Working paper

2016-14

Statistics and Econometrics
ISSN 2387-0303

Accurate Subsampling Intervals of Principal Components Factors

Javier Maldonado, Esther Ruiz

uc3m | Universidad Carlos III de Madrid

Accurate Subsampling Intervals of Principal Components Factors

Javier Maldonado and Esther Ruiz *

Department of Statistics, Universidad Carlos III de Madrid

December 2017

Abstract

In the context of Dynamic Factor Models (DFMs), one of the most popular procedures for factor extraction is Principal Components (PC). Measuring the uncertainty associated to PC factor estimates should be part of interpreting them. However, the asymptotic distribution of PC factors could not be an appropriate approximation to the finite sample distribution for the sample sizes and cross-sectional dimensions usually encountered in practice. The main problem is that parameter uncertainty is not taken into account. We show that several bootstrap procedures proposed in the context of DFM with goals related to inference are not appropriate to measure the uncertainty of PC factor estimates. In this paper, we propose an asymptotically valid subsampling procedure designed with this purpose. The finite sample properties of the proposed procedure are analyzed and compared with those of the asymptotic and alternative extant bootstrap procedures. The results are empirically illustrated obtaining confidence intervals of the underlying factor in a system of Spanish macroeconomic variables.

Keywords: Bootstrap, Dynamic Factor Models, Parameter uncertainty, Resampling Procedures, Unobserved Components.

*Financial Support from the Spanish Goverment contract grant ECO2015-70331-C2-2-R (MINECO/FEDER) is gratefully acknowledged. We are also grateful for very helpful comments to participants at Computational and Financial Econometrics meeting (CFE 2016, Seville), Time Series Workshop meeting (2017, Zaragoza), Oxmetrics User Conference (2017, Paris) and Workshop on New Developments in Econometrics and Time Series (2017, Rome) and at seminars at the Joint Research Center (JRC) of the European Commision at Ispra and the Joint Statistics and Econometrics Workshop at London School of Economics. Any remaining errors are obviously our responsability.

1 Introduction

Currently, large systems of macroeconomic variables are easily accessible, and the consequent extraction of the underlying common factors is an important issue for econometricians and policy decision makers. The latent factors are useful instruments for a wide range of applications: i) to represent economic cycles, trends and structural shocks; see Arouba et al. (2009), Camacho et al. (2015) and Breitung and Eickmeier (2016) for some references; ii) to serve as instrumental variables; see Favero et al. (2005), Bai and Ng (2010) and Kapetanios and Marcellino (2010); iii) as regressors for the construction of Factor-Augmented Vector Autorregresive models (FAVAR) or Factor-Augmented Error Correction models (FECM); see, for example, Bernanke et al. (2005), Banerjee et al. (2014), Abbate et al. (2016) and Bai et al. (2016) or iv) in the context of factor-augmented predictive regressions; see, for example, Stock and Watson (2006), Ludvigson and Ng (2007, 2009), Ando and Tsay (2014), Bräuning and Koopman (2014) and Neely et al. (2014).

In this context, Dynamic Factor Models (DFMs), originally introduced by Geweke (1977) and Sargent and Sims (1977), have received a great deal of attention; see Breitung and Eickmeier (2006), Bai and Ng (2008), Stock and Watson (2011), Breitung and Choi (2013) and Bai and Wang (2016) for excellent surveys. The main goal of DFMs is to explain the dynamics of the system using a reduced number of unobservable common factors. Although, several factor extraction methods have been proposed in the DFM literature, the most popular procedures for large data sets are still based on Principal Components (PC) techniques; see, for example, Ludvigson and Ng (2007, 2009, 2010), Wang (2009), Foester et al. (2011), Ando and Tsay (2014), Gonçalves and Perron (2014), Neely et al. (2014), Djogbenov et al. (2015), Fossati (2016) and Jackson et al. (2016) just to name a few recent references. The popularity of PC factor extraction relies on its good theoretical properties and on its computational simplicity which allows dealing with very large systems of economic or financial variables. However, in practice, it is crucial to obtain not only accurate point estimates of the latent factors, but also of their associated uncertainty. For example, Bai (2003) remarks the importance of constructing confidence intervals of the extracted factors in empirical applications in which they represent economic indices. Boivin and Ng (2006) also pay attention to the uncertainty of factor estimates in the context of predictive regressions while Bai and Ng (2006) argue about the importance of measuring correctly the uncertainty of factors in FAVAR models. More recently, Jackson et al. (2016) argue that measures of factor uncertainty should always accompany applied work in order to establish the statistical legitimacy of the results.

The asymptotic distribution of the factors extracted using PC is derived by Bai (2003) assuming weak dependence in the idiosyncratic term while Bai and Ng (2006) propose estimators of the asymptotic covariance matrix of the factors. More recently, Bai and Ng (2013) derive the limiting distribution of the factors and its corresponding covariance matrix estimation for different identification restrictions. However, results on the performance of the asymptotic distribution to approximate the finite sample distribution of the estimated factors are scarce. Ouysse (2006) shows that, if the factor is static, the asymptotic variance is underestimated while Poncela and Ruiz (2016) show that PC intervals based on the asymptotic distribution could underestimate the uncertainty of the extracted factors [1]. The poor performance of the asymptotic distribution could be attributed to the fact that parameter uncertainty is not considered. Alternatively, the finite sample distribution of the estimated factors can be obtained using resampling procedures which could incorporate parameter uncertainty. Several authors propose using bootstrap in the context of DFMs with other objectives than obtaining the distribution of the underlying factors. For example, Yamamoto (2016) obtains bootstrap bands for impulse response functions (IRF) in the context of FAVAR models; see also Barigozzi et al. (2016) and Forni et al. (2014) for empirical applications. Ludvigson and Ng (2007, 2009 and 2010), Gospodinov and Ng (2013), Gonçalves and Perron (2014), Djogbenou et al. (2015), Jackson et al. (2016) and Gonçalves et al. (2017) implement bootstrap procedures in the context of the parameters of factor-augemented predictive regression models; see also Alonso et al. (2008) and Alonso et al. (2011) who use bootstrap procedures for constructing forecasting intervals for population projections and electricity prices, respectively. Finally, Shintani and Guo (2015) also propose using bootstrap to test about the autoregressive parameter governing the dependence of the latent factor. However, the procedures proposed in these papers obtain either the marginal Mean Squared Errors (MSEs) of the underlying estimated factors and/or do not incorporate parameter uncertainty. Furthermore, none of these papers analyze the performance of the bootstrap procedures when they are used to obtain confidence bands for the extracted factors.

This paper has three main contributions. First, we provide extensive Monte Carlo experiments in order to asses the conditions under which the asymptotic distribution of the factors extracted using PC is a good approximation of the finite sample distribution. In concordance with the results in Poncela and Ruiz (2016), we show that the asymptotic confidence intervals of the estimated factors are unrealistically tiny when the time series size is not large relative to the cross-sectional size. However, if the temporal dimension is large relative to the cross-sectional dimension with the latter being large enough, the asymptotic distribution is appropiate to approximate the finite sample distribution of PC factors. Note that, in this latter case, parameter uncertainty is not relevant while a large cross-sectional dimension minimizes the disturbance

[1]In the context of inference for the OLS estimator of the parameters of factor-augmented predictive regression models, Gonçalves and Perron (2014) show that the finite sample properties of the asymptotic approach proposed by Bai and Ng (2006) can be poor, especially if the cross-sectional dimension is not sufficiently large relative to the temporal dimension.

noise. The presence of serial dependence or heteroscedasticity of the idiosyncratic noises only have mild effects on the properties of asymptotic intervals. However, when the idiosyncratic noises are cross-sectionally correlated, the undercoverage of asymptotic intervals could be very severe if the signal to noise ratio is small. We also analyze the performance of the main available bootstrap methods mentioned above when implemented to obtain confidence bands of PC factors. We show that, if they obtain the marginal distribution of the factors, the corresponding intervals are too wide as to be informative. On the other hand, if they do not incorporate parameter uncertainty, their performance is similar to that of asymptotic intervals.

The second and main contribution of this paper is to propose an asymptotically valid subsampling procedure designed to construct conditional confidence bands for PC factors. The proposed procedure takes into account parameter uncertainty incorporating simultaneously the uncertainty attributed to the fact that the factors are unobserved. The finite sample performance of the proposed procedure is analyzed and compared with that of the asymptotic approach and alternative bootstrap procedures. We show that the converages of the intervals based on the proposed procedure are very close to the nominal coverages.

Finally, the last contribution of this paper is an empirical illustration of the implications of using different procedures to construct confidence intervals for the Spanish economic cycle extracted using PC implemented to a system of macroeconomic variables.

The rest of the paper is organized as follows. Section 2 describes the PC factor extraction procedure and its asymptotic distribution. Monte Carlo experiments are carried out to asses the adequacy of the asymptotic distribution to approximate the finite sample distribution of the factors. Section 3 describes available bootstrap procedures proposed for DFM and analyzes their finite sample performance. In Section 4, the new resampling procedure is proposed and its asymptotic validity and finite sample performance are analyzed. Section 5 illustrates the results with an empirical illustration to compute the uncertainty associated with the Spanish economic cycle. Finally, Section 6 concludes.

2 Factor extraction

In this section, we describe the DFM considered in this paper and introduce notation. We also describe the asymptotic properties of PC factor estimates. Finally, we carry out Monte Carlo experiments to analyze the finite sample performance of asymptotic confidence intervals for the extracted factors.

2.1 The Dynamic Factor Model

We consider the following stationary DFM in which the latent factors and the idiosyncratic components are VAR(1) processes

$$Y_{\cdot t} = PF_t + \varepsilon_{\cdot t}, \tag{1}$$

$$F_t = \Phi F_{t-1} + \eta_t, \tag{2}$$

$$\varepsilon_{\cdot t} = \Theta \varepsilon_{\cdot t-1} + a_{\cdot t} \tag{3}$$

where $Y_{\cdot t} = (y_{1t}, ..., y_{Nt})'$ is the $N \times 1$ vector of observed variables at time t for $t = 1, ..., T$, P is the $N \times r$ matrix of factor loadings, $F_t = (f_{1t}, ..., f_{rt})'$ is the $r \times 1$ vector of unobservable factors and $\varepsilon_{\cdot t} = (\varepsilon_{1t}, ..., \varepsilon_{Nt})'$ is the $N \times 1$ vector of idiosyncratic noises. To uniquely fix the $T \times r$ matrix of factors, $F = (F_1, ..., F_T)'$, and P (up to a column sign change), we assume that $\frac{1}{T}F'F = I_r$ and $P'P$ is a diagonal matrix with its main diagonal values ordered in decreasing order; see Bai and Ng (2013) for an extensive discussion on identification issues. The disturbances $\eta_t = (\eta_{1t}, ..., \eta_{rt})'$ and $a_{\cdot t} = (a_{1t}, ..., a_{Nt})'$ are mutually independent Gaussian white noise vectors with finite covariance matrices Σ_η and Σ_a, respectively. The matrices Φ and Γ are diagonal with their parameters restricted so that $Y_{\cdot t}$ is stationary. The number of factors, r, is assumed to be known and fixed as the cross-sectional and temporal dimensions, N and T, respectively, grow. The DFM in equations (1) to (3) has been frequently considered in the related literature; see, for example, Jungbacker and Koopman (2015), Alvarez et al. (2016) and Jackson et al. (2016) for some recent references.

Note that, according to (2) and assuming that $E\left(F_t F_t'\right) = I_r$, the point-wise marginal (unconditional) distribution of the factors is given by

$$F_t \sim N(0, I_r), \tag{4}$$

and, consequently, one can always construct confidence intervals for the unobserved factors using this distribution. However, the corresponding confidence intervals will be uninformative. Confidence intervals with less uncertainty can be constructed conditional on $Y_{\cdot t}$. Also, it is obvious that the marginal MSE in (4) is not appropriate when the intervals are not centered at the marginal mean (zero) but in a estimation of the factor based on $Y_{\cdot t}$.

2.2 Principal Components Factor Extraction

In the context of iid data, PC is justified because it is optimal in the sense that is the best linear MSE dimension reduction from N to r generating mutually orthogonal factors. However, in a time series context, PC fails to exploit the information contained in the leads and lags of $Y_{\cdot t}$. It still provides the best static r-dimensional approximation but has not minimum MSE as alternative linear procedures involving the past will have smaller MSE. Furthermore, in a dynamic context, PC factors will still be mutually orthogonal at lag zero but correlated at other lags. Consequently, the resulting PC factors cannot be analysed component-wise but need to be considered as vector time series, which are less easy to handle and interpret; see Brillinger (1981). However, PC is still among the most popular factor extraction procedures due to its simplicity and low computational burden when dealing with very large systems of macroeconomic or financial variables. The method of PC minimizes the following sum of squares:

$$V(r) = (NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(y_{it} - P_i' F_t \right)^2, \tag{5}$$

where P_i' is the $i'th$ row of P. Mechanically speaking, the factor estimates can be obtained in one of two ways. The first solution is obtained concentrating out the matrix of weights P. Using the normalization $F'F/T = I_r$, the estimated factors, $\widetilde{f}$, are $\sqrt{T}$ times the eigenvectors corresponding to the r largest eigenvalues of YY' and $\widetilde{P}' = \frac{1}{T}\widetilde{f}'Y'$, with $\widetilde{P}'\widetilde{P}$ being diagonal and Y being the $N \times T$ matrix of observations. The second solution is obtained after concentrating out the factors, F. Then, $\overline{P}$ is $\sqrt{N}$ times the eigenvectors corresponding to the r largest eigenvalues of $Y'Y$. Using the normalization $\frac{1}{N}\overline{P}'\overline{P} = I_r$, yields

$$\overline{f} = \frac{1}{N} Y' \overline{P}. \tag{6}$$

Note that the matrices YY' and $Y'Y$ have identical nonzero eigenvalues and, consequently,

$$\frac{1}{T}\overline{f}'\overline{f} = \frac{1}{N}\widetilde{P}'\widetilde{P} = \widetilde{V}, \tag{7}$$

where $\widetilde{V}$ is the $r \times r$ diagonal matrix consisting of the first r eigenvalues of the matrix $\frac{1}{TN}YY'$ arranged in decreasing order. Then, $\overline{f} = \widetilde{f}\widetilde{V}^{1/2}$ and $\widetilde{P} = \overline{P}\widetilde{V}^{1/2}$; see Bai and Ng (2008). Let $\widehat{f} = \overline{f}\left(\frac{1}{T}\overline{f}'\overline{f}\right)^{1/2} = \overline{f}\widetilde{V}^{1/2}$. From the results above, we can see that $\widehat{f} = \frac{1}{N}Y'\overline{P}\widetilde{V}^{1/2} = \frac{1}{N}Y'\widetilde{P}$, and, consequently,

$$\widehat{f}_t = \frac{1}{N}\widetilde{P}'Y_{\cdot t}. \tag{8}$$

The interest in expression (8) relyes on the fact that the factor estimates are expressed as a linear filter of the original observations as in (6) while, simultaneously, they satisfy the restriction $\frac{1}{T}\widehat{f}'\widehat{f} = I_r$.

It is well known that the extracted factors, $\widehat{f}_t$, estimate only a rotation of the true factors, HF_t, where $H = \left(\frac{P'P}{N}\right)$. Given that the filter used to estimate the factors at time t is based on $Y_{\cdot t}$, the MSE should also be computed conditional on this information. The MSE of the estimated factors can be obtained as follows:

$$\underset{t}{E}\left[\left(\widehat{f}_t - HF_t\right)\left(\widehat{f}_t - HF_t\right)'\right] =$$

$$\underset{t}{E}\left[\left(\widehat{f}_t - f_t\right)\left(\widehat{f}_t - f_t\right)'\right] + \underset{t}{E}\left[\left(f_t - HF_t\right)\left(f_t - HF_t\right)'\right] + 2\underset{t}{E}\left[\left(\widehat{f}_t - f_t\right)\left(f_t - HF_t\right)'\right], \quad (9)$$

where f_t is the factor extracted if the loadings were known, i.e.

$$f_t = \frac{1}{N}P'Y_{\cdot t}, \quad (10)$$

and the t bellow the expectation means that it is conditional on $Y_{\cdot t}$. Note that the total MSE of $\widehat{f}_t$ in expresion (9) is decomposed into the uncertainty due to parameter estimation which represents the difference between the estimates PC factors obtained with known and unknown parameters, the disturbance uncertainty which is due to the process of separating signal and noise and it is inherent to the factor extraction and the cross-product between both. First, using (8) and (10), we can obtain the following expression of the MSE attributed to parameter uncertainty

$$\underset{t}{E}\left[\left(\widehat{f}_t - f_t\right)\left(\widehat{f}_t - f_t\right)'\right] = \frac{1}{N^2}\underset{t}{E}\left[\left(\widetilde{P} - P\right)' Y_{\cdot t}Y_{\cdot t}'\left(\widetilde{P} - P\right)\right]. \quad (11)$$

On the other hand, from equation (1) we can obtain the following expression for the rotated true factors

$$HF_t = \frac{1}{N}P'Y_{\cdot t} - \frac{1}{N}P'\varepsilon_{\cdot t} \quad (12)$$

and, consequently, the disturbance uncertainty is given by

$$\underset{t}{E}\left[\left(f_t - HF_t\right)\left(f_t - HF_t\right)'\right] = \frac{1}{N^2}\underset{t}{E}\left[P'\varepsilon_{\cdot t}\varepsilon_{\cdot t}'P\right]. \quad (13)$$

Finally, the expectation of the cross-product in (9) is zero under the assumption of conditional Normality; see Rodriguez and Ruiz (2012).

2.3 Asymptotic distribution of PC factors

The first asymptotic result on PC factor estimates in the context of strict DFM, is due to Connor and Korajczyk (1986) who prove consistency of PC factors when N goes to infinity and T is fixed. Bai (2003) shows that, in this case, consistency requires to assume asymptotic orthogonality and homoscedasticity of the idiosyncratic components. Only under large N and T, Bai (2003) establishes consistency in the presence of serial correlation and heteroscedasticity; see also Stock and Watson (2002) who show that the space spanned by the estimated factors is consistent when both N and T tend simultaneously to infinity if the serial and cross-sectional correlations of the idiosyncratic noises are weak and the factors are pervasive. Furthermore, if $\frac{\sqrt{N}}{T} \to 0$, Bai (2003) derives the limiting distribution of the factors. Under the restrictions $\frac{1}{T}F'F = I_r$ and the diagonal elements of $P'P$ being distinct and positive and arranged in decreasing order, Bai and Ng (2013) show that

$$\sqrt{N}\left(\tilde{f}_t - F_t\right) \xrightarrow{d} N\left(0, \Sigma_p^{-1}\Gamma_t\Sigma_p^{-1}\right), \tag{14}$$

where $\Sigma_p = \lim_{N\to\infty} \frac{1}{N}P'P$ and $\frac{1}{\sqrt{N}}\sum_{i=1}^{N} P_i\varepsilon_{it} \xrightarrow{d} N(0, \Gamma_t)$. Furthermore, Bai (2003) shows that, if the idyosincratic noises are serially uncorrelated, the limiting distributions are asymptotically independent across t. From (14), the asymptotic MSE can be estimated as follows

$$MSE_t = \left(\frac{\tilde{P}'\tilde{P}}{N}\right)^{-1} \frac{\tilde{\Gamma}_t}{N} \left(\frac{\tilde{P}'\tilde{P}}{N}\right)^{-1}, \tag{15}$$

where, according to Bai and Ng (2006), $\tilde{\Gamma}_t$ can be estimated assuming that the idyiosincratic errors are cross-sectionaly uncorrelated, as follows [2],

$$\widetilde{\Gamma}_t = \frac{1}{N}\sum_{i=1}^{N} \tilde{P}_i\tilde{P}_i'\tilde{\varepsilon}_{it}^2 \tag{16}$$

where, $\tilde{P}_i$ is the $i-th$ row of the estimated factor loading matrix $\tilde{P}$ and $\tilde{\varepsilon}_{it} = y_{it} - \tilde{P}_i'\tilde{F}_t$.

In the single factor model, when $r = 1$, approximated $(1-\alpha)\%$ asymptotic confidence bands for F_t can be constructed as follows

$$[L_t, U_t] = \left[\tilde{f}_t - z_{\alpha/2}MSE_t^{1/2}, \tilde{f}_t + z_{\alpha/2}MSE_t^{1/2}\right] \tag{17}$$

where $z_{\alpha/2}$ is the $\alpha/2$ quantile of the standard normal distribution. Given that $\hat{f} = \tilde{f}\tilde{V} = \tilde{f}\frac{1}{N}\tilde{P}'\tilde{P}$, $(1-\alpha)\,\%$ confidence bands can also be written in terms of $\hat{f}$ as follows

$$[L_t, U_t] = \left[\left(\frac{\tilde{P}'\tilde{P}}{N}\right)^{-1}\hat{f}_t - z_{\alpha/2}MSE_t^{1/2}, \left(\frac{\tilde{P}'\tilde{P}}{N}\right)^{-1}\hat{f}_t + z_{\alpha/2}MSE_t^{1/2}\right]. \tag{18}$$

On the other hand, if $r \geq 2$ the asymptotic $(1-\alpha)\%$ ellipsoids are given by

$$\left[\left(\frac{\tilde{P}'\tilde{P}}{N}\right)^{-1}\hat{f}_t\right] MSE_t^{-1} \left[\left(\frac{\tilde{P}'\tilde{P}}{N}\right)^{-1}\hat{f}_t\right]' \leq \chi^2_\alpha(r), \tag{19}$$

where $\chi^2_\alpha(r)$ is the α quantile of a Chi-squared distribution with r degrees of freedom.

As an illustration, we have generated a system of $N = 100$ variables of size $T = 50$ by the DFM in equations (1) to (3) with idiosyncratic errors being serial and cross-sectionally uncorrelated,

[2]Bai and Ng (2006) propose this estimator of the asymptotic covariance matrix arguing that, if the cross-correlation in the errors is small, assuming that they are zero could be convenient because the sampling variability from their estimation could cause an efficiency loss.

i.e. $\Gamma = 0$ and $\Sigma_a = \sigma_a^2 I$ with $\sigma_a^2 = 1$. The number of factors is $r = 1$ with $\phi = 0.7$ and $\sigma_\eta^2 = (1 - \phi^2)$. Finally, the weights, P, have being generated by an $U(0,1)$ distribution with $\sum_{i=1}^{N} p_{i1}^2 = 31.27$. The top left panel of Figure 1 plots the simulated factor, F_t, together with the factor extracted by PC, $\hat{f}_t$, and the corresponding point-wise 95% asymptotic confidence bands computed as in (18). We can observe that, in this particular realization, the asymptotic bands are rather thin with the true factor being outside the intervals more often than expected. Additionally, a system of variables with the same structure than that described above but with $r = 2$ factors, $\phi_{11} = \phi_{22} = 0.7$ and $T = 25$ has also been generated with $\sum_{i=1}^{N} p_{i1}^2 = 29.39$ and $\sum_{i=1}^{N} p_{i2}^2 = 4.92$. Figure 2 plots the simulated factor, and the corresponding 95% confidence contours constructed as in (19) for $t = 1, ..., 25$. We can observe that the asymptotic contours are too narrow, and leave the factors outside more often than they should.

Note that the estimated finite sample approximation of the asymptotic covariance matrix of $\tilde{f}_t$ (and, consequently, of $\hat{f}_t$) in expression (15) is asymptotically equivalent to that of a least squares (LS) estimator in which P is treated as if it were known explanantory variables. This asymptotic approximation underestimates the covariance of $\tilde{f}_t$ as it does not take into account the MSE attributed to parameter uncertainty in (11). Consequently, unless T is very large relative to N, the asymptotic MSE will underestimate the finite sample MSE and the corresponding coverage of the confidence regions of F_t will be bellow the nominal; see, for example, Poncela and Ruiz (2016).

2.4 Finite sample performance

We carry out Monte Carlo experiments in order to assess the finite sample adequacy of the asymptotic distribution when constructing confidence regions for the latent unobserved factors. These experiments complement those carried out by Poncela and Ruiz (2016) and are carried out for the shake of completeness [3]. The Monte Carlo experiments are performed using DFM of increasing complexity. The first model considered is the ubiquitous single factor model with temporal and cross-sectionally independent idiosyncratic components. Then, we consider the single factor model with the idiosyncratic components being either cross-correlated, temporally dependent or heteroscedastic. Finally, we generate simulated systems by a DFM with $r = 2$. We consider $N, T = 20, 50$ and 100 and the number of Monte Carlo replicates is $R = 1000$.

The first data generating process considered (DGP1) is the DFM in equations (1)-(3) with $r = 1$, and the idiosyncratic noises being homoscedastic and cross-sectionally uncorrelated white noises. The matrix of factor loadings, P, is generated once from a uniform distribution in

[3] Note that our Monte Carlo design is different from that in Ouysee (2006) as she considers F_t as fixed.

[0,1] with $\sum_{i=1}^{N} p_i^2 = 6.62, 15.87$ and 33.91 for $N = 20, 50$ and 100, respectively. In order to analyze the effect of the temporal dependence of the factor, we consider several values of the autorregresive parameter, $\phi = 0.3, 0.5$ and 0.7. In each case, the noise in equation (2), η_t, has variance such that $Var(F_t) = 1$. Finally, the covariance matrix of the idiosyncratic noises is given by $\Sigma_a = q^{-1} I$. Note that, given $Var(F_t) = 1$, the signal to noise ratio is given by $qN^{-1} \sum_{i=1}^{N} p_i^2$. We consider $q = 2, 1$ and 0.5 and, consequently, regardless of N, the signal to noise ratios are approximately given by $0.66, 0.33$ and 0.16, respectively; see Breitung and Eickmeier (2016) who point out that the accuracy of factor estimates can depend on the signal to noise ratio. For each replicate, $i = 1, ..., R$, and moment of time, $t = 1, ..., T$, we construct asymptotic point-wise intervals, $\left(L_t^{(i)}, U_t^{(i)}\right)$ as in (18) with nominal coverages 70% and 95% [4]. Then, at each moment of time, the empirical coverage is computed by counting how many true factors, $F_t^{(i)}, i = 1, ..., R$, lie inside the corresponding interval through the Monte Carlo replicates as $C_t = \frac{1}{R} \sum_{i=1}^{R} I\left(F_t^{(i)} \in \left[L_t^{(i)}, U_t^{(i)}\right]\right)$ where $I(\cdot)$ is the indicator function. We should mention that, in our Monte Carlo experiments, regardless of N and T, the coverages are rather constant over time. Finally, we also compute the length of each interval at each moment of time and for each replicate. Table 1 reports the average coverage across time and the average length across time and Monte Carlo replicates for different temporal and cross-sectional dimensions when $\phi = 0.7$ and $q = 1$[5]. We also report the Monte Carlo average of the scoring rule proposed by Gneiting and Raftery (2007) to measure interval accuracy which is given by

$$SR_t^{(i)} = (U_t^{(i)} - L_t^{(i)}) + \frac{2}{\alpha}(L_t^{(i)} - F_t^{(i)}) I(F_t^{(i)} < L_t^{(i)}) + \frac{2}{\alpha}(F_t^{(i)} - U_t^{(i)}) I(F_t^{(i)} > U_t^{(i)}). \tag{20}$$

Table 1 shows that, regardless the cross-sectional and temporal dimensions, N and T respectively, the coverages of the asymptotic bands are always well bellow the nominal coverages. Furthermore, we can observe that, for fixed T, the undercoverage is larger as N increases. On the other hand, for fixed N, increasing T reduces the undercoverage.

In order to analyze the role of q in the performance of the asymptotic bands, Table 2 reports the coverages, lengths and SRs when the systems are generated by DGP1 with $\phi = 0.7$ and $q = 2, 1$ and 0.5 when $N = T = 50$. Note that, although the coverages are approximately constant (around 0.6 and 0.85 when the nominals are 0.7 and 0.95, respectively), the length and SRs of the asymptotic intervals increase when q decreases. This result could be expected given that, when q is small, the uncertainty around the estimated factors is larger.

Finally, to have a better understanding of the finite sample properties of the asymptotic PC confidence bands with more realistic structures of the idiosincratyc components, we also

[4]Forni et al. (2014) and Barigozzi et al. (2016) construct 64% confidence bands for IRFs. Forni et al. (2014) also consider a nominal coverage of 90% while Bai (2003) considers 95%.

[5]Results for other values of ϕ and q are similar. They are available upon request.

simulate systems with the same parameters as DGP1 but with serially dependent idiosyncratic components generated by equation (3) with $\Gamma = \gamma I_N$ and $\gamma = 0.5$ and 0.7 (DGP2) [6,7], cross-sectionally heteroscedastic idiosyncratic components with $\Sigma_a = diag\left[q^{-1}U(0.1, 2)\right]$ (DGP3) and cross-correlated idiosyncratic components with Σ_a being a Toeplitz matrix with parameter 0.5 (DGP4). Table 2 reports the Monte Carlo coverages, the average lengths and SRs for DGP2 with $\gamma = 0.7$, DGP3 and DGP4. We can observe that the results when the idiosyncratic terms are heteroscedastic[8] are quite similar to the results when the systems were generated by DGP1. When the idiosyncratic component is serially correlated, the coverages reported in Table 2 are slightly smaller than those reported for iid idiosyncratic components. Note that this further undercoverage is more pronounced when q is small. Finally, when the idiosyncratic components are cross-sectionally correlated, the asymptotic coverages are extremely low when q is small. Recall that the asymptotic covariance matrix of the factors is computed as recommended by Bai and Ng (2006) assuming that the idiosyncratic noises are cross-sectionally uncorrelated. According to the results in Table 2, this wrong simplifying assumption may badly affect the construction of confidence intervals for the factors when q is small.

Jackson et al. (2016) show that the conclusions for $r = 1$ could not always be generalized to cases with $r > 1$. Consequently, we also perform Monte Carlo experiments in a DFM in equations (1)-(3) with $r = 2$ where $\Phi = diag(0.7, 0.7)$ and Σ_η is diagonal and such that $E(F_t F_t') = I$. The idiosyncratic noises are defined as in DGP1 being homoscedastic and serial and cross-sectionally uncorrelated. Finally, the matrix of factor loadings, P, is generated once from a uniform distribution in [0,1] with $P'P$ being diagonal. The sums of squared loadings of the first factor are $11.40, 28.54$ and 58.53 when $N = 20, 50$ and 100, respectively, while the sums corresponding to the second factor are $2.41, 4.24$ and 8.65. Consequently, regardless of N, the signal to noise ratios of the first factor are approximately $1.14, 0.57$ and 0.29 when $q = 2, 1$ and 0.5 while for the second factor, the corresponding signal to noise ratios are $0.2, 0.1$ and 0.05. For each Monte Carlo replicate and moment of time, the asymptotic ellipsoid is computed as in (19). Then, at each moment of time, we compute the coverage of the ellipsiods by counting how many realizations $\left(F_{1,t}^{(i)}, F_{2,t}^{(i)}\right)$ ly within the corresponding ellipsoids. Table 3, which reports the average across time of these coverages, shows that the coverages of the asymptotic ellipsoids can be extremelly low. If $q = 1$, even when $T = 100$, the average coverages are around 0.34 and 0.65 when the corresponding nominals are 0.7 and 0.95, respectivelly. The undercoverage when $q = 0.5$ is even more severe.

[6] According to Bai (2003), the limiting distributions are only asymptotically independent if the idiosyncratic noises are serially uncorrelated. However, we still analyze the performance of point-wise intervals as an approximation.

[7] The signal to noise ratio is given by $q\left(1-\gamma^2\right)N^{-1}\sum_{i=1}^{N} p_i^2$.

[8] Results for other sample sizes and idiosyncratic structures are available from the authors upon request.

Finally, note that, according to our experience in simulations with DFMs with $r = 1$, only when both T and N are larger than 100 and the ratio T/N is larger than 2.5, the asymptotic coverages are close to the nominal. We expect that for $r > 1$ the sample sizes should be even larger for the asymptotic distribution of the factors to be appropriate to approximate their finite sample distribution.

3 Extant bootstrap procedures for PC factors

Several alternative bootstrap procedures have been proposed in the context of DFMs with other objectives than constructing confidence bands for extracted factors [9]. In this section, we describe these extant bootstrap algorithms and carry out Monte Carlo experiments to assess their adequacy when implemented to construct confidence bands for extracted PC factors. The extant algorithms can be classified into two main groups: i) Block bootstrap and ii) residual bootstrap.

3.1 Block bootstrap

Gospodinov and Ng (2013) propose a moving block bootstrap of the original vector of observations. Denoting by $B_{t,m} = (Y_{\cdot t}, Y_{\cdot t+1}, ..., Y_{\cdot t+m-1})$ a block of m $(1 \leq m < T)$ consecutive observations of $Y_{\cdot t}$, bootstrap replicates $Y_{\cdot t}^{*(b)}$ are obtained by drawing with replacement $K = T/m$ blocks from $(B_{1,m}, B_{2,m}, ..., B_{T-m+1,m})$, for $b = 1, ..., B$, and m growing at a slower rate than T. PC estimates $\tilde{f}_t^{*(b)}$ are obtained as $\sqrt{T}$ times the eigenvectors corresponding to the r largest eigenvalues of $Y^{*(b)}Y^{*(b)\prime}$. Denote by $\tilde{G}_t^{*}(x)$ the empirical distribution of $\tilde{f}_t^{*(b)}$ given by

$$\tilde{G}_t^{*}(x) = \#\left(\tilde{f}_t^{*(b)} \leq x\right)/B. \tag{21}$$

For each $t = 1, ..., T$ and $r = 1$ [10], $(1-\alpha)\,\%$ block bootstrap confidence bands for the extracted factors can be constructed as follows

$$[L_t, U_t] = \left[\tilde{q}^{*}_{(\alpha/2)t}, \tilde{q}^{*}_{(1-\alpha/2)t}\right], \tag{22}$$

where $\tilde{q}^{*}_{it}$ is the *ith* empirical quantile of $\tilde{G}_t^{*}(x)$. Alternatively, it is possible to compute the bootstrap MSE at time t, as follows

$$MSE_t = \frac{1}{B}\sum_{b=1}^{B}\left(\tilde{f}_t^{*(b)} - \frac{1}{B}\sum_{b=1}^{B}\tilde{f}_t^{*(b)}\right)^2. \tag{23}$$

[9] Several authors propose implementing resampling techniques in the context of PC for iid observations; see, for example, Beran and Srivastra (1985), Stauffer et al. (1985), Timmerman et al. (2007), Babamoradi et al. (2013), Van Aelst et al. (2013) and Fisher et al. (2015).

[10] Scenarios with $r > 1$ are not considered since we will see that even when $r = 1$, this procedure is not appropriate to construct confidence intervals for the estimated factors.

Assuming normality of the factors, $(1-\alpha)\%$ block bootstrap confidence intervals are constructed by

$$[L_t, U_t] = \left[\tilde{f}_t - z_{\alpha/2} MSE_t^{1/2}, \tilde{f}_t + z_{\alpha/2} MSE_t^{1/2}\right]. \tag{24}$$

It is important to note that, when bootstrapping $Y_{\cdot t}^{*(b)}$ as proposed by Gospodinov and Ng (2013), one obtains replicates of the marginal distribution of $\{Y_{\cdot t}\}$ and, consequently, of the marginal distribution of F_t in (4). If confidence bands are constructed as in (22), they will be centered at zero with MSE given by (23). Although, they will have the correct coverages, they are uninformative. On the other hand, when the intervals are computed as in (24), the MSE is marginal while the intervals are centered at $\tilde{f}_t$. Note that these intervals will be too wide with coverages expected to be above the nominal. As an illustration, we consider again the same simulated system described above and construct confidence bands for the factor using (22) and (24) with, as suggested by Gospodinov and Ng (2013), $m = 4$ and $B = 1000$ bootstrap replicates; see Ludvigson and Ng (2007, 2009, 2010) for $B = 1000$. Figure 1 plots the true and PC estimated factors together with 95% confidence bands. We can observe that, when the bands are constructed as in (22), they are approximately constant around ± 2 as expected given that the factor is normally distributed with zero mean and variance 1. As mentioned above, these bands have the assumed coverage but they are not informative about the evolution of the factor. On the other hand, when the bands are constructed as in (24), they are much wider than those based on the asymptotic approximation and the true factor is always within the bands. Obviously, these bands are too wide.

The finite sample performance of the block bootstrap bands are analyzed by Monte Carlo experiments using DGP1 described above. Even this idealized setting is sufficient to demonstrate that the block bootstrap has a poor performance when implemented to obtain confidence intervals for the factors. Consequently, we do not consider any of the other DGPs considered in the previous section. Table 1 reports the coverages through Monte Carlo experiments, average lengths and SRs for 70% and 95% block bootstrap confidence intervals constructed as in (22) and (24) and denoted by block bootstrap 1 and block bootstrap 2, respectively. Consider first the intervals constructed as in (22). Regardless of N and T, the coverages are close to the nominal but the lengths and SRs are extremely large. The intervals are conservative to the point of being non-informative. On the other hand, when the intervals are constructed as in (24), they are not appropiate with coverages close to 1 even when the nominal coverage is 0.7. Observe that the length is similar to that observed for the confidence intervals in (22). Furthermore, the average SR measure of the block bootstrap intervals for the factors is larger than those of the asymptotic intervals except when $N = T = 20$ and 95% confidence intervals are considered.

Regardless of whether they are based on (22) or (24), the block bootstrap intervals are not appropriate to obtain a measure of the uncertainty of the estimated PC factors.

3.2 Residual bootstrap

Bootstrapping DFM using residual bootstrap schemes is very popular. Ludvigson and Ng (2007, 2009 and 2010) obtain bootstrap replicates of $Y_{\cdot t}$ as follows

$$Y_{\cdot t}^{*(b)} = \tilde{P}\tilde{f}_t + \tilde{\varepsilon}_{\cdot t}^{*(b)} \tag{25}$$

where $\tilde{\varepsilon}_{\cdot t}^{*(b)}$ are random extractions with replacement from $\tilde{G}_{\varepsilon}$[11], the empirical distribution of $\tilde{\varepsilon}_{\cdot t} = Y_{\cdot t} - \tilde{P}\tilde{f}_t$. PC estimates of the factors, $\tilde{f}_t^{*(b)}$, are obtained as $\sqrt{T}$ times the eigenvectors corresponding to the r largest eigenvalues of $Y^{*(b)}Y^{*(b)\prime}$. The residual bootstrap confidence intervals can be constructed as in (22) or as in (24) based on the corresponding empirical bootstrap density or MSE, respectively. When the intervals are constructed as in (24), they are called time-residual bootstrap intervals. It is important to note that all bootstrap replicates of $Y_{\cdot t}$ in equation (25) are centered in the estimated common factor $\tilde{P}\tilde{f}_t$ and incorporate uncertainty about the idiosyncratic noises but not about the parameters. Consequently, although the corresponding intervals are adequately centered, they are expected to have coverages bellow the nominal. As an illustration, we consider again the same simulated factor described when constructing asymptotic and block bootstrap intervals. Figure 1, which plots the factor together with 95% point-wise time-residual bootstrap intervals[12], shows that they are very similar to the asymptotic intervals with the true factor lying very often outside their limits.

Table 1, which reports the Monte Carlo results of the time-residual bootstrap confidence intervals for the same designs described above, shows that the average coverages are even lower than those of the asymptotic intervals. Furthermore, they decrease when T increases. This is due to the fact that, as T increases, the PC factor estimate is consistent and therefore the bootstrap factors are very similar in all bootstrap replicates.

Shintani and Guo (2015) propose two alternative residual bootstrap procedures. For $i = 1, ..., N$, $Y_{i\cdot} = (Y_{i1}, Y_{i2}, ..., Y_{iT})$ is the ith row of Y and $\tilde{\varepsilon}_{i\cdot} = (\tilde{\varepsilon}_{i1}, \tilde{\varepsilon}_{i2}, ..., \tilde{\varepsilon}_{iT})$ the corresponding vector of residuals. The first algorithm proposed by Shintani and Guo (2015) is based on generating bootstrap replicates of $Y_{i\cdot}$, for $i = 1, ..., N$, as follows

$$Y_{i\cdot}^{*(b)} = \tilde{P}_i^{*(b)}\tilde{f} + \tilde{\varepsilon}_{i\cdot}^{*(b)} \tag{26}$$

where $\left(\tilde{P}_i^{*(b)}, \tilde{\varepsilon}_{i\cdot}^{*(b)}\right)$ are joint random extractions with replacement from pairs $\left(\tilde{P}_i, \tilde{\varepsilon}_{i\cdot}\right)$. PC

[11] Gonçalves and Perron (2014) and Djogbenou et al. (2015) propose a wild bootstrap algorithm to obtain replicates of $\varepsilon_{\cdot t}^{*(b)}$ that take into account potential heteroscedasticity while Breitung and Eickmeier (2016) propose a block bootstrap scheme to account for the serial correlation of the idiosyncratic noises.

[12] The results when the residual bootstrap intervals are constructed as in (22) are almost identical.

estimates of the factors, $\tilde{f}_t^{*(b)}$, are obtained as $\sqrt{T}$ time the eigenvectors corresponding to the r largest eigenvalues of $Y_{\cdot t}^{*(b)} Y_{\cdot t}^{*(b)\prime}$. Note that the bootstrap replicates in (26) are based on random draws obtained from the cross-sample pairs of weights and residuals instead of bootstrapping in the time dimension as in (25). This procedure is called cross-residual bootstrap. Given that the estimated weights are also bootstrapped, the corresponding bands for the factors are expected to be larger than those obtained using the time residual bootstrap in (25). However, all replicates of $Y_{i\cdot}$ are constructed based on the same estimated factors. Therefore, given that they do not incorporate the uncertainty associated with the estimation of the factors, it is expected that the coverages of cross-residual bootstrap intervals will be bellow the nominal. As an illustration, we consider again the same simulated factor described above. Figure 1 plots the factor together with its PC estimation and 95% cross-residual bootstrap intervals constructed as in (24) [13]. We can observe that, as explained before, the confidence bands are slightly larger than those obtained using the asymptotic approach and the time-residual bootstrap. However, there are still too many moments of time in which the true factors are outside the bands. Table 1 reports the Monte Carlo results of the cross-residual bootstrap confidence intervals. We can observe that the coverages are better than when the time-residual bootstrap is implemented but still well bellow the nominal. Table 1 also reports the average SR interval accuracy measures when the intervals are constructed using the cross-residual bootstrap. We can observe that the average values of the SR statistic are even larger than those observed for the asymptotic intervals.

The second bootstrap algorithm proposed by Shintani and Guo (2015). Consider that $r = 1$. In this case, bootstrap replicates are obtained as follows

$$\tilde{F}_t^{*(b)} = \hat{\phi}\tilde{F}_{t-1}^{*(b)} + \tilde{\eta}_t^{*(b)} \tag{27}$$

$$Y_{\cdot t}^{*(b)} = \tilde{P}^{*(b)}\tilde{F}_t^{*(b)} + \tilde{\varepsilon}_{\cdot t}^{*(b)} \tag{28}$$

where $\hat{\phi}$ is the OLS estimator of the autorregresive parameter of an AR(1) model fitted to $\tilde{f}_t$ and $\tilde{\eta}_t^*$ are random extractions with replacement from the empirical distribution function of the centered residuals, $\hat{\eta}_t = \tilde{f}_t - \hat{\Phi}\tilde{f}_{t-1}$ and $\tilde{P}^{*(b)}$ and $\tilde{\varepsilon}_{\cdot t}^{*(b)}$ are defined as in (26). The bands, based on the factors extracted using $Y_{\cdot t}^{*(b)}$ defined as in (28), are marginal given that they are based on bootstrap replicates of the factors in (27) which are not based on the available information set. Therefore, we expect a similar behaviour as that of the bands constructed using the block bootstrap procedure [14].

Finally, Yamamoto (2016) considers two further residual-based bootstrap procedures. The first one is based on factor estimation based on bootstrap replicates generated as in (25) while

[13]The results when the residual bootstrap intervals are constructed as in (22) are almost identical.

[14]Monte Carlo results are available upon request.

the second one treats the original factor as in (27) [15]. The performance of the first procedure is the same as that of the residual bootstrap procedure proposed by Ludvigson and Ng (2007, 2009 and 2010) and, therefore, we do not consider it further in this paper. The second one obtain marginal bands. Therefore, we expect a similar behaviour as that of the bands constructed using the block bootstrap algorithm [16].

Summarizing the results in this section, we can conclude that none of the residual bootstrap procedures available in the context of PC factor extraction in DFM, are adequate to construct confidence bands of the factors with coverages close to the nominal ones.

4 Conditional Subsampling for Factors

Given that the asymptotic and bootstrap procedures described in previous sections are not adequate when the aim is to construct confidence intervals of the underlying factors, in this section, we propose a resampling strategy designed for this purpose. Its asymptotic validity is stablished and its finite sample performance is analyzed through extensive Monte Carlo experiments.

4.1 Subsampling Procedure

In previous sections, we have seen that neither the asymptotic approximation nor the available bootstrap procedures are adequate to measure the uncertainty associated with factors extracted using PC when the temporal and cross-sectional sizes are not large enough. The main problem associated with asymptotic intervals and regions is that they do not incorporate the parameter estimation uncertainty. On the other hand, there are two main problems associated with the failure of boostrap procedures. First, as explained above, these procedures either compute the marginal MSE of the factors and/or do not incorporate paramater uncertainty. Second, there is evidence about the bootstrap being fraught with problems when implemented in models with high dimensions. For example, El Karoui and Purdom (2015) show that both residual bootstrap and pairs bootstrap give poor inference on the LS estimator of the parameters of a regression model when the number of regressors is large relative to the sample size. They show that the residual bootstrap tend to give anti-conservative estimates while the pairs bootstrap gives very conservative estimates as the ratio between the number of regressors and the sample size grows. Recall that the PC estimator is related to the LS estimator and, as such, we expect the same problems observed in regression models to affect PC factor extraction.

[15] Alonso et al. (2008 and 2011) propose a bootstrap with the same structure for forecasting purposes.

[16] Monte Carlo results are available upon request.

First, to solve the problem of incorporating parameter uncertainty, in this paper, we propose to compute the uncertainty associated with PC factor extraction by considering the decomposition of the total MSE into the noise uncertainty and the parameter estimation uncertainty as in (9). While the noise uncertainty can be computed using the asymptotic covariance in (16), we propose computing the parameter estimation uncertainty in (11) by first computing the MSE of $\hat{f}_t$ conditional on the parameter estimates and then computing the average of this conditional MSE over the distribution of the parameter estimator; see Hamilton (1986), Pfeffermann and Tiller (2005) and Rodriguez and Ruiz (2012) for the same strategy in the context of state space models. Consequently, the MSE attributed to parameter uncertainty is given by

$$\underset{\tilde{P}}{E}\left[\underset{t}{E}\left[\left(\widehat{f}_t - f_t\right)\left(\widehat{f}_t - f_t\right)'|\tilde{P}\right]\right] = \frac{1}{N^2}\underset{\tilde{P}}{E}\left[\left(\widetilde{P} - P\right)' Y_{\cdot t}Y_{\cdot t}'\left(\tilde{P} - P\right)\right], \tag{29}$$

where the expected value $\underset{\tilde{P}}{E}$ is taken over the sampling distribution of $\tilde{P}$. Note that because the MSE in (29) depends on $Y_{\cdot t}$, it is conditional on the particular observed sample and it is not marginal with respect to all possible realizations of $Y_{\cdot t}$.

Second, in order to deal with the lack of adequacy of the bootstrap in the context of high-dimensional LS problems, we propose to estimate the sampling distribution of $\tilde{P}$ using subsampling as proposed by Politis and Romano (1994); see Politis (2003) for the advantages of subsampling. The basic idea is to approximate the sampling distribution of $\tilde{P}$ based on estimates of the loadings computed over subsets of data of cross-sectional size $N^* < N$. Under weak hypothesis, the sample distribution of the PC estimator of the loadings based on N^* and that based on N should be close. Consequently, using subsampling, it is possible to accurately estimate the sampling distribution of the PC estimator of the loadings.

The subsampling algorithm is given next. For $b = 1, ..., B$, obtain the $N^* \times T$ matrix $Y^{*(b)}$ by drawing N^* vectors randomly without replacement from $Y_{i\cdot} = (y_{i1}, y_{i2}, ..., y_{iT})$, where $N^* = p \times N$ with $0 \leq p \leq 1$. Using $Y^{*(b)}$, obtain PC estimates $\widetilde{P}^{*(b)}$ and compute

$$\hat{f}_t^{*(b)} = \frac{1}{N^*}\widetilde{P}^{*(b)\prime} Y_{\cdot t}^{*(b)}. \tag{30}$$

The subsampling analog of the MSE due to parameter uncertainty conditional on the parameter estimates given by $\underset{t}{E}\left[\left(\widehat{f}_t - f_t\right)\left(\widehat{f}_t - f_t\right)'|\tilde{P}\right]$ is given by $\left(\widehat{f}_t^{*(b)} - \widehat{f}_t\right)\left(\widehat{f}_t^{*(b)} - \widehat{f}_t\right)'$ and, consequently,

$$\underset{\tilde{P}}{E}\left[\underset{t}{E}\left[\left(\widehat{f}_t - f_t\right)\left(\widehat{f}_t - f_t\right)'|\tilde{P}\right]\right] = \frac{1}{B}\sum_{b=1}^{B}\left(\left(\widehat{f}_t^{*(b)} - \widehat{f}_t\right)\left(\widehat{f}_t^{*(b)} - \widehat{f}_t\right)'\right). \tag{31}$$

Finally, the subsampling analog of the MSE_t^* of $\widehat{f}_t$ is given by

$$MSE_t^* = \left(\frac{\tilde{P}'\tilde{P}}{N}\right)^{-1}\left(\frac{1}{B}\sum_{b=1}^{B}\left(\left(\widehat{f}_t^{*(b)} - \widehat{f}_t\right)\left(\widehat{f}_t^{*(b)} - \widehat{f}_t\right)'\right) + \tilde{\Gamma}_t\right)\left(\frac{\tilde{P}'\tilde{P}}{N}\right)^{-1}, \tag{32}$$

where $\tilde{\Gamma}_t$ is defined as in (16).

Subsampling works well under weak assumptions because each subset of size N^* (taken without replacement from the original data $(Y_{1\cdot}, ..., Y_{N\cdot})$ is indeed a sample of size N^* from the true DGP and, consequently, the sampling distributions of $\tilde{P}$ based on samples of size N^* and N should be close. As a result, choosing an adequate subsampling size, N^*, is very important for both the asymptotic and finite sample validity of the procedure. Note that if N^* is too large, there is not enough variability in the estimated loadings as all of them will be obtained using similar samples. On the other hand, if N^* is too small, the variability could be too large and not similar to that corresponding to the estimator based on the cross-sectional dimension N. For the asymptotic validity of the subsampling estimator, $\tilde{P}^*$, $\frac{N^*}{N} \to 0$ and $N^* \to \infty$ when $N \to \infty$; see Politis et al. (2001). However, the optimal block size is unknown in practice. In finite samples, we carry out extensive simulation experiments and conclude that if $q = 1$ and $p = 0.8 + 0.09\log_{10}\left(\frac{T}{N}\right)$, the coverages of the estimated factors are optimal. On the other hand if $q < 1$, i.e. the signal to noise ratio decreases, then p should be smaller while if $q > 1$ then p should be larger for the subsampling coverages to be optimal.

The proposed procedure is computationally very simple [17]. Furthermore, its asymptotic validity can be stablished by using the results in Politis et al. (2001) who show that under extremely weak conditions, which include dependent data, if $\frac{N^*}{N} \to 0$ and $N^* \to \infty$ when $N \to \infty$, the subsampling distribution of the estimated loadings reproduce the sampling distribution. Therefore, when computing the expectation in (31) for all posible values of $\widetilde{P}$, the parameter uncertainty is properly approximated.

In the context of Gaussian DFM, as that considered in this paper, the PC extracted factors are normally distributed. Consequently, when $r = 1$, the corresponding subsampling $(1-\alpha)\,\%$ point-wise confidence interval for the true factor, $F_{t,}$, is given by

$$[L_t, U_t] = \left[\left(\frac{\tilde{P}'\tilde{P}}{N}\right)^{-1}\hat{f}_t - z_{\alpha/2}MSE_t^{*1/2}, \left(\frac{\tilde{P}'\tilde{P}}{N}\right)^{-1}\hat{f}_t + z_{\alpha/2}MSE_t^{*1/2}\right], \tag{33}$$

where MSE_t^* is defined as in (32). When $r > 1$, the subsampling regions are given by

[17] For $B = R = 500$ and $N = T = 50$, it takes 6 minutes and 52 seconds to compute the subsampling MSE on Intel i7-6700 (4 cores - 2.6 GHz).

$$\left[\left(\frac{\tilde{P}'\tilde{P}}{N}\right)^{-1}\hat{f}_t\right] MSE_t^{*-1}\left[\left(\frac{\tilde{P}'\tilde{P}}{N}\right)^{-1}\hat{f}_t\right]' \leq \chi^2_\alpha(r). \tag{34}$$

Bai (2003) shows that normality of the factors holds even without assuming normality. Consequently, we guess that the coverages of the intervals and regions given by (33) and (34) can be close to the nominals even in the context of non-Gaussian DFMs.

As an illustration of the behaviour of the subsampling intervals in (33), Figure 1 plots the same simulated and PC estimated factor considered above together with the corresponding 95% point-wise subsampling confidence bands.[18] When compared with the asymptotic or the (time) (cross) residual bootstrap bands, we can observe that the subsampling bands are wider. On the other hand, the subsampling bands are more informative than the marginal bootstrap bands and than the wrong bands constructed using the block bootstrap MSEs and centered in the estimated factors.

We also illustrate the new proposed procedure to construct regions for estimated PC factors when $r = 2$. With this purpose, we consider the factors simulated by the same DFM with $r = 2$ described in subsection 2.3 when dealing with the asymptotic confidence regions. Figure 2 plots the point-wise subsampling and asymptotic 95% contours obtained for $t = 1, ..., 25$. It can be observed that the subsampling regions are considerably wider than the asymptotic ones and contain the true factors in a larger proportion of times.

It is important to note that subsampling is carried out in the cross-sectional dimension. Consequently, the information on the temporal dependence is keeped. This is why this procedure can be valid even if the factors are non-stationary as far as the idiosyncratic noises are stationary; see Bai (2004) for the consistency of PC non-stationary factors.

Finally, note that the subsampling procedure proposed in this paper could also be easily extended to compute the parameter uncertainty of the common component if it were of interest.

4.2 Finite Sample Performance

We carry out Monte Carlo experiments in order to asses the adequacy of the propose subsampling procedure to approximate the MSEs of PC factors and, consequently, to construct confidence intervals and regions. The Monte Carlo experiments are performed using the same DGPs considered above. The number of subsampling replicates is $B = 1000$. For each Monte Carlo replicate,

[18] The subsampling has been carried out with $B = 500$. The asymptotic results in Politis et al. (2001) are stablished for all possible samples of size N^*. However, in practice, it is unfeasible to estimate the loadings for all possible samples as its number is too large. The results based on $B = 500$ are already very reliable.

$i = 1, ..., R$, we construct point-wise intervals as in equation (33). Table 1 reports, for nominal coverages of 70% and 95%, the average coverage across time and the average length across time and Monte Carlo replicates for different temporal and cross-sectional dimensions when $\phi = 0.7$ and $q = 1$[19]. Observe that, regardless N and T, the proposed subsampling procedure estimates correctly the uncertainty of PC factors with coverages always close to the nominal. Furthermore, the SR measure of the subsampling intervals is also considerably smaller than those of the asymptotic intervals and of the extant bootstrap procedures.

Table 2 reports the Monte Carlo results when $T = N = 50$ for DGP2, DGP3 and DGP4 described before, with serial ($\gamma = 0.7$), cross-sectional heteroscedasticity and cross-sectional dependence in the idiosyncratic term, respectively. It can be observed that, the presence of serial correlation and heteroscedasticity of the idiosyncratic term does not affect the finite sample performance of the proposed subsampling procedure. Regardless of the signal to noise ratio, the coverages are rather close to the nominal and much larger than those obtained when using the asymptotic approximation. The same is true when there is cross-dependence and the signal to noise ratio is large enough, $q = 1, 2$. However, when the signal to noise ratio is small ($q = 0.5$), the proposed procedure has smaller coverages than the nominal. This could be due to the way the covariance matrix is computed.

DFMs with $r = 2$ have also been considered. In particular, we consider the same DGP1 described when dealing with asymptotic regions. For each Monte Carlo replicate, we construct the point-wise subsampling regions as in equation (34). Table 3, which reports the average coverages across time, shows that, regardless the system dimensions and the value of q, the subsampling coverages are very close to the nominal ones. Therefore, the new procedure estimates correctly the uncertainty for more than one factor.

5 Empirical Illustration

In this section, we illustrate the importance of a proper measurement of the uncertainty associated with PC estimates of the factors by analyzing a system of $N = 60$ seasonally adjusted macroeconomic Spanish variables observed quarterly from the first quarter of 1980 to the last of 2015 with $T = 144$ [20]. The variables are converted to stationary. The list of all variables and their stationary transformations are reported in the Appendix. After centering and standardizing each of the variables in the system, the number of common factors is determined using the criteria by

[19]For brevity, we give only brief descriptions of the simulations in what follows. Detailed descriptions are available upon request.

[20]The Database considered is built by the Ministry of Treasure and Public Administration, "Base de datos trimestrales de la economía española".

Ahn and Horenstein (2013) as one. The factor is extracted by PC and confidence intervals are constructed using the asymptotic approximation and the subsampling procedure proposed in this paper. The sum of squared weights is $\sum_{i=1}^{N} \tilde{p}_i^2 = 9.71$ with estimated weights larger than 0.8 in absolute value corresponding to: gross capital formation, capital stock, imports, unemployment rate, rest of the word clients' GDP and total resources of public administrations. The estimated autorregressive parameter is $\hat{\phi} = 0.6$, $\hat{\sigma}_a^2 = [0.29, 0.99]$ with the mode close to 0, and serial dependence with $\hat{\gamma} = [-0.75, 0.96]$ distributed uniformly in this interval. Figure 3 plots the estimated PC factor together with 95% confidence bands constructed using the asymptotic approach and the subsampling procedure proposed in this paper. Additionally, a line representing a scenario of zero-growth has been included in order to facilitate the interpretation of the cycles. As expected, the asymptotic confidence intervals are narrower than those constructed following the procedure proposed in this paper. Figure 4 plots the asymptotic and the subsampling MSEs. It can be seen how the uncertainty estimated under both methods is fairly similar, with the exception of periods of economic crisis (1993, 2001 and 2008). It can also be observed that the new procedure detects phases of high macroeconomic uncertainty several periods in advance. If practitioners and policy decision makers use the asymptotic approximation for constructing confidence bands for the latent factors, it could lead to a wrong interpretation of the economic reality -cycles and recessions-. The conclusion of a favourable economic situation could be drawn when both the extracted factor and its confidence intervals have positive values. However, when the intervals are constructed using subsampling, they include negative values. Consequently, it is not possible to confirm a period of economic growth at the established level of confidence.

6 Conclusions

This paper explores different methods for computing the uncertainty associated to factors extracted using PC in DFMs. By means of extensive Monte Carlo experiments, the finite sample performance of the asymptotic approximation is investigated. We show that it does not incorporate parameter uncertainty and, consequently, underestimates the uncertainty of PC factors, causing narrower confidence intervals and regions than desired. Moreover, we show that the extant bootstrap procedures proposed in the context of PC extraction in DFM are not capable of measuring correctly the uncertainty associated to the factors. Some of them compute the marginal MSE instead of the conditional one, while others do not take into account the parameter uncertainty. We propose a subsampling algorithm to compute the uncertainty of PC factors and to construct confidence intervals. The subsampling intervals and regions are computationally very simple and asymptotically valid. Furthermore, they have better finite sample coverages than those constructed using the asymptotic approximation or the bootstrap procedures available in the literature. Finally, we construct confidence intervals for the factor extracted from a system of Spanish macroeconomic variables and show the importance of having adequate intervals when interpreting wether the growth is truly positive.

References

[1] Abbate, A., Eickmeier, S., Lemke, W., Marcellino, M. (2016). The changing international transmission of financial shocks: evidence from a classical time-varying FAVAR. *Journal of Money, Credit and Banking* 48(4):573-601.

[2] Ahn, S., Horenstein A. (2013). Eigenvalue ratio test for the number of factors. *Econometrica* 81(3):1203-1227.

[3] Alonso, A.M., Peña, D., Rodríguez, J. (2008). A methodology for population projections: an application to Spain. *Statistics and Econometrics Series 12.* WP 08-45. Universidad Carlos III de Madrid.

[4] Alonso, A.M., Rodriguez, J., Sanchez, M. J., García-Martos, C. (2011). Dynamic factor analysis and bootstrap inference: application to electricity market forecasting. *Technometrics* 53(2):137-151.

[5] Alvarez, R., Camacho, M., Perez-Quiros, G. (2016). Aggregate versus disaggregate information in dynamic factor models. *International Journal of Forecasting* 32:680-694.

[6] Ando, T., Tsay, R.S. (2014). A predictive approach for selection of diffusion index models. *Econometric Reviews* 33(1-4):68-99.

[7] Arouba, S.B., Diebold, F.X., Scotti, C. (2009). Real-time measurement of business conditions. *Journal of Business and Economic Statistics* 27(4):417-427.

[8] Babamoradi, H., Van den Berg, F., Rinnan, A. (2013). Bootstrap based confidence limits in principal component analysis - A case study. *Chemometrics and Intelligent Laboratory Systems* 120:97-105.

[9] Bai, J. (2003). Inferential theory for factor models of large dimension. *Econometrica* 71(1):135-171.

[10] Bai, J. (2004). Estimating cross-section common stochastic trends in nonstationary panel data. *Journal of Econometrics* 122(1):137-183.

[11] Bai, J., Ng, S. (2006). Confidence intervals for diffusion index forecast and inference for factor-augmented regressions. *Econometrica* 74(4):1133-1150.

[12] Bai, J., Ng, S. (2008). Large dimensional factor analysis. *Foundations and Trend in Econometrics* 3: 89-163.

[13] Bai, J., Ng, S. (2010). Instrumental variable estimation in a data rich enviroment. *Econometric Theory* 26:1577-1606.

[14] Bai, J., Ng, S. (2013). Principal components estimation and identification of static factors. *Journal of Econometrics* 176:18-29.

[15] Bai, J., Kunpeng, L., Lina, L. (2016). Estimation and inference of FAVAR models. *Journal of Business & Economic Statistics* 34(4):620-641.

[16] Bai, J., Wang, P. (2016). Econometric analysis of large factor models. *Annual Review of Economics* 8: 53-80.

[17] Banerjee, A., Marcellino, M., Masten, I. (2014). Forecasting with factor-augmented error correction models. *International Journal of Forecasting* 30(3):589-614.

[18] Barigozzi, M., Lippi, M., Luciani, M. (2016). Non-stationary dynamic factor models for large datasets, Finance and Economics Discussion Series; Division of Research and Statistics and Monetary Affairs, Federal Reserve Bank: Washington, DC.

[19] Beran, R., Srivastra, M.S. (1985). Bootstrap tests and confidence regions for functions of a covariance matrix. *Annals of Statistics* 13:95-115.

[20] Bernanke, B.S., Boivin, J., Eliasz, P. (2005). Measuring the effects of monetary policy: a factor-augmented vector autoregressive (FAVAR) approach. *The Quarterly Journal of Economics* 120:387-422.

[21] Boivin, J., Ng, S. (2006), Are more data always better for factor analysis?, *Journal of Econometrics*, 132, 169-194.

[22] Bräuning, F., Koopman, S.J. (2014). Forecasting macroeconomic variables using collapsed dynamic factor analysis. *International Journal of Forecasting* 30:573-584.

[23] Breitung, J., Choi, I. (2013). Factor Models. In: *Handbook of Research Methods and Applications in Empirical Macroeconomics.* Cheltenham:Edwar Elgar, pp. 249-265.

[24] Breitung, J., Eickmeier, S. (2006). Dynamic Factor Models. In: *Modern Econometric Analysis.* Heidelberg: Springer, pp. 25-40.

[25] Breitung, J., Eickmeier, S. (2016). Analyzing international business and financial cycles using multi-level factor models: A comparison of alternative approaches. In: *Advances in Econometrics.* Bingley: Emerald Group Publishing Limited, pp. 177-214.

[26] Brillinger, D.R. (1981). *Time Series Data Analysis and Theory.* San Francisco: Holden-Day.

[27] Camacho, M., Perez-Quiros G., Poncela, P. (2015). Extracting nonlinear signals from several economic indicators. *Journal of Applied Econometrics* 30:1073-1089.

[28] Connor, G., Korajczyk, R.A. (1986). Performance measurement with arbitrage pricing theory: A new framework for analysis. *Journal of Financial Economics* 15:373-394.

[29] Djogbenou, A., Gonçalves, S., Perron, B. (2015). Bootstrap inference in regressions with estimated factors and serial correlation. *Journal of Time Series Analysis* 36: 481-502.

[30] El Karoui, N., Purdom, E. (2015). Can we trust the bootstrap in high-dimension?. *Manuscript.*

[31] Favero, C.A., Marcellino M., Neglia F. (2005). Principal components at work: the empirical analysis of monetary policy with large datasets. *Journal of Applied Econometrics* 20(5):603-620.

[32] Fisher, A., Caffo, B., Schwartz, B., Zipunnikov, V. (2015). Fast, exact bootstrap principal component analysis for p > 1 million. *Journal of the American Statistical Association* 111(514):846-860.

[33] Foester, A., Sarte, P.D.G., Watson M.H. (2011). Sectoral vesus aggregate shocks: a structural factor analysis of industrial production. *Journal of Political Economy* 119:1-38.

[34] Forni, M., Gambetti, L., Sala, L. (2014). No news in business cycles. *Economic Journal* 124(581):1168-1191.

[35] Fossati, S. (2016). Dating US business cycle with macroeconomic factors. *Studies in Nonlinear Dynamics of Econometrics* 20(5):529-549.

[36] Geweke, J. (1977). The dynamic factor analysis of economic time series. In: *Latent Variables in Socio-Economic Models.* North-Holland:Amsterdam, pp. 365-382.

[37] Gneiting, T., Raftery A.E. (2007). Strictly proper scoring rules, prediction and estimation. *Journal of the American Statistical Association* 102(477):359-378.

[38] Gonçalves, S., Perron, B. (2014). Bootstrapping factor-augmented regression models. *Journal of Econometrics* 182:156-173.

[39] Gonçalves, S., Perron, B., Djogbenou, A. (2017). Prediction intervals for factor models. *Journal of Business & Economic Statistics* 35(1):53-69.

[40] Gospodinov, N., Ng, S. (2013). Commodity prices, convenience yields and inflation. *The Review of Economics and Statistics* 95(1):206-219.

[41] Hamilton, J.D. (1986). A standard error for the estimated state vector of a state-space model. *Journal of Econometrics* 33(3):387-397.

[42] Jackson, L.E., Kose, M.A., Otrok, C., Owyan, M.T. (2016). Specification and estimation of bayesian dynamic factor models: A monte Carlo analysis with an application to global house price comovement. In: *Advances in Econometrics.* Bingley:Emerald Group Publishing Limited, pp. 361 - 400.

[43] Jungbacker, B., Koopman, S. J. (2015). Likelihood-based dynamic factor analysis for measurement and forecasting. *Econometric Journal* 18:1-21.

[44] Kapetanios, G., Marcellino, M. (2010). Factor-GMM estimation with large sets of possibly weak instruments. *Computational Statistics and Data Analysis* 54(11): 2655-2675.

[45] Ludvigson, S.C., Ng, S. (2007). The empirical risk-return tradeoff: A factor analysis approach. *The Journal of Financial Economics* 83:171-222.

[46] Ludvigson, S.C., Ng, S. (2009). Macro Factors in Bond Risk Premia. *The Review of Financial Studies* 22(12): 5027-5067.

[47] Ludvigson, S.C., Ng, S. (2010). A Factor Analysis of Bond Risk Premia. In: *Handbook of Empirical Economics and Finance.* Boca Raton: Chapman and Hall, pp. 313 - 372.

[48] Neely, C.J., Rapach, D.E., Tu, J., Zhan, G. (2014). Forecasting the equity risk premium: the role of technical indicators. *Management Science* 60(7):1772-1791.

[49] Ouysse, R. (2006). Approximate factor models: finite sample distribution. *Journal of Statistical Computation and Simulation* 76(4):279-303.

[50] Pfeffermann, D., Tiller, R. (2005). Bootstrap approximation to prediction MSE for state-space models with estimated parameters, *Journal of Time Series Analysis* 26:893-916.

[51] Politis, D.N. (2003). The impact of bootstrap methods on time series analysis. *Statistical Science* 18(2):219-230.

[52] Politis, D. N., Romano J. P. (1994). Large sample confidence regions based on subsamples under minimal assumptions. *Annals of Statistics* 22:2031-2050.

[53] Politis, D. N., Romano J. P., Wolf, M. (2001). On the asymptotic theory of subsampling. *Statistica Sinica* 11:1105-1124.

[54] Poncela, P., Ruiz, E. (2016). Small versus big data factor extraction. In: *Advances in Econometrics.* Bingley:Emerald Group Publishing Limited, pp. 401-434.

[55] Rodriguez, A., Ruiz, E. (2012). Bootstrap prediction mean squared errors of unobserved states based on the Kalman filter with estimated parameters. *Computational Statistics and Data Analysis* 56:62–74.

[56] Sargent, T.J., Sims, C.A. (1977). Business cycle modeling without pretending to have too much a-priori economic theory, New Methods in Business Cycle Research. Federal Reserve Bank of Minneapolis: Minneapolis.

[57] Shintani, M., Guo, Z. (2015). Improving the finite sample performance of autoregression estimators in dynamic factor models: A bootstrap approach. *Econometric Reviews*, forthcoming.

[58] Stauffer, D. O., Garton, E.O., Kirk Steinhorst R. (1985). A comparison of Principal Components from real and random Data. *Ecology*, 66(6):1693-1698.

[59] Stock, J.H., Watson, M.W. (2002). Forecasting using Principal Components from a large number of predictors. *Journal of the American Statistical Association*, 97(460):1167-79.

[60] Stock, J.H., Watson, M.W. (2006). Forecasting with many predictors. In: *Handbook of Economic Forecasting*. Elsevier, pp. 515-554.

[61] Stock, J.H., Watson, M.W. (2011). Dynamic factor models. In:*The Oxford Handbook of Economic Forecasting*. Oxford: Oxford University Press.

[62] Timmerman, M.E., Kiers, H.A.L., Smilde, A. K. (2007). Estimating confidence intervals for principal component loadings: a comparison between the bootstrap and asymptotic results. *British Journal of Mathematical and Statistical Psychology* 60:295-314.

[63] Van Aelst, S., Willems, G. (2013). Fast and robust bootstrap for multivariate inference: the R package FRB. *Journal of Statistical Software* 53(3):1-32.

[64] Wang, M-C. (2009). Comparing the DSGE model with the factor model: An out of sample forecasting experiment. *Journal of Forecasting* 28:167-182.

[65] Yamamoto, Y. (2016). Bootstrap inference for impulse response functions in factor-augmented vector autoregressions. HIAS Discussion Paper E-26.

Table 1: Monte Carlo coverages (C), lengths (L) and Scoring Rule (SR) of asymptotic, extant bootstrap procedures and new resampling bands when the idiosyncratic component is homoscedastic and serial and cross-sectionally uncorrelated with $r = 1$, $\phi = 0.7$ and $q = 1$.

		T=20			T=50			T=100		
		N=20	N=50	N=100	N=20	N=50	N=100	N=20	N=50	N=100
Asymptotic										
70%	C	0.50	0.50	0.47	0.59	0.59	0.54	0.61	0.62	0.62
	L	0.73	0.50	0.35	0.77	0.52	0.36	0.77	0.51	0.37
	SR	1.87	1.27	1.05	1.49	1.03	0.84	1.41	0.91	0.7
95%	C	0.77	0.78	0.74	0.86	0.87	0.83	0.88	0.90	0.90
	L	1.38	0.94	0.67	1.45	0.98	0.67	1.46	0.96	0.64
	SR	4.02	2.77	2.55	2.67	1.86	1.72	2.41	1.48	1.21
Block Bootstrap 1										
70%	C	0.77	0.77	0.79	0.76	0.76	0.76	0.74	0.74	0.75
	L	2.01	2.01	2.01	2.04	2.02	2.04	2.06	2.05	2.03
	SR	2.59	2.61	2.58	2.74	2.72	2.8	2.89	2.82	2.78
95%	C	0.95	0.95	0.95	0.95	0.95	0.95	0.95	0.95	0.95
	L	3.55	3.52	3.58	3.74	3.68	3.73	3.8	3.74	3.76
	SR	3.67	3.65	3.69	3.85	3.79	3.86	3.93	3.86	3.85
Block Bootstrap 2										
70%	C	0.93	0.97	0.98	0.98	0.99	0.99	0.98	0.99	0.99
	L	1.97	1.96	1.97	2.03	2.00	2.03	2.05	2.04	2.04
	SR	1.97	1.99	1.99	2.05	2.01	2.03	2.07	2.04	2.04
95%	C	0.99	0.99	1.00	1.00	1.00	1.00	0.99	1.00	1.00
	L	3.73	3.71	3.74	3.85	3.79	3.85	3.89	3.86	3.85
	SR	3.79	3.71	3.74	3.85	3.79	3.85	3.89	3.86	3.85
Time-Residual Bootstrap										
70%	C	0.48	0.5	0.59	0.37	0.39	0.39	0.35	0.37	0.33
	L	0.66	0.51	0.50	0.39	0.3	0.23	0.33	0.27	0.16
	SR	1.66	1.18	0.98	1.44	1.06	0.84	1.35	1.01	0.75
95%	C	0.76	0.78	0.85	0.64	0.66	0.66	0.61	0.64	0.58
	L	1.24	0.97	0.95	0.75	0.57	0.44	0.63	0.52	0.31
	SR	3.57	2.5	1.98	3.77	2.76	2.24	3.79	2.63	2.27
Cross-Residual Bootstrap										
70%	C	0.63	0.61	0.54	0.65	0.66	0.62	0.67	0.67	0.64
	L	1.61	1.48	1.26	1.67	1.28	1.06	1.43	1.18	0.83
	SR	2.64	2.57	2.36	2.66	1.99	1.98	2.27	1.69	1.43
95%	C	0.86	0.85	0.82	0.83	0.87	0.87	0.91	0.91	0.88
	L	2.84	2.68	2.38	2.84	2.45	2.03	2.89	2.24	1.50
	SR	3.14	3.00	2.72	3.05	2.56	2.24	2.91	2.35	1.69
Subsampling										
70%	C	0.68	0.71	0.71	0.72	0.71	0.70	0.73	0.73	0.71
	L	1.12	0.80	0.62	1.02	0.67	0.50	100	0.64	0.46
	SR	1.71	1.18	0.95	1.47	1.00	0.79	1.39	0.90	0.69
95%	C	0.93	0.94	0.93	0.94	0.94	0.94	0.95	0.96	0.95
	L	2.12	1.51	1.17	1.93	1.26	0.95	1.89	1.22	0.87
	SR	2.73	1.92	1.54	2.35	1.62	1.34	2.23	1.4	1.07

Table 2: Monte Carlo coverages (C), lengths (L) and Scoring Rule (SR) of intervals based on the asymptotic approximation and on subsampling for different idiosyncratic structures with $r = 1$, $T = N = 50$.

	q	Nominal		Independence	Serial Dependence	Cross-Dependence	Heteroscedasticity
Asymptotic	2	70	C	0.57	0.55	0.58	0.58
			L	0.38	0.37	0.38	0.53
			SR	0.81	0.83	0.81	0.83
		95	C	0.84	0.83	0.85	0.85
			L	0.72	0.70	0.72	0.73
			SR	1.58	1.49	1.39	1.47
	1	70	C	0.58	0.55	0.56	0.58
			L	0.51	0.48	0.5	0.52
			SR	1.03	1.04	1.01	1.05
		95	C	0.86	0.83	0.85	0.86
			L	0.96	0.91	0.95	1.01
			SR	1.86	1.85	1.89	1.84
	0.5	70	C	0.58	0.52	0.39	0.57
			L	0.67	0.61	0.58	0.71
			SR	1.38	1.44	1.87	1.35
		95	C	0.86	0.81	0.66	0.85
			L	1.26	1.16	1.10	1.34
			SR	2.45	2.65	4.91	2.53
Subsampling	2	70	C	0.69	0.68	0.69	0.69
			L	0.5	0.49	0.49	0.5
			SR	0.81	0.78	0.75	0.79
		95	C	0.93	0.93	0.94	0.94
			L	0.94	0.92	0.93	0.95
			SR	1.27	1.15	1.09	1.47
	1	70	C	0.7 0	0.69	0.70	0.70
			L	0.65	0.64	0.66	0.69
			SR	1.03	0.99	0.96	1.00
		95	C	0.93	0.92	0.93	0.94
			L	1.24	1.21	1.25	1.31
			SR	1.62	1.57	1.62	1.55
	0.5	70	C	0.71	0.67	0.58	0.71
			L	0.88	0.88	0.98	0.97
			SR	1.38	1.35	1.59	1.30
		95	C	0.95	0.93	0.86	0.94
			L	1.66	1.66	1.84	1.83
			SR	2.1 0	2.16	2.69	2.21

Table 3: Monte Carlo averages of coverages of asymptotic and subsampling ellipsoids when the idiosyncratic component is homoscedastic and serial and cross-sectionally uncorrelated.

			T=20			T=50			T=100		
	q	Nominal	N=20	N=50	N=100	N=20	N=50	N=100	N=20	N=50	N=100
Asymptotic	2	70	0,22	0,19	0,15	0,33	0,32	0,30	0,40	0,40	0,41
		95	0,45	0,37	0,28	0,59	0,58	0,55	0,69	0,71	0,70
	1	70	0,17	0,16	0,10	0,31	0,27	0,25	0,34	0,34	0,35
		95	0,45	0,33	0,23	0,54	0,51	0,48	0,65	0,66	0,68
	0,5	70	0,17	0,11	0,08	0,19	0,15	0,16	0,20	0,21	0,24
		95	0,44	0,23	0,17	0,47	0,31	0,29	0,47	0,46	0,48
Subsampling	2	70	0,72	0,71	0,70	0,71	0,70	0,70	0,70	0,70	0,70
		95	0,91	0,91	0,90	0,89	0,90	0,90	0,91	0,90	0,91
	1	70	0,70	0,70	0,71	0,70	0,7	0,71	0,70	0,71	0,71
		95	0,94	0,95	0,92	0,94	0,93	0,93	0,95	0,93	0,91
	0,5	70	0,70	0,70	0,73	0,71	0,71	0,71	0,70	0,70	0,70
		95	0,92	0,89	0,89	0,91	0,91	0,93	0,92	0,92	0,92

Figure 1: Factor generated by a DFM (black continous lines) together with its PC estimates (blue discontinous lines) and 95% confidence bands (red continous lines) constructed using the asymptotic approximation (first row, first column), the subsampling procedure (first row, second column), block bootstrap based on bootstrap quantiles (second row, first column) and on Gaussian densities with bootstrap MSEs (second row, second column), time-residual bootstrap (third row, first column) and cross-residual bootstrap (third row, second column).

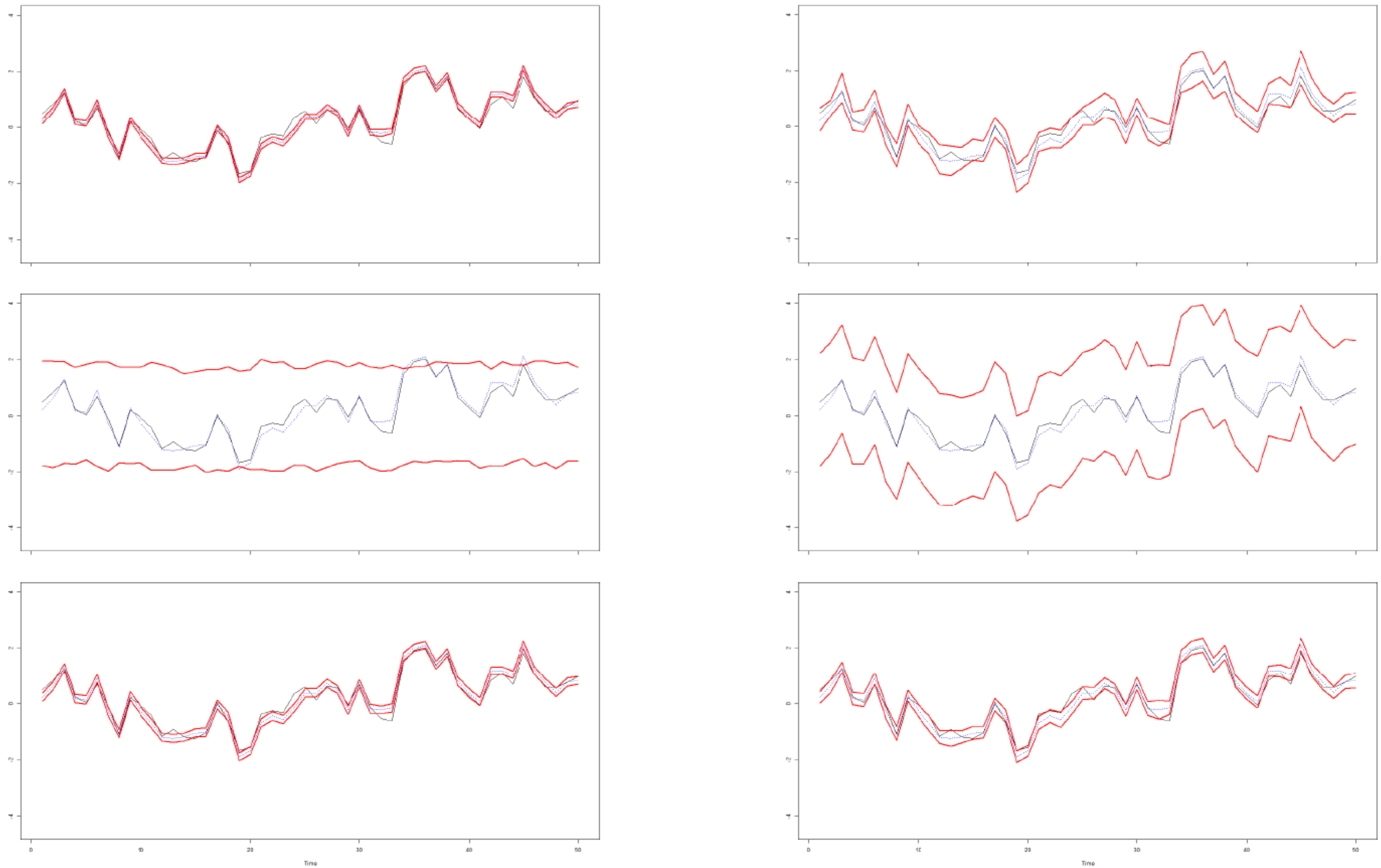

Figure 2: Factor generated by a DFM (black points) together with 95% confidence contours constructed using the asymptotic approximation (blue lines) and the subsampling procedure (red line) for every $t = 1, ..., 25$.

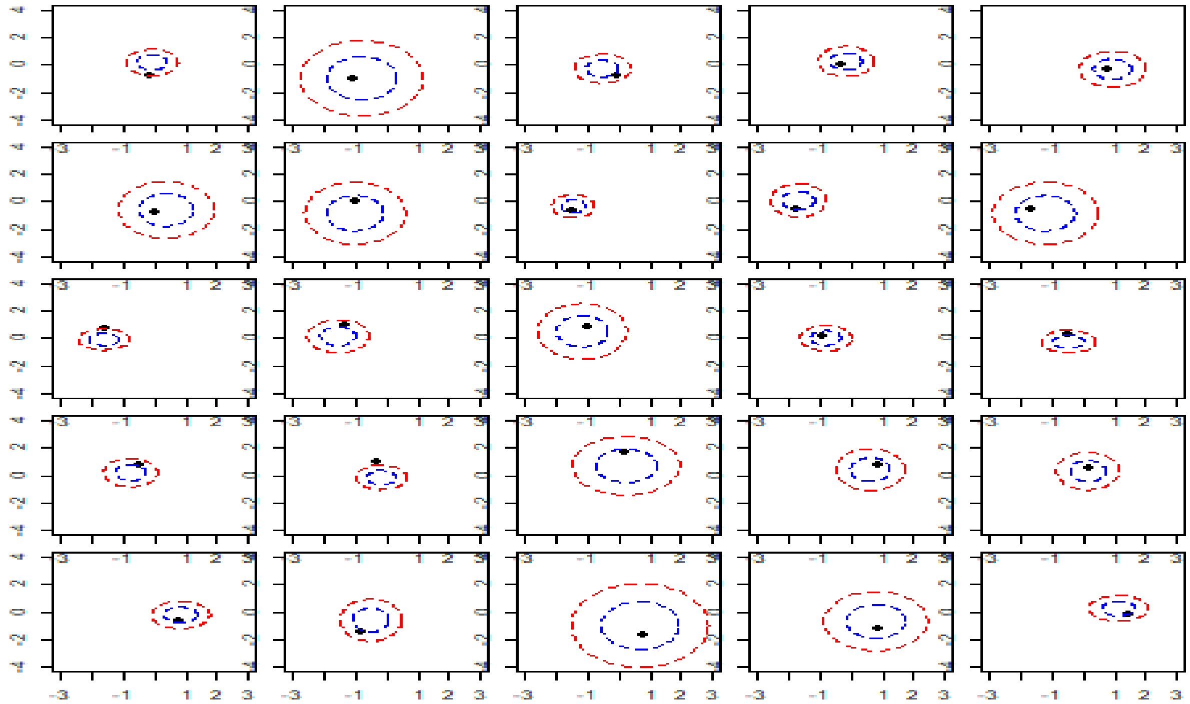

Figure 3: Asymptotic (blue continous lines) and resampling 95% intervals (red discontinous lines) for estimated economic cycle in Spain (black line).

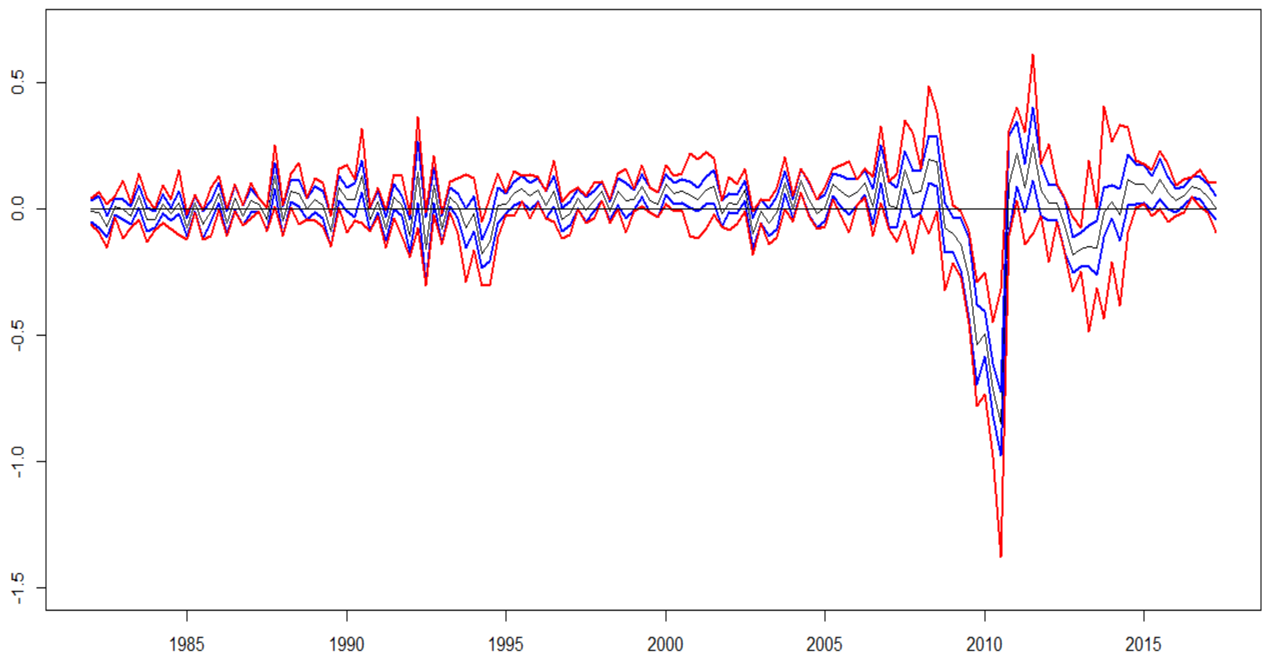

Figure 4: Asymptotic (blue lines) and resampling MSE (red lines) for estimated economic cycle in Spain.

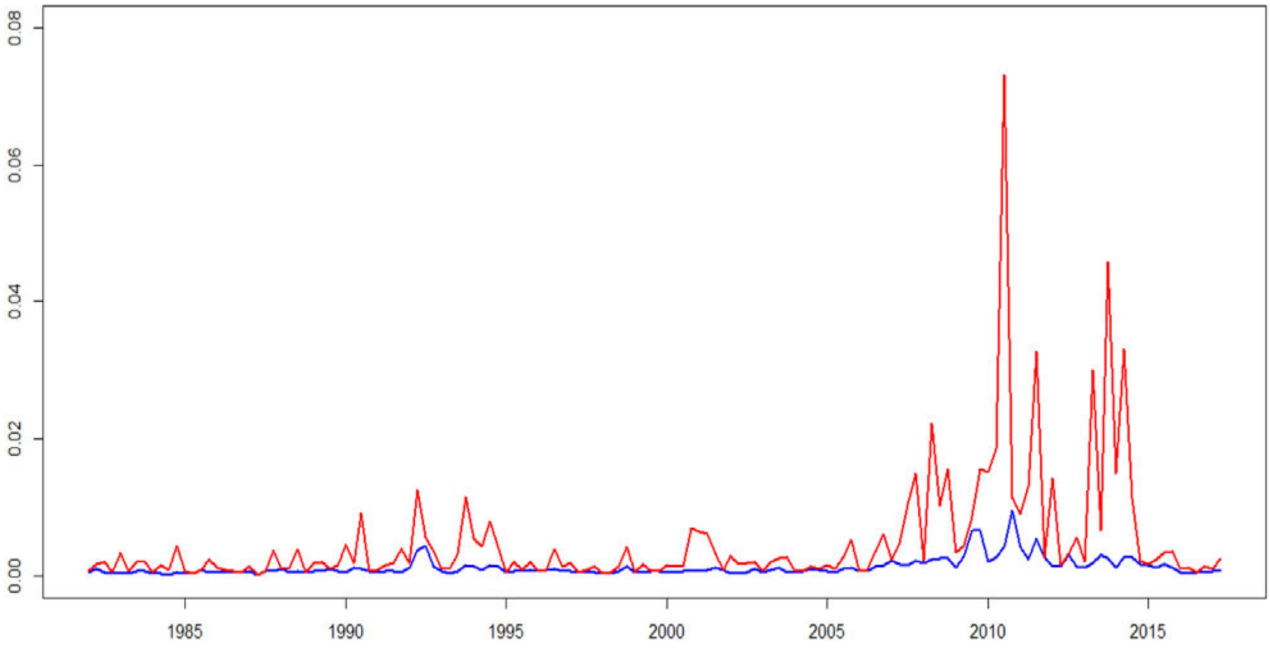

Table 4: List of the macroeconomic Spanish variables and their stationary transformations.

Variable	**Stationarity**
Gross Domestic Product mp	I(2)
Non-Market Servive Sector GAV	I(2)
Private GDP	I(2)
Households and non-profit institutions serving households final consumption expenditure	I(1)
Final consumption expenditure of Public Administrations	I(2)
Gross Capital Formation	I(1)
Gross Fixed Capital Formation	I(2)
Stock variations	I(0)
Exports of goods and services	I(1)
Imports of goods and services	I(1)
Imports of goods	I(1)
Imports of consumer goods	I(1)
Imports of capital goods	I(1)
Imports of intermediate goods	I(1)
Capital stock	I(2)
GDPmp deflator	I(2)
Labour market	I(2)
Workers in employment	I(2)
Workers in employment: full-time job equivalents	I(2)
Ratio full-time job equivalents/workers in employment	I(1)
Employees	I(2)
Employees: full-time job equivalents	I(2)
Workers in non-market services	I(2)
Workers in non-market services: full-time job equivalents	I(2)
Unemployment rate	I(2)
Number of hours worked	I(2)
Total compensation of employees (cp)	I(2)
Net taxes on products	I(2)
Private GDP at basic prices	I(2)
GDP deflator at basic prices (2010=1)	I(1)
Energy prices index	I(1)
Spain's Monetary Supply (M1)	I(1)
Spain's Monetary Supply (M3)	I(2)
US 3-month interest rates	I(1)
Vacancies	I(2)
Nominal exchange rate	I(1)
Debt of public administrations	I(2)
Net financial assets national economy	I(2)
Total resources of the public administrations	I(1)
Market production (P. 11)	I(1)
Non-market payments	I(1)
Taxes on production and imports	I(1)
Property income	I(1)
Current taxes on income, wealth, etc.	I(1)
Social contributions	I(2)
Other current transfers	I(1)
Capital transfers	I(1)
Overall employement in public administrations	I(2)
Intermediate consumption	I(2)
Other taxes on production	I(1)
Subsidies	I(1)
Social benefits other than transfers in kind	I(2)
Social transfers in kind related to the expenses in products supplied to households by market producers	I(1)
Purchases minus Transfers of non-financial Assests	I(0)
Lending (+)/Borrowing (-) capacity	I(1)
Unemployment benefits	I(2)
Stock of public capital	I(2)
Current taxes onincome	I(1)
3-month yields	I(1)