mathematics

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A Look at Generalized Degenerate Bernoulli and Euler Matrices

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Abstract: In this paper, we consider the generalized degenerate Bernoulli/Euler polynomial matrices and study some algebraic properties for them. In particular, we focus our attention on some matrix-inversion formulae involving these matrices. Furthermore, we provide analytic properties for the so-called generalized degenerate Pascal matrix of the first kind, and some factorizations for the generalized degenerate Euler polynomial matrix.

Keywords: generalized degenerate Bernoulli polynomials; generalized degenerate Euler polynomials; generalized degenerate Bernoulli matrix; generalized degenerate Euler matrix; generalized degenerate Pascal matrix

MSC: 33E20; 11B83; 11B68

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Citation: Hernández, J.; Peralta, D.; Quintana, Y. A Look at Generalized Degenerate Bernoulli and Euler Matrices. *Mathematics* **2023**, *11*, 2731. https://doi.org/10.3390/math11122731

Academic Editor: Sitnik Sergey

Received: 24 May 2023
Revised: 14 June 2023
Accepted: 14 June 2023
Published: 16 June 2023

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1. Introduction

Matrices play an important role in all branches of science, engineering, social science, and management. In many settings (see, e.g., [1–4] and the references therein), a number of interesting and useful identities involving binomial (q-binomial or λ-binomial) coefficients can be obtained from a matrix representation of a particular counting sequence. Such a matrix representation provides a powerful computational tool for deriving identities and an explicit formula related to the sequence.

There are many special types of matrices such as Pascal, Vandermonde, Stirling, Riordan arrays, and others. These matrices are of specific importance in many scientific and engineering applications. For instance, Pascal matrices appear in combinatorics, image processing, signal processing, numerical analysis, probability, and surface reconstruction.

In the case of generalized Pascal matrices of the first kind, extensive research has been devoted to them (cf., e.g., [3–10] and the references therein). Situations with a matrix representation—including analogs of generalized Pascal matrices of the first kind and degenerate versions of special classes of polynomials (e.g., Bernstein, Bernoulli, and Euler polynomials, etc.)—are of particular interest.

Motivated by recent articles [1–4,11–14] that consider degenerate Bernstein polynomials, degenerate Euler polynomials, generalized degenerate Euler–Genocchi polynomials of order α, and algebraic properties of the generalized Euler and generalized Apostol-type polynomial matrices, in the present article, we consider the generalized degenerate Bernoulli/Euler polynomial matrix. In particular, we focus our attention on some inversion-type formulae from a matrix framework. Furthermore, we show some analytic properties for the so-called generalized degenerate Pascal matrix of the first kind. Furthermore, some factorizations for the generalized degenerate Euler polynomial matrix in terms of such a matrix are given.

The paper is organized as follows. Section 2 is a preliminary section containing the definitions, notations, and terminology needed. Section 3 contains the main results of this paper. First, we provide the corresponding inversion-type formulae for the degenerate Bernoulli and Euler polynomials, respectively (Theorems 1 and 2). Second, we show that the generalized degenerate Pascal matrix of the first kind is a matrix exponential (Theorem 4), and, as a consequence, we obtain an Appell-type property for this matrix (Corollary 5). In addition, factorizations for the generalized degenerate Pascal matrix of the first kind in terms of the degenerate Bernoulli/Euler matrices are given (Theorems 6 and 7, respectively). The remainder of this section is devoted to establishing the corresponding product formulae for generalized degenerate Euler polynomial matrices and their factorizations in terms of generalized degenerate Pascal matrices of the first kind (Theorems 8 and 9).

2. Background and Previous Results

Throughout this paper, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote, respectively, the set of all natural numbers, the set of all non-negative integers, the set of all integers, the set of all real numbers, and the set of all complex numbers. As usual, we will always use the principal branch for complex powers, in particular, $1^\alpha = 1$ for $\alpha \in \mathbb{C}$. Furthermore, the convention $0^0 = 1$ will be adopted.

For $w \in \mathbb{C}$ and $k \in \mathbb{Z}$, we use the notations $w^{(k)}$ and $(w)_k$ for the rising and falling factorials, respectively, i.e.,

$$w^{(k)} = \begin{cases} 1, & \text{if } k = 0, \\ \prod_{i=1}^{k}(w+i-1), & \text{if } k \geq 1, \\ 0, & \text{if } k < 0, \end{cases}$$

and

$$(w)_k = \begin{cases} 1, & \text{if } k = 0, \\ \prod_{i=1}^{k}(w-i+1), & \text{if } k \geq 1, \\ 0, & \text{if } k < 0. \end{cases}$$

Any matrix is assumed an element of $M_{n+1}(\mathbb{R})$, the set of all $(n+1)$-square matrices over the real field $\mathbb{R}$. Moreover, for i, j, any nonnegative integers, and any matrix $A \in M_{n+1}(\mathbb{R})$ we adopt, respectively, the following conventions

$$\binom{i}{j} = 0, \text{ whenever } j > i, \quad \text{and} \quad A^0 = I_{n+1} = \mathrm{diag}(1, 1, \ldots, 1),$$

where I_{n+1} denotes the identity matrix of order $n+1$.

For $\lambda, x \in \mathbb{R}$ and $z \in \mathbb{C}$, the degenerate exponentials are defined as follows (cf., [15]):

$$e_\lambda^x(z) = \begin{cases} (1+\lambda z)^{\frac{x}{\lambda}}, & \text{if } \lambda \in \mathbb{R} \setminus \{0\}, \\ e^{xz}, & \text{if } \lambda = 0. \end{cases} \tag{1}$$

As usual, for $x = 1$, we use the notation $e_\lambda(z) = e_\lambda^x(z)$.

It follows immediately from (1) that

$$e_\lambda^x(z) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{z^n}{n!}, & |\lambda z| < 1, \quad \text{if } \lambda \in \mathbb{R} \setminus \{0\}, \\ \displaystyle\sum_{n=0}^{\infty} x^n \frac{z^n}{n!}, & \text{if } \lambda = 0. \end{cases} \tag{2}$$

where the generalized falling factorials $(x)_{n,\lambda}$, are given by (cf., [1,2,12–15]):

$$(x)_{n,\lambda} = \begin{cases} 1, & \text{if } n = 0, \\ \prod_{i=1}^{n}(x-(i-1)\lambda), & \text{if } n \geq 1, \\ 0, & \text{if } n < 0, \end{cases}$$

where $x, \lambda \in \mathbb{R}$ and $n \in \mathbb{Z}$.

It is clear that $\lim_{\lambda \to 0} e_\lambda^x(z) = e_0^x(z) = e^{xz}$, and for $n \in \mathbb{N}_0$, the polynomial in two variables $Q_n(x,\lambda)$, given by

$$Q_n(x,\lambda) = \begin{cases} 1, & \text{if } n = 0, \\ \prod_{i=1}^{n}(x-(i-1)\lambda), & \text{if } n \geq 1, \end{cases}$$

is a continuous function on $\mathbb{R}^2$, and consequently, $(x)_{n,0} = x^n$.

In [16,17], Carlitz introduced the degenerate Bernoulli (Euler) and the generalized degenerate Bernoulli (Euler) polynomials of order $\alpha \in \mathbb{C}$, respectively, by means of the generating functions and series expansions:

$$\frac{z}{e_\lambda(z)-1}e_\lambda^x(z) = \sum_{n=0}^{\infty} \mathscr{B}_{n,\lambda}(x)\frac{z^n}{n!}, \tag{3}$$

$$\frac{2}{e_\lambda(z)+1}e_\lambda^x(z) = \sum_{n=0}^{\infty} \mathscr{E}_{n,\lambda}(x)\frac{z^n}{n!}, \tag{4}$$

$$\left(\frac{z}{e_\lambda(z)-1}\right)^{\alpha} e_\lambda^x(z) = \sum_{n=0}^{\infty} \mathscr{B}_{n,\lambda}^{(\alpha)}(x)\frac{z^n}{n!}, \tag{5}$$

$$\left(\frac{2}{e_\lambda(z)+1}\right)^{\alpha} e_\lambda^x(z) = \sum_{n=0}^{\infty} \mathscr{E}_{n,\lambda}^{(\alpha)}(x)\frac{z^n}{n!}. \tag{6}$$

These are valid in a suitable neighborhood of $z = 0$ and represent degenerate versions of the classical Bernoulli and Euler polynomials, respectively. In [8], the notation $\beta_n(\lambda, x)$ is used for the degenerate Bernoulli (3).

Since the degenerate exponentials (1) satisfy the same exponent product law as the exponentials functions, i.e.,

$$e_\lambda^{x+y}(z) = e_\lambda^x(z)\, e_\lambda^y(z),$$

we can use the generating relations (2), (5) and (6) to deduce the following addition formulas:

$$(x+y)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} (x)_{k,\lambda}(y)_{n-k,\lambda}, \quad n \geq 0, \tag{7}$$

$$\mathscr{B}_{n,\lambda}^{(\alpha+\beta)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \mathscr{B}_{k,\lambda}^{(\alpha)}(x)\mathscr{B}_{n-k,\lambda}^{(\beta)}(y), \quad n \geq 0, \tag{8}$$

$$\mathscr{E}_{n,\lambda}^{(\alpha+\beta)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \mathscr{E}_{k,\lambda}^{(\alpha)}(x)\mathscr{E}_{n-k,\lambda}^{(\beta)}(y), \quad n \geq 0. \tag{9}$$

For a treatment of diverse aspects of some summation formulas and their applications, the interested reader is referred to the relatively recent works [18–20].

For $r \in \mathbb{N}_0$, $\lambda \in \mathbb{R}$, and $\alpha \in \mathbb{C}$, definitions of generalized degenerate Euler–Genocchi and generalized degenerate Euler–Genocchi polynomials of order α, respectively, have recently been introduced in [14] (Section 2):

$$\frac{2z^r}{e_\lambda(z)+1}e_\lambda^x(z) = \sum_{n=0}^{\infty} \mathscr{A}_{n,\lambda}^{(r)}(x)\frac{z^n}{n!}, \tag{10}$$

$$z^r\left(\frac{2}{e_\lambda(z)+1}\right)^{\alpha} e_\lambda^x(z) = \sum_{n=0}^{\infty} \mathscr{A}_{n,\lambda}^{(r,\alpha)}(x)\frac{z^n}{n!}. \tag{11}$$

Remark 1. *Notice that:*

(i) *If $r \in \mathbb{N}$, then it follows immediately from (2), (4) and (10), that* $\mathscr{A}_{0,\lambda}^{(r)}(x) = \mathscr{A}_{1,\lambda}^{(r)}(x) = \cdots = \mathscr{A}_{r-1,\lambda}^{(r)}(x) = 0$, *and*

$$\mathscr{A}_{n,\lambda}^{(r)}(x) = \frac{n!}{(n-r)!}(x)_{n,\lambda} = n^{(r)}\mathscr{E}_{n-r,\lambda}^{(0)}(x), \quad n \geq r.$$

Furthermore, $\mathscr{A}_{n,\lambda}^{(0)}(x) = \mathscr{E}_{n,\lambda}(x), \quad n \geq 0.$
The first above identities guarantee that, up to multiplicative constants, it suffices to take generalized degenerate Euler polynomials of order 0 *instead of the so-called generalized degenerate Euler–Genocchi polynomials as the main family to study. Similarly, the last identity tells us that the generalized degenerate Euler polynomials coincides with the generalized degenerate Euler–Genocchi polynomials of order* 0.

(ii) *In [14], Theorem 4 proves the following reduction formula:*

$$\mathscr{A}_{n,\lambda}^{(r,\alpha)}(x) = n^{(r)}\mathscr{E}_{n-r,\lambda}^{(\alpha)}(x), \quad n \geq r,\ n,r \in \mathbb{N}_0.$$

In particular, we obtain that up to multiplicative constants, the generalized degenerate Euler–Genocchi polynomials of order $\alpha = 1$ *can be reduced to the generalized degenerate Euler polynomials (4).*

Hence, in order to avoid essentially redundant definitions (cf., [21]), the families of polynomials eqrefeul-gen1 and (11) will not be considered in this paper.

3. The Generalized Degenerate Bernoulli and Euler Matrices and Their Properties

In this section, we present some novel properties for the generalized degenerate Bernoulli and Euler matrices. Before that, we show the corresponding inversion-type formulae for the generalized degenerate Bernoulli and Euler polynomials, respectively.

Theorem 1. *For every $n \geq 0$ and $\lambda \in \mathbb{R}$, the degenerate Bernoulli polynomials satisfy the following inversion-type formula:*

$$(x)_{n,\lambda} = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}(1)_{k+1,\lambda}\mathscr{B}_{n-k,\lambda}(x) \tag{12}$$

$$= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}(1-\lambda)_{k,\lambda}\mathscr{B}_{n,\lambda}(x). \tag{13}$$

Proof. Let $\lambda \in \mathbb{R}$. In view of (2) and (3), and the identity

$$z\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{z^n}{n!} = \sum_{n=0}^{\infty}(n+1)(x)_{n,\lambda}\frac{z^{n+1}}{(n+1)!},$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)(x)_{n,\lambda}\frac{z^{n+1}}{(n+1)!} &= \left[\sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{z^n}{n!}-1\right]\left[\sum_{n=0}^{\infty}\mathscr{B}_{n,\lambda}(x)\frac{z^n}{n!}\right] \\ &= \left[\sum_{n=0}^{\infty}(1)_{n+1,\lambda}\frac{z^{n+1}}{(n+1)!}\right]\left[\sum_{n=0}^{\infty}\mathscr{B}_{n,\lambda}(x)\frac{z^n}{n!}\right]. \end{aligned} \tag{14}$$

From the use of the Cauchy product rule on the right-hand side of (14), it follows that

$$\sum_{n=0}^{\infty}(n+1)(x)_{n,\lambda}\frac{z^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n+1}{k+1}(1)_{k+1,\lambda}\mathscr{B}_{n-k,\lambda}(x)\right]\frac{z^{n+1}}{(n+1)!}. \tag{15}$$

Hence, comparing the coefficients of z^{n+1} on both sides of (15), we obtain (12).

Finally, (13) is a simple consequence of the identity $(1)_{k+1,\lambda} = (1-\lambda)_{k,\lambda}$, for all $k \in \mathbb{N}_0$. □

Remark 2. *Notice that the substitution of $\lambda = 0$ into (12) recovers the inversion formula for the classical Bernoulli polynomials (cf., [22] (Equation (9))).*

From a matrix framework, Theorem 1 has the following consequence.

Corollary 1. *For $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$, the matrix $\mathbf{T}_\lambda(x) = \begin{pmatrix}1 & (x)_{1,\lambda} & \cdots & (x)_{n,\lambda}\end{pmatrix}^T$ can be expressed as follows:*

$$\begin{aligned}\mathbf{T}_\lambda(x) &= \mathbf{M}_\lambda\mathbf{B}_\lambda(x) \\ &= \begin{pmatrix} \binom{1}{1}(1)_{1,\lambda} & 0 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{2}{2}(1)_{2,\lambda} & \frac{1}{2}\binom{2}{1}(1)_{1,\lambda} & 0 & \cdots & 0 \\ \frac{1}{3}\binom{3}{3}(1)_{3,\lambda} & \frac{1}{3}\binom{3}{2}(1)_{2,\lambda} & \frac{1}{3}\binom{3}{1}(1)_{1,\lambda} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1}\binom{n+1}{n+1}(1)_{n+1,\lambda} & \frac{1}{n+1}\binom{n+1}{n}(1)_{n,\lambda} & \frac{1}{n+1}\binom{n+1}{n-1}(1)_{n-1,\lambda} & \cdots & \frac{1}{n+1}\binom{n+1}{1}(1)_{1,\lambda} \end{pmatrix}\mathbf{B}_\lambda(x) \\ &= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{2}(1)_{2,\lambda} & 1 & 0 & \cdots & 0 \\ \frac{1}{3}(1)_{3,\lambda} & (1)_{2,\lambda} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1}(1)_{n+1,\lambda} & (1)_{n,\lambda} & \frac{1}{2}(1)_{n-1,\lambda} & \cdots & 1 \end{pmatrix}\mathbf{B}_\lambda(x), \end{aligned} \tag{16}$$

where $\mathbf{B}_\lambda(x) = \begin{pmatrix}\mathscr{B}_{0,\lambda}(x) & \mathscr{B}_{1,\lambda}(x) & \cdots & \mathscr{B}_{n,\lambda}(x)\end{pmatrix}^T$.

Theorem 2. *For every $n \geq 0$ and $\lambda \in \mathbb{R}$. The degenerate Euler polynomials satisfy the following inversion-type formula:*

$$(x)_{n,\lambda} = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}(1+a_k(\lambda))(1)_{k,\lambda}\mathscr{E}_{n-k,\lambda}(x) \tag{17}$$

where

$$a_k(\lambda) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } 1 \leq k \leq n, \end{cases}$$

Proof. From (2) and (4) we have

$$\begin{aligned}
2\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{z^n}{n!} &= \left[\sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{z^n}{n!}+1\right]\left[\sum_{n=0}^{\infty}\mathscr{E}_{n,\lambda}(x)\frac{z^n}{n!}\right] \\
&= \left[\sum_{n=0}^{\infty}(1+a_k(\lambda))(1)_{n,\lambda}\frac{z^n}{n!}\right]\left[\sum_{n=0}^{\infty}\mathscr{E}_{n,\lambda}(x)\frac{z^n}{n!}\right] \\
&= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(1+a_k(\lambda))\binom{n}{k}(1)_{k,\lambda}\mathscr{E}_{n-k,\lambda}(x)\right]\frac{z^n}{n!},
\end{aligned}$$

where

$$a_k(\lambda) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } 1 \le k \le n. \end{cases}$$

Therefore, by comparing the coefficients of z^n on both sides, we obtain the identity. □

Remark 3. *Notice that if $\lambda = 0$ in (17), then we recover the inversion formula for the classical Euler polynomials (cf., [22] (Equation (27))).*

Theorem 2 has the following consequence.

Corollary 2. *For $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$, the matrix $\mathbf{T}_\lambda(x) = \begin{pmatrix}1 & (x)_{1,\lambda} & \cdots & (x)_{n,\lambda}\end{pmatrix}^T$ can be expressed as follows:*

$$\begin{aligned}
\mathbf{T}_\lambda(x) &= \frac{1}{2}\mathbf{N}_\lambda\mathbf{E}_\lambda(x) \\
&= \frac{1}{2}\begin{pmatrix}
\binom{0}{0}(1+a_0(\lambda))(1)_{0,\lambda} & 0 & \cdots & 0 \\
\binom{1}{1}(1+a_1(\lambda))(1)_{1,\lambda} & \binom{1}{0}(1+a_0(\lambda))(1)_{0,\lambda} & \cdots & 0 \\
\binom{2}{2}(1+a_2(\lambda))(1)_{2,\lambda} & \binom{2}{1}(1+a_1(\lambda))(1)_{1,\lambda} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\binom{n}{n}(1+a_n(\lambda))(1)_{n,\lambda} & \binom{n}{n-1}(1+a_{n-1}(\lambda))(1)_{n-1,\lambda} & \cdots & \binom{n}{0}(1+a_0(\lambda))(1)_{0,\lambda}
\end{pmatrix}\mathbf{E}_\lambda(x) \\
&= \frac{1}{2}\begin{pmatrix}
2 & 0 & 0 & 0 & \cdots & 0 \\
(1)_{1,\lambda} & 2 & 0 & 0 & \cdots & 0 \\
(1)_{2,\lambda} & 2(1)_{1,\lambda} & 2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(1)_{n,\lambda} & n(1)_{n-1,\lambda} & \frac{(n)_2}{2!}(1)_{n-2,\lambda} & \frac{(n)_3}{3!}(1)_{n-3,\lambda} & \cdots & 2
\end{pmatrix}\mathbf{E}_\lambda(x),
\end{aligned} \tag{18}$$

where $\mathbf{E}_\lambda(x) = \begin{pmatrix}\mathscr{E}_{0,\lambda}(x) & \mathscr{E}_{1,\lambda}(x) & \cdots & \mathscr{E}_{n,\lambda}(x)\end{pmatrix}^T$ and $a_k(\lambda) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } 1 \le k \le n. \end{cases}$

Clearly, when $\lambda \in \mathbb{R}$, the matrix $\mathbf{N}_\lambda$ is an invertible matrix.

Corollary 3. *For $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$, we have*

$$\mathbf{E}_\lambda(x) = 2(\mathbf{N}_\lambda)^{-1}\mathbf{M}_\lambda\mathbf{B}_\lambda(x).$$

The degenerate Pascal matrices corresponding to the generalized falling factorials can be defined as follows:

Definition 1. *Let x be any nonzero real number. For $\lambda \in \mathbb{R}$, the generalized degenerate Pascal matrix of the first kind $P_\lambda[x]$, is an $(n+1) \times (n+1)$ matrix whose entries are given by*

$$p_{i,j,\lambda}(x) := \begin{cases} \binom{i}{j}(x)_{i-j,\lambda}, & i \geq j, \\ 0, & \text{otherwise}. \end{cases} \tag{19}$$

Remark 4.

(i) *It is clear that the matrix $P_\lambda[x]$ tends to the generalized Pascal matrix of the first kind $P[x]$ as $\lambda \to 0$.*

(ii) *For $n \in \mathbb{N}_0$, $x \in \mathbb{R} \setminus \{0\}$, $\lambda \in \mathbb{R}$, it is clear that $P_{-\lambda}[x] = P_{n,\lambda}[x]$, where $P_{n,\lambda}[x]$ is the Pascal functional matrix introduced in [5]. Hence, all results corresponding to $P_{-\lambda}[x]$ given in [5] hold in this setting.*

(iii) *It is worth mentioning that the matrix entries (19) coincide with the entries of the variation of Pascal functional matrix $\mathscr{P}_n[x, \lambda]$ introduced by Can and Cihat-Dağli in [8]. Hence, all results corresponding to factorizing the matrix $\mathscr{P}_n[x, \lambda]$ by the summation matrices also hold for $P_\lambda[x]$, taking into account the suitable shift on the respective order for these matrices (cf., [8] (Lemma 1 and Theorem 2)).*

(iv) *If for $x \in \mathbb{R} \setminus \{0\}$, $\lambda \in \mathbb{R}$ we consider the truncated exponential generating function for the binomial-type polynomial sequence $\{(x)_{n,\lambda}\}_{n\geq 0}$ (cf., [9]):*

$$f(t;x) = \sum_{k=0}^{n} (x)_{k,\lambda} \frac{t^k}{k!},$$

then, it is easy to see that

$$P_\lambda[x] = \mathscr{P}_n[f(x,t)]|_{t=0} = \mathscr{P}_n\left[\sum_{k=0}^{n} (x)_{k,\lambda} \frac{t^k}{k!}\right]\Bigg|_{t=0},$$

where $\mathscr{P}_n[f(t;x)]$ denotes the generalized Pascal functional matrix introduced by Yang and Micek in [9].

From now on, we denote $P_\lambda = P_\lambda[1]$. The following theorem summarizes some properties of $P_\lambda[x]$.

Theorem 3. *Let $P_\lambda[x] \in M_{n+1}(\mathbb{R})$ be the generalized degenerate Pascal matrix of the first kind. Then, the following statements hold.*

(a) *Special value. If the convention $(0)_{0,\lambda} = 1$ is adopted, then it is possible to define*

$$P_\lambda[0] := I_{n+1}.$$

(b) *For $x, y \in \mathbb{R}$, we have*

$$P_\lambda[x+y] = P_\lambda[x]P_\lambda[y]. \tag{20}$$

(c) *$P_\lambda[x]$ is an invertible matrix and its inverse is given by*

$$P_\lambda^{-1}[x] := (P_\lambda[x])^{-1} = P_\lambda[-x]. \tag{21}$$

Proof. Since part (a) is a straightforward consequence of the extension of Definition 1 for the case $x = 0$, we shall omit its proof. Thus, we focus our efforts on the proof of parts (b) and (c).

Let $A_{i,j,\lambda}(x,y)$ be the (i,j)-th entry of the matrix product $P_\lambda[x]P_\lambda[y]$. Then, by (7), we have

$$\begin{aligned} A_{i,j,\lambda}(x,y) &= \sum_{k=0}^{n} \binom{i}{k}(x)_{i-k,\lambda}\binom{k}{j}(y)_{k-j,\lambda} \\ &= \sum_{k=j}^{i} \binom{i}{k}(x)_{i-k,\lambda}\binom{k}{j}(y)_{k-j,\lambda} \\ &= \sum_{k=j}^{i} \binom{i}{j}\binom{i-j}{i-k}(x)_{i-k,\lambda}(y)_{k-j,\lambda} \\ &= \binom{i}{j}\sum_{k=0}^{i-j} \binom{i-j}{k}(x)_{i-j-k,\lambda}(y)_{k,\lambda} \\ &= \binom{i}{j}(x+y)_{i-j,\lambda}, \end{aligned}$$

which implies (20).

The substitution $y=-x$ into (20) yields

$$P_\lambda[0] = P_\lambda[x]P_\lambda[-x] = P_\lambda[-x]P_\lambda[x].$$

By part (a), we have $P_\lambda[0] = I_{n+1}$, thus

$$P_\lambda[x]P_\lambda[-x] = I_{n+1} = P_\lambda[-x]P_\lambda[x],$$

and (21) follows. □

Corollary 4. *For any $\lambda \in \mathbb{R}$, $r \in \mathbb{Z}$ and $s \in \mathbb{Z}\setminus\{0\}$ we have*

(a) $P_\lambda^r = P_\lambda[r]$.

(b) $\left(P_\lambda\left[\frac{r}{s}\right]\right)^s = P_\lambda^r$.

Proof. Making the corresponding modifications, we apply the same reasoning as in the proof of [7] (Corollary 3). Since $P_\lambda = P_\lambda[1]$, $P_\lambda[0]$, and P_λ^0 coincide with the identity matrix, it follows from Theorem 3, by induction on r, that $P_\lambda[r] = P_\lambda^r$, for all $r \in \mathbb{N}_0$. Again, by Theorem 3, we have that $P_\lambda[-1] = P_\lambda^{-1}$, and a similar induction on $|r|$ shows $P_\lambda[r] = P_\lambda^r$, for all $r < 0$.

Next, by Theorem 3 and part (a), we obtain $\left(P_\lambda\left[\frac{r}{s}\right]\right)^s = P_\lambda[r] = P_\lambda^r$. □

Remark 5. *Part (b) of Corollary 4 shows that for a fixed $\lambda \in \mathbb{R}$ and any rational number x, $P_\lambda[x]$ is the x-th power of P_λ. Indeed, this property could be expected in the sense that it is satisfied for the generalized Pascal matrix of the first kind $P[x]$ (cf., [7]).*

From the addition Formula (20), we proceed according to [7] and conclude that the degenerate Pascal matrix $P_\lambda[x]$ has an exponential form as follows: Assume that for $\lambda \in \mathbb{R}$, there is a matrix L_λ, such that $P_\lambda[x] = e^{xL_\lambda}$. Then,

$$\frac{d}{dx}P_\lambda[x] = L_\lambda e^{xL_\lambda} = L_\lambda P_\lambda[x],$$

and

$$\left.\frac{d}{dx}P_\lambda[x]\right|_{x=0} = L_\lambda P_\lambda[0] = L_\lambda I_{n+1} = L_\lambda.$$

Thus, there is at most one matrix L_λ such that $P_\lambda[x] = e^{xL_\lambda}$. For instance, in the case $n = 3$, we can find the only possible value as follows:

$$\frac{d}{dx}P_\lambda[x] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -\lambda + 2x & 2 & 0 & 0 \\ x(-2\lambda + x) + x(-\lambda + x) + (-2\lambda + x)(-\lambda + x) & 3(-\lambda + 2x) & 3 & 0 \end{bmatrix},$$

and

$$L_\lambda = \frac{d}{dx}P_\lambda[x]\bigg|_{x=0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -\lambda & 2 & 0 & 0 \\ 2\lambda^2 & -3\lambda & 3 & 0 \end{bmatrix}.$$

While, in the case $n = 7$, we have

$$L_\lambda = \frac{d}{dx}P_\lambda[x]\bigg|_{x=0} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\lambda^2 & -3\lambda & 3 & 0 & 0 & 0 & 0 & 0 \\ -6\lambda^3 & 8\lambda^2 & -6\lambda & 4 & 0 & 0 & 0 & 0 \\ 24\lambda^4 & -30\lambda^3 & 20\lambda^2 & -10\lambda & 5 & 0 & 0 & 0 \\ -120\lambda^5 & 144\lambda^4 & -90\lambda^3 & 40\lambda^2 & -15\lambda & 6 & 0 & 0 \\ 720\lambda^6 & -840\lambda^5 & 504\lambda^4 & -210\lambda^3 & 70\lambda^2 & -21\lambda^2 & 7 & 0 \end{bmatrix}.$$

This suggests a general way of choosing L_λ. More precisely, the entries of L_λ are given by

$$(L_\lambda)_{i,j} = \begin{cases} s_\lambda(i-j,1)\binom{i}{j}, & \text{if } i \geq j+1, \\ \\ 0, & \text{otherwise,} \end{cases}$$

where $s_\lambda(n,k)$ denotes the degenerate Stirling number of the first kind, defined as follows (cf., [17,23] or [24] (Ch. 5)):

$$\sum_{k=0}^{n} s_\lambda(n,k)x^k = (x)_{n,\lambda}. \tag{22}$$

Furthermore, the entries of the matrix L_λ^k, for $1 \leq k \leq n$ and $n \in \mathbb{N}$ can be explicitly represented as follows.

Lemma 1. *For every $n \in \mathbb{N}$ and $1 \leq k \leq n$, the entries of L_λ^k are given by the formula*

$$\left(L_\lambda^k\right)_{i,j} = \begin{cases} k!s_\lambda(i-j,k)\binom{i}{j}, & \text{if } i \geq j+k, \\ \\ 0, & \text{otherwise,} \end{cases}$$

where $s_\lambda(n,k)$ is the degenerate Stirling number of the first kind (22).

Proof. It suffices to proceed by induction on k, taking into account that for $k > n$, we have $L_\lambda^k = 0$. □

Theorem 4. *For every real numbers $x, \lambda \in \mathbb{R}$, $P_\lambda[x] = e^{xL_\lambda}$.*

Proof. By part (a) of Theorem 3, if $x = 0$, then $e^{xL_\lambda} = I_{n+1} = P_\lambda[x]$. Now, assume that $x \neq 0$ since $L_\lambda^k = 0$ for $k > n$, the infinite series for e^{xL_λ} reduces to the finite sum

$$e^{xL_\lambda} = I_{n+1} + xL_\lambda + \frac{x^2}{2}L_\lambda^2 + \cdots + \frac{x^n}{n!}L_\lambda^n. \tag{23}$$

Applying Lemma 1, we can now read off the entries in e^{xL_λ}. Clearly, it is a lower triangular matrix, and the diagonal entries are all 1. Now suppose $i > j$, and let $0 \leq k \leq i - j$. Then, using (22), we have that the (i,j)-th entry in the sum (23) is

$$\left(e^{xL_\lambda}\right)_{i,j} = \sum_{k=0}^{i-j} \frac{x^k}{k!}\left(L_\lambda^k\right)_{i,j} = \binom{i}{j}\sum_{k=0}^{i-j} s_\lambda(i-j,k)x^k = \binom{i}{j}(x)_{i-j,\lambda} = p_{i,j,\lambda}(x).$$

This completes the proof. □

As a consequence of Lemma 1 and Theorem 4, we obtain the following Appell-type property.

Corollary 5. *The generalized degenerate Pascal matrix of the first kind $P_\lambda[x]$ satisfies the following differential equations:*

$$D_x^k P_\lambda[x] = L_\lambda^k P_\lambda[x], \quad 1 \leq k \leq n, \tag{24}$$

where $D_x^k P_\lambda[x]$ is the matrix resulting from the k-th derivative with respect to x of each entry of $P_\lambda[x]$.

Definition 2. *The generalized degenerate $(n+1) \times (n+1)$ Bernoulli matrix $\mathscr{B}_\lambda^{(\alpha)}(x)$ of (real or complex) order α is defined by the entries*

$$\mathscr{B}_{i,j,\lambda}^{(\alpha)}(x) = \begin{cases} \binom{i}{j}\mathscr{B}_{i-j,\lambda}^{(\alpha)}(x), & i \geq j, \\ 0, & \text{otherwise}. \end{cases}$$

Remark 6.

(i) *It is worth mentioning that the entries (2) of $\mathscr{B}_\lambda^{(\alpha)}(x)$ coincide with the entries of the generalized degenerate Bernoulli matrix $\mathscr{B}_m^{(\alpha)}[\lambda, x]$ introduced in [8], when these matrices are the same order.*

(ii) *We denote by $\mathscr{B}_\lambda(x)$ the degenerate Bernoulli matrix $\mathscr{B}_\lambda^{(1)}(x)$.*

The following result was established in [8] (Theorem 4).

Theorem 5. *The generalized degenerate Bernoulli matrices $\mathscr{B}_\lambda^{(\alpha)}(x)$ satisfy the following product formulas.*

$$\begin{aligned} \mathscr{B}_\lambda^{(\alpha+\beta)}(x+y) &= \mathscr{B}_\lambda^{(\alpha)}(x)\,\mathscr{B}_\lambda^{(\beta)}(y) = \mathscr{B}_\lambda^{(\beta)}(x)\,\mathscr{B}_\lambda^{(\alpha)}(y) \\ &= \mathscr{B}_\lambda^{(\alpha)}(y)\,\mathscr{B}_\lambda^{(\beta)}(x). \end{aligned} \tag{25}$$

Definition 2 and the inversion-type Formula (12) lead to the following result:

Theorem 6. *The generalized degenerate Pascal matrix of the first kind $P_\lambda[x]$ can be factorized in terms of $\mathscr{B}_\lambda(x)$ as follows:*

$$P_\lambda[x] = \mathscr{B}_\lambda(x)\mathscr{H}_\lambda, \tag{26}$$

where $\mathscr{H}_\lambda$ is an $(n+1) \times (n+1)$ invertible matrix with entries

$$\mathscr{H}_{i,j,\lambda} = \begin{cases} \binom{i}{i-j} \dfrac{(1)_{i-j+1,\lambda}}{i-j+1}, & i \geq j, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Let us consider $n \in \mathbb{N}_0$ and $0 \leq i, j \leq n$ such that $i \leq j$. From Definition 2 and the inversion-type Formula (12), we have

$$\begin{aligned} p_{i,j,\lambda}(x) &= \binom{i}{j}(x)_{i-j,\lambda} = \frac{\binom{i}{j}}{i-j+1} \sum_{k=0}^{i-j} \binom{i-j+1}{k+1} (1)_{k+1,\lambda} \mathscr{B}_{i-j-k,\lambda}(x) \\ &= \sum_{k=0}^{i-j} \left[\binom{i-j}{k} \mathscr{B}_{i-j-k,\lambda}(x) \right] \left[\binom{i}{i-j} \frac{(1)_{k+1,\lambda}}{k+1} \right]. \end{aligned} \tag{27}$$

Since the right hand member of (27) is the (i,j)-th entry of matrix product $\mathscr{B}_\lambda(x)\mathscr{H}_\lambda$, we conclude that (26) holds. □

The following example shows the validity of Theorem 6.

Example 1. *Let us consider $n = 2$. It follows from Definition 1, (26), and a simple computation that*

$$\begin{aligned} P_\lambda[x] &= \begin{bmatrix} \binom{0}{0}(x)_{0,\lambda} & 0 & 0 \\ \binom{1}{0}(x)_{1,\lambda} & \binom{1}{1}(x)_{0,\lambda} & 0 \\ \binom{2}{0}(x)_{2,\lambda} & \binom{2}{1}(x)_{1,\lambda} & \binom{2}{2}(x)_{0,\lambda} \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} \binom{0}{0}\mathscr{B}_{0,\lambda}(x) & 0 & 0 \\ \binom{1}{0}\mathscr{B}_{1,\lambda}(x) & \binom{1}{1}\mathscr{B}_{0,\lambda}(x) & 0 \\ \binom{2}{0}\mathscr{B}_{2,\lambda}(x) & \binom{2}{1}\mathscr{B}_{1,\lambda}(x) & \binom{2}{2}\mathscr{B}_{0,\lambda}(x) \end{bmatrix}}_{\mathscr{B}_\lambda(x)} \underbrace{\begin{bmatrix} \binom{0}{0}(1)_{1,\lambda} & 0 & 0 \\ \binom{1}{1}\frac{(1)_{2,\lambda}}{2} & \binom{1}{0}(1)_{1,\lambda} & 0 \\ \binom{2}{2}\frac{(1)_{3,\lambda}}{3} & \binom{2}{1}\frac{(1)_{2,\lambda}}{2} & \binom{2}{0}(1)_{1,\lambda} \end{bmatrix}}_{\mathscr{H}_\lambda} \end{aligned}$$

Definition 3. *The generalized degenerate $(n+1) \times (n+1)$ Euler matrix $\mathscr{E}_\lambda^{(\alpha)}(x)$ is defined by the entries*

$$\mathscr{E}_{i,j,\lambda}^{(\alpha)}(x) = \begin{cases} \binom{i}{j}\mathscr{E}_{i-j,\lambda}^{(\alpha)}(x), & i \geq j, \\ 0, & \text{otherwise}. \end{cases}$$

We denote by $\mathscr{E}_\lambda(x)$ the degenerate Euler matrix $\mathscr{E}_\lambda^{(1)}(x)$.

Definition 3 and the inversion-type Formula (17) lead to the following result:

Theorem 7. *The generalized degenerate Pascal matrix of the first kind $P_\lambda[x]$ can be factorized in terms of $\mathscr{E}_\lambda(x)$ as follows:*

$$P_\lambda[x] = \mathscr{E}_\lambda(x)\mathscr{T}_\lambda, \tag{28}$$

where $\mathscr{T}_\lambda$ is an $(n+1)\times(n+1)$ invertible matrix with entries

$$\mathscr{T}_{i,j,\lambda} = \begin{cases} \binom{i}{i-j}\dfrac{(1+a_{i-j}(\lambda))(1)_{i-j,\lambda}}{2}, & i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let us consider $n \in \mathbb{N}_0$ and $0 \leq i,j \leq n$ such that $i \leq j$. From Definition 3 and the inversion-type Formula (17), we have

$$\begin{aligned} p_{i,j,\lambda}(x) &= \binom{i}{j}(x)_{i-j,\lambda} = \frac{1}{2}\binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}(1+a_k(\lambda))(1)_{k,\lambda}\mathscr{E}_{i-j-k,\lambda}(x) \\ &= \sum_{k=0}^{i-j}\left[\binom{i-j}{k}\mathscr{E}_{i-j-k,\lambda}(x)\right]\left[\binom{i}{j}\frac{(1+a_k(\lambda))(1)_{k,\lambda}}{2}\right]. \end{aligned} \tag{29}$$

Since the right-hand member of (29) is the (i,j)-th entry of matrix product $\mathscr{E}_\lambda(x)\mathscr{T}_\lambda$, we conclude that (28) holds. □

Combining Theorems 6 and 7 gives the following connection formula.

Corollary 6. *For any $\lambda, x \in \mathbb{R}$, we have*

$$\mathscr{E}_\lambda(x) = \mathscr{B}_\lambda(x)\mathscr{H}_\lambda\mathscr{T}_\lambda^{-1}.$$

The next result is an immediate consequence of Definition 3 and the addition Formula (9).

Theorem 8. *The generalized degenerate Euler matrices $\mathscr{E}_\lambda^{(\alpha)}(x)$ satisfy the following product formulas.*

$$\begin{aligned} \mathscr{E}_\lambda^{(\alpha+\beta)}(x+y) &= \mathscr{E}_\lambda^{(\alpha)}(x)\,\mathscr{E}_\lambda^{(\beta)}(y) = \mathscr{E}_\lambda^{(\beta)}(x)\,\mathscr{E}_\lambda^{(\alpha)}(y) \\ &= \mathscr{E}_\lambda^{(\alpha)}(y)\,\mathscr{E}_\lambda^{(\beta)}(x). \end{aligned} \tag{30}$$

Proof. Let $C_{i,j,\lambda}^{(\alpha,\beta)}(x,y)$ be the (i,j)-th entry of the matrix product $\mathscr{E}_\lambda^{(\alpha)}(x)\,\mathscr{E}_\lambda^{(\beta)}(y)$, then, by the addition Formula (9), we have

$$\begin{aligned} C_{i,j,\lambda}^{(\alpha,\beta)}(x,y) &= \sum_{k=0}^{n}\binom{i}{k}\mathscr{E}_{i-k,\lambda}^{(\alpha)}(x)\binom{k}{j}\mathscr{E}_{k-j,\lambda}^{(\beta)}(y), \quad n \geq 0 \\ &= \sum_{k=j}^{i}\binom{i}{j}\binom{i-j}{i-k}\mathscr{E}_{i-k,\lambda}^{(\alpha)}(x)\mathscr{E}_{k-j,\lambda}^{(\beta)}(y) \\ &= \binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}\mathscr{E}_{i-j-k,\lambda}^{(\alpha)}(x)\mathscr{E}_{k,\lambda}^{(\beta)}(y), \\ &= \binom{i}{j}\mathscr{E}_{i-j,\lambda}^{(\alpha+\beta)}(x+y), \quad \text{for } i \geq j, \end{aligned}$$

which implies the first equality of (30). The second and third equalities of (30) can be derived in a similar way. □

Corollary 7. *Let $(x_1, \ldots, x_k) \in \mathbb{R}^k$. For α_j real or complex parameters, the generalized degenerate Euler matrices $\mathscr{E}_\lambda^{(\alpha)}(x)$ satisfy the following product formulas, $j = 1, \ldots, k$.*

$$\mathscr{E}_\lambda^{(\alpha_1+\alpha_2+\cdots+\alpha_k)}(x_1 + x_2 + \cdots + x_k) \;=\; \prod_{j=1}^{k} \mathscr{E}_\lambda^{(\alpha_j)}(x_j).$$

Proof. The application of induction on k gives the desired result. □

Taking $x = x_1 = x_2 = \cdots = x_k$ and $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_k$, we obtain the following simple formula for the powers of the generalized degenerate Euler matrices $\mathscr{E}_\lambda^{(\alpha)}(x)$.

Corollary 8. *The generalized degenerate Euler matrices $\mathscr{E}_\lambda^{(\alpha)}(x)$ satisfy the following identity.*

$$\left(\mathscr{E}_\lambda^{(\alpha)}(x)\right)^k \;=\; \mathscr{E}_\lambda^{(\alpha)}(kx), \quad k \in \mathbb{N}.$$

Remark 7. *Analogously, the above corollaries hold, mutatis mutandis, for the generalized degenerate Bernoulli matrices. More precisely, from Theorem 5, and using the same assumptions as Corollaries 7 and 8, we obtain*

$$\begin{aligned}
\mathscr{B}_\lambda^{(\alpha_1+\alpha_2+\cdots+\alpha_k)}(x_1 + x_2 + \cdots + x_k) &= \prod_{j=1}^{k} \mathscr{B}_\lambda^{(\alpha_j)}(x_j), \\
\left(\mathscr{B}_\lambda^{(\alpha)}(x)\right)^k &= \mathscr{B}_\lambda^{(\alpha)}(kx).
\end{aligned}$$

Theorem 9. *The generalized degenerate Euler matrices $\mathscr{E}_\lambda^{(\alpha)}(x)$ satisfy the following relations.*

$$\begin{aligned}
\mathscr{E}_\lambda^{(\alpha)}(x + y) &= \mathscr{E}_\lambda^{(\alpha)}(x)\, P_\lambda[y] = P_\lambda[x]\, \mathscr{E}_\lambda^{(\alpha)}(y) \\
&= \mathscr{E}_\lambda^{(\alpha)}(y)\, P_\lambda[x].
\end{aligned} \tag{31}$$

Proof. The substitution $\beta = 0$ into (30) yields

$$\begin{aligned}
\mathscr{E}_\lambda^{(\alpha)}(x + y) &= \mathscr{E}_\lambda^{(\alpha)}(x)\, \mathscr{E}_\lambda^{(0)}(y) = \mathscr{E}_\lambda^{(0)}(x)\, \mathscr{E}_\lambda^{(\alpha)}(y) \\
&= \mathscr{E}_\lambda^{(\alpha)}(y)\, \mathscr{E}_\lambda^{(0)}(x).
\end{aligned}$$

Since $\mathscr{E}_\lambda^{(0)}(x) = P_\lambda[x]$, we obtain

$$\mathscr{E}_\lambda^{(\alpha)}(x + y) = P_\lambda[x]\mathscr{E}_\lambda^{(\alpha)}(y).$$

A similar argument allows us to show that $\mathscr{E}^{(\alpha)}(x + y) = \mathscr{E}^{(\alpha)}(x)P_\lambda[y]$ and $\mathscr{E}_\lambda^{(\alpha)}(x + y) = \mathscr{E}_\lambda^{(\alpha)}(y)P_\lambda[x]$. This completes the proof of (31). □

4. Conclusions

The aim of our research was to determine novel properties of generalized degenerate Bernoulli and Euler matrices. First, we focused our attention on some matrix-inversion formulae involving these matrices. Secondly, we showed some analytic properties for the generalized degenerate Pascal matrix of the first kind and provided some factorizations for the generalized degenerate Euler polynomial matrix in terms of the generalized degenerate Pascal matrix of the first kind.

Finally, it is worth mentioning that the use of the Cauchy product of the power series is the technique behind some of our formulations. This approach is not a novelty; however, it has been useful for generating new families of special polynomials (satisfying or not

Appell-type conditions), even very recently. In this regard, we refer the interested reader to [25,26] and the references therein for a detailed exposition about very recent trends in this broad field.

Author Contributions: Conceptualization, J.H. and Y.Q.; methodology, J.H., D.P. and Y.Q.; formal analysis, J.H., D.P. and Y.Q.; investigation, J.H., D.P. and Y.Q.; writing—original draft preparation, Y.Q.; writing—review and editing, J.H., D.P. and Y.Q.; supervision, Y.Q.; project administration, Y.Q.; funding acquisition, J.H. and Y.Q. All authors have read and agreed to the published version of the manuscript.

Funding: The research of J. Hernández has been partially supported by the Fondo Nacional de Innovación y Desarrollo Científico y Tecnológico (FONDOCYT), Dominican Republic, under grant 2020-2021-1D1-135. The research of Y. Quintana has been partially supported by the grant CEX2019-000904-S funded by MCIN/AEI/10.13039/501100011033, and by the Madrid Government (Comunidad de Madrid-Spain) under the Multiannual Agreement with UC3M in the line of Excellence of University Professors (EPUC3M23), in the context of the Fifth Regional Programme of Research and Technological Innovation (PRICIT).

Data Availability Statement: Data sharing is not applicable to this article.

Acknowledgments: The authors would like to thank the reviewers for their valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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