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OPTIMAL CONTROL OF PARTIALLY OBSERVABLE LINEAR QUADRATIC SYSTEMS WITH ASYMMETRIC OBSERVATION ERRORS

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Keywords: Partially observable control systems; LQG optimal control; Adaptive control; Separation principle; Asymmetric Kalman filter.

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This paper deals with the optimal quadratic control problem for non-Gaussian discrete-time stochastic systems. Our main result gives explicit solutions for the optimal quadratic control problem for partially observable dynamic linear systems with asymmetric observation errors. For this purpose an asymmetric version of the Kalman filter based on asymmetric least squares estimation is used. We illustrate the applicability of our approach with numerical results

Key words: Partially observable control systems; LQG optimal control; adaptive control; separation principle; asymmetric Kalman filter.

1. Introduction

The Linear Gaussian control problem with Quadratic cost functional (LQG) is probably the most well-known quadratic control problem in both cases: discrete-time and continuous-time. Unfortunately, no explicit controller design has been obtained in general for optimal control problems with unobserved states. However, in adaptive control problems, the separation or certainty equivalence principle provides a specific solution, and the control is determined by two steps: construct the optimal control as if the system were observable and replace the unknown state in the control by the corresponding estimate (online); see [3] and [9]. In recent terminology, this model can be viewed as a special case of a controlled HMM (Hidden Markov Model); see [6].

In control engineering there is an increasing interest in non-Gaussian systems because a realistic statistical description of the quantities of interest, includes bimodal distributions, heavy tailed distributions or in general non-Gaussian behaviours; see [7] and [8]. The Kalman Filter (KF) has widely been used to construct the optimal solution for linear quadratic controlled systems; see [1]. Departures from the Gaussianity of the noise distribution can drastically disturb standard estimation and prediction procedures in dynamic linear systems.. In the recent statistical literature several alternatives to the standard KF have been suggested; see [4]. Unexpected observed values (outliers) of LQG problems has been treated in [13].

In this paper we give an explicit solution for the optimal control problem of discrete time LQG system in the presence of asymmetry in the observation errors. It means that unexpected asymmetric observation values are obtained, and then the Gaussian hypothesis does not hold. We develop our approach provided the estimation step in the separation principle is carried out through a modified version of the KF.

The article is organized as follows. Section 2 discusses the model and summarises the main results under Gaussian hypotheses. Our main result (Theorem 2.1) is then formulated. Section

3 contains the proof of Theorem 2.1. For this purpose an asymmetric version of the KF is introduced. In Section 4, numerical results showing best performance of the linear control based on the Asymmetric KF are presented. Conclusions are in Section 5.

2. Model and set-up

We consider the discrete stochastic system

$$x_{t+1} = F_t\, x_t + C_t\, u_t + w_t, \tag{2.1}$$
$$y_t = H_t\, x_t + v_t, \tag{2.2}$$

where $x_t \in R^n$ is the state vector, $y_t \in R^m$ is the measurement or observation vector and the exogenous variable $u_t \in R^p$ is selected according to some policy based on the past observations (separated policy). F_t ,C_t and H_t are non-stochastic time-varying known matrices of appropriate dimensions.

The noise terms w_t and v_t are independent random variables with multinormal distributions $N(0,Q_t)$ and $N(0,R_t)$, respectively. We assume that the covariance matrices Q_t and R_t are known. The initial conditions are given by the Gaussian variable x_0 following the multinormal distribution $N(0,P_0)$ and the covariance matrix P_0.

The information available at time t is given by $z^t = (Y^t, U^{t-1}) = (y_0,..., y_t, u_0,...,u_{t-1})$. The sequences of variables z^t and Y^t generate the same σ-field. The error term considered here is a martingale difference with respect to the increasing sequence of σ- fields z^t. We observe that the control variable u_t influences both the future state x_{t+1} and the future available information z^{t+1}.

From this general formulation some particular cases can be considered. For instance, if $H_t = I$ and $v_t = 0$, we have a perfectly observable stochastic system. The filtering problem is obtained in the non-controlled case ($C_t = 0$ or $u_t = 0$).

Let us consider the case $u_t = 0$. We have a linear Gaussian system (LG) for which the standard KF gives explicit expressions for the estimator and the predictor. The random variables x_t, x_{t+1} and Y^t are jointly Gaussian. The conditional density functions of the random variables (x_t/z^t) and (x_t/z^{t-1}) are given by the multinormal distributions $N(x_t^t, P_t^t)$ and $N(x_t^{t-1}, P_t^{t-1})$ respectively, where,

$$x_t^t := E\,\{x_t/Y^t\}, \tag{2.3}$$
$$P_t^t := E\,\{(x_t-x_t^t)(x_t-x_t^t)^T/Y^t\}. \tag{2.4}$$

We have similar expression for the one-step-ahead predictor

$$x_t^{t-1} := E\,\{x_t/Y^{t-1}\}, \tag{2.5}$$
$$P_t^{t-1} := E\,\{(x_t-x_t^{t-1})(x_t-x_t^{t-1})^T/Y^{t-1}\}, \tag{2.6}$$

Notice that x_t^t is sufficient to characterise the probabilistic behaviour of the information state.

Thus the estimate and the covariance error matrix are given by

$$x_t^t = x_t^{t-1} + K_t\,(y_t - H_t\, x_t^{t-1}), \tag{2.7}$$
$$P_t^t = P_t^{t-1} - K_t\, H_t\, P_t^{t-1}, \tag{2.8}$$

where the gain matrix K_t is

$$K_t := P_t^{t-1} H_t^T [H_t P_t^{t-1} H_t^T + R_t]^{-1}. \qquad (2.9)$$

The one-step-ahead predictor and the covariance error matrix are given by

$$x_t^{t-1} = F_{t-1} x_{t-1}^{t-1}, \qquad (2.10)$$
$$P_t^{t-1} = F_{t-1} P_{t-1}^{t-1} F_{t-1}^T + Q_{t-1}, \qquad (2.11)$$

In the general case, u_t is selected according to some feedback policy $g=\{g_0, g_1....\}$. Then the system is described by equations (2.1) and (2.2) with state and observation variables given by $x_{g,t}$ and $y_{g,t}$ respectively. The control variable is given by $u_{g,t} = g_t (z_g^t) = g_t (y_{g,0}, y_{g,1}, \ldots, y_{g,t}, u_{g,0}, u_{g,1}, \ldots, u_{g,t-1})$. If the function g_t is not linear, the corresponding processes $x_{g,t}$, $y_{g,t}$ and $u_{g,t}$ are not Gaussian even if x_0, w_t and v_t are Gaussian,. However, we can ensure a Gaussian conditional density for the information state even if the feedback function g_t is nonlinear. Moreover, the conditional density functions of the random variables $(x_{g,t}/z_g^t)$ and $(x_{g,t}/z_g^{t-1})$ are given by the multinormal distributions $N(x_{g,t}^t, P_t^t)$ and $N(x_{g,t}^{t-1}, P_t^{t-1})$ respectively.

Thus the estimate and the predictor are given by

$$x_{g,t}^t = x_{g,t}^{t-1} + P_t^t [y_{g,t} - H_t x_{g,t}^{t-1}], \qquad (2.12)$$
$$x_{g,t}^{t-1} = F_{t-1} x_{g,t-1}^{t-1} + C_{t-1} u_{g,t-1}. \qquad (2.13)$$

The error covariance matrices P_t^t and $P_t^{t-1}$and the gain matrix K_t are given by (2.8), (2.11) and (2.9) respectively.

The state estimator has an intuitive interpretation. It reproduces the structure of the linear model including a third term in the output-prediction error weighted with the gain matrix. Notice that (2.12) and (2.13) follow because $z_{g,t}$ and Y_g^t generate the same σ-field. The covariance error matrix does not depend on the feedback law and can be precomputed as in the classical KF for LG systems. Also notice that the control law g affects the conditional mean $x_{g,t}^t$, but not the covariance matrix P_t^t. This drastically simplifies the optimal control problem.

If the basic variables x_0, w_t and v_t are non-Gaussian, but their moments are as above, the moments of the processes $\{x_{g,t}\}$ and $\{y_{g,t}\}$remain the same as if the basic variables were Gaussian. Then $x_{g,t}^t$ as in (2.12) is only the best linear estimate and not the conditional mean which is the best nonlinear estimate if the Gaussian hypothesis does not hold.

We consider feasible control policies of the form $u_t = g_t (z^t)$. Let $g = (g_0,\ldots,g_{N-1})$. Then the finite horizon quadratic control problem is to find, for a finite time horizon N, the optimal policy g that minimises the functional

$$J(g) := E^g \{ \sum_{t=0}^{N-1} c_t(x_t,u_t) + c_N(x_N)\}, \qquad (2.14)$$

where the one-stage cost functions c_t are given by

$$c_t(x_t,u_t) = x_t^T A_t x_t + u_t^T T_t u_t,$$
$$c_N = x_N A_t x_N^T,$$

with A_t and T_t symmetric positive definite matrices.

In this section we state our main result. Our purpose now is to give the optimal control for the partially observed LQG system when observation errors are detected not normally distributed because they present asymmetry. The problem arises because at the estimation step, the standard KF can not be applied. Thus, we propose a useful procedure to be used at the estimation step. We provide a closed solution for the optimal control strategy of the LQG control problem when the observation errors are non-Gaussian. Our main contribution can be summarised as follows

Theorem 2.1. For the Linear Quadratic system given by (2.1) and (2.2) with asymmetric observation errors v_t (non-Gaussian)

i) the optimal control strategy minimizing the cost functional (2.14) is given by

$$u_{g,t} = L_t\, x_{g,t}^{\,t}, \tag{2.15}$$

where the state estimate $x_{g,t}^{\,t}$ is constructed from the recursive Asymmetric KF (this is an asymmetric version of the KF developed in section 3) and

$$L_t = -\,[T_t + C_t^T S_{t+1} C_t]^{-1} C_t^T S_{t+1} F_t, \tag{2.16}$$

with S_t recursively obtained backwards from F_t, C_t, A_t and T_t, with the boundary condition

$$S_N = A_N.$$

ii) the value function is given by

$$V_t = (x_{g,t}^{\,t})^T S_t\, x_{g,t}^{\,t} + s_t\ ,\ 0 \le t \le N, \tag{2.17}$$

where S_t is recursively obtained as above, and s_t is recursively obtained backwards from S_{t+1}, A_t, $P_t^{\,t}$, $P_{t+1}^{\,t}$ and $P_{t+1}^{\,t+1}$, with the boundary condition

$$s_N = \mathrm{Tr}\,(A_N\, P_N^{\,N})$$

Theorem 2.1, which is proved in Section 3, is essentially known except for the fact that we assume asymmetric observation errors v_t (non-Gaussian). As far as we know, our proposal is new, and general enough to cover a wide variety of non-Gaussian real situations.

Remark. Notice that the disturbance variables w_t and v_t do not appear in the control law. There is a great similarity between this solution and the optimal control law of the LQG stochastic control problem for the case of accurate observation of the state ($w_t = v_t = 0$ almost surely). The only difference is that the estimator $x_{g,t}^{\,t}$ replaces in our case the state x_t itself in the control law. This fact is known as the *Separation (or Certainty Equivalence) Principle*.

3. Proof of Theorem 2.1

To proof Theorem 2.1 let us first consider the following result for the Gaussian case.

Lemma 3.1. For the LQG problem defined by (2.1), (2.2) and (2.14), the optimal control policy depending on $x_{g,t}^{t}$, is obtained by the linear feedback relation

$$u_{g,t} = L_t\, x_{g,t}^{t}, \tag{3.1}$$

where

$$L_t = -\,[T_t + C_t^T S_{t+1} C_t]^{-1} C_t^T S_{t+1} F_t, \tag{3.2}$$

$x_{g,t}^{t}$ is given by (2.12) and (2.13), and S_t is recursively obtained backwards by

$$S_t = A_t + F_t\,\{S_{t+1} - S_{t+1}C_t\,[T_t + C_t^T S_{t+1} C_t]^{-1} C_t^T S_{t+1}\}F_t\ ,\ t = N\text{-}1, N\text{-}2,\ldots,0 \tag{3.3}$$

with the boundary condition

$$S_N = A_N. \tag{3.4}$$

Proof. Applying dynamic programming methods, this result can be obtained as in [10]. ■

We develop now the asymmetric version of the KF to be used in our approach.

Asymmetric Kalman filter and asymmetric prediction errors

The KF implementation is basically a sequence of filtering-prediction equations. The filtering step is obtained through a least squares minimisation and the prediction step is an update procedure based on the state equation. It is well known, for instance see [2], that the estimate $x_{g,t}^{t}$ in (2.12), for each policy $g = \{g_0, \ldots g_{N-1}\}$, can be obtained through the following weighted least squares minimisation, based on predictive values $x_{g,t}^{t-1}$ and P_t^{t-1} constructed one-step before

$$x_{g,t}^{t} = \underset{x_t \in \mathbb{R}^n}{\operatorname{argmin}}\{(x_{g,t}^{t-1} - x_t)^T (P_t^{t-1})^{-1} (x_{g,t}^{t-1} - x_t) + (y_t - H_t x_t)^T R_t^{-1} (y_t - H_t x_t)\} \tag{3.5}$$

We concentrate ourselves in the case of univariate observations.

Following [12], we replace the least squares estimation by the asymmetric least squares estimation. For this purpose we apply the asymmetric least squares estimator at the filtering step and we solve the following minimisation problem

$$x_{g,t}^{t} = \underset{x_t \in \mathbb{R}^n}{\operatorname{argmin}}\ (x_{g,t}^{t-1} - x_t)^T (P_t^{t-1})^{-1} (x_{g,t}^{t-1} - x_t) + [(y_t - H_t x_t)^-]^T (R^1{}_t)^{-1} [(y_t - H_t x_t)^-]$$

$$+ [(y_t - H_t x_t)^+]^T (R^2{}_t)^{-1} [(y_t - H_t x_t)^+]\ \}, \tag{3.6}$$

where

$$v_t^- = \min(v_t, 0) \text{ and } v_t^+ = \max(v_t, 0), \qquad (3.7)$$

where variances R^i_t, $i = 1,2$ are the unknown variances of the positive and negative observation errors and should be estimated at each step. The explicit solution of (3.5) is the Asymmetric KF

$$x_{g,t}^{\ t} = x_{g,t}^{\ t-1} + P_t^{t-1} H_t^T (H_t P_t^{t-1} H_t^T + R^1_t)^{-1} [(y_t - H_t x_t^{t-1} - C_t u_{g,t})^-]$$

$$+ P_t^{t-1} H_t^T (H_t P_t^{t-1} H_t^T + R^2_t)^{-1} [(y_t - H_t x_t^{t-1} - C_t u_{g,t})^+] \qquad (3.8)$$

where the error covariance matrix P_t^{t-1} is still given by (2.11). Notice that if $R^1_t = R^2_t = R_t$ we have the standard KF.

The next question is how to update R^1_t and R^2_t.

We face now the one-step-ahead prediction errors with asymmetric observation errors. Having in mind the strong dependence of the KF on the Gaussianity of the errors, we select an approach successfully used in forecasting based on the split-normal distribution. The split normal distribution is the most natural asymmetric version of the normal distribution. This generalization is inspired by [11], which includes a discussion about accuracy of forecasting with asymmetric errors using several methods including a split-normal modelisation. In [5], a similar approach is suggested for asymmetric recursive estimation of AR(p) models.

The split-normal distribution

Let consider the following asymmetric probability density function

$$f(x) = \begin{cases} (2\sigma_2/\sigma_1(\sigma_1+\sigma_2))\, \phi((x-\mu)/\sigma_1) & \text{for } x \le \mu \\ (2\sigma_1/\sigma_2(\sigma_1+\sigma_2))\, \phi((x-\mu)/\sigma_2) & \text{for } x > \mu \end{cases}$$

where ϕ stands for the standard normal density. This density is known as the split-normal density $N(\mu;\sigma_1^2, \sigma_2^2)$. For $\sigma_1 = \sigma_2$ we have the classical normal distribution. The marginal and conditional moments of the centred split-normal distribution $N(0; \sigma_1^2, \sigma_2^2)$ are

$$E[X^k] = ((-1)^k \sigma_1^k \sigma_2 + \sigma_2^k \sigma_1)\, \Gamma((k+1)/2)\, 2^{k/2} / (\pi^{1/2} (\sigma_1+\sigma_2)), \text{ for } k \ge 1,$$
$$Var[X] = \sigma_1 \sigma_2,$$
$$E[(X^-)^2 / X<0] = \sigma_1^2,$$
$$E[(X^+)^2 / X>0] = \sigma_2^2,$$

where X^- and X^+ are defined as in (3.7).

Using the split-normal $N(0; \sigma_1^2, \sigma_2^2)$ we can adaptively estimate the prediction error variance one-step-ahead. Let us denote the prediction error by

$$e_t = y_t - H_t x_t^{t-1}, \qquad (3.9)$$

We use a centred split-normal distribution because we predict without bias. Our proposal is to estimate variances $(\sigma_1^2)_t$ and $(\sigma_2^2)_t$ by R^1_t and R^2_t at each step as follows:

$$R^i_t = R^i_{t-1} + 1(e_{t-1}<0).\delta.(e_{t-1}^2 - R^i_{t-1}),\ \ i =1,2, \qquad (3.10)$$

where 1(.) is the indicator function and δ is a damping constant with $0<\delta<1$. Notice that using this approach we can derive probability limits on prediction based on the split-normal distribution function. We need starting values R^1_0 and R^2_0 provided along with δ.

To illustrate the behaviour of both, the KF and the Asymmetric KF, numerical simulations have been performed. We filter observations generated by the system (2.1) and (2.2) with perturbation errors v_t generated by the asymmetric distribution log- $(\times^2_1)$. We fix $H_t = 1$ and $u_t = 0$. Figure 1 displays the performance of the standard KF and the Asymmetric KF. It is clearly seen that de AKF is closer to the true state than the standard KF.

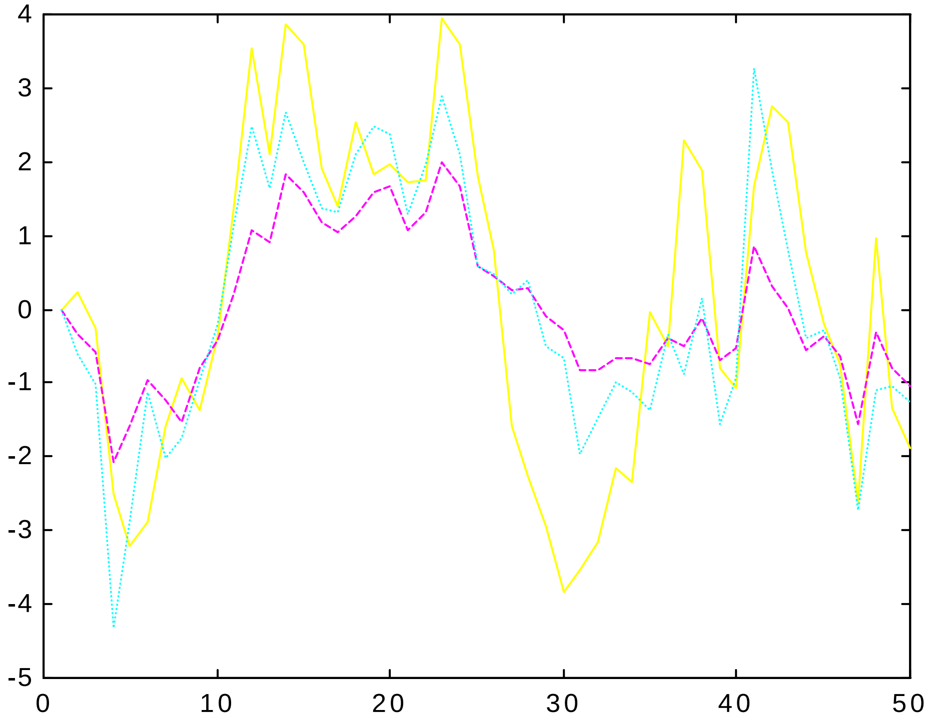

Figure 1.Behaviour of the KF (doted line) and the AKF (dashed line) on 50 iterations. The simulated state (solid line) corresponds to the system (2.1) and (2.2) with perturbation errors v_t generated by the asymmetric distribution log- $(\times^2_1)$.

Proof of Theorem 2.1. The best linear estimate of $x_{g,t}$ given z_g^t minimizing (3.6) depends on the second order properties of $x_{g,t}$ and z_g^t, and, moreover, this estimate and the corresponding error covariance are the same as if $x_{g,t}$ and z_g^t were jointly Gaussian. The explicit expression of this linear estimate is given by (3.8) which is the Asymmetric KF . The corresponding error covariance matrices P_t^t and P_{t+1}^t are also obtained through the Asymmetric KF.

The linear feedback law of the optimal control (3.1) is constructed following Theorem 7.1.11 in [10]; then i) holds. The explicit expression of the value function (3.3) is obtained as in Lemma 7.5.5 in [10], thus ii) holds. ■

Notice that the disturbance variables w_t and v_t do not appear in the control law, which depends linearly on the state estimate. This control law is obtained by some certainty equivalence principle.

5. Numerical results

We consider the following system

$$x_{t+1} = F_t\, x_t + C_t\, u_t + w_t,$$
$$y_t = H_t\, x_t + v_t,$$

as in (2.1) and (2.2), where $x_t \in 3$, $y_t \in 3$, and $u_t \in 3$, and $F_t = 0.99$, $C_t = 1$, $H_t = 1.1$, $Q_t = 1.5$ and $R_t = 4$. We assume that the state errors $\{w_t\}$ are independent N(0,1) random variables and the observations errors are independent log-($X^2{}_1$) random (asymmetric) variables.

In order to initialise the standard KF and the Asymmetric KF, the initial values are taken $x_0 = 0$, $P_0 = 1$, $R^1{}_0 = 4.94$ and $R^2{}_0 = 1.5$. The value x_0 is the mean of the random unobservable initial state x_t at t = 0 and P_0 is the variance of this initial state. We assume that this variable is independent of the error variables w_t. The cost functional is given by (2.14) with $A_t \equiv A$ and $T_t \equiv T$. We fix the damping constant $\delta = 0.25$. We implement the optimal linear regulator (3.1) for this system based on the standard KF, and the optimal linear regulator (2.15 based on the Asymmetric KF. To check the performance of these optimal controls, 500 replicated Monte Carlo simulations were obtained under the same conditions in both cases. Our experimental design changes the values (A, T) for each time horizon. Table 2 shows averaged index J values for each design. Each cell gives the index $J_{standard}$ at the first row and the index $J_{asymmetric}$ at the second row.

	Horizon = 20	Horizon = 50	Horizon = 100	Horizon = 500
A=1 T=0.1	56.8363 37.3368	149.4878 97.89.9	303.3282 202.1712	2.2097e+003 1.4901e+003
A=1	74.3893	196.1582	403.6085	2.0118e+003

T=0.5	47.5302	124.7092	250.4215	1.287e+003
A=1	88.4834	209.5412	483.8569	2.4469e+003
T=0.9	53.7725	190.5484	290.1906	1.4849e+003
A=1.5	83.8060	219.2159	436.2675	2.2106e+003
T=0.1	55.0692	144.3561	290.3682	1.4898e+003

Table 2. Monte Carlo simulations on optimal linear regulator based on standard KF (first value) and Asymmetric KF (second value) of the LQG system given by (2.1), (2.2) and (2.14), with asymmetric observation errors generated by a log-X^2_1 distribution. 500 replicates. Reported the averaged index J for each design: the first value in each cell is $J_{standard}$ and the second value is $J_{asymmetric}$.

Finally, the cost percentage reduction $((J_{standard} - J_{asymmetric}) / J_{standard}) \times 100$ gives the economy obtained if the Asymmetric approach is used. From Table 2, the cost percentage reduction takes values in the interval (32,16 , 40,02). Thus, as expected, the best behaviour corresponds to the asymmetric approach..

6. Conclusions.

The paper presents a useful construction of the optimal linear control for LQG systems with asymmetric observation errors. It has been obtained in a closed form given by a recursive algorithm. This is a very attractive computational feature of our proposal. The Separation Principle drives the feedback control law. The estimation step is based on an asymmetric version of the Kalman Filter. High performance of the asymmetric optimal control versus the standard optimal control has been shown. There is no extra computational cost using the asymmetric optimal control solution, even if a nominal gaussian hypothesis holds. Although we simulate asymmetric errors by a log-X^2_1 distribution, our approach is general enough to be used with any other asymmetric model.

This methodology is also useful in estimation and smoothing problems such as autoregressive time series models.

For numerical purposes we develop a Matlab code, which is available on request to the author.

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