Indeed, from (4.4)-(4.5) we have

$$\begin{aligned} h_n^{\alpha,\beta}(s,N)_q &= \frac{q^{\frac{n}{2}(\alpha+N)}(q^{\beta+1};q)_n(q^{1-N};q)_n}{\kappa_q^n(q;q)_n}\,{}_3\varphi_2\left(\begin{array}{c} q^{-n}, q^{-s}, q^{n+\alpha+\beta+1} \\ q^{\beta+1}, q^{1-N} \end{array}; q, q\right) \\ &= \frac{(q^{\beta+1};q)_n(q^{N+\alpha+\beta+1};q)_n}{q^{-\frac{n}{2}(N+\alpha+2\beta+n+1)}\kappa_q^n(q;q)_n}\,{}_3\varphi_2\left(\begin{array}{c} q^{-n}, q^{s+\beta+1}, q^{n+\alpha+\beta+1} \\ q^{\beta+1}, q^{N+\alpha+\beta+1} \end{array}; q, q\right). \end{aligned}$$

In addition, for λ_n we find

$$\lambda_n = q^{\frac{1}{2}(\alpha+\beta+2)}[n]_q[n+\alpha+\beta+1]_q.$$

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6. Conclusions and remarks. The method presented in this paper connects in a precise way (see equation (4.24)) the q-Hahn polynomials with q-CGC, i.e., the Nikiforov-Uvarov approach and the quantum group representation one. The Pearson type approach to q-Hahn polynomials used here is based on a specific kind of lattices $x(s)$. In fact, this approach is very effective since the lattice $x(s)$ is always the same for each relation used through the paper, i.e., the lattice is the same for Pearson-type difference equation, hypergeometric-type difference equation, structure relation, difference-recurrence relation, and orthogonality relation which could not occur in other approaches. For example, it could happen that the difference equation is given on the lattice $x(s) = q^s$ and the orthogonality relation on $x(s) = q^{-s}$. Thus, our approach avoids this and unifies the election of the lattice.

As the reader has observed the limit $q \to 1$ do not transform directly the polynomials on $x(s) = q^s$ into the classical discrete Hahn polynomials since a rescaling factor before taking limits is needed. However, one of the contributions contained in the paper is precisely to avoid this. In such a way we avoid to answer the following questions: Is the rescaling factor the same for all the aforementioned relations/characterizations? or different rescaling factor should be introduced depending on the specific relation for which one is looking for the classical analogue? With the lattice $x(s) = \frac{q^s-1}{q-1}$ proposed in the paper one can avoid this; therefore the q-Hahn polynomials on $x(s)$ just becomes directly into the classical Hahn polynomials. Moreover, there is also another reason. When one computes all the characteristics of the q-Hahn polynomials on $x(s) = q^s$ such as the three-term recurrence relation, structure relations, etc, and takes the limit $q \to 1$ one realizes that all these formulas becomes zero. For getting a non trivial result one should use not only a rescaling factor and some times the higher (usually 2) order Taylor expansion. One can avoid this by using an appropriate lattice.

On the other hand, notice that the q-Hahn polynomials obtained in this paper are new ones despite the fact they are included in the Nikiforov-Uvarov approach. Regarding this to avoid confusions let us comment a couple of things. First, any change of variable carried out to transform the weight (measure) which leads to a non-linear change of the support of the measure never will produce up to a constant depending on the degree the same polynomials. For instance, if the orthogonality condition of a discrete orthogonal polynomial P_n are given on the lattice $\{x_k = x(k) \mapsto \Re^+ : k = 0, 1, \ldots, N-1\}$, being the support of the measure the closure of $\{x_k\}_{k=0}^{N}$. Then, for such a support the orthogonality condition

$$\sum_{s=0}^{N-1} P_n(x(s)) x^k(s)\, \rho(s) \bigtriangleup x(s - \tfrac{1}{2}) = 0, \quad k = 0, 1, \ldots, n-1,$$

by replacing s by $N-1-t$ leads us to a completely different polynomial family. Notice that the support is not the set $\{0, \ldots, N-1\}$ to which belongs the variable s.

Second, the q-Hahn polynomials studied in this paper are the solution of the corresponding second order difference equation of hypergeometric type on the non-uniform lattice $x(s)$. Since this equation is linear one can always make a change of variable to convert the lattice $x(s) = c_1 q^s + c_2$ into $x(s) = q^s$, in fact if one writes the solution of the equation in terms of the basic series it looks very similar (see [29], or the paper by Nikiforov and Uvarov [31]). Actually, this representation does not depend on c_2. Nevertheless, the polynomials are quite different since they are polynomials on different lattice. This means that when one writes the expansions of $P_n(s)$ as polynomials in $x(s)$ the coefficients are completely different, that is not evident when one compares the basic hypergeometric function —the reason is that in both cases the polynomial can be expanded in the $(q^s; q)_k$ basis just in the same manner–.

Finally, we would like to point out two remarkable benefits (among others) obtained from the approach presented in this paper:

1. A unique lattice is involved in all relations.

2. A simple limit $q \to 1$ without a rescaling factor is computed wherever is needed to obtain any classical property.

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