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STOCHASTIC MEASURES OF FINANCIAL MARKETS EFFICIENCY AND INTEGRATION

Alejandro Balbás* and María José Muñoz**

Abstract

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The purpose or this paper is to extend the refereed notions to a more general context. This is accomplished by introducing a methodology which may be applied in any intertemporal dynamic asset pricing model and without special assumptions on the assets prices stochastic process. Then, the integration measures introduced here are stochastic processes testing different relative arbitrage profits and depending on the state of nature and on the date. The measures are introduced in a single financial market. When this market is not a global market from different ones, the measures simply test the degree of market efficiency.

Transaction costs can be discounted in our model. Therefore, one can measure efficiency and integration in models with frictions.

The main results are also interesting form a mathematical pint of view, since some topics of Operational Research are involved. We provide a procedure to solve a vector optimization problem with a non differentiable objective function and prove some properties about its sensitivity.

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Stochastic Measures of Financial Markets Efficiency and Integration

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Abstract

The notion of integration of different financial markets is often related to the absence of cross-market arbitrage opportunities. Under the appropriated assumptions and in absence of cross-market arbitrage opportunities, a risk-neutral probability measure, shared by both markets, must exist. Some authors have considered this to provide some integration measures when the markets do not share any pricing rule, but always in static (or one period) asset pricing models.

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The measures are introduced in a single financial market. When this market is not a global market from different ones, the measures simply test the degree of market efficiency.

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1 Introduction

Most of the theory of portfolio choice and asset pricing models under uncertainty is built around one basic constraint on asset prices: absence of arbitrage. However, some of the empirical research that has been carried out presents evidence supporting the existence of arbitrage opportunities. For instance, Protopadakis and Stoll (1983) identify situations where the Law of One Price is violated in dealing with spot and future prices. Kamara and Miller's (1995) tests of put-call parity using data on European options lead them to conclude that arbitrage opportunities exist and are available to some traders. Moreover, analysis of integration between two or more markets is usually implemented by testing the presence of cross-market arbitrages, e.g., Kleidon and Whaley (1992), Lee and Nayar (1993), Chen and Knez (1995), Harris, et al.(1995), and Kempf and Korn (1996). Thus, it is important to develop a formal measurement theory of arbitrage opportunities. There are some approaches on this subject in the existing literature. For example, Chen and Knez (1995) develop a measurement theory of market integration based on the notion that two markets cannot be integrated if there are cross-market arbitrage opportunities. They start with two markets such that there is no arbitrage opportunity on either market. Then, they use the characterizations of the fulfillment of the Law of One Price and the absence of arbitrage by the existence of some stochastic discount factors, see Chamberlain and Rothschild (1983) and Hansen and Richard (1987), to define two measures of the extent of segmentation for the two markets as the minimum mean-square distance between the respective sets of stochastic discount factors. In Balbás and Muñoz (1996) we develop a measurement theory of arbitrage opportunities. The measures proposed there test some maximum relative profits obtained from arbitrage opportunities in a financial market. They can also be applied to measure market integration. So, they are an alternative to the one introduced by Chen and Knez (1995).

All the measures in the existing literature are defined for static models. A purpose of this paper is to define a measure in a dynamic setting. Attention is directed to measure in a discrete (finite or infinite) time model. We start with the idea that there is an arbitrage opportunity in the model if there exist two consecutive dates such that there is an arbitrage opportunity in this period. This leads us to consider a single period where the prices of the assets and the strategies depend on the state of nature at both dates.

Another purpose of this paper is to measure arbitrage opportunities in monetary terms following the approach of Balbás and Muñoz (1996). In a financial market, violation of the no arbitrage condition creates an opportunity for infinite arbitrage profits. Thus, we must look for relative profits. We proceed in two ways.

The first assumes the absence of short sale restrictions. The central idea of the stochastic measure l is to test in each period the minimum initial investment over the price of the interchanged assets, needed to purchase a portfolio that generates nonnegative payoff in almost every state of nature.

The second way takes place in the presence of short sale restrictions. We first introduce for each period a dual pair of infinite linear programs. A solution to these programs yields to the maximum expected profit obtained by an agent in presence of short sale restrictions in all the assets, i.e, the agent cannot sell what he/she does not have. We then consider the maximum of the found maxima among all the portfolios of the bounds of short position with price one at the first date of the period. This leads to a measure $\mathfrak{M}$ which depends only on the period.

The results and some of their proofs are interesting from two points of view: Financial Economics and Optimization Theory. An optimal arbitrage portfolio x leading to the maximum expected relative profit $\mathfrak{M}$ also leads to a stochastic measure m . This measure can be interpreted as the maximum arbitrage profit relative to the price of the sold assets in almost every state of nature. To show this, we prove that x solves a vector optimization problem in which the objective function is non differentiable and takes values in an infinite-dimensional L^2-space. Finally, we prove that x also solves the multiobjective optimization program introduced to define l. This allows us to relate l, m and $\mathfrak{M}$.

This article is organized as follows. Section 2 introduces the optimization programs leading to define the measures l and $\mathfrak{M}$. In Section 3 we prove the solvability of the programs associated to the measure $\mathfrak{M}$ and define the stochastic measure m. Section 4 provides some dual optimization problems such that their solutions lead to the measures m and $\mathfrak{M}$. This allows us to relate m and $\mathfrak{M}$ with the stochastic discount factors, and therefore, with some dynamic measures which extends static ones of the existing literature. We provide other interpretations of the measure m in Section 5. We also state the solvability of the optimization problem leading to the measure l and relate both measures m and l. The concluding section contains miscellaneous remarks about and extensions of the analysis.

2 Measurement of the arbitrage opportunities

Consider a market for trading n securities at a countable number of times $0,\ 1\ ,\ \cdots$. As usual, there is some finite or infinite set Ω of states of the world. For each date t, a σ-algebra Σ_t of subsets of Ω denotes the set of events corresponding to the information available at time t. We adopt the usual convention that $\Sigma_t \subseteq \Sigma_s$ whenever $t \leq s$. Finally, P is a probability measure defined on a σ-algebra of subsets of Ω containing all the σ-algebras Σ_t . A strategy trading at time t is $x = (x_1, x_2, \cdots, x_n) \in (L^2(\Sigma_t))^n$. This restriction of attention to square integrable random variables is made for expositional and mathematical ease. Let $p^t = (p_1^t, p_2^t, \cdots, p_n^t) \in (L^2(\Sigma_t))^n$ the prices vector at time t. Then, for a strategy trading at time t, the value at date t is the sum $\sum_{i=1}^n x_i p_i^t$ and the value at date $t+1$ is the the sum $\sum_{i=1}^n x_i p_i^{t+1}$.

Definition 2.1 *The portfolio $x \in (L^2(\Sigma_t))^n$ is said to be an arbitrage opportunity (strong form) at date t if there exists $A \in \Sigma_t$ with $P(A) > 0$ such that $\sum_{i=1}^n x_i p_i^{t+1} \geq 0$ a.e. in A and $\sum_{i=1}^n x_i p_i^t < 0$ a.e. in A.*

If $x \in (L^2(\Sigma_t))^n$ is an arbitrage opportunity at date t, it allows an agent to increase with positive probability consumption at date t, and increase (or at least not decrease) consumption at date $t+1$. Latter definition extends the notion of arbitrage opportunity of the second type (Ingersoll(1987)).We do not consider the opportunities of the first type which can be introduced as the portfolios $x \in (L^2(\Sigma_t))^n$, with nonnegative price at time $t+1$ that are positive with positive probability, and with zero price at time t.
In the existing literature, the notion of absence of arbitrage in the different models varies from some authors to others. Under very natural assumptions and for a finite number of dates, it may be proved that the absence of arbitrage of both types at every date is equivalent to the absence of simple free lunches in the sense of Harrison and Kreps (1979). However, for an infinite number of trading dates the absence of free lunches is a stronger property. In this paper we turn our attention to measure relative profits from arbitrage strategies of the second type and thus we adopt the following definition.

Definition 2.2 *We say that the model is arbitrage free if there are not arbitrage opportunities in every date.*

We are interested in defining two stochastic processes to analyze attainable relative arbitrage profits in each date and in almost every state of nature. For expositional reasons, a brief synopsis of these approaches to measure arbitrage opportunities follows. Most technical details have been left to below sections, so all definitions stated in this synopsis are subjects to results to be proved.
From now on, fix the period $t,\ t+1$. Let us introduce the following assumptions:

A_1. For every $i = 1, \cdots, n,\ p_i^t \in L^2(\Sigma_t)$ and there exists $k_i \in \mathbb{R}$ such that $p_i^t(\omega) > k_i > 0$

A_2. For every $i = 1, \cdots, n,\ p_i^{t+1} \in L_+^\infty(\Sigma_{t+1})$, and there exists $k \in \mathbb{R}$ such that $p_1^{t+1}(\omega) > k > 0$.

The assumptions imposed to p_1 are verified if we assume that the first security is a riskless asset. As Harrison and Kreps (1979) show, this is not a very restrictive assumption. In fact, assuming that one of the securities always has strictly positive price, we can use the price of this security as the numeraire (see Section 6).

FIRST MEASURE.

Consider for every $x \in (L^2(\Sigma_t))^n$ the function,

$$g(x,\omega) = \begin{cases} \dfrac{-\sum\limits_{i=1}^{n} x_i(\omega)p_i^t(\omega)}{\sum\limits_{i=1}^{n} |x_i(\omega)|p_i^t(\omega)} & \text{if} \quad x(\omega) \neq \mathbf{0} \\ 0 & \text{if} \quad x(\omega) = \mathbf{0} \end{cases}$$

If x is an arbitrage opportunity at date t and $A \in \Sigma_t$ is as in Definition 2.1, then $g(x,\omega)$ is the quotient whenever $\omega \in A$ between the profit generated by x and the price of all interchanged assets, both computed at date t.
Since $|g(x,\omega)| \leq 1$ holds for every $\omega \in \Omega$ and every portfolio x, and $g(x, \quad)$ is Σ_t-measurable, it follows that $g(x, \quad) \in L^2(\Sigma_t)$ for every $x \in (L^2(\Sigma_t))^n$. Consider the following non-differentiable vector optimization problem:

$$(I) \qquad \max \; g(x, \quad) \qquad \text{s.t} \qquad \sum_{i=1}^{n} x_i p_i^{t+1} \geq 0$$

The problem (I) describes the process of identifying the portfolio which maximizes in almost every state of nature the arbitrage profit at date t in relation to the price of all interchanged assets.

Definition 2.3 *Assume that problem (I) is solvable. We define the first measure of the level of arbitrage opportunities by*

$$l^t(\omega) = g(x,\omega)$$

where $x \in (L^2(\Sigma_t))^n$ is an optimal solution in (I).

It is important to point out that l^t gives an information useful to analyze the absence of arbitrage in markets with transaction costs. Therefore, we have an alternative to the approaches of Prisman (1986) or Jouini and Kallal (1995) in markets with frictions. In fact, we may assume that the transaction costs are a.e. determined by the price $V(\omega) = \sum_{i=1}^{n} |x_i(\omega)|p_i^t(\omega)$ of the exchanged assets. Once solving (I), we find out the maximum profit an investor can obtain for $V(\omega)$. Then, we can discount the transaction costs.

We must prove the consistency of the above definition or, equivalently, that problem (I) is solvable. From a point of view of Optimization Theory, problem (I) presents some difficulties. Note that the objective function is non linear and non differentiable. Moreover, it is an $L^2(\Sigma_t)$-valued function and we are looking for a strong maximum, i.e., an upper bound for the usual partial order in $L^2(\Sigma_t)$. In sharp contrast to the vast body of literature on efficiency in multiobjective optimization, see for instance Khanh (1995), little is known about strict optimality. There are some conditions for the existence of strong optima in vector optimization problems, see for instance Zowe (1975), but they usually do not apply in practical situations. In much of the cases, an strong optimum does not necessarily exist. However, we prove in Section 5 that there exists an optimal strong solution of problem (I).

SECOND MEASURE.

We now assume that we are in presence of short sale restrictions. We introduce a couple of combined optimization programs to analyze arbitrage profits. The first one is a pair of primal-dual programs for each short sale restriction $h = (h_1, \cdots h_n) \in (L^2_+(\Sigma_t))^n$:

$$(II_h) \qquad \max \; -\sum_{i=1}^{n} \int_\Omega x_i p_i^t \, dP \;\; \text{s.t.} \; \left\{ \begin{array}{ll} \sum_{i=1}^n x_i p_i^{t+1} \geq 0 & \\ x_i \geq -h_i & i = 1, \cdots, n \\ x_i \in L^2(\Sigma_t) & i = 1, \cdots, n \end{array} \right.$$

and its dual problem,

$$(III_h) \qquad \min \; \sum_{i=1}^{n} \int_\Omega h_i \lambda_i \, dP \;\; \text{s.t.} \; \left\{ \begin{array}{l} E(f p_i^{t+1} \mid \Sigma_t) + \lambda_i = p_i^t \\ f \in L^2_+(\Sigma_{t+1}),\ \lambda_i \in L^2_+(\Sigma_t) \quad \text{for every } i = 1, \cdots, n \end{array} \right.$$

where f and λ_i, $i = 1, \cdots, n$ are the decision variables and $E(f p_i^{t+1} \mid \Sigma_t)$ denotes the conditional expectation of $f p_i^{t+1}$ relative to Σ_t.
The problem (II_h) describes the process of identifying the portfolio (constrained by the bounds in a short position $h_i \geq 0$) which minimizes the initial expected investment needed to purchase a portfolio that generates a nonnegative payoff in almost every state of nature. Thus, a non zero optimal value in (II_h), provided its existence, represents the maximum expected profit obtained by an agent implementing arbitrage in such a way that he/she cannot sell more than h_i units in each asset i.
Problems (II_h) and (III_h) avoid some difficulties encountered when dealing with problem (I) since their objective functions are linear and real valued. However, the variables take values in an L^2-space, with an empty interior of the positive cone unless Σ_t is a finite σ-algebra. So, most of the usual conditions do no apply to establish the absence of duality gap. In Section 4 we establish conditions on Σ_t under which there is no duality gap. These conditions are fulfilled in practical situations. When this absence of duality gap could not be stated, some considerations in Section 3 prove that we can consider an equivalent formulation in which the primal constraint space is an L^∞-space; this is the only one of the L^p-spaces to have a positive cone with interior points (in its norm topology). In this case the dual variable space must be the dual Banach space for L^∞, and thus will not be a function space. Nevertheless, the absence of duality gap can be always stated. This is the reason for considering in Section 4 both topological frameworks for the pair dual (II_h) and (III_h).
We denote by $\varphi(h)$ the optimum value in (II_h). We try to define a measure as the maximum attainable expected profit from an arbitrage opportunity among all the possible investors holding portfolios h priced one. Then we look for $h = (h_1, \cdots h_n) \in (L^2_+(\Sigma_t))^n$ so as to solve

$$(IV) \qquad \max \; \varphi(h) \;\; \text{s.t.} \; \left\{ \begin{array}{ll} \sum_{i=1}^{n} h_i p_i^t = 1 & \\ h_i \geq 0 & i = 1, \cdots, n \end{array} \right.$$

Definition 2.4 *We define the second measure $\mathfrak{M}^t$ of the level of arbitrage opportunities as the optimum value achieved in problem (IV).*

Note first that $\mathfrak{M}^t$ is scalar for every t. The consistency of this definition relies on the solvability of problem (IV). We prove it in Section 3.

3 Measuring with short sale restrictions

In this section we prove the solvability of problem (IV), and hence the consistency of $\mathfrak{M}^t$. Furthermore, if h^* is an optimal solution in problem (IV) and x^* is a corresponding optimal solution

in problem (II_{h^*}) we prove that the function $-\sum_{i=1}^n x_i^* p_i^t$ does not depend on the chosen optimal solutions h^* and x^*. Thus, such a function defines a third measure of the arbitrage opportunities. From now on, we adopt the following notations:
$\mathcal{H}^t$ denotes the set $\{h \in (L_+^2(\Sigma_t))^n \mid \sum_{i=1}^n h_i p_i^t \in L^2(\Sigma_t)\}$ and F_h the feasible set in (II_h).

Lemma 3.1 *Problem (II_h) is solvable for every $h \in \mathcal{H}^t$.*

Proof. The proof of the lemma relies on the fact that we do not need to bound the whole feasible set, but only a subset of it containing the optimal solution.
Fix $h \in \mathcal{H}^t$. First note that the feasible set F_h is non void since $x = 0 \in F_h$. Furthermore, the value of (II_h) is bounded by $\sum_{i=1}^n \int_\Omega h_i p_i^t \, dP$ and hence finite. Moreover, it is nonnegative.
For every $x \in F_h$, let $A = \{\omega \in \Omega \mid -\sum_{i=1}^n x_i(\omega) p_i^t(\omega) > 0\}$ and $x' = x\chi_A$, where χ_A denotes the characteristic function of A.
Then, $x' \in F_h$ and $-\sum_{i=1}^n \int_\Omega x_i' p_i^t \, dP \geq -\sum_{i=1}^n \int_\Omega x_i p_i^t \, dP$. Consequently we can take as feasible set for (II_h) the subset $F_h' = \{x \in F_h \mid \sum_{i=1}^n x_i(\omega) p_i(\omega) \leq 0\}$ of F_h. F_h' is an order bounded set since $-h_i \leq x_i \leq (\sum_{j \neq i} h_j p_j^t) \ / \ p_i^t$ whenever $x \in F_h'$. Thus, F_h' is a weakly-compact set. Then, the weak-continuity of the objective function guarantees the solvability of Problem (II_h). □

The above lemma allows us to properly define $\varphi(h)$ as the optimal value in (II_h) , that is,

$$\varphi(h) = \max \left\{ -\sum_{i=1}^n \int_\Omega x_i p_i^t \, dP \mid x \in F_h \right\}.$$

It is easily verified that

$$\varphi(h + h') \geq \varphi(h) + \varphi(h') \qquad \text{and} \qquad \varphi(\delta h) = \delta \varphi(h)$$

for every $h, h' \in \mathcal{H}^t$ and $\delta > 0$, so φ is a concave function.

Also, $\varphi(h) \geq \varphi(h')$ if $h \geq h'$ in the usual partial order of $L^2(\Sigma_t)$.

Finally, $\varphi(h) \leq \sum_{i=1}^n \int_\Omega h_i p_i^t \, dP$ since $-\sum_{i=1}^n \int_\Omega x_i p_i^t \, dP \leq \sum_{i=1}^n \int_\Omega h_i p_i^t \, dP$ for every feasible x.

Lemma 3.2 *Suppose that $x \in (L^2(\Sigma_t))^n$ is an arbitrage opportunity and let be $A \in \Sigma_t$ with $P(A) > 0$ and such that $\sum_{i=1}^n x_i p_i^{t+1} \geq 0$ a.e. in A and $\sum_{i=1}^n x_i p_i^t < 0$ a.e. in A. Then for every $\varepsilon > 0$ there exists $B \in \Sigma_t$, $B \subseteq A$ with $P(A - B) < \varepsilon$ such that $\hat{x} = x.\chi_B \in (L^\infty(\Sigma_t))^n$, $\sum_{i=1}^n \hat{x}_i p_i^{t+1} \geq 0$ and $\sum_{i=1}^n \hat{x}_i p_i^t < 0$ a.e. in B.*

Proof. For every $k \in \mathbb{N}$ take $A_k = \{\omega \in A \mid |x_i(\omega)| \leq k \quad i = 1, 2 \cdots, n\}$. Since $\lim_k P(A_k) = P(A)$, then for every $\varepsilon > 0$, there exists $k' \in \mathbb{N}$ such that $P(A - A_k) < \varepsilon$ whenever $k \geq k'$. Then $B = A_{k'}$ is the required subset. □

Proposition 3.3 *No arbitrage opportunity exists at date t if and only if $\varphi(h) = 0$ for every $h \in \mathcal{H}^t$.*

Proof. Suppose there exists an arbitrage portfolio $x \in (L^2(\Sigma_t))^n$ and let $A \in \Sigma_t$ such that $P(A) > 0$, $\sum_{i=1}^n x_i p_i^{t+1} \geq 0$ a.e. in A and $\sum_{i=1}^n x_i p_i^t < 0$ a.e. in A.
From Lemma 3.2, let be B and $\tilde{x}(\omega) = \begin{cases} x & \text{if} \quad \omega \in B \\ 0 & \text{otherwise} \end{cases}$ such that $P(B) > 0$ and $\tilde{x} \in (L^\infty(\Sigma_t))^n$. Hence, $\sum_{i=1}^n |\tilde{x}_i| p_i^t \in L^2(\Sigma_{t+1})$. Set $h = (|\tilde{x}_1|, \cdots, |\tilde{x}_n|)$. Then, $\tilde{x} \in F_h$ and consequently

$$\varphi(h) \geq -\sum_{i=1}^n \int_\Omega \tilde{x}_i p_i^t \, dP = -\sum_{i=1}^n \int_B x_i p_i^t \, dP > 0$$

Assume now that no arbitrage opportunity exists. Then for every $h \in (L^2_+(\Sigma_t))^n$ and $x \in F_h$ one has that,

$$P\left\{\omega \mid \sum_{1=1}^{n} x_i p_i^t < 0\right\} = 0.$$

Consequently $\varphi(h) = 0$ for every $h \in \mathcal{H}^t$. □

In order to prove that Problem (IV) is solvable, we introduce the following optimization program,

$$(V) \qquad \max \ -\sum_{i=1}^{n} \int_\Omega x_i p_i^t \, dP \quad \text{s.t.} \quad \begin{cases} \sum_{i=1}^n x_i p_i^{t+1} \geq 0 \\ \sum_{i=1}^n h_i p_i^t = 1 \\ x_i \geq -h_i & i = 1, \cdots, n \\ x_i \in L^2(\Sigma^t), \quad h_i \in L^2_+(\Sigma^t) & i = 1, \cdots, n \end{cases}$$

Note that Problem (V) is equivalent to the combined problem from (II_h) and (V). Furthermore, Problem (V) is just a linear program where the decision variable (x, h) takes values in $(L^2(\Sigma_t))^n \times (L^2_+(\Sigma_t))^n$.

Theorem 3.4 *Problem (V) is solvable. Consequently, Problem (IV) is also solvable.*

Proof. Just proceed as in the proof of Lemma 3.1 to choose a subset F of the feasible set of V, such that $(-1) \;/\; p_i \leq -h_i \leq x_i \leq 1 \;/\; p_i$ whenever $(x, h) \in F$. Then Problem (V) is solvable in a such weakly compact feasible region. However, the choice of F yields an optimal solution for Problem (V) in the whole feasible set.
Finally, note that an optimal solution (x^*, h^*) of Problem (V) yields an optimal solution h^* of Problem (IV) and an optimal solution x^* of Problem (II_{h^*}). □

Remarks.

1) In empirical applications an optimal solution of (IV) may be obtained by solving (V). Problem (V) is just a linear program which can be solved by the classical optimization techniques (see Anderson and Nash (1987)). Furthermore, in many specially interesting situations, the σ-algebras Σ_t and Σ_{t+1} will be finite and then (V) can be solved by the simplex method.

2) It can be deduced from the proof of Theorem 3.4 that an optimal solution (x^*, h^*) of Problem (V) verifies

$$(-1) \;/\; p_i \leq -h_i^* \leq x_i^* \leq 1 \;/\; p_i.$$

Thus, assuming A_1, one gets that $(x^*, h^*) \in (L^\infty(\Sigma_t))^{2n}$. So, programs (II_h), (IV) and (V) can be reformulated in such a way that all the spaces involved are L^∞-spaces.

3) Finally, it should be noted that assumption A_2 has not been needed for the proofs of Lemma 3.1 and Theorem 3.4. More precisely, the condition $p_i^{t+1} \in L^\infty_+(\Sigma_{t+1})$ can be relaxed to $p_i^{t+1} \in L^2(\Sigma_{t+1})$ without modifying the results about the solvability of programs (II_h), (IV) and (V). The condition $p_i^{t+1} \in L^\infty_+(\Sigma_{t+1})$ is only needed for the results obtained in Section 4.

Recall that we denote by $\mathfrak{M}^t$ the optimum value achieved in (IV) (see Definition 2.4) or (V), and therefore, it follows from Proposition 3.3 that $\mathfrak{M}^t = 0$ if and only if there are no arbitrage portfolios at date t.
Our next purpose is to prove that an optimal solution h^* which maximizes an attainable expected payment, also maximizes an attainable profit in almost every state of nature.
Let $x^h \in F_h$ be a solution where the optimum value $\varphi(h)$ is achieved. We first prove that x^h solves

a multiobjective optimization program. For every $A \in \Sigma_t$ consider the restricted optimization programs:

$$(II_h^A) \qquad \max \quad -\sum_{i=1}^{n} \int_A x_i p_i^t \, dP \qquad \text{s.t.} \quad \begin{cases} \sum_{i=1}^n x_i p_i^{t+1} \geq 0 & \text{a.e. in A} \\ x_i \geq -h_i & \text{a.e. in A}, \; i = 1, \cdots, n \end{cases}$$

Lemma 3.5 *Suppose that x^h solves (II_h) for $h \in \mathcal{H}^t$. Then, x^h solves (II_h^A) for every $A \in \Sigma_t$.*

Proof. Proceeding by contradiction suppose there exists x feasible in (II_h^A) such that

$$-\sum_{i=1}^{n} \int_A x_i p_i^t \, dP > -\sum_{i=1}^{n} \int_A x_i^h p_i^t \, dP$$

Define $x' = \begin{cases} x^h & \text{if } \omega \notin A \\ x & \text{if } \omega \in A \end{cases}$. Obviously x' is feasible in (II_h) and $-\sum_{i=1}^n \int_\Omega x_i p_i^t \, dP > \varphi(h)$ and the proof is concluded. □

Theorem 3.6 *Suppose that x^h solves (II_h) for $h \in \mathcal{H}^t$.*
Then x^h solves the following vector optimization program:

$$(VI_h) \qquad max \quad -\sum_{i=1}^{n} x_i p_i^t \qquad s.t. \quad \begin{cases} \sum_{i=1}^n x_i p_i^{t+1} \geq 0 & \\ x_i \geq -h_i & i = 1, \cdots, n \end{cases}$$

Furthermore, $-\sum_{i=1}^{n} x_i^h p_i^t$ is a strong maximum in (VI_h), i.e., an upper bound for (VI_h), and not only a maximal value.

Proof. It follows immediately from Lemma 3.5. □

Theorem above allows to define for every $h \in \mathcal{H}^t$ a function Υ of the maximum arbitrage profit obtained by an an agent holding a portfolio h. Concretely $\Upsilon(h, \;)$ is the strong maximum in (VI_h) and

$$\Upsilon(h, \omega) = -\sum_{i=1}^{n} x_i^h(\omega) p_i^t(\omega).$$

From $x_i^h \geq -h_i$ it follows that $\Upsilon(h, \;)$ takes values in $L^2(\Sigma_t)$ whenever $h \in \mathcal{H}^t$.

Hereafter, we denote by H^t the feasible set in (IV),i.e.,

$$H^t = \{h \in \mathcal{H}^t \mid \sum_{i=1}^{n} h_i p_i^t = 1\}.$$

It follows from A_1 that H^t is a subset of $(L^\infty(\Sigma_t))^n$.

Theorem 3.7 *Let $h^* \in H^t$ such that $\mathfrak{M}^t = \varphi(h^*)$. Then, h^* solves the following vector optimization problem*

$$(VII) \qquad max \;\; \Upsilon(h, \;) \qquad s.t. \qquad h \in H^t$$

Furthermore, $\Upsilon(h^*, \;)$ is a strong maximum in (VII).

Proof. Let $x^* \in F_{h^*}$ such that $\varphi(h^*) = -\sum_{i=1}^n \int_\Omega p_i^t x_i^* \, dP$. From Theorem 3.6 we get that

$$\Upsilon(h^*, \omega) = -\sum_{i=1}^n x_i^*(\omega) p_i^t(\omega).$$

Proceeding by contradiction suppose that $\Upsilon(h^*, \ )$ is not a strong maximum in (VII). Then, there exist $\hat{h} \in H^t$, $\hat{x} \in F_{\hat{h}}$ and $A \in \Sigma_t$ with $P(A) > 0$ such that

$$-\sum_{i=1}^n \hat{x}_i(\omega) p_i^t(\omega) = \Upsilon(\hat{h}, \omega) > \Upsilon(h^*, \omega)$$

for every $\omega \in A$. Setting,

$$x_i(\omega) = \begin{cases} \hat{x}_i(\omega) & \text{if } \omega \in A \\ x_i^*(\omega) & \text{otherwise} \end{cases} \quad \text{and} \quad h_i(\omega) = \begin{cases} \hat{h}_i(\omega) & \text{if } \omega \in A \\ h_i^*(\omega) & \text{otherwise} \end{cases}$$

then, h is feasible in (IV) and $x \in F_h$ and consequently,

$$\varphi(h) \geq -\sum_{i=1}^n \int_\Omega x_i p_i^t \, dP = -\sum_{i=1}^n \int_A x_i p_i^t \, dP - \sum_{i=1}^n \int_{\Omega - A} x_i p_i^t \, dP > -\sum_{i=1}^n \int_\Omega x_i^* p_i^t \, dP = \varphi(h^*)$$

This strict inequality is in contradiction with the choice of h^* as a solution in problem (IV). □

Theorem 3.7 shows that $-\sum_{i=1}^n x_i^* p_i^{t+1}$ does not depend on the optimal solution (x^*, h^*) of (V) and consequently, leads to the following definition.

Definition 3.8 *The measure m^t of the level of arbitrage opportunities is defined by*

$$m^t(\omega) = -\sum_{i=1}^n x_i^*(\omega) p_i^t(\omega) = \Upsilon(h^*, \omega)$$

where $h^* \in H^t$, and $x^* \in F_{h^*}$ are such that

$$\mathfrak{M}^t = \varphi(h^*) = -\sum_{i=1}^n \int_\Omega x_i^* p_i^t \, dP.$$

Observe that $0 \leq m^t(\omega) = -\sum_{i=1}^n x_i^*(\omega) p_i^t(\omega) \leq \sum_{i=1}^n h_i^*(\omega) p_i^t(\omega) = 1$ and hence $m^t \in L^2(\Sigma_t)$.
The measure m^t, therefore, reflects in almost every state of nature the maximum attainable profit from an arbitrage opportunity, obtained among all the investors holding a priced one portfolio. Theorem 3.7 shows that m^t is obtained as a strong optimum in a multiobjective optimization program (Problem (VII)).

One can check that, with this definition, m^t verifies the first requirement to be a measure of the level of arbitrage opportunities, that is,

Theorem 3.9 *The following conditions are equivalent.*
i) No arbitrage opportunity exists on the market.
ii) $\mathfrak{M}^t = 0$.
iii) $m^t = 0$

The no existence of arbitrage opportunities is thus made testable by estimating m^t directly. The closer the value of $\mathfrak{M}^t = \int_\Omega m^t \, dP$ to zero, the lower the maximum expected quotient between the profit and the total price of short-selling restrictions

Such a test is also valid for a measurement of market integration: for two or more not integrated markets, treated as parts of one combined market, $\mathfrak{M}^t$ also indicates the level of arbitrage opportunities across the markets (see Section 6).

4 The state prices and the dual approach

In this section we turn our attention to the dual problem (III_h). Before starting our discussion of duality theory we shall study some properties of the dual problem (III_h).

Problem (III_h) is consistent since $(f, \lambda) = (0, p)$ is feasible. Furthermore, problem (III_h) has a finite value since problem (II_h) is consistent. Nevertheless, solvability of (III_h) presents a problem: its feasible set is not bounded and solvability cannot be settled by using Alaoglu's theorem. This is easily overcome by re-posing the problem as

$$(VIII_h) \qquad \min \sum_{i=1}^{n} \int_\Omega h_i \lambda_i \, dP \qquad \text{s.t.} \qquad \lambda \in \overline{\Lambda}$$

where

$$\Lambda = \left\{ \lambda \in (L^2_+(\Sigma_t))^n \mid E(f p_i^{t+1} \mid \Sigma_t) + \lambda_i = p_i^t, \ i = 1, \cdots, n, \ f \in L^2_+(\Sigma_{t+1}) \right\}$$

and $\overline{\Lambda}$ denotes its closure (in any topology consistent with the duality since Λ is a convex set). It is easy to prove that (III_h) and $(VIII_h)$ have the same value and Problem $(VIII_h)$ is solvable since $\overline{\Lambda}$ is a weakly-compact set.

The dual problem also presents an associated vector optimization with strong solutions.

Theorem 4.1 *Suppose that $\lambda \in \overline{\Lambda}$ solves $(VIII_h)$. Then, λ solves the following optimization problem*

$$(IX_h) \qquad min \sum_{i=1}^{n} h_i \lambda_i \qquad s.t \qquad \lambda \in \overline{\Lambda}$$

Proof. Proceeding by contradiction suppose there exists $\lambda' \in \overline{\Lambda}$ such that $P(A) > 0$ where $A = \{\omega \in \Omega \mid \sum_{i=1}^n h_i(\omega)\lambda'_i(\omega) < \sum_{i=1}^n h_i(\omega)\lambda_i(\omega)\}$. Take $\bar{\lambda} = \lambda'.\chi_A + \lambda.\chi_{\Omega - A}$. It is easily proved that $\bar{\lambda} \in \overline{\Lambda}$. Besides, the inequality

$$\sum_{i=1}^{n} \int_\Omega h_i \bar{\lambda}_i \, dP = \sum_{i=1}^{n} \int_A h_i \lambda'_i \, dP + \sum_{i=1}^{n} \int_{\Omega - A} h_i \lambda_i \, dP < \sum_{i=1}^{n} \int_\Omega h_i \lambda_i \, dP$$

holds and yields a contradiction. □

In particular, Theorem 4.1 states that an optimal solution of $(VIII_h)$ also solves the restricted program to every $A \in \Sigma_t$, that is,

$$(VIII_h^A) \qquad \min \quad \sum_{i=1}^{n} \int_A h_i \lambda_i \, dP \qquad \text{s.t.} \qquad \lambda \in \overline{\Lambda_A}$$

where

$$\Lambda_A = \left\{ \lambda \in (L^2_+(\Sigma_t \cap A))^n \mid E(f p_i^{t+1} \mid \Sigma_t \cap A) + \lambda_i = p_i^t \ \text{ in } \ A, \ i = 1, \cdots, n, \ f \in L^2_+(\Sigma_{t+1}) \right\}.$$

From the weak-duality relation between (II_h^A) and (III_h^A) one gets that

$$-\sum_{i=1}^{n}\int_A x_i p_i^t \, dP \leq \sum_{i=1}^{n}\lambda_i h_i \, dP$$

for every $A \in \Sigma_t$ and for every $\lambda \in \Lambda$. Then, this inequality remains true for every $\lambda \in \overline{\Lambda}$. Consequently a weak-duality relation can be stated for the associated vector optimization problems. More precisely, one has that

$$-\sum_{i=1}^{n} x_i p_i^t \leq \sum_{i=1}^{n}\lambda_i h_i$$

for every x feasible in (II_h) and for every $\lambda \in \overline{\Lambda}$.

For a finite linear program the values of the primal and dual programs are always equal. This is not the case for infinite-dimensional linear programs. The usual conditions under which this property holds do not apply here. This leads us to assume in the remainder of this section the no existence of duality gap for (II_h) and its dual (III_h). To motivate this assumption, we begin by proving that there is no duality gap when the σ-algebra Σ_t is generated by a finite or countable partition of Ω. This is an important case since in particular it englobes almost all the practical situations where Σ_t can be thought of as a finite σ-algebra.

Theorem 4.2 *Suppose that A_0, $A_1, \cdots$ is a finite or countable partition of Ω generating the σ-algebra Σ_t. Then, there is no duality gap for (II_h) and (III_h).*

Proof. Let us prove that there is no duality gap for (II_h) and $(VIII_h)$. Consider the restricted optimization programs to each A_j. Note that every Σ_t-measurable function must have some constant value over A_j. In particular, p_i^t, h_i have a constant value over A_j, say p_i^j and h_i^j. The restricted problems to each A_j can be posed as

$$(II_h^j) \qquad \max \; -\sum_{i=1}^{n} x_i p_i^j \;\text{ s.t. }\; \begin{cases} \sum_{i=1}^n x_i p_i^{t+1} \geq 0 & \text{a.e in } A_j \\ x_i \geq -h_i^j & i = 1,\cdots,n \\ x_i \in \mathbb{R} & i = 1,\cdots,n \end{cases}$$

and

$$(VIII_h^j) \qquad \min \quad \sum_{i=1}^{n} h_i^j \lambda_i \quad \text{s.t.} \quad \lambda \in \overline{\Lambda_j}$$

where

$$\Lambda_j = \left\{ \lambda \in \mathbb{R}_+^n \mid \int_{A_j} f p_i^{t+1} \, dP + \lambda_i = p_i^j, \; i = 1,\cdots,n, \; f \in L_+^2(\Sigma_{t+1}) \right\}.$$

In the inequality-constrained program (II_h^j) the associated positive cone P is $\mathbb{R}^n$ while the inequality constraints take values in $L^2(\Sigma_{t+1}) \times \mathbb{R}^n$, where the associated positive cone Q is $L_+^2(\Sigma_{t+1}) \times \mathbb{R}_+^n$. P is a cone with compact sole (since $B = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ is a compact set in P such that 0 is not in B and B spans P.) Besides, if $x \in \mathbb{R}^n$ is such that $(\sum_{i=1}^n x_i p_i^{t+1}, \; x) \in Q$ and $-\sum_{i=1}^n x_i p_i^j = 0$, then $x = 0$.
These two facts allow us to ensure that

$$D' = \left\{ \left(\sum_{i=1}^{n} x_i p_i^{t+1}, \, x - y, \, \sum_{i=1}^{n} x_i p_i^j \right) \mid x \in \mathbb{R}^n, \, y \in \mathbb{R}_+^n \right\}$$

is a closed set (see Theorem 3.19 of Anderson and Nash(1987)). Now, from Theorem 3.10 of Anderson and Nash(1987) it can be deduced that for every $j \in \mathbb{N}$ there exists $\lambda^j \in \overline{\Lambda_j}$ and x^j feasible in (II_h^j) such that

$$-\sum_{i=1}^{n} p_i^j x_i^j = \sum_{i=1}^{n} h_i^j \lambda_i^j.$$

Define now for every $q \in \mathbb{N}$ and every $i = 1, \cdots, n$ the Σ_t-measurable functions,

$$x_i^{(q)}(\omega) = \begin{cases} x_i^j & \text{if } \omega \in A_j \text{ and } j \leq q \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \lambda_i^{(q)}(\omega) = \begin{cases} \lambda_i^j & \text{if } \omega \in A_j \text{ and } j \leq q \\ p_i^t(\omega) & \text{otherwise} \end{cases}$$

It is easily checked that $x^{(q)}$ is feasible in (II_h) and $\lambda^{(q)} \in \overline{\Lambda}$. Besides, denoting by $A^q = \bigcup_{j=1}^q A_j$, one has that the relation

$$\begin{aligned} \textstyle\sum_{i=1}^n \int_\Omega h_i \lambda_i^{(q)}\, dP &= \textstyle\sum_{i=1}^n \int_{A^q} h_i \lambda_i^{(q)}\, dP + \sum_{i=1}^n \int_{\Omega - A^q} h_i \lambda_i^{(q)}\, dP \\ &= -\textstyle\sum_{i=1}^n \int_{A^q} p_i^t x_i^{(q)}\, dP + \sum_{i=1}^n \int_{\Omega - A^q} h_i p_i^t\, dP \\ &= -\textstyle\sum_{i=1}^n \int_{\Omega} p_i^t x_i^{(q)}\, dP + \sum_{i=1}^n \int_{\Omega - A^q} h_i p_i^t\, dP \end{aligned}$$

holds for every $q \in \mathbb{N}$. Finally, since $\lim_{q \to \infty} \sum_{i=1}^n \int_{\Omega - A^q} h_i p_i^t\, dP = 0$ it follows that $\sum_{i=1}^n \int_\Omega h_i \lambda_i\, dP = -\sum_{i=1}^n \int_\Omega x_i p_i^t\, dP$, which concludes the proof of the theorem. □

Next theorem ensures that in absence of duality gap for (II_h) and (III_h), the stochastic measure m^t can be formulated in terms of the optimal solution $\lambda^* \in \overline{\Lambda}$ of $(VIII_h)$.

Theorem 4.3 *Assume that there is not duality gap between (II_h) and (III_h) for every $h \in H^t$. Then, the equality $-\sum_{i=1}^n x_i^h p_i^t = \sum_{i=1}^n \lambda_i h_i$ holds whenever x^h solves (II_h) and λ solves $(VIII_h)$. In particular, one has that*

$$m^t(\omega) = -\sum_{i=1}^{n} x_i^*(\omega) p_i^t(\omega) = \sum_{i=1}^{n} \lambda_i^*(\omega) h_i^*(\omega)$$

where $h^* \in H^t$, $x^* \in F_{h^*}$ are as in Definition 3.8 and $\lambda^* \in \overline{\Lambda}$ solves Problem $(VIII_{h^*})$.

Proof. Just apply Theorem 4.1 and the weak-duality relation stated for the vector optimization problems (VI_h) and $(VIII_h)$. □

Latter theorem is a very surprising result in Operational Research. First, both problems, (VI_h) and (IX_h), have a strong solution. Second, we can conclude that there is no duality gap for (VI_h) and (IX_h) whenever there is no duality gap for the scalar problems (II_h) and (III_h). However, (IX_h) is not the dual problem of (VI_h) since its dual variables must be an element of the space of all bounded linear operators from $L^2(\Sigma_t)$ into itself, see for instance Balbás and Heras(1993). Furthermore, the absence of duality gap for scalarized dual vector problems does not guarantee in general the absence of duality gap for the vector problems. This is not our case even considering that in (IX_h) the feasible set is a strong simplification (and then a subset) of the feasible set of the dual of (VI_h). Finally, since Problem (IX_h) also has a strong solution, the primal sensitivity (given by λ) can be easily studied, what would be far more difficult without strong solutions (see Balbás and Guerra (1996) or Kuk et al. (1996)).

In absence of duality gap, $\phi(h)$, the optimal value of (II_h), can be obtained as

$$\phi(h) = \inf_{\lambda \in \Lambda} \sum_{i=1}^{n} \int_\Omega h_i \lambda_i\, dP = \min_{\lambda \in \overline{\Lambda}} \sum_{i=1}^{n} \int_\Omega h_i \lambda_i\, dP$$

Hence, the problem of finding $\mathfrak{M}^t$ can be expressed by a max-min problem

$$\mathfrak{M}^t = \max_{h \in H^t} \inf_{\lambda \in \Lambda} U(\lambda, h),$$

where U is defined by $U(\lambda, h) = \sum_{i=1}^{n} \int_\Omega h_i \lambda_i \, dP$. A min-max theorem is now established.

Theorem 4.4 *The equality* $\max_{h \in H^t} \inf_{\lambda \in \Lambda} U(\lambda, h) = \inf_{\lambda \in \Lambda} \max_{h \in H} U(\lambda, h)$ *holds.*

Proof. Note that H^t and Λ are convex subsets, H^t is a weak-compact set, $U(\lambda, .)$ is quasiconcave and weakly upper-semicontinuous for every $\lambda \in \Lambda$, and $U(., h)$ is quasiconvex and weakly below-semicontinuous for every $h \in H$. Now, just proceed as in Sion's theorem (Moulin(1979)) to get the result. □

In absence of duality gap for (II_h) and (III_h) for every $h \in H^t$, the above theorem states that

$$\mathfrak{M}^t = \max_{h \in H^t} \inf_{\lambda \in \Lambda} U(\lambda, h) = \inf_{\lambda \in \Lambda} \max_{h \in H} U(\lambda, h).$$

In game theoretic terminology latter equality expresses a two-person zero-sum game of the investor against the "market". Since $\lambda_i = p_i^t - E(f p_i^{t+1} \mid \Sigma_t)$ could be interpreted as the error committed by the "market" in the price of each asset for the state prices f, the sum $\sum_{i=1}^n \int_\Omega h_i \lambda_i$ would be the expected payment from the "market" to the investor due to h and λ. Thus, the investor chooses a priced one portfolio of short-selling bounds in such a way that it maximizes the minimal expected payment desired by the "market" and solves $\max_{h \in H^t} \inf_{\lambda \in \Lambda} U(\lambda, h)$. The problem, $\inf_{\lambda \in \Lambda} \max_{h \in H^t} U(\lambda, h)$ describes the process by which the "market" counteracts the goal of the investor by choosing the feasible λ which minimizes the maximal expected payment desired by the investor.

We conclude this section with some interesting results whenever the absence of duality gap for (II_h) and (III_h) cannot be stated.
As already said, see Remark 2 after Theorem 3.4, an optimal solution (x^*, h^*) leading to $\mathfrak{M}^t$ verifies that $(x^*, h^*) \in (L^\infty(\Sigma_t))^{2n}$. Then, program (II_h) can be reformulated as

$$(II_h^\infty) \qquad \max \quad -\sum_{i=1}^{n} \int_\Omega x_i p_i^t \, dP \quad \text{s.t.} \quad \left\{ \begin{array}{ll} \sum_{i=1}^n x_i p_i^{t+1} \geq 0 & \\ x_i \geq -h_i & i = 1, \cdots, n \\ x_i \in L^\infty(\Sigma_t) & i = 1, \cdots, n \end{array} \right.$$

for every $h \in (L_+^\infty(\Sigma_t))^n$. In this topological framework, we get its dual problem

$$(III_h^\infty) \qquad \min \sum_{i=1}^{n} \Gamma_i(h_i) \quad \text{s.t.} \quad \left\{ \begin{array}{l} \Gamma(p_i^{t+1}) + \Gamma_i = p_i^t \\ \Gamma \in (L^\infty(\Sigma_{t+1}))'_+, \, \Gamma_i \in (L^\infty(\Sigma_t))'_+ \quad \text{for every } i = 1, \cdots, n \end{array} \right.$$

where $(L^\infty(\Sigma_{t+1}))'_+$ and $(L^\infty(\Sigma_t))'_+$ denote the positive cones of the associated dual Banach spaces. The dual constraints must be understood as equalities in $(L^\infty(\Sigma_t))'_+$. More precisely, we identify each p_i^t and each $\Gamma(p_i^{t+1})$ with the elements of $(L^\infty(\Sigma_t))'_+$ such that $p_i^t(z) = \int_\Omega z p_i^t \, dP$ and $\Gamma(p_i^{t+1})(z) = \Gamma(z p_i^{t+1})$ for every $z \in L^\infty(\Sigma_t)$.
The interest of a such topological framework in which to pose our problems is that the conditions of Lagrange duality theorem (see Luenberger(1969)) hold for (II_h^∞) and (III_h^∞). Consequently, there is no duality gap for (II_h^∞) and (III_h^∞). Moreover, (III_h^∞) is solvable.
The absence of duality gap allows to characterize the absence of arbitrage by the existence of state prices that belong to $(L^\infty(\Sigma_{t+1}))'$.

Theorem 4.5 *No arbitrage opportunity exists at date t if and only if there exists $\Gamma \in (L^\infty(\Sigma_{t+1}))'_+$ such that $\Gamma(p_i^{t+1}) = p_i^t$ for every $i = 1, \cdots, n$.*

Proof. From Lemma 3.2 and Proposition 3.3 it follows that no arbitrage opportunity exists if and only if $\varphi(h) = 0$ for every $h \in (L_+^\infty(\Sigma_t))^n$. From the absence of duality gap between (II_h^∞) and (III_h^∞) we conclude that for every $h \in (L_+^\infty(\Sigma_t))^n$ there exists $(\Gamma_i)_{i=1}^n$ feasible in (III_h^∞) such that $\sum_{i=1}^n \Gamma_i(h_i) = 0$. Take an arbitrary interior point h of $(L_+^\infty(\Sigma_t))^n$. Then, the associated $(\Gamma_i)_{i=1}^n$ verifies that $\Gamma_i(h_i) = 0$ for every $i = 1, \cdots, n$. Consequently, $\Gamma_i = 0$ since h_i is an interior point of $L_+^\infty(\Sigma_t)$ and $\Gamma_i \in (L^\infty(\Sigma_t))'_+$. Thus, $\Gamma_i = 0$ $i = 1, \cdots n$ is feasible in (III_h^∞) and Theorem 4.5 is proved. □

From the absence of duality gap and solvability for both programs when working in L^∞-spaces, not only the absence of arbitrage portfolios can be characterized by state prices or dual variables but also the function φ and the measure $\mathfrak{M}^t$. This will allow to relate $\mathfrak{M}^t$ with some measures defined by means of state prices, and particularly, with a dynamic measure which extends the one of Chen and Knez (1995) (see Section 6).

5 Measuring without short sale restrictions

The purpose of this section is to prove that the measure m^t tests different relative arbitrage profits in almost every state of nature without short selling restrictions. Of special interest is the equality between the maximum expected profit with short-selling restrictions with total price one at the first date of the period and the maximum arbitrage profit obtained relative to the price of the sold assets.
From now on, denote by $\Omega_1 = \{\omega \in \Omega \mid m^t(\omega) \neq 0\}$ and by $\varphi_A(h)$ the optimal value attained in (II_h^A) for every $A \in \Sigma_t$. Note that from Theorem 3.7 one can deduce that an optimal solution h^* also maximizes φ_A in $H_A^t = \{h \in (L_+^2(\Sigma_t))^n \mid \sum_{i=1}^n h_i p_i^t = 1 \text{ a.e. in } A\}$.

Lemma 5.1 *Let $h^* \in H^t$ and $x^* \in F_{h^*}$ such that*

$$\mathfrak{M}^t = \varphi(h^*) = -\sum_{i=1}^n \int_\Omega x_i^* p_i^t \, dP$$

Then,

$$P(\{\omega \in \Omega_1 \mid x_i^*(\omega) > -h_i^*(\omega) \neq 0\}) = 0$$

for every $i = 1, \cdots, n$

Proof. In order to simplify the notation and without loss of generality, we will prove for $i = n$. Proceeding by contradiction, suppose that $P(\{\omega \in \Omega_1 \mid x_n^*(\omega) > -h_n^*(\omega) \neq 0\}) > 0$. Then, setting $A = \{\omega \in \Omega_1 \mid h_n^*(\omega) \neq 0\}$, $B = \{\omega \in A \mid x_n^*(\omega) \geq 0\}$ and $C = \{\omega \in A \mid 0 > x_n^*(\omega) > -h_n^*(\omega)\}$, one has $P(B) > 0$ or $P(C) > 0$ and $B, C \subseteq \Omega_1$.

First assume that $P(B) > 0$, and let $h_0 = (h_1^*, \cdots, h_{n-1}^*, 0)$ and $\gamma = \sum_{i=1}^{n-1} p_i^t h_i^*$.

The inequality $\varphi_B(h_0) \leq \varphi_B(h^*)$ holds, since the feasible set of $(II_{h_0}^B)$ is a subset of the feasible set of $(II_{h^*}^B)$.
Since x^* is feasible in $(II_{h_0}^B)$ we get that $\varphi_B(h^*) = \varphi_B(h_0)$.
Besides, $P(E) = 0$ holds, where $E = \{\omega \in B \mid p_n^t(\omega) h_n^*(\omega) = 1\}$, since $\varphi_E(h_0) = \varphi_E(h^*) = \int_\Omega m^t \, dP = 0$ and $E \subseteq \Omega_1$. Then, $\gamma > 0$ in B. Taking into account that $h_n^* > 0$ and $p_n^t > 0$ in B ,

we get $\gamma < 1$ in B. Then, there exists $D \subseteq B$, $D \in \Sigma_t$ such that $P(D) > 0$ and $0 < b < \gamma(\omega) < a < 1$ in D and consequently $\frac{1}{\gamma} \in L^\infty(\Sigma_t)$ and hence $\frac{1}{\gamma} h_0 \in (L^2_+(\Sigma_t))^n$. Finally, we have

$$\varphi_D(h^*) = \varphi_D(h_0) = a\varphi_D(\frac{1}{a}h_0) < \varphi_D(\frac{1}{a}h_0) \leq \varphi_D(\frac{1}{\gamma}h_0) \leq \varphi_D(h^*), \qquad \textbf{(1)}$$

where the last inequality follows from the fact that $\frac{1}{\gamma}h_0 \in H^t_D$. Since **(1)** leads to a contradiction, we get $P(B) = 0$.

Assume now that $P(C) > 0$ and let $h^0 = (h_1^*, \cdots, h_{n-1}^*, |x_n^*|)$ and $\delta = \sum_{i=1}^{n-1} p_i^t h_i^* + p_n^t |x_n^*|$. As above, we get $0 < \delta < 1$ and $G \subseteq C$, $G \in \Sigma_t$ such that $P(G) > 0$ and $0 < b < \delta(\omega) < a < 1$ in G and consequently $\frac{1}{\delta} \in L^\infty(\Sigma_t)$ and hence $\frac{1}{\delta}h_0 \in (L^2_+(\Sigma_t))^n$: Thus, we derive that

$$\varphi_G(h^*) = \varphi_G(h^0) = a\varphi_G(\frac{1}{a}h^0) < \varphi_G(\frac{1}{a}h^0) \leq \varphi_G(\frac{1}{\delta}h^0) \leq \varphi_G(h^*)$$

and we get again a contradiction. $\square$

Lemma 5.1 says basically that the portfolio where the maximum expected profit is achieved either sells all the stock or purchases in each asset.

Given a portfolio $x \in (L^2(\Sigma_t))^n$, we denote by $x_i^-(\omega) = \max(-x_i(\omega), 0)$ and we define the function

$$f(x,\omega) = \begin{cases} \dfrac{-\sum_{i=1}^{n} x_i(\omega)p_i^t(\omega)}{\sum_{i=1}^{n} x_i^-(\omega)p_i^t(\omega)} & \text{if } \sum_{i=1}^{n} x_i^-(\omega)p_i^t(\omega) \neq \mathbf{0} \\ 0 & \text{if } \sum_{i=1}^{n} x_i^-(\omega)p_i^t(\omega) = \mathbf{0} \end{cases}$$

If x is an arbitrage opportunity at date t and $A \in \Sigma_t$ is as in Definition 2.1, the function $f(x, \ )$ is the quotient in A between the profit generated by x and the price of all the sold assets, both computed at date t. We now consider the following non differentiable optimization problem

$$(X) \qquad \max\ f(x,\ ) \qquad \text{s.t} \quad \sum_{i=1}^{n} x_i(\omega)p_i^{t+1}(\omega) \geq 0$$

Theorem 5.2 *Assume that the market satisfies A_1 and A_2 and let $x^* \in (L^2(\Sigma_t))^n$ such that $m^t(\omega) = -\sum_{i=1}^n x_i^*(\omega)p_i^t(\omega)$. Then, x^* solves problem (X) and $m^t(\omega) = f(x^*, \omega)$ almost everywhere.*

Proof. We first prove that for every x feasible in (X) one has that $f(x,\omega) \leq m^t(\omega)$ a.e. in Ω. Proceeding by contradiction suppose that $P(D) > 0$, where $D = \{\omega \in \Omega \mid f(x,\omega) > m^t(\omega)\}$. Note that $f(x,\omega) \leq 1$ whenever $\omega \in D$. Set $\varepsilon(\omega) = \sum_{i=1}^{n} x_i^-(\omega)p_i^t(\omega) > 0$. Then, $\varepsilon > 0$ in D since $f(x,\omega) \geq 0$ in D and $\varepsilon \in L^1(\Sigma_t)$, from where it is deduced the existence of $C \subseteq D$, $C \in \Sigma_t$ such that $P(C) > 0$ and $0 < a < \varepsilon(\omega) < b$ in C. Consequently, the functions $y_i = \dfrac{1}{\varepsilon}x_i \in L^2(\Sigma_t)$. Define $h' = (h'_1, h'_2 \cdots h'_n)$ by $h'_i = \sup(-y_i, 0) = y_i^-$ for every $i = 1, \cdots, n$.

It is easily verified that $\varphi_C(h') \leq \int_C m\,dP$.

Besides, from y feasible in $(II_{h'}^C)$ we get $\displaystyle\int_C f(x,\omega)\,dP(\omega) = -\int_C \sum_{i=1}^n p_i^t y_i\,dP \leq \varphi_C(h')$.

Thus, combining both inequalities, we obtain a contradiction. Then, $f(x,\omega) \leq m^t(\omega)$ a.e in A.

Assume now that $x = x^*$. Since $m^t(\omega) = -\sum_{i=1}^n x_i^*(\omega)p_i^t(\omega)$, and the fact that $x_i^*(\omega) = -h_i(\omega)$ whenever $i \in S_{x^*,\omega}$ and $h_i^*(\omega) = 0$ otherwise B a.e., we get

$$\sum_{i=1}^n (x_i^*)^-(\omega)p_i^t(\omega) = \sum_{i=1}^n h_i^*(\omega)p_i^t(\omega) = 1$$

Consequently, $f(x^*,\omega) = m^t(\omega)$ □

Finally, the following theorem states that the same portfolio x^* leading to the measures $\mathfrak{M}^t$ and m^t also leads to the measure l^t

Theorem 5.3 *Let $x^* \in (L^2(\Sigma_t))^n$ be such that*

$$m^t(\omega) = -\sum_{i=1}^n x_i^*(\omega)p_i^t(\omega)$$

Then, x^* yields a strong maximum in problem (I) and the equality

$$l^t(\omega) = \frac{m^t(\omega)}{2 - m^t(\omega)}$$

holds.

Proof. We may assume without loss of generality that $x^*(\omega) = 0$ whenever $\sum_{i=1}^n (x_i^*)^-(\omega)p_i^t(\omega) = 0$. Manipulating g it is easy to prove that

$$g(x,\omega) = \frac{f(x,\omega)}{2 - f(x,\omega)}$$

for every $x \in (L^2(\Sigma_t))^n$ and whenever $\sum_{i=1}^n x_i^-(\omega)p_i^t(\omega) \neq 0$. In particular, $g(x^*,\omega) = f(x^*,\omega)\,/\,(2 - f(x^*,\omega))$ holds for almost every $\omega \in \Omega$. Let x be feasible in problem (I). If $\sum_{i=1}^n x_i^-(\omega)p_i^t(\omega) = 0$ and $x(\omega) \neq 0$, then $g(x,\omega) = -1 \leq g(x^*,\omega)$.
If $\sum_{i=1}^n x_i^-(\omega)p_i^t(\omega) \neq 0$ then, from Theorem 5.5 the inequality $f(x,\omega) \leq f(x^*,\omega)$ holds. Besides, $\frac{t}{2-t}$ is a increasing continuous function in $(-\infty, 1]$ $(f(x,\omega), f(x^*,\omega) \leq 1)$. Thus, $g(x,\omega) \leq g(x^*,\omega)$ Finally, the equality $l^t(\omega) = m^t(\omega)\,/\,(2 - m^t(\omega))$ comes from the equality $g(x^*,\omega) = f(x^*,\omega)\,/\,(2 - f(x^*,\omega))$. □

6 Remarks and conclusions

It is of interest to ask how the results given here must be modified if the model does not verify some of the assumptions.

As we noticed in Section 2, the assumptions imposed to p_1^t $(t \in \mathbb{N})$ can be replaced by the assumption that one of the securities always has strictly positive price. We can then measure in the security market model with prices so normalized. Passing from the original to the primed model involves only a change in units on the prices. Since these changes do not affect problems (I) and (X), the same measures are obtained for both models.

We also noticed in Section 3 (see Remark 3 after Theorem 3.4) that we can relax the constraints imposed to p_i^{t+1}. If we only assume that $p_i^{t+1} \in L^2(\Sigma_{t+1})$ for $i = 2, \cdots n$, all the results stated here extend to this new setting except for those derived from the established duality (Section 4).

Suppose now that in A_1 we relax the constraint $p_i^t(\omega) > k_i > 0$ assuming only that $p_i^t(\omega) > 0$. For every $k \in \mathbb{N}$ denote by

$$A^k = \{\omega \in \Omega \mid p_i^t(\omega) > \frac{1}{k},\ i = 1, \cdots n\}$$

Obviously, $\lim_k P(A^k) = 1$.
Set $(p_i^t)_k = p_i^t . \chi_{A^k}$ and consider the programs (II_h^k), (III_h^k) and (IV^k) where each p_i^t is replaced by $(p_i^t)_k$. Choose corresponding zero valued in $\Omega - A^k$ functions h_{ik}^*, λ_{ik}^*, x_{ik}^* and m_k^t. It follows from Theorem 3.7 that $m_j^t . \chi_{A^k} = m_k^t$ for every $j \geq k$. Hence $(m_k^t)_{k \in \mathbb{N}}$ is an a.e. pointwise convergent sequence such that $0 \leq m_k^t \leq 1$. Then,

$$m^t = \lim_{k \in \mathbb{N}} m_k^t \in L^2(\Sigma_t).$$

Note that $\int_\Omega m^t \, dP$ is the optimum value in Program (IV), but this optimum is not necessarily attained in H^t. Nevertheless, for every $\varepsilon > 0$ there exist $A \in \Sigma_t$ such that $P(A) \geq 1 - \varepsilon$, $h_\varepsilon \in H$ such that Program (IV) restricted to A achieves its optimum $\int_A m \, dP$ in h_ε and $x_\varepsilon \in F_{h_\varepsilon}$ such that Program $(II_{h_\varepsilon}^A)$ achieves its optimum in x_ε.

Applications to financial market integration

Chen and Knez (1995) develop a measurement theory of market integration for two markets A and B whenever there exist arbitrage opportunities across them. They assume that the price of each security at time 1 is constant (1 indeed) and that there is no arbitrage opportunity on each market. They use the characterization of the absence of arbitrage (Hansen and Richard (1987)) by the existence of strictly positive admissible stochastic discount factors. Then, they define a strong integration measure for the markets A and B as the L^2-distance between the respective L^2-closures of the sets of strictly positive admissible stochastic discount factors.
Following this idea, Theorem 4.5 allows us to extend their measure to the more general setting of this paper. More precisely, for the markets A and B with respective prices of the n_A and n_B securities,

$$\begin{array}{l} p_i^t \text{ and } p_i^{t+1},\ i = 1, \cdots, n_A \\ q_j^t \text{ and } q_j^{t+1},\ j = 1, \cdots, n_B \end{array}$$

verifying Assumptions A_1 and A_2, the sets

$$\begin{array}{l} F_A = \{\Gamma \in (L^\infty(\Sigma_{t+1}))'_+ \mid \Gamma(p_i^{t+1}) = p_i^t,\ i = 1, \cdots, n_A\} \\ F_B = \{\Gamma \in (L^\infty(\Sigma_{t+1}))'_+ \mid \Gamma(q_j^{t+1}) = q_j^t,\ j = 1, \cdots, n_B\} \end{array}$$

are closed and nonvoid. We extend their measure by

$$a(A, B) = \min_{\Gamma_A \in F_A, \Gamma_B \in F_B} \|\Gamma_A - \Gamma_B\|.$$

Consequently the duality theory established here provides a method to extend their measure to a dynamical setting.
In absence of duality gap for (II_h) and (III_h) and provided the solvability of (III_h) this extension adopts a more convenient expression since we can restrict F_A and F_B to $L^2(\Sigma_t)$, that is,

$$F_A = \{f \in L^2_+(\Sigma_{t+1}) \mid E(fp_i^{t+1} \mid \Sigma_t) = p_i^t,\ i = 1, \cdots, n_A\}$$
$$F_B = \{f \in L^2_+(\Sigma_{t+1}) \mid E(fq_j^{t+1} \mid \Sigma_t) = q_j^t,\ j = 1, \cdots, n_B\}$$

The measure m^t is an alternative to the strong integration measure. In fact, treating both markets A and B as parts of the one combined market, we can consider the measure m^t of this global market as a measure of integration between both markets.

Having in mind that the results of Section 4 show that m^t can be described from state prices and proceeding as in Theorem 13 of Balbás and Muñoz (1996), it can be proved that the measure $\mathfrak{M}^t$ is continuous with respect to the given extension of the Chen and Knez measure.

Conclusions

Two new stochastic measures of the arbitrage opportunities have been provided. The methodology is quite general and may be applied in any discrete time model. No special assumptions are imposed on the stochastic process applied to price the different securities.

The measures quantify the lack of the absence of arbitrage in (relative) monetary terms. Then, transaction costs can be discounted. Hence, the existence of arbitrage can be tested in practical situations taking into account the transaction costs.

The theory may be applied to measure the integration between two or more financial markets. We follow the approach of many authors in which two markets are perfectly integrated if there is no cross-market arbitrage opportunity. We then consider the measure of the efficiency of the combined market as a measure of the integration. Consequently, we get two stochastic measures of the level of integration. This is an important fact if we have in mind that the usual models applied to price derivatives are dynamic models, and the underlying assets are available in different markets.

The measures can be also obtained from the discount factors. This allows to relate them with some extensions of other measures appeared in previous literature.

The main results are also interesting from a mathematical point of view. In particular some topics in Operational Research are involved: we prove some results about the absence of duality gap for a dual pair of infinite-dimensional linear optimization problems and we solve a vector optimization problem with a non differentiable objective function and prove some properties about its sensitivity.

References

[1] Anderson, E.J., and P. Nash, 1987, *Linear Programming in Infinite-Dimensional Spaces*, John Wiley & Sons, New York.

[2] Balbás, A., and P.J. Guerra, 1996, "Sensitivity Analysis for Convex Multiobjective Programming in Abstract Spaces," *Journal of Mathematical Analysis and Applications*, 202, 645-658.

[3] Balbás, A., and A. Heras, 1993, "Duality Theory for Infinite-Dimensional Multiobjective Linear Programming," *European Journal of Operational Research*, 68, 379-388.

[4] Balbás, A., and M.J. Muñoz, 1996, "Measuring the Degree of Fulfillment of the Law of One Price. Applications to Financial Markets Integration," Working Paper 75, Economic Series 31, Universidad Carlos III de Madrid.

[5] Chamberlain, G., and M. Rothschild, 1983, "Arbitrage, Factor Structure, and Mean-Variance Analysis on Large Assets," *Econometrica*, 51, 1281-1304.

[6] Chen, Z., and P.J. Knez, 1995, "Measurement of Market Integration and Arbitrage," *The Review of Financial Studies*, Vol. 8, 2, 545-560.

[7] Hansen, L., and S. Richard, 1987, "The Role of Conditioning Information in Deducing Testable Restrictions Implied by Dynamic Asset Pricing Models," *Econometrica*, 55, 587-613.

[8] Harris, F.H., T. H. McInish, G.L. Shoesmith, and R.A. Wood, 1995, "Cointegration, Error Correction, and Price Discovery on Informationally Linked Security Markets," *Journal of Financial and Quantitative Analysis*, 30, 4, 563-579.

[9] Harrison, J., and D. Kreps, 1979, "Martingales and Arbitrage in Multiperiod Securities Markets." *Journal of Economic Theory*, 20, 381-408.

[10] Ingersoll, J.E., Jr, 1987, *Theory of Financial Decision Making*, Rowman & Littlefield Publishers, Inc.

[11] Jouini, E., and H. Kallal, 1995, "Martingales and Arbitrage in Securities Markets with Transaction Costs," *Journal of Economic Theory*, 66, 178-197.

[12] Kamara, A., and T.W. Miller, Jr, 1995, "Daily and Intradaily Tests of European Put-Call Parity," *Journal of Financial and Quantitative Analysis*, 30, 4, 519-541.

[13] Kempf, A., and O. Korn, 1996, "Trading System and Market Integration," *Proceedings of the 6th International AFIR-Colloquium*, 2, 1709-1728.

[14] Khanh, P.O., 1995, "Sufficient Optimality Conditions and Duality in Vector Optimization With Invex-Convexlike Functions," *Journal of Optimization. Theory and Applications*, 87, 359-368.

[15] Kleidon, A.W., and R.E. Whaley, 1992, "One market? Stocks, Futures, and Options During October 1987," *The Journal of Finance*, 83, 8, 851-877.

[16] Kuk, H., J. Tanino, and M. Tanaka, 1996, "Sensitivity Analysis in Vector Optimization," *Journal of Optimization. Theory and Applications*, 89, 713-730.

[17] Lee, J.H., and N. Nayar, 1993, " A Transactions Data Analysis of Arbitrage between Index Options and Index Futures," *The Journal of Future Markets*, 13, 8, 889-902.

[18] Luenberger, D.G.,1969, *Optimization by Vector Spaces Methods*, John Wiley & Sons, New York.

[19] Moulin, H., 1979, *Fondation de la Théorie de Jeux*, Hermann, Paris.

[20] Prisman, E.Z, 1986, "Valuation of Risky Assets in Arbitrage Free Economies with Frictions," *The Journal of Finance*, 41, 3, 545-56.

[21] Protopapadakis A., and H.R. Stoll, 1983, "Spot and Future Prices and the Law of One Price," *The Journal of Finance*, 38, 5, 1431-1455.

[22] Zowe S. ,1975, "A Duality Theorem for a Convex Programming Problem in Order Complete Vector Lattices," *Journal of Mathematical Analysis and Applications*, 50, 283-287.

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