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Successful uninformed bidding^{*}

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Abstract

This paper provides some striking results that arise in the unique symmetric equilibrium of common value multi-unit auctions in which some bidders have more information than others. We show that in a generalized second price auction with single-unit demand, bidders with less information do surprisingly well: they can have a greater probability of winning than bidders with more information do, and may even have a higher expected utility. We also find a positive relationship between the success of *less-informed* bidders and a ratio of units for sale to bidders.

JEL classification D44

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1. Introduction

In this paper we study auctions of one or more homogeneous units of a common value good, about which some of the bidders have more information than the others do. Such auctions are of theoretical importance, since they model a number of real-life auctions reasonably well. Typical examples of such auctions are those for oil and gas leases, treasury-bill auctions, and auctions of parts of the radio spectrum.

We shall show that the equilibrium behaviour in a generalized second price auction in which bidders have a single-unit demand is quite surprising: *less-informed* bidders can bid

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very aggressively and beat *more-informed* bidders with high probability. We argue that this result can be explained in terms of two effects: the *winner's curse* and the *loser's curse*.

In order to illustrate our results we start with the following example. Suppose that one auctioneer offers one unit of a good for sale through a sealed bid second price auction to a pool of three bidders. One of these bidders, say Bidder A, knows the common value of the good, whereas the other two bidders, say Bidder B and Bidder C, only know that this value is drawn from a given bounded set according to a probability distribution.

The same reasons as in a second price private value auction show that Bidder A has a unique weakly dominant strategy, to bid the true value of the good. Assuming that Bidder A follows this strategy, Bidders B and C's unique weakly dominant strategy¹ is to bid the minimum value of the good, i.e. Bidder A's minimum bid. Suppose that Bidder B submits a bid higher than Bidder A's minimum bid. Bidder B can win in either of two cases:

- (i) when B bids above A and A bids above C, or
- (ii) when B bids above C, and C bids above A.

In case (i), Bidder B pays a price that is equal to Bidder A's bid, i.e. the true value of the good, whereas in case (ii), Bidder B pays a price that is equal to Bidder C's bid, i.e. a price above the true value of the good. In this sense, we can say that Bidder B suffers a winner's curse.

Let us now suppose that the auctioneer puts for sale two identical units of the good, instead of one. In this case, we assume that the auction's format is a generalisation of the sealed-bid second price auction to a two unit sale. The bidders with the two highest bids win one unit each and the price that they pay is the third highest bid, i.e. the loser's bid. Note that this auction setup is, in fact, the Vickrey auction for multi-unit sales and bidders with a single-unit demand.

Once again, Bidder A's unique weakly dominant strategy is to bid the true value of the good. But, in this case, if Bidder A follows this strategy, Bidders B and C's unique weakly dominant strategy is to bid the maximum value of the good, i.e. Bidder A's maximum bid. Suppose that Bidder B bids below Bidder A's maximum bid. Bidder B can win in either of two cases:

- (i) when C bids above A and A bids above B, or
- (ii) when A bids above C and C bids above B.

In case (i), had Bidder B bid high enough, she would have won at a price that is equal to the true value of the good. Whereas in case (ii), had Bidder B bid high enough, she would have won at a price below the true value of the good. Thus, we can say that Bidder B suffers a loser's curse.

Hence, if there is one unit of the good for sale, the uninformed bidders (B and C) lose with probability one and the perfectly informed bidder (A) wins with probability one at

¹ Note that what we are actually doing is solving the game by using two steps of iterated elimination of weakly dominated strategies.

a minimum price. Whereas, if there are two units of the good for sale, the uninformed bidders win with probability one at a price that is equal to the true value of the good, and the informed bidder loses with probability one. Note that the auctioneer's revenue is zero with one unit of the good, whereas he gets full surplus extraction when he sells two units.

In this paper we show how the results of this simple example can be extended to more general situations. We learn from our models that when the number of units for sale is sufficiently large with respect to the number of bidders, less-informed bidders tend to bid very aggressively and win more often than more-informed ones, and in some cases with a higher expected utility.² Moreover, we show that our results are not a pathological equilibrium of the game but rather the unique symmetric equilibrium. Symmetric in the sense that bidders of the same class use the same strategy.

Our results provide new ideas about the bidders' incentives to choose their optimal bid. For instance, our analysis explains why it is not at all obvious that less-informed bidders should bid more conservatively than more-informed ones. We also analyse a bidder's incentives to acquire information, and to do it either openly or covertly. We argue that the greater the number of units that are offered for sale relative to the number of buyers, the lower the incentives are to acquire information openly.

Our models also provide new ideas about the optimal design and the efficiency of auctions.³ For instance, we show that the expected selling price can increase with the number of units offered for sale, and decrease with the number of bidders. We also provide some results on the expected probability with which bidders with different quality of information win the auction.

For the case in which the bidders have a unit-demand and that the number of units offered for sale is smaller than the number of well-informed bidders, Milgrom (1981) has displayed an equilibrium of a generalisation of the second price auction in which bidders who have no relevant private information lose to more-informed bidders with probability one.

In this paper, we focus on the opposite case, i.e. that there are at least as many units for sale as there are well-informed bidders. It is in this case where we show that Milgrom's result is, in a certain sense, reversed. Actually, we find a kind of monotonicity, increasing a ratio of units for sale to bidders increases the probability that less-informed bidders bid higher than more-informed ones. In practice, as for example in the case of the auctions

² Nevertheless, one property of the equilibria in our models is that bidders with no private information get zero expected utility. This result is consistent with the intuition that bidders with no private information have no informational rents in an auction, and thus do not get any positive expected utility. This is also a feature of many existing models, like those presented by Engelbrecht-Wiggans et al. (1983) and Milgrom and Weber (1982). Daripa (1997), however, shows that, in some circumstances, an uninformed bidder can achieve a positive expected utility.

³ Although efficiency is not an issue in a pure common value setup, if there are small private value differences the final allocation matters. Moreover, there can be situations in which the auctioneer can have preferences for bidders that are either more or less informed. An example of this case is the one in which the auctioneer wishes to encourage (or discourage) the acquisition of information. Another possibility is the case of auctions in which there are "incumbents" and "entrants." The incumbents will be typically more-informed than the entrants are. Hence, we could conjecture that the more successful the less-informed bidders are, the more attractive the auction will be to the entrants.

cited in the first paragraph of this introduction, it often seems realistic that well-informed bidders form only a small fraction of the total market.

It is important to emphasise that, although we consider multi-unit auctions we maintain the assumption that each bidder, individually, demands just one unit of the good. As such, our results are unrelated to the difficult problems that arise in auctions in which bidders are allowed to submit multi-unit demands. Because we maintain the unit-demand assumption, it is also obvious how the second price auction needs to be defined in the multi-unit case, say with k units for sale: the bidders with the k highest bids win and pay the $(k + 1)$ -th highest bid.

The reason for our interest in the second price format is that it allows us to develop the intuition for our findings particularly clearly. We explain our results in terms of the different effect of the winner's curse and the loser's curse on the incentives of bidders with private information of different quality to bid.

In the (generalized) second price auction a bidder will want to raise his bid by a small amount, say from b to $b + \epsilon$, if the expected value of a unit, conditional on its price being $p \in (b, b + \epsilon)$, is higher than p . The price is p if and only if the k -th highest bid of the other bidders is p . This event is the intersection of two events, one of which implies good news, whereas the other implies bad news. The good news is that at least k other bidders have been willing to bid p or more. If these bidders had any private information at all, it must have been favourable. This is good news. This effect has been called the *loser's curse*⁴ as a bidder who neglects this effect will regret losing. For a total number of bidders $n + 1$, the bad news is that at least $n + 1 - k$ other bidders have bid p or less, and hence, if they had any private information at all, it must have been unfavourable. This effect has been called the *winner's curse* as a bidder who neglects this effect will regret winning.⁵

The bad news of the winner's curse reduces the incentives to bid higher, whereas the good news of the loser's curse raises the incentives to bid higher. Moreover, both effects are stronger for less-informed bidders because of two different reasons. The more-informed bidders' estimation is more accurate, and hence, less sensitive to new information.⁶ The average informational content of the other bidders' signals is of a lower quality from the view point of a more-informed bidder than from that of a less-informed one. A more-informed bidder faces one less more-informed bidder and one more less-informed bidder than a less-informed bidder.

The balance between the loser's curse and the winner's curse will obviously depend on p . More interestingly, it will also depend on the exogenous parameters k and n , and more precisely on the ratio $(k - 1)/(n - k)$ or equivalently on $(k - 1)/(n - 1)$. This is so, because

⁴ See for instance Holt and Sherman (1994) in the context of a bargaining game, and Pesendorfer and Swinkels (1997) in an auction setup.

⁵ The winner's curse has traditionally been defined as an out-of-equilibrium outcome that typically shapes the equilibrium strategies. We opt for an alternative definition. We call winner's curse to a statistical event which, if not taken into account by a bidder, will move her to the winner's curse out-of-equilibrium outcome. We also adopt the same convention with regard to the loser's curse. This definition of the winner's and the loser's curses was first used by Pesendorfer and Swinkels (1997). To avoid confusions, whenever we do not refer to the statistical event we use italics.

⁶ This is true only under the usual assumption, in auction theory, that the bidders' signals are informational substitutes, see the brief discussion in Milgrom and Weber (1982).

in the intersection between the loser's and the winner's curse events one bidder bids p , and among the other $n - 1$ bidders, exactly $k - 1$ bid above p and exactly $n - k$ bid below it.

Thus, an increase in the ratio of number of units (minus one) to number of bidders (minus two) will increase the relative strength of the loser's curse with respect to the winner's curse. As a consequence, the incentives to bid higher of less-informed bidders will increase relatively to those of more-informed bidders. This explains why increases in the number of units offered for sale or decreases in the number of bidders make less-informed bidders more eager to beat the more-informed ones in equilibrium.⁷

The observation that uninformed bidders may win auctions is not original to our paper. In fact, Engelbrecht-Wiggans et al. (1983) showed that this may happen in the single unit case if the format is a first price auction. Daripa (1997) extends Engelbrecht-Wiggans et al.'s result to a multi-unit setup, using a generalisation of the first price auction. Daripa also shows that an uninformed bidder can have a higher expected utility than a perfectly informed one.

However, Daripa's setup is quite different from ours. The auction format that he studies is more complicated, making it difficult to provide a clear intuition, or to extend it in different directions, as for instance, allowing the less-informed bidders to hold some private information, as we do. Moreover, he allows for *multi-demand bids* and hence, faces the severe problems of multiplicity of equilibria that usually arises under this assumption. Finally, our results differ from Daripa's, since, in his model, only one of the uninformed bidders does well, whereas in ours, it is a common feature to all the uninformed bidders.

Engelbrecht-Wiggans and Weber (1983) have also shown that uninformed bidding can be more profitable than informed bidding in a multi-unit setup. Their framework differs in that they study sales through a sequence of auctions.

Another related study is that of Pesendorfer and Swinkels (1997), which like ours, studies the generalisation of the second price auction to the multi-unit case when bidders have unit-demand bids. There are two main differences between our paper and theirs. First, they assume that all of the bidders have equally informative signals, whereas we focus on the case in which some of the bidders have signals that are more informative than the others have. Second, they focus on the case in which the number of units offered for sale and the number of bidders are both large. Our focus, in contrast, is on the case of a fixed, finite number of bidders and units.

This paper is structured as follows. In Section 2, we study a basic model in which there is one bidder with relevant information, and several other, completely uninformed bidders. In Section 3, we extend the model of Section 2 to allow the bidders who were uninformed to hold some pieces of information. We assume, however, that their information is less

⁷ Other authors, for instance Bulow and Klemperer (2002), have remarked that the winner's curse also has a "multiplier" effect. According to this effect, when one bidder shifts her bid upwards, it increases the effect of the winner's curse on the other bidders. This decreases their incentives to bid higher, and as a result, they bid lower. This, however, implies a decrease in the effect of the winner's curse on the first bidder, and hence, an increase in her incentives to bid higher. As a result this bidder bids higher and this induces a new round of effects. Note, however, that the effect of the loser's curse is opposite, and decreases the power of this "multiplier" effect. When one bidder bids higher, it also increases the loser's curse on the other bidders, and this induces the other bidders to bid higher as well.

significant than that of the other bidder. We also provide an appendix with all the proofs that do not appear in the main text.

2. An auction with one informed bidder and many uninformed bidders

An auctioneer puts k indivisible units of a good up for sale by auction. There are $n + 1$ bidders,⁸ $n \geq 2$. Each bidder can bid for one or zero units of the good.⁹ We assume that the number of bidders is greater than the number of units offered for sale, $n + 1 > k$.

We also assume bidders to be risk neutral and to put a monetary value of v in the consumption of the good. The value v is common to all bidders and we assume that it is a random variable with a continuous distribution function $F(v)$ and a bounded convex support that we normalise to $[0, 1]$. We assume that one¹⁰ bidder, the *informed* one, observes the value¹¹ of v privately, whereas the others, the *uninformed* bidders, do not have any private information.

We restrict to (generalized) second price auctions with neither a reserve bid nor an entry fee. In this auction format, all the bidders submit simultaneously one non-negative bid each. The bidders who make the k -th highest bids win one unit of the good each. The price they have to pay is the $(k + 1)$ -th highest bid. If the k -th highest bid and the $(k + 1)$ -th highest bid have the same value b , then the price in the auction is b , all bidders who make a bid strictly higher than b get one unit with probability one, and the remaining winners are randomly selected among all bidders who have made bid b , whereby all such bidders have the same probability of being selected.

We shall say that an equilibrium is *symmetric* if all the uninformed bidders use the same strategy (possibly mixed). By focusing on symmetric strategies, we disregard some pathological equilibria that typically exist in (generalized) second price auctions. In the one-unit-for-sale case, these equilibria are such that a given bidder bids sufficiently high, and thus, he wins with probability one, while all the others bid sufficiently low, and lose with probability one.¹² We could use the same logic to construct similar equilibria in the

⁸ In the case $n = k = 1$ the auction game which we are considering here has numerous equilibria. Since an analysis of these equilibria would distract us from the main point of this paper, we restrict our attention to the case of $n \geq 2$.

⁹ Equivalently, we could assume that a perfectly divisible good is for sale, that all of the bidders have a constant marginal utility for the good, and that the auctioneer splits the good into k identical lots and allows each bidder to bid for, at the most, one of such lots.

¹⁰ The model can be easily generalized to more than one perfectly informed bidder. If we suppose that the number of perfectly informed bidders is strictly less than the number of uninformed bidders, we could show that, in equilibrium, (i) the uninformed bidders lose with probability one if the number of units for sale is equal to, or is less than, the number of informed bidders; (ii) all of the uninformed bidders win with probability one if the number of units for sale is greater than the number of perfectly informed bidders and less than, or equal to, the number of uninformed bidders; otherwise, (iii) each uninformed bidder will win with a strictly positive probability strictly less than one.

¹¹ Our results also hold if we assume that the informed bidder only observes a noisy signal informative about the value of the good.

¹² Such equilibria are explained by Milgrom (1981) for the case of ex ante symmetric bidders. It is easy to show that these arguments can also be extended to ex ante asymmetric bidders.

multi-unit-for-sale case. Then, a number of bidders, equal to the number of units for sale, bid sufficiently high and all the other bidders bid sufficiently low.

Although under our assumptions, the former equilibria imply that the informed bidder uses a weakly dominated strategy, we can provide other equilibria, based on the same logic, in which the informed bidder uses a weakly dominant strategy. For instance, the informed bidder bids the true value, $k - 1$ uninformed bidders bid 1, one uninformed bidder uses any mixed strategy with support in the interval $[0, 1]$ and the other uninformed bidders bid 0. Nevertheless, these equilibria do not seem very realistic since they require a great capacity for coordination among the uninformed bidders.

Proposition 1. *There exists a unique symmetric equilibrium in strategies that are not weakly dominated. The informed bidder bids the true value of the good, and the uninformed bidders use a mixed strategy (possibly a degenerate one) with support in $[0, 1]$ and a distribution function:*

$$G^*(b) \equiv \frac{(n - k) \int_0^b F(s) ds}{(n - k) \int_0^b F(s) ds + (k - 1) \int_b^1 [1 - F(s)] ds}, \quad (1)$$

for all $b \in (0, 1)$.

Corollary 1.

- *If there is one unit for sale ($k = 1$), the uninformed bidders bid zero with probability one in equilibrium. Hence (i) the informed bidder wins with probability one the only unit offered for sale; (ii) the informed bidder has a positive expected utility, whereas the uninformed bidders have an expected utility of zero; (iii) the price is completely uninformative, since it is always equal to zero.*
- *If there are n units for sale ($k = n$), the uninformed bidders bid one with probability one in equilibrium. Hence (i) with probability one all units are won by uninformed bidders; (ii) all bidders have an expected utility of zero; (iii) the price reveals the true value of the good.*
- *If the number of units for sale is between 1 and n ($1 < k < n$), the uninformed bidders randomise their bids on the interval $[0, 1]$ in equilibrium. Hence (i) all bidders have a positive probability of winning; (ii) the informed bidder has a positive expected utility, but the uninformed bidders have an expected utility of zero; (iii) the price contains information about the conditional true value, but it is an imperfect signal.*

The equilibrium can be explained in terms of the effect of the events winner's curse and loser's curse on the incentives of the uninformed bidders to increase their bids. In equilibrium, both events provide information about the probability that the informed bidder bids either above or below the final price in the auction. Thus, if the informed bidder follows a monotone bid function, as it is the case here, both events convey information about the value of the good.

The incentives of the informed bidder, however, are not affected by either the winner's curse or the loser's curse. There are two reasons for this: first, she is completely informed

about the value of the good; and secondly, the other bidders do not hold any relevant information about the value of the good. The informed bidder's incentives, therefore, are determined by her private information.

Let us suppose, for instance, that there is only one unit of the good for sale, $k = 1$. In such a case, the winner's curse for an arbitrary bid p is especially strong: it means that all the other bidders bid (weakly) below p . An uninformed bidder learns from this event that the informed bidder is bidding (weakly) below p . This information is less favourable than the private information that the informed bidder bidding p has. This implies that the informed bidder will have greater incentives to increase the bid above p than the uninformed bidders will.

The opposite situation arises when the number of units offered for sale is equal to the number of uninformed bidders, $k = n$. The loser's curse for an arbitrary bid p is especially strong: it means that all the other bidders bid (weakly) above p . An uninformed bidder learns from this event that the informed bidder is bidding (weakly) above p . This information is more favourable than the private information that the informed bidder bidding p has. This implies that the uninformed bidders will have greater incentives to increase their bids above p than the informed bidder has.

In general, an increase in the number of units offered for sale strengthens the loser's curse and weakens the winner's curse. Likewise, a decrease in the number of uninformed bidders weakens the winner's curse. As such, both changes increase the uninformed bidders' incentives to increase their bids.

Proposition 2. *Either an increase in the number of units for sale or a decrease in the number of uninformed bidders shifts the equilibrium distribution of uninformed bidders' bids (in the sense of first order stochastic dominance) to the right.*

Proof. Direct from the expression of G^* , see Eq. (1). \square

This is a striking result, as we would have expected, instead, that an alleviation in the excess of demand would have eased the competition. Note that this result implies that increases in the number of units being offered for sale, or decreases in the number of bidders, have a double effect on both the probability that a given bidder wins the auction and on the expected price. The first effect is a direct one: there are more units for sale relative to the number of bidders. The second effect is an indirect one, which is due to the increased aggressiveness of the bidding behaviour of the uninformed bidders.

Both effects go in the same direction when we consider the probability that a given uninformed bidder wins the auction. But they have opposite effects on the probability that the informed bidder will win, and on the expected selling price. We provide some numerical examples that show that the indirect effect can be dominant, producing some surprising outcomes. We have computed these examples for F uniform. Figures 1 and 2 illustrate, respectively, that the probability that the informed bidder wins can decrease with the number of units offered for sale, k , and increase with the number of uninformed bidders, n .

Somewhat more surprising are the results presented in Figs. 3 and 4. They show, respectively, that the expected price in the auction can increase with the number of units

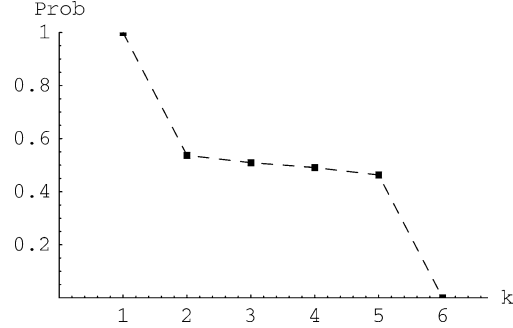


Fig. 1. Probability that the informed bidder wins as a function of k ($n = 6$).

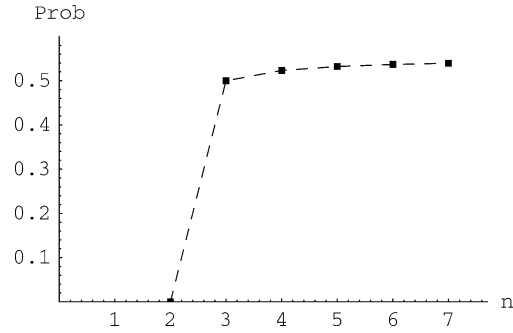


Fig. 2. Probability that the informed bidder wins as a function of n ($k = 2$).

offered for sale and decrease with the number of bidders.¹³ Such results challenge the standard economics view that any increases in the supply or decreases in the demand should decrease the price of the good. Our results also support those presented by Bulow and Klemperer (2002), who show that, in asymmetric auctions with a common value component, increasing the number of units being offered for sale, or decreasing the number of bidders, can raise the expected selling price. Note, however, that our model differs from Bulow and Klemperer's in the kind of asymmetries considered. Those authors assume that the asymmetries arise from a private value component, whereas, we assume that the asymmetries are due to differences in the quality of the information.

Proposition 1 also has implications for the bidders' incentives to acquire information. To see why this is so, let us suppose, first, that all the bidders are uninformed. In this case, it is weakly dominant for each bidder to bid the expected value of the good. When all of the bidders follow this strategy, they all get an expected utility equal to zero. Let us suppose

¹³ These numerical examples and others that we have produced suggest that the indirect effect always dominates over the direct effect. It is unclear, however, whether this result holds with generality. In fact, we cannot ensure that an increase in our ratio of units to bidders increases the probability that the informed bidder wins either more or less than the increase in the ratio of units to bidders. The reason is that even if we keep the strategies fixed, an increase in this ratio will affect the ex ante probability of winning in a different way for bidders with different distributions of their bids.

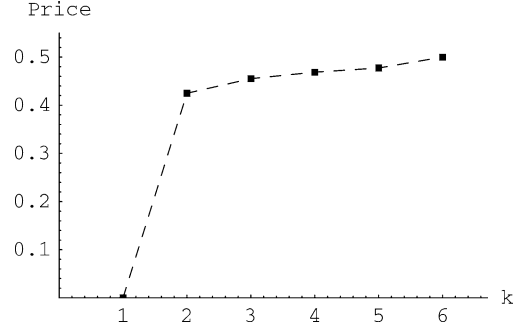


Fig. 3. Expected price in the auction as a function of k ($n = 6$).

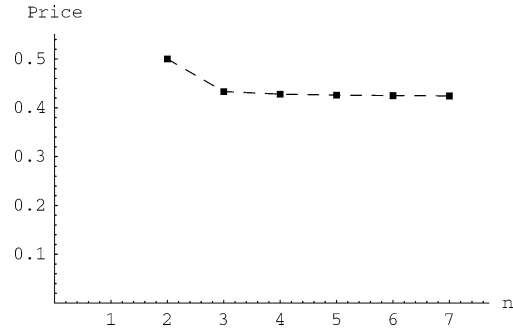


Fig. 4. Expected price in the auction as a function of n ($k = 2$).

now that just one bidder, say Bidder I, can become informed at a given cost which is strictly positive but sufficiently small. Suppose as well, that the acquisition of such information is observed by all of the other bidders, and then, the equilibrium in Proposition 1 is played.¹⁴ The following result follows directly from Corollary 1.

Proposition 3. *Bidder I finds it profitable to become informed if $k = 1$, but finds it unprofitable if $k = n$.*

In many examples, as those illustrated in Fig. 5 and in Fig. 6 for F uniform, the informed bidder's expected utility is decreasing in k and increasing in n . Thus, we could argue that the bidders' incentives to acquire information, in the former sense, are decreasing in the number of units offered for sale, but increasing in the number of bidders.

Another related question is whether Bidder I has incentives to reveal that she has become informed or not. Let us suppose that when Bidder I does not reveal anything, the other bidders follow the unique weakly dominant strategy for the game in which all bidders are

¹⁴ We do not study covert acquisition of information. This decision has no strategic effect and, thus, it always increases the bidder's expected utility. Consequently, if the cost of acquiring information is sufficiently low, the bidder finds it profitable to acquire information covertly.

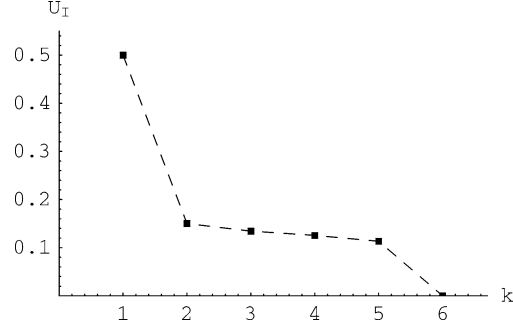


Fig. 5. Informed bidder's expected utility as a function of k ($n = 6$).

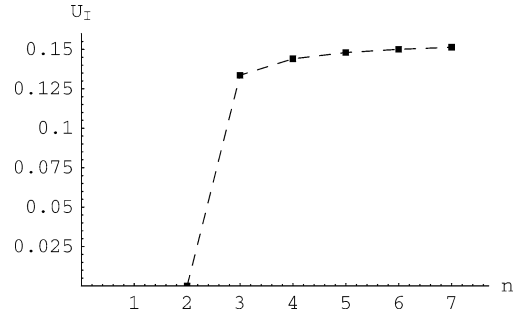


Fig. 6. Informed bidder's expected utility as a function of n ($k = 2$).

uninformed. We shall assume as well, that when Bidder I does reveal the fact that she has become informed, the equilibrium of Proposition 1 is played. Then, we have the following result.

Proposition 4. *If $k = 1$, Bidder I increases her expected utility by revealing that she has become informed. However, if $k = n$, her expected utility decreases on revealing that she has become informed.*

Proof. To prove the proposition, note that if Bidder I does not reveal that she is informed, her only weakly dominant strategy is still to bid the true value. Bidder I will therefore win the auction if, and only if, the true value of the good is greater than its expected value, in which case she pays the expected value of the good. The proposition then follows from the results in Corollary 1. \square

This result shows that the revenue rankings provided by Milgrom and Weber (1982, Theorem 4 (ii)–(iii)), for first price sealed bid auctions with just one unit for sale, also hold in our setup for one unit for sale but they are reversed when the number of units for sale is maximum. In fact, it can be shown that, for many examples, there exists a cut-off point such that the informed bidder prefers to reveal that she has private information, if and only

if, the number of units offered for sale is lower than this cut-off point. For instance, it can be shown that if F is uniform and $n = 6$, this cut-off point is equal to 4.

3. An auction with one informed and many poorly informed bidders

The simplicity of the model presented in the previous section hinges on quite an extreme assumption: that the bidders either have a perfectly informative private signal, or no private information at all. This assumption has its drawbacks. First, as uninformed bidders get no informational rents, they have zero expected utility in equilibrium. The implications of this are clear, the uninformed bidders never get a strictly higher expected utility than the informed bidder does, in spite of winning more often. Furthermore, the uninformed bidders have no strict incentive to submit bids. Hence, we can conjecture that they will not submit any bid in the presence of bidding costs, even when such costs are small.

The second drawback with this assumption is that the intuition we present in the introduction applies only in a degenerate fashion. Since the informed bidder is the only one who holds relevant private information, her incentives are not affected by either the winner's curse or the loser's curse. Only the uninformed bidders' incentives are affected by both events, and, in fact, their effects are quite extreme.

In this section we extend the analysis of Section 2 by allowing the less-informed bidders to hold some relevant information. We thus talk of one bidder with more information (the *informed bidder*) and a few other bidders with less information (the *poorly informed bidders*), but not entirely uninformed.

We shall now show that this extension overcomes the drawbacks of the model presented in the previous section.¹⁵ In the unique equilibrium of the game, poorly informed bidders get a strictly positive expected utility, and sometimes, they even get a greater expected utility than the informed bidder does. Moreover, the intuition presented in the introduction applies naturally.

In this section we keep all of the assumptions introduced in Section 2 except for the information structure, which is modified to allow for less informative signals. We assume that the value of the good v is a simple arithmetic mean of some $n + 1$ signals s_i ($i = 0, 1, 2, \dots, n$). The signals s_i are assumed to be statistically independent and to follow the same continuous distribution function F with a bounded support of $[0, 1]$.

We assume that the informed bidder observes one of these signals (say s_0), which we shall call s^I in what follows, whereas, each of the poorly informed bidders observes a garble of a different signal s_i , that we shall call s_i^P ($i = 1, 2, \dots, n$). These garbles are generated by the following procedure: with a probability that is independent of the other random variables in the model, say λ ($0 < \lambda < 1$), s_i^P equals s_i , and with the complementary probability, $1 - \lambda$, s_i^P is equal to another random variable which is statistically independent

¹⁵ We could also use the model presented in this section to check the robustness of the equilibrium in the previous section. This is shown in the working paper version of this paper. We show that the unique symmetric equilibrium with one informed bidder and several poorly informed bidders converges in an appropriate sense to the unique symmetric equilibrium with one informed bidder and several uninformed bidders when the informativeness of the poorly informed bidders' signals vanishes.

of the other random variables of the model and which follows the same distribution as s_i , i.e. F .

We can say that the signal s^I is more informative of v than each of the signals s_i^P in the following sense. Clearly, s^I and s_i are equally informative of v , but since s_i^P is a garble of s_i , s_i is more informative than s_i^P in Blackwell's sense. Moreover, when λ tends towards one, each signal s_i^P becomes as informative about v as s^I , and when λ tends towards zero, each signal s_i^P becomes completely uninformative about v . Note also, that we have defined the signals s_i^P in such a way that they have a marginal distribution function F .

The reader may find it unnatural to assume that the private signals are independent. This is, however, a simplifying assumption. Our arguments do not depend on independence, and hence our results should also hold when signals have some type of correlation. We only need assumptions that assure that Lemma 1 still works. This lemma is necessary for the existence of an equilibrium in continuous and strictly increasing bid functions. Our reason for using independent signals is that in this case, these assumptions are quite simple and intuitive:

Monotonic Assumption. *The functions $s - E[s_i | s_i \leq s]$ and $E[s_i | s_i \geq s] - s$ are respectively strictly increasing and strictly decreasing in s , for $s \in [0, 1]$.*

The above assumption is satisfied by many distribution functions, e.g. the uniform distribution function. If F has a continuously differentiable density, see Lemma 3 in Bikhchandani and Riley (1991), a sufficient condition for the first part of the assumption is that F is strictly log-concave (i.e. f/F strictly decreasing). Likewise, if F has a continuously differentiable density, a sufficient condition for the second part is that $1 - F$ is strictly log-concave, this is to say that F has a strictly increasing hazard rate (i.e. $f/(1 - F)$ strictly increasing).

We define a *symmetric* equilibrium of the game as a bid function $b_I : [0, 1] \rightarrow \mathbb{R}_+$ for the informed bidder and a bid function $b_P : [0, 1] \rightarrow \mathbb{R}_+$ for the poorly informed bidders that form a Nash equilibrium of the game in strategies that are not weakly dominated.¹⁶ Note that we study symmetric equilibrium in the sense that all poorly informed bidders use the same bid function.

For the sake of simplicity we shall only consider equilibria in continuous and strictly increasing strategies. It is difficult to provide uniqueness results when one of these two assumptions does not hold, see for instance Bikhchandani and Riley (1991). We also restrict our attention to equilibria in which all the bidders have an unconditional positive probability of winning.¹⁷ This assumption disregards pathological equilibria that always exist in (generalized) second price auctions.¹⁸

¹⁶ We rule out weakly dominated strategies to avoid some trivial multiplicity of equilibria that arises in the cases of $k = 1$ and $k = n$. See also the explanations after Eqs. (5) and (6).

¹⁷ It seems reasonable to argue that, in the presence of bidding costs, although they might be small, the bidders who do not get any positive expected utility from the auction will not participate in it. If they do participate, therefore, it must be because they get some positive expected utility. In the model presented in this section, this is equivalent to saying that they win with a positive expected utility.

¹⁸ Since we assume that all the poorly informed bidders use the same strategy, these equilibria only exist in the case $k = 1$ and $k = n$. For instance, if $k = 1$, it is an equilibrium that the informed bidder bids a sufficiently

We next propose a function that we shall use to define the equilibrium strategies. We shall show that this function maps types of the poorly informed bidder into types of the informed bidder that submit the same bid in the unique equilibrium of the game. We call this function ϕ and it is implicitly defined in the domain $[0, 1]$ by the condition that the following conditional expected values are equal. We shall explain the intuitive meaning of this condition after Eqs. (3) and (4):¹⁹

(i) If $k = 1$,

$$E[v \mid s^I = \phi, s_{(1)}^P = s] = E[v \mid s^I \leq \phi, s_{(1)}^P = s_{(2)}^P = s]. \quad (2a)$$

(ii) If $k = n$,

$$E[v \mid s^I = \phi, s_{(n)}^P = s] = E[v \mid s^I \geq \phi, s_{(n-1)}^P = s_{(n)}^P = s]. \quad (2b)$$

(iii) If $1 < k < n$,

$$\begin{aligned} E[v \mid s^I = \phi, s_{(k)}^P = s] \\ = E[v \mid \{s^I \geq \phi, s_{(k-1)}^P = s_{(k)}^P = s\} \cup \{s^I \leq \phi, s_{(k)}^P = s_{(k+1)}^P = s\}]. \end{aligned} \quad (2c)$$

The next lemma shows that the function ϕ is well defined in the above equations and can play the role that we had suggested above.

Lemma 1. *Eqs. (2a)–(2c) define implicitly a unique function $\phi : [0, 1] \rightarrow [0, 1]$, which is continuous and strictly increasing.*

We use the function ϕ to define a bid strategy for the informed bidder and another one for the poorly informed bidders:

$$b_I^*(s) \equiv E[v \mid s^I = s, s_{(k)}^P = \phi^{-1}(s)] \quad (3)$$

for $s \in [\phi(0), \phi(1)]$, and

$$b_P^*(s) \equiv E[v \mid s^I = \phi(s), s_{(k)}^P = s] \quad (4)$$

for $s \in [0, 1]$.

To understand the underlying intuition, suppose that the bidders follow the above bid functions. In such a case, ϕ gives us the type of the informed bidder that submits the same bid as a given type of the poorly informed bidder.

We next argue that each bidder bids the expected value of the good conditional on her private information and on the information that she can learn from the event that she is

high value and the poorly informed bidders bid a sufficiently low value. Furthermore, some of these equilibria cannot be ruled out by restricting ourselves to strategies that are not weakly dominated. See also Section 2 and footnote 12.

¹⁹ Here and in what follows, we denote by $s_{(r)}^P$ the r -th highest signal of the poorly informed bidders. Likewise, $E[\cdot \mid \cdot]$ will denote the expected value of the random variable in front of the vertical line, conditional on the event that is defined after the vertical line. To simplify the notation, we also drop the dependence of ϕ on s in Eqs. (2a)–(2c).

winning and that the price is equal to her bid. That is, she conditions on the event that the k -th highest bid of the other bidders is equal to her bid. Obviously, to an informed bidder with type s this event means that the k -th highest type of the poorly informed bidders is equal to $\phi^{-1}(s)$.

In the case of poorly informed bidders, the explanation is less obvious, as the former event could imply one of two different scenarios:

- (i) the informed bidder has submitted the k -th highest bid, or
- (ii) another poorly informed bidder has done so.

A poorly informed bidder with type s learns from (i) that she has the k -th highest type among the poorly informed bidders and that the type of the informed bidder equals $\phi(s)$. This is actually the event on which we condition the expected value that defines b_p^* . The definition of the function ϕ assures that the bid of the poorly informed bidder is also equal to the expected value conditional on event (ii).

For instance, suppose that $1 < k < n$, then a poorly informed bidder with type s learns from event (ii) that either the informed bidder is of a higher type than $\phi(s)$ and hence the $(k-1)$ -th and k -th highest type of the poorly informed bidders are equal to s ; or that the informed bidder is of a lower type than $\phi(s)$ and hence the k -th and $(k+1)$ -th highest type of the poorly informed bidders are equal to s . Equation (2c) ensures that the expected value conditional on this event equals the bid of the poorly informed bidder.

Note that Eq. (3) only defines b_1^* for types in the range of ϕ . Next lemma shows that this range does not coincide with the space of types when $k = 1$ or $k = n$.

Lemma 2. (i) If $k = 1$, then $\phi(0) = 0$ and $\phi(1) < 1$.
(ii) If $k = n$, then $\phi(0) > 0$ and $\phi(1) = 1$.
(iii) If $1 < k < n$, then $\phi(0) = 0$ and $\phi(1) = 1$.

We complete the definition of b_1^* in the case $k = 1$,

$$b_1^*(s) \equiv E[v \mid s^I = s, s_{(1)}^P = 1], \quad (5)$$

for all s in $(\phi(1), 1]$.

It is remarkable that the informed bidder is actually indifferent among all the bids above $b_1^*(\phi(1))$. Since they are all above $b_p^*(1)$, they all win with probability one. However, as we show later on in the proof of Proposition 5, b_1^* as defined in Eq. (5) is the only bid function that it is neither so high that it is weakly dominated, nor so low that gives an incentive to the poorly informed bidders to deviate by increasing their bids.

Similarly, we complete the definition of b_1^* in the case $k = n$,

$$b_1^*(s) \equiv E[v \mid s^I = s, s_{(n)}^P = 0], \quad (6)$$

for all s in $[0, \phi(0))$.

Similarly, the informed bidder is indifferent among all the bids below $b_1(\phi(0))$. Since they are all below $b_p(0)$, they all lose with probability one. However, as we show in the proof of Proposition 5, b_1^* as defined in Eq. (6) is the only bid function that it is neither so

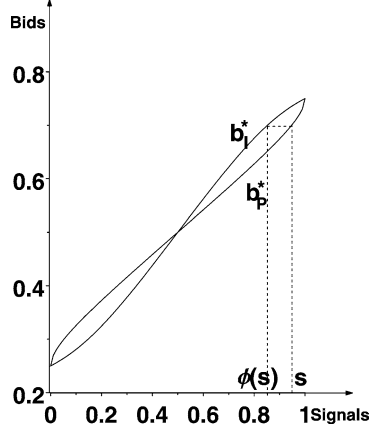


Fig. 7. Equilibrium bid functions with $k = 2$, $n = 3$ and $\lambda = 0.5$.

low that it is weakly dominated nor so high that gives an incentive to the poorly informed bidders to deviate by decreasing their bids.

We provide in Fig. 7 an example of (b_I^*, b_P^*) for the case in which F is uniform. This example also illustrates the role of function ϕ .

Proposition 5. *The strategies (b_I^*, b_P^*) happen to be the only symmetric equilibrium in strategies that are not weakly dominated.*

The former equilibrium has the following feature:²⁰

Proposition 6. *The probability that a given poorly informed bidder with type $s \in (0, 1)$ bids higher than the informed bidder strictly increases when the number of units offered for sale increases.*

Just as in the case of Proposition 2, an increase in the number of units offered for sale makes the poorly informed bidders bid relatively more aggressively than the informed bidder does. Once again, this result can be explained by the effect that the events winner's curse and loser's curse have on the bidders' incentives. The more units that are offered for sale, the stronger the loser's curse and the weaker winner's curse, leading to an increase in the bidders' incentives to bid higher that it is relatively stronger for less-informed bidders, see the arguments in the introduction.

We next show that poorly informed bidders can sometimes have a greater probability of winning than the informed bidder does, and that they may even achieve a higher expected utility as well. To provide such results, we shall focus on the case in which there is a

²⁰ The following result provides comparative static results, equivalent to Proposition 2, but only with respect to the number of units for sale. The corresponding results with respect to the number of (poorly informed) bidders also hold.

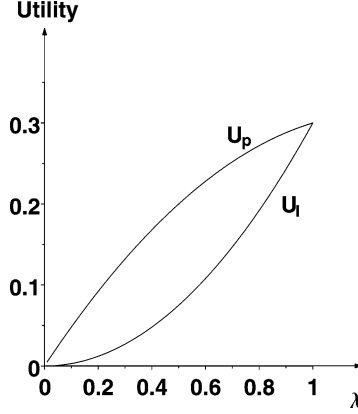


Fig. 8. Ex ante expected utilities of the informed bidder (U_I) and the poorly informed bidders (U_P), for $k = n = 3$.

maximum number of units being offered for sale, which is the most favourable scenario for the poorly informed bidders:

Proposition 7. *Let $k = n$ and $F(x) = x^\alpha$ for $\alpha \geq 1$.²¹ Under these assumptions, each poorly informed bidder has higher expected probability of winning and gets a higher (ex ante) expected utility than the informed bidder does.*

Figure 8 illustrates the utility comparison stated in the above proposition for the case in which F is uniform.

4. Conclusions

In this paper we have provided some natural common value auction models with asymmetrically-informed bidders, that have a striking equilibrium behaviour. Basically, we have shown that less-informed bidders can bid more aggressively, win with a higher expected probability, and even achieved a greater expected utility than better informed bidders do.

Our model suggests new ideas about the optimal bid behaviour. It also has important implications for different aspects of auction design, such as how many bidders should be allowed to participate in an auction or how many lots the good being offered for sale should be divided into. Furthermore, it poses new questions about entry in auctions, about the bidders' acquisition of information, and hence, about information aggregation in auctions.

Our model also provides a rationale for the aggressive bidding behaviour of the less-informed bidders. Our results suggest that we could expect this sort of outcome in multi-unit auctions with little excess of demand. Indeed, this is actually the case in the final phase of open ascending auctions in which the number of bidders who are still active is equal to

²¹ The restriction to $\alpha \geq 1$ is required as otherwise our Monotonic Assumption would not be satisfied

the number of units on sale plus one. Furthermore, if the bidders anticipate this, we can also expect to see aggressive bidding by the less-informed bidders in previous phases, and thus, throughout the entire game.

One of the limitations of our analysis is that we consider just one auction format, a (generalized) second price auction. In spite of the common difficulties generally seen in asymmetric auctions, however, the analysis of a (generalized) first price-auction has many features in common with our format, at least for the case in which there is one perfectly-informed bidder and several completely-uninformed bidders.

We can also wonder how our analysis extends to multi-demand auctions. Daripa (1997) has shown that aggressive less-informed bidding can be even more common when multi-demand bids are allowed at least in uniform price auctions. However, the connection between our results and those of Daripa remains unclear.

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Appendix

A. Proofs of Section 2

Proof of Proposition 1. To analyse equilibrium bidding in this auction we begin with the following observation: the informed bidder has a unique weakly dominant strategy, to bid the true value of the good v . This follows from the standard argument that is used to show that in a single-object, private-value, second price auction bidding one's true value is the unique weakly dominant strategy. Next note that bids which are not in the interval $[0, 1]$ are weakly dominated, hence, the support of the equilibrium randomisation of the uninformed bidders must be contained in $[0, 1]$.

We next consider two different cases:

Case A: $k = 1$ or $k = n$.

We only provide the proof for $k = 1$ and include between brackets the required modifications for $k = n$.

Note that when $k = 1$ ($k = n$) the equilibrium randomisation, G^* imposes a probability of one on the minimum value of the good, i.e. 0 (maximum value of the good, i.e. 1).

Proof that the proposed strategy is an equilibrium strategy. This part of the proof follows from the arguments provided in the introduction for computing the equilibrium in a second (third) price auction, with one (two) unit(s) for sale, a perfectly-informed bidder and two completely-uninformed bidders.

Proof that there are no other equilibrium strategies. Let us suppose that all of the uninformed bidders choose the same mixed strategy, and assume, furthermore, that this strategy assigns a positive probability to bids above 0 (below 1). As such, each uninformed bidder can gain by changing her strategy and bidding 0 (1) with probability one. To demonstrate this, we distinguish between the two events:

- (i) the highest (lowest) of all the uninformed bidders' bids is greater (less) than the informed bidders' bid, and
- (ii) the highest (lowest) of all the uninformed bidders' bids is less (greater) than or equal to the informed bidders' bid.

Observe that both events occur with a positive probability. In event (ii), all of the bids provide an expected utility of zero. Thus, the change in bidding strategy has no effect. In event (i), however, there is a strict incentive to be among the losers (winners) of the auction. In other words, there is a winner's curse (loser's curse). If the bidder adopts the same mixed strategy as all of the other uninformed bidders, there is a positive probability that she will be among the winners (losers). Thus, she can strictly gain by deviating to 0 (1).

Case B: $1 < k < n$.

Note that when $1 < k < n$ the support of the equilibrium randomisation G^* is $[0, 1]$.

Proof that the proposed strategy is an equilibrium strategy. We shall show that, in the proposed equilibrium, an uninformed bidder gets an expected utility of zero with any bid in $[0, 1]$. As such, she has no incentive to deviate. We assume, in what follows, that all of the uninformed bidders play G^* and the informed bidder plays her weakly dominant strategy.

An uninformed bidder can win the auction in either of two different events:

- (i) if the auction price is equal to the bid of the informed bidder, and
- (ii) if the auction price is equal to the bid of another uninformed bidder.

In event (i), the expected utility of winning is trivially zero, as the price is equal to the value of the good.

To compute the expected utility of winning in event (ii), we introduce, for an arbitrary b in $[0, 1]$, the notation $\rho(b)$. This represents the probability that the informed bidder's bid, v , is greater than b , conditional on event (ii). This is conditional on the event that there are exactly $k - 1$ bids above b among $n - 2$ uninformed bids and the informed bid. This is the probability that an uninformed bidder suffers a loser's curse at price b . Similarly, $1 - \rho(b)$ is the probability that an uninformed bidder suffers a winner's curse at price b . Using this notation, we can compute the uninformed bidder's expected utility under event (ii) as

$$\rho(b)E[v \mid v \geq b] + (1 - \rho(b))E[v \mid v \leq b] - b, \quad (\text{A.1})$$

where $\rho(b)$ equals by definition

$$\frac{\binom{n-2}{k-2} [1 - F(b)][1 - G^*(b)]^{k-2} G^*(b)^{n-k}}{\binom{n-2}{k-2} [1 - F(b)][1 - G^*(b)]^{k-2} G^*(b)^{n-k} + \binom{n-2}{k-1} F(b)[1 - G^*(b)]^{k-1} G^*(b)^{n-k-1}}. \quad (\text{A.2})$$

Simple algebra shows that this expected utility equals zero for G^* as defined in Eq. (1).

Proof that there are no other equilibrium strategies. This proof is broken down into two steps.

Step 1. In the first step we consider mixed strategies of the uninformed bidder that have a continuous distribution function. A necessary condition for such strategies to be an equilibrium is that each uninformed bidder is indifferent between all the bids in the support, if she takes as given that all the other uninformed bidders adopt the proposed strategy, and that the informed bidder plays her weakly dominant strategy.

Note that the above indifference condition is satisfied only if each uninformed bidder gets zero expected utility with any bid in the support of her equilibrium strategy. To see why, note that the number of units for sale is less than the number of uninformed bidders. Thus, the lowest bid in the support of the uninformed bidders' strategy must lose with probability one.

Hence, the expected value of the good conditional on an uninformed bidder winning at a price b in the support of the equilibrium strategy of the uninformed bidders must be equal to b . We can argue, as in the sufficient part of the proof, that an equilibrium requirement for a continuous distribution function is that, for each b in its support, expression (A.1) is equal to zero. It is easy to see that G^* is the only function that satisfies this condition.

Step 2. In this second step, we study mixed strategies that have a discontinuous distribution function. Assume that G is one such strategy, with an atom at \hat{b} . We next look at an uninformed bidder's incentives to deviate, say that of Bidder l . Let $b_{(k)}$ be the k -th highest bid of all the bidders but l . Define the event " \hat{b} wins" as the event in which Bidder l in making a bid \hat{b} wins one unit, and the event " \hat{b} loses" the complement of " \hat{b} wins," this is the event in which Bidder l in making a bid \hat{b} loses the auction.

We begin by arguing that we must have: $E[v \mid b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ wins}] \geq \hat{b}$. Let us suppose, instead, that $E[v \mid b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ wins}] < \hat{b}$. If this were the case, Bidder l could gain by shifting the entire probability mass that is placed on \hat{b} to some bid $\hat{b} - \epsilon$ where $\epsilon > 0$ is close to zero. This change would obviously make no difference to Bidder l 's utility in the case that $b_{(k)} > \hat{b}$, nor would it affect l 's utility in the case that $b_{(k)} = \hat{b}$ and \hat{b} loses. Finally, it would obviously not make any difference either in the event that $b_{(k)} < \hat{b} - \epsilon$. In the event that $b_{(k)} = \hat{b}$ and \hat{b} wins, which has positive probability, the change in strategy would lead to a strict increase in Bidder l 's utility. Finally, the probability of the event that $\hat{b} - \epsilon \leq b_{(k)} < \hat{b}$ can be made arbitrarily small by choosing a sufficiently small ϵ , so that it does not affect the advantageous nature of the proposed deviation. Likewise, it can be argued that we must have $E[v \mid b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ loses}] \leq \hat{b}$.

If $\hat{b} = 1$, the event $b_{(k)} = \hat{b}$ means that the bid of the informed bidder is below 1. As a consequence, the first of the conditions above cannot be satisfied. It can similarly be shown that $\hat{b} = 0$ violates the second of the above conditions.

We can complete our indirect proof by arguing that if $\hat{b} \in (0, 1)$, then

$$E[v \mid b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ wins}] < E[v \mid b_{(k)} = \hat{b}] < E[v \mid b_{(k)} = \hat{b} \text{ and } \hat{b} \text{ loses}].$$

In other words, there is a winner's curse and a loser's curse at price \hat{p} . This last inequality obviously contradicts the other two inequalities. Suppose you knew that $b_{(k)} = \hat{b}$, but you did not know whether the informed bidder is bidding above or below \hat{b} . If you learned that the informed bidder is bidding above \hat{b} , then the probability that \hat{b} wins would drop. Hence, \hat{b} wins has strictly negative correlation with the event that the informed bidder is bidding above \hat{b} , conditional on $b_{(k)} = \hat{b}$. This implies that whenever \hat{b} wins it is ex post more likely that the informed bidder is bidding below \hat{b} , and vice versa when \hat{b} loses. \square

B. Proofs from Section 3

For the sake of clarity in our arguments we introduce two functions: $\eta : [0, 1] \rightarrow \mathbb{R}_+$, where $\eta(s) = s - E[s_i \mid s_i \leq s]$, and $\mu : [0, 1] \rightarrow \mathbb{R}_+$, where $\mu(s) = E[s_i \mid s_i \geq s] - s$. These functions have the following properties.

Lemma 3. *Function η is continuous and strictly increasing. Moreover, $\eta(0) = 0$ and $\eta(1) > 0$. Function μ is continuous and strictly decreasing. Moreover, $\mu(1) = 0$ and $\mu(0) > 0$.*

Proof. Continuity follows from the continuity of F , the monotonic properties from our Monotonic Assumption, and the value of η and μ at 0 and 1 is direct from their definitions. \square

With the help of these two functions and using the equivalence

$$\begin{aligned} E[v \mid s^1 = s, s_{(k)}^p = s'] \\ = \frac{s + \lambda(s' + (k-1)E[s_i \mid s_i \geq s']) + (n-k)E[s_i \mid s_i \leq s']) + (1-\lambda)nE[s_i]}{1+n}, \end{aligned}$$

and other similar expressions that also hold for the other expected values in Eqs. (2a)–(2c), we can simplify these equations after some algebra to:

- (i) If $k = 1$, then $\phi(s) = \eta^{-1}(\lambda\eta(s))$.
- (ii) If $k = n$, then $\phi(s) = \mu^{-1}(\lambda\mu(s))$.
- (iii) If $1 < k < n$, then

$$\begin{aligned} (k-1)(1-F(\phi(s)))F(s)[\mu(\phi(s)) - \lambda\mu(s)] \\ - (n-k)F(\phi(s))(1-F(s))[\eta(\phi(s)) - \lambda\eta(s)] = 0. \end{aligned} \tag{B.1}$$

We next include the proofs from Section 3.

Proof of Lemma 1. If $k = 1$ or $k = n$, ϕ is continuous and strictly increasing because, respectively, η and μ are continuous and strictly monotone. The case of $1 < k < n$ is more

subtle. We begin by remarking that $\mu(\phi(s)) \geq \lambda\mu(s)$ and $\eta(\phi(s)) \geq \lambda\eta(s)$. To see why, note that if one of these inequalities were not satisfied, the other would not be satisfied by Eq. (B.1). But, from the first inequality, $\phi > s$ as μ is strictly decreasing, and from the second inequality, $\phi < s$ as η is increasing, which is a contradiction. As a consequence, the left-hand side of Eq. (B.1) must be decreasing in ϕ and increasing in s around the solutions of Eq. (B.1). Therefore, the continuity of Eq. (B.1) in ϕ and s , implies that ϕ is uniquely defined, continuous and strictly increasing. \square

Proof of Lemma 2. In this proof we repeatedly use the results in Lemma 3 about η and μ .

- (i) If $k = 1$, then $\phi(0) = \eta^{-1}(\lambda\eta(0)) = \eta^{-1}(0) = 0$, and $\phi(1) = \eta^{-1}(\lambda\eta(1)) < \eta^{-1}(\eta(1)) = 1$.
- (ii) If $k = n$, then $\phi(0) = \mu^{-1}(\lambda\mu(0)) > \mu^{-1}(\mu(0)) = 0$, and $\phi(1) = \mu^{-1}(\lambda\mu^{-1}(1)) = \mu^{-1}(0) = 1$.
- (iii) If $1 < k < n$, then $\phi(0) = 0$ and $\phi(1) = 1$ follow from the unique solution of Eq. (B.1). \square

Proof of Proposition 5.

Proof that the proposed strategies are equilibrium strategies. We show that no type of any bidder has strict incentives to decrease her bid when all the bidders follow the proposed strategies. Upward deviations can be checked in a symmetrical fashion. A bidder with a given type does not have any incentive to decrease her bid if the expected utility of winning at any price below her proposed bid is non-negative. Since lower types get less expected utility when they win, it is sufficient to show that, for any price below the proposed bid there is always a lower type that gets zero expected utility if she wins at that price. This is true for prices within the range of the bidder's bid function by definition of the proposed bid functions. These are such that a bidder gets zero expected utility when she wins and the price equals her bid. Poorly informed bidders, and only in case $k = n$, may end up paying prices out of the range of their bid function. This may happen when the price is fixed by the bid of the informed bidder below $b_p^*(0)$. But then, Condition (6) ensures that a poorly informed bidder with type 0 gets zero expected utility.

Proof that there are no other equilibrium strategies. Suppose that b_I, b_P are two strictly increasing and differentiable bid functions that form an equilibrium of the game. Define σ_I and σ_P the inverse of b_I and b_P , respectively. As such, the necessary conditions for maximum imply that, for any bid $b \in (b_I(0), b_I(1)) \cap (b_P(0), b_P(1))$, assuming by now that this intersection is not empty, $b_I(\sigma_I(b)) = b_I^*(\sigma_I(b))$, $b_P(\sigma_P(b)) = b_P^*(\sigma_P(b))$, and $\sigma_I(b) = \phi(\sigma_P(b))$. By continuity, $\sigma_I(\underline{b}) = \phi(\sigma_P(\underline{b}))$, where \underline{b} is the infimum of $(b_I(0), b_I(1)) \cap (b_P(0), b_P(1))$. Since b_I and b_P are continuous and strictly increasing, either $\underline{b} = b_I(0)$ or $\underline{b} = b_P(0)$. By Lemma 2, $\phi(0) \geq 0$, hence, $\underline{b} = b_P(0)$. We can argue similarly that $\bar{b} = b_P(0)$, where \bar{b} is the supremum of $(b_I(0), b_I(1)) \cap (b_P(0), b_P(1))$. Consequently, $b_I(s) = b_I^*(s)$ for any s in the range of ϕ , and $b_P(s) = b_P^*(s)$ for any $s \in [0, 1]$.

When $1 < k < n$, Lemma 2 ensures that the range of ϕ is $[0, 1]$, consequently, $b_I(s) = b_I^*(s)$ for all $s \in [0, 1]$. If $k = 1$, the same arguments imply that $b_I(s) = b_I^*(s)$

for all $s \in [0, \phi(1)]$. We next show that $b_I(s) = b_I^*(s)$ for all $s \in (\phi(1), 1]$. First, note that it cannot be that $b_I(s) > b_I^*(s)$ for $s \in (\phi(1), 1]$ as in such a case, $b_I(s)$ is weakly dominated by $b_I^*(s)$. Suppose next that $b_I(s) \leq b_I^*(s)$ for all $s > \phi(1)$. Then, there cannot be a positive measure of types for which the inequality is strict. Otherwise, poorly informed bidders with a type close to 1 would have strict incentives to deviate upwards, say to bid $b_I(1)$. By deviating, the bidder wins in some additional cases in which the price is strictly lower than the expected value of the good, conditional on winning.

Similarly, if $k = n$, the above arguments only imply that $b_I(s) = b_I^*(s)$ for all $s \in [\phi(0), 1]$. In this case, it cannot be that $b_I(s) < b_I^*(s)$ for $s \in [0, \phi(0))$, as, in such a case, $b_I(s)$ would be weakly dominated by $b_I^*(s)$. If we next suppose that $b_I(s) \geq b_I^*(s)$, for all $s \in [\phi(0), 1]$, then there cannot be a positive measure of types for which the inequality is strict. Otherwise, poorly informed bidders with a type close to 0 have strict incentives to deviate downwards, say to bid $b_I(0)$. By deviating, they avoid winning in cases in which the price is strictly higher than the expected value of the good conditional on winning.

We complete our proof by showing that the intersection of the interior of the ranges of b_I and b_P cannot be empty. Since we restrict ourselves to equilibrium in continuous and strictly increasing functions, this intersection can be empty if, and only if, either $b_I(0) \geq b_P(1)$ or $b_P(0) \geq b_I(1)$. We, therefore, only need to verify that none of these two possibilities can happen in equilibrium.

We start with $b_I(0) \geq b_P(1)$. If $k = 1$ this possibility is ruled out by our assumption that we are restricted to equilibria in which all the bidders have an ex ante positive probability of winning. If $k > 1$, then the informed bidder gets one unit with probability one (independently of her signal) and the poorly informed bidders compete for the remaining $k - 1$ units. We can use an analysis similar to Harstad and Levin (1986) to show that there is a unique symmetric equilibrium strategy for the poorly informed bidders. In this equilibrium, the bid function is equal to $b_P(s) = E[v \mid s_{(k-1)}^P = s_{(k)}^P = s]$. If all the poorly informed bidders follow this strategy, the informed bidder with a type 0 (or arbitrarily close to 0) has incentives to deviate lowering her bid slightly below $b_P(1)$. This deviation only changes the informed bidder's payoffs when the k -th highest type of the poorly informed bidders is arbitrarily close to 1. In this case, the informed bidder's expected utility when she wins is strictly negative. Hence, the deviation is profitable since it avoids winning in these cases.

The case of $b_P(0) \geq b_I(1)$ is ruled out by assumption as the informed bidder loses with probability one. Note, however, that for the case of $1 < k < n$, we could also use an argument that it is symmetrical to the one in the former paragraph. \square

Proof of Proposition 6. To prove the proposition it is sufficient to show that the type of the informed bidder who, in equilibrium, submits the same bid as a given type of the poorly informed bidders, increases when the number of units increases. Since by Proposition 5, $b_P^*(s) = b_I^*(\phi(s))$, the aforementioned statement follows if $\phi(s)$ shifts upwards when we increase k . We can use the same arguments as in the proof of Lemma 1 to show that the left-hand side of Eq. (B.1) increases with k around the solutions of Eq. (B.1). The proof then follows since for any given s , the left-hand side of Eq. (B.1) decreases with ϕ around the solutions of Eq. (B.1). \square

Proof of Proposition 7. To prove the first claim, it suffice to show that $\phi(s) \geq s$ for all $s \in [0, 1]$. This holds, since $\phi(s) = \mu^{-1}(\lambda\mu(s))$, where μ is strictly decreasing and $\lambda < 1$.

To prove the second claim, note that, since we have assumed independence across signals, we can use the arguments of Myerson (1981) to show that in equilibrium, the expected utility of the informed bidder with type s is a differentiable function with derivative the probability that this type wins the auction multiplied by the derivative of the expected value of the good with respect to the type of the bidder. Hence, the ex ante expected utility of the informed bidder (U_I) equals

$$U_I = \frac{1}{n+1} \int_{\phi(0)}^1 \int_{\phi(0)}^s [1 - (1 - F(\phi^{-1}(x)))^n] dx dF(s).$$

Recall that when $k = n$, $\mu(\phi(s)) = \lambda\mu(s)$, hence $\phi'(s) = \lambda\mu'(s)/\mu'(\phi(s))$. We change the variable of integration using this result and after a bit of algebra we get

$$U_I = \frac{\lambda}{n+1} \int_0^1 [1 - F(\phi(x))] [1 - (1 - F(x))^n] \frac{\mu'(x)}{\mu'(\phi(x))} dx.$$

We can also deduce that the ex ante expected utility of a poorly informed bidder (U_P) equals

$$U_P = \frac{\lambda}{n+1} \int_0^1 (1 - F(x)) [1 - (1 - F(\phi(x)))(1 - F(x))^{n-1}] dx.$$

Since for $\lambda \in (0, 1)$, $\phi(x) > x$, a sufficient condition for $U_P > U_I$ is that $\mu'(x)$ is a weakly increasing function. For the uniform case this condition holds trivially ($\mu(x) = (1 - x)/2$). We can show that for $F(x) = x^\alpha$, with $\alpha > 1$,

$$\mu''(x) = \frac{\alpha^2 x^{\alpha-2} \Psi(x)}{(\alpha+1)(1-x^\alpha)^3},$$

where $\Psi(x) = (\alpha-1)(1-x^{\alpha+1}) - (\alpha+1)(x-x^\alpha)$.

To complete our proof, we only need to show that $\Psi(x) \geq 0$, for all $x \in [0, 1]$. It is easy to verify that $\Psi''(x) \geq 0$, and $\Psi'(x) = 0$, thus $\Psi'(x) \leq 0$. Since, $\Psi(1) = 0$, this implies that $\Psi(x) \geq 0$ for all $x \in [0, 1]$. \square

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