ARTICLE IN PRESS

Available online at www.sciencedirect.com
ScienceDirect

Journal of Economic Theory ■■■ (■■■■) ■■■–■■■

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

Note

On the generic finiteness of equilibrium outcome distributions in bimatrix game forms☆

Nikolai S. Kukushkin[a], Cristian M. Litan[b], Francisco Marhuenda[b,*]

[a] *Russian Academy of Sciences, Dorodnicyn Computing Center, Moscow, Russia*
[b] *Department of Economics, University Carlos III de Madrid, Madrid, Spain*

Received 23 May 2007; final version received 30 May 2007

Abstract

We provide an example of an outcome game form with two players for which there is an open set of utilities for both players such that, in each of the associated games, the set of Nash equilibria induces a continuum of outcome distributions.

JEL classification: C72

Keywords: Generic finiteness; Nash equilibrium

1. Introduction

Harsanyi [2] shows that, generically in the space of utilities, a finite normal form game has a finite (and odd) number of mixed equilibria. Kreps and Wilson [3] note that when the normal form is derived from an extensive form, Harsanyi's result has no immediate implications, because many strategies lead to the same final node. Even if the payoffs at the final nodes can be perturbed independently, the finiteness of the number of equilibria need not be a generic property; appropriate counterexamples are easy to construct. Thus, Harsanyi's result cannot be applied directly to many economic models of interest.

☆ This problem was communicated to us by Francesco De Sinopoli.

* Corresponding author.
E-mail addresses: ququns@pochta.ru (N.S. Kukushkin), clitan@eco.uc3m.es (C.M. Litan), marhuend@eco.uc3m.es (F. Marhuenda).

0022-0531/$ - see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.jet.2007.05.007

For games in extensive form, Kreps and Wilson [3] prove that for generic utility payoffs on the terminal nodes, the Nash equilibria induce finitely many probability distributions on the same set of terminal nodes. Based on techniques from the theory of semi-algebraic sets, Govindan and Wilson [4] provide a more direct proof of the results above, for normal as well as for extensive form games.

On the other hand, Mas-Colell [1] and later Govindan and McLennan [5] point out that, in many economic instances, different strategy profiles generate the same outcome; moreover, this may be true for different terminal nodes of an extensive game. Thus, the same criticism can be reiterated for the Kreps and Wilson [3] result.

A suitable general setup to address the above problem is obtained with the notion of an outcome game form, i.e., a mapping from strategy profiles to outcomes. We can now pose the question as follows: assuming that the utility functions on outcomes may be independently perturbed, is it true that, generically, the equilibrium strategies of the associated game induce a finite number of probability distributions on the set of outcomes?

When there are at least three players, Govindan and McLennan [5] provide a negative answer to the above question. They show an example of a game form with six outcomes where, for an open set of utility functions, the Nash equilibria induce a continuum of outcome distributions. On the positive side and for games with any number of players, Govindan and McLennan [5] prove that if there are only two outcomes, generic finiteness of equilibrium outcome distributions is restored.

So far, for two person game forms, partial evidence had been obtained that suggested a possible positive answer to the question above. Mas-Colell [1] shows that the equilibrium payoffs are generically finite for two player game forms. Govindan and McLennan [6] prove the generic finiteness of equilibrium distributions on outcomes for zero sum games and common interest games. González-Pimienta [7] argues that the same result holds for games with two players and three outcomes.

This note closes the last remaining gap in the case of two players, in a rather unexpected way. We provide an example of a game form (with four outcomes) and an open set of utility functions such that in each of the associated games there is a continuum of outcome distributions induced by Nash equilibria. By introducing dummy players, our example can be extended, in an obvious way, to any number of players and four outcomes. The results in González-Pimienta [7] and Govindan and McLennan [5] show that this is the minimum number of outcomes needed for a counterexample.

2. The conjecture

Let Ω be a finite outcome space and let S_1, S_2 denote the finite strategy spaces of each of the two players. An outcome game form is a mapping $\theta : S_1 \times S_2 \to \Omega$. The utilities of the players are defined by the functions $u_i : \Omega \to \mathbb{R}$, $i = 1, 2$. For each $i = 1, 2$, let $\Delta_i = \{\mu \in \mathbb{R}^{S_i}_{++} : \sum_{x \in S_i} \mu(x) = 1\}$. A pair of strategy vectors $p_i \in \Delta_i$ of the players induces a probability distribution in Ω.

An outcome game form θ and the utilities of the players u_i, $i = 1, 2$ determine a bimatrix game. We denote by $A(u_i)$ the associated matrices of utilities of the players in this game. That is, the entry a_{jl} of $A(u_i)$ is $u_i(\theta(s_j, s_l))$. A completely mixed Nash equilibrium in this game consists of two strategy vectors $p_1 \in \Delta_1$ and $p_2 \in \Delta_2$ such that

$$A(u_2)p_1 = \beta e \quad \text{for some } \beta \in \mathbb{R}$$

Please cite this article as: N.S. Kukushkin, et al., On the generic finiteness of equilibrium outcome distributions in bimatrix game forms Journal of Economic Theory (2007), doi: 10.1016/j.jet.2007.05.007

and

$$p_2 A(u_1) = \alpha e \quad \text{for some } \alpha \in \mathbb{R},$$

where e denotes the vector (in the appropriate Euclidean space) with all of its entries equal to 1.

Conjecture 1. For every bimatrix game form θ, there is a generic set of utilities in $\mathbb{R}^\Omega \times \mathbb{R}^\Omega$ for which the set of distributions induced on outcomes by the *completely mixed Nash equilibria* is finite.

For the case of two person games, this conjecture corresponds to Conjecture T in Govindan and McLennan [5]. They disprove the conjecture for games with at least three players.

3. The example

We now give an example that shows that Conjecture 1 does not hold. Let there be four outcomes, denoted by $\Omega = \{a, b, c, d\}$. And consider the outcome matrix

$$A(a,b,c,d) = \begin{pmatrix} c & a & b & b \\ d & a & a & b \\ c & d & b & c \end{pmatrix}.$$

Let us use the notation $a_i = u_i(a)$, $b_i = u_i(b)$, $c_i = u_i(c)$, $d_i = u_i(d)$, and $u^i = (a_i, b_i, c_i, d_i) \in \mathbb{R}^4$ for the utilities of agent $i = 1, 2$. We denote $G = \{(u^1, u^2) \in \mathbb{R}^8 \mid d_1, b_1 < a_1, c_1 \text{ and } d_2 < b_2 < a_2, c_2\}$, an open subset in the space of utilities. For each $(u^1, u^2) \in G$ and $t \in \mathbb{R}$, we define

$$p_1(u^2) = \frac{1}{a_2 - b_2 + c_2 - d_2}(b_2 - d_2, c_2 - b_2, a_2 - b_2) \in \mathbb{R}^3$$

and

$$p_2(u^1; t) = \left(\frac{a_1 - b_1}{a_1 - b_1 + c_1 - d_1} - \frac{(a_1 - b_1)t}{a_1 - d_1}, \frac{(c_1 - b_1)t}{a_1 - d_1}, \right. \\ \left. \frac{c_1 - d_1}{a_1 - b_1 + c_1 - d_1} - \frac{(c_1 - d_1)t}{a_1 - d_1}, t \right) \in \mathbb{R}^4.$$

One checks immediately that the pair $\langle p_1(u^2), p_2(u^1; t) \rangle$ is a completely mixed Nash equilibrium provided t is positive and small enough. The probability of outcome a induced by the equilibrium is computed easily as

$$p_a = \frac{(b_2 - c_2)(c_1 - d_1)}{(a_1 - b_1 + c_1 - d_1)(-a_2 + b_2 - c_2 + d_2)} \\ + \frac{b_2(d_1 - c_1) + b_1(c_2 - d_2) + c_1 d_2 - c_2 d_1}{(a_1 - d_1)(-a_2 + b_2 - c_2 + d_2)} t.$$

Therefore, there is a continuum of equilibrium probability distributions (i.e. for t positive and small enough) on the set of outcomes as long as $(u^1, u^2) \in G$ and $b_2(d_1 - c_1) + b_1(c_2 - d_2) + c_1 d_2 - c_2 d_1 \neq 0$; such values of u^1 and u^2 form an open subset in the space of utilities. Remark that when $u^1 = \pm u^2$, the coefficient of t in the above expression for p_a (as well as for the other outcomes) disappears and there is a unique probability distribution on outcomes, as proved in Govindan and McLennan [6] and Litan [8].

Please cite this article as: N.S. Kukushkin, et al., On the generic finiteness of equilibrium outcome distributions in bimatrix game forms Journal of Economic Theory (2007), doi: 10.1016/j.jet.2007.05.007

Acknowledgments

We thank Andreu Mas-Colell for providing us with the manuscript [1]. We are grateful to an anonymous associate editor for pointing out some typos in a previous version of the manuscript. The authors acknowledge financial support from the project No. SEJ2004-00968/ECON from the Spanish Ministry of Education and Science. C. Litan acknowledges financial support from "Plan de Formación de Personal Investigador" of the Comunidad de Madrid.

References

[1] A. Mas-Colell, Generic finiteness of equilibrium payoffs for bimatrix games, Mimeo, Harvard University.
[2] J.C. Harsanyi, Oddness of the number of equilibrium points: a new proof, Int. J. Game Theory 2 (1973) 235–250.
[3] D.M. Kreps, R. Wilson, Sequential equilibria, Econometrica 50 (1982) 863–894.
[4] S. Govindan, R. Wilson, Direct proofs of generic finiteness of Nash equilibrium outcomes, Econometrica 69 (2001) 765–769.
[5] S. Govindan, A. McLennan, On the generic finiteness of equilibrium outcome distributions in game forms, Econometrica 69 (2001) 455–471.
[6] S. Govindan, A. McLennan, Generic finiteness of outcome distributions for two person game forms with zero sum and common interest utilities, Mimeo, University of Western Ontario.
[7] C. González-Pimienta, Generic finiteness of outcome distributions for two person game forms with three outcomes, Mimeo, University Carlos III.
[8] C. M. Litan, On the generic determinacy in bimatrix game forms, Mimeo, University Carlos III.

Please cite this article as: N.S. Kukushkin, et al., On the generic finiteness of equilibrium outcome distributions in bimatrix game forms Journal of Economic Theory (2007), doi: 10.1016/j.jet.2007.05.007