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Conditions for equivalence between sequentiality and subgame perfection

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Abstract We characterize the class of finite extensive forms for which the sets of Subgame Perfect and Sequential equilibrium strategy profiles coincide for any possible payoff function. In addition, we identify the class of finite extensive forms for which the outcomes induced by these two solution concepts coincide.

Keywords Extensive form · Sequential equilibrium · Subgame perfect equilibrium

JEL Classification Number C72

1 Introduction

Analysis of backward induction in finite extensive form games provides useful insights for a wide range of economic problems. The basic idea of backward induction is that each player uses a best reply to the other players' strategies, not only at the initial node of the tree, but also at any other information set.

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To capture this type of rationality Selten (1965) defined the subgame perfect equilibrium concept. While subgame perfection has some important applications, it does not always eliminate irrational behavior at every information set. In order to solve this problem, Selten (1975) introduced the more restrictive notion of "trembling-hand" perfection.

Sequential equilibrium, due to Kreps and Wilson (1982), requires that every player maximizes her expected payoff at every information set, according to some consistent beliefs. They showed that "trembling-hand" perfection implies sequentiality, which in turn implies subgame perfection. They also proved that for generic payoffs, almost all sequential equilibrium strategies are "trembling-hand" perfect, a result that was strengthen by Blume and Zame (1994) who proved that for a fixed extensive form and generic payoffs it is the case that the two concepts coincide.

Although it is a weaker concept than Selten's perfection, Kohlberg and Mertens (1986) note that "sequential equilibrium seems to be the direct generalization [of backward induction] to games of imperfect information". It fulfills all the properties that characterize subgame perfection (backward induction) in games of perfect information. This is no longer true with different concepts like perfect or proper equilibrium.[1]

In this paper we find the maximal set of finite extensive forms (extensive games without any payoff assignment) for which sequential and subgame perfect equilibrium yield the same set of equilibrium strategies, for every possible payoff function (Proposition 1). It can be characterized as the set of extensive forms, such that for any behavior strategy profile every information set is reached with positive probability conditional on the smallest subgame that contains it. Whenever the extensive form does not have this structure, payoffs can be assigned such that the set of subgame perfect equilibria does not coincide with the set of sequential equilibria.

However, it may still happen that the set of equilibrium outcomes of both concepts coincides for any possible assignment of the payoff function. Thus, we also identify the maximal set of finite extensive forms for which subgame perfect and sequential equilibrium always yield the same equilibrium outcomes (Proposition 2).

Notice that, unlike related results on equivalence between refinements of Nash equilibrium, where the object of analysis is the payoff space (e.g. Kreps and Wilson 1982; Blume and Zame 1994), we find conditions on the game form. Our results characterize the information structures where applying sequential rationality does not make a relevant difference with respect to subgame perfection. We consider them as tools for economic modelling. They allow us to know if, for the extensive game under study, subgame perfect and sequential equilibrium are always equivalent, either in equilibrium strategies or in equilibrium outcomes.

The paper is organized as follows: in Sect. 2 we briefly introduce the main notation and terminology of extensive form games. This closely follows van Damme (1991). Section 3 contains definitions. Results are formally stated and proved in Sect. 4. In Sect. 5 we give some examples where our results can be applied.

[1] See Kohlberg and Mertens (1986) for details.

2 Notation and terminology

The analysis is restricted to finite extensive form games with perfect recall. Since our characterization is based on the structural properties of extensive games, we cannot dispose of a complete formal description of extensive form games. However, and in consideration with those readers who are already familiar with extensive games, we relegate such a long discussion to the appendix and only offer in Table 1 a brief list with very terse explanations of the symbols that we require.

We need the following definitions before moving to the next section.

If $x \in X$, let $\mathbb{P}_x^b$ denote the probability distribution on Z if the game is started at x and the players play according to the strategy profile b. Given a system of beliefs μ, a strategy profile b and an information set u, we define the probability distribution $\mathbb{P}_u^{b,\mu}$ on Z as $\mathbb{P}_u^{b,\mu} = \sum_{x \in u} \mu(x)\mathbb{P}_x^b$.

These probability distributions allow us to compute expected utilities at parts of the extensive game other than the initial node, already considered in $R_i(b)$. Define $R_{ix}(b) = \sum_{z \in Z} \mathbb{P}_x^b(z) r_i(z)$ as player i's expected payoff at node x. In a similar fashion, $R_{iu}(b) = \sum_{z \in Z} \mathbb{P}^b(z|u) r_i(z) = \sum_{x \in u} \mathbb{P}^b(x|u) R_{ix}(b)$ is player i's expected payoff at every information set u such that $\mathbb{P}^b(u) > 0$. Furthermore, under the system of beliefs μ, $R_{iu}^{\mu}(b) = \sum_{z \in Z} \mathbb{P}_u^{b,\mu}(z) r_i(z)$ denotes player i's expected payoff at the information set u.

Table 1 Notation and terminology of finite extensive games with perfect recall

Notation	Terminology	Comments
Ξ	Extensive form	Extensive game without payoff assignment
T	Set of nodes in Ξ	Typical elements $x, y \in T$
$\leq$	Precedence relation on T	$\leq$ partially orders T
U_i	Player i's information sets	Typical elements $u, v, w \in U_i$
C_u	Choices available at u	Typical elements $c, d, e \in C_u$
Z	Set of final nodes	$\{z \in T : \nexists x \in T \text{ s.t. } z < x\}$
X	Set of decision nodes	$X = T \setminus Z$
r_i	Player i's payoff function	$r_i : Z \to \mathbb{R}, r = (r_1, \ldots, r_n)$
Γ	n-player extensive game	$\Gamma = (\Xi, r)$
b_i	Player i's behavioral strategy	$b_i \in B_i, b = (b_1, \ldots, b_n)$
$\mathbb{P}^b$	Probability measure on Z	Induced by b
$R_i(b)$	Player i's expected utility at b	$\sum_{z \in Z} \mathbb{P}^b(z) r_i(z)$
$Z(A)$	Final nodes coming after A	$A \subseteq T$
$\mathbb{P}^b(A)$	Probability of $A \subseteq T$	$\mathbb{P}^b(Z(A))$
Ξ_y	Subform starting at y	Subgame without payoff assignment
Γ_y	Subgame starting at y	$\Gamma_y = (\Xi_y, \hat{r})$
μ	System of beliefs	$\mu(\cdot) \geq 0, \sum_{x \in u} \mu(x) = 1, \forall u$

3 Definitions

We use the substitution notation $b\backslash b'_i$ to denote the strategy profile in which all players play according to b, except player i who plays b'_i. The strategy b_i is said to be a best reply against b if it is the case that $b_i \in \arg\max_{b'_i \in B_i} R_i(b\backslash b'_i)$. If $\mathbb{P}^b(u) > 0$, we say that the strategy b_i is a best reply against b at the information set $u \in U_i$ if it maximizes $R_{iu}(b\backslash b'_i)$ over the domain where it is well defined.

The strategy b_i is a best reply against (b, μ) at the information set $u \in U_i$ if $b_i \in \arg\max_{b'_i \in B_i} R^{\mu}_{iu}(b\backslash b'_i)$. If b_i prescribes a best reply against (b, μ) at every information set $u \in U_i$, we say that b_i is a sequential best reply against (b, μ). The strategy profile b is a sequential best reply against (b, μ) if it prescribes a sequential best reply against (b, μ) for every player.

With this terminology at hand we define several equilibrium concepts.

Definition 1 (Nash equilibrium) A strategy profile $b \in B$ is a Nash equilibrium of Γ if every player is playing a best reply against b.

We denote by $\mathrm{NE}(\Gamma)$ the set of Nash equilibria of Γ. Subgame perfection refines the Nash equilibrium concept by requiring a Nash equilibrium in every subgame. Formally,

Definition 2 (Subgame perfect equilibrium) A strategy profile b is a subgame perfect equilibrium of Γ if, for every subgame Γ_y of Γ, the restriction b_y constitutes a Nash equilibrium of Γ_y.

We denote by $\mathrm{SPE}(\Gamma)$ the set of subgame perfect equilibria of Γ. We write $\mathrm{SPEO}(\Gamma) = \{\mathbb{P}^b : b \in \mathrm{SPE}(\Gamma)\}$ for the set of subgame perfect equilibrium outcomes, and $\mathrm{SPEP}(\Gamma) = \{R(b) : b \in \mathrm{SPE}(\Gamma)\}$ for the set of subgame perfect equilibrium payoffs, where $R(b) = (R_1(b), \ldots, R_n(b))$.

Sequential rationality is a refinement of subgame perfection. Every player must maximize at every information set according to her beliefs about how the game has evolved so far. If b is a completely mixed strategy profile, beliefs are perfectly defined by Bayes' rule. Otherwise, beliefs should meet a consistency requirement. A sequential equilibrium is an assessment that satisfies such a consistency requirement together with an optimality requirement. This is formalized by the next two definitions.

Definition 3 (Consistent assessment) An assessment (b, μ) is consistent if there exists a sequence $\{(b_t, \mu_t)\}_t$, where b_t is a completely mixed strategy profile and $\mu_t(x) = \mathbb{P}^{b_t}(x|u)$ for $x \in u$, such that $\lim_{t\to\infty}(b_t, \mu_t) = (b, \mu)$.

Definition 4 (Sequential equilibrium) A sequential equilibrium of Γ is a consistent assessment (b, μ) such that b is a sequential best reply against (b, μ).

If Γ is an extensive game, we denote by $\mathrm{SQE}(\Gamma)$ the set of strategies b such that (b, μ) is a sequential equilibrium of Γ for some μ. Moreover, $\mathrm{SQEO}(\Gamma) = \{\mathbb{P}^b : b \in \mathrm{SQE}(\Gamma)\}$ denotes the set of sequential equilibrium outcomes and $\mathrm{SQEP}(\Gamma) = \{R(b) : b \in \mathrm{SQE}(\Gamma)\}$ the set of sequential equilibrium payoffs. Recall that $\mathrm{SQE}(\Gamma) \subseteq \mathrm{SPE}(\Gamma)$ for any game Γ.

We now introduce some new definitions that are needed for the results.

Definition 5 (Minimal subform of an information set) Given an information set u, the minimal subform that contains u, to be denoted $\Xi(u)$, is the subform Ξ_y that contains u and does not properly include any other subform that contains u.

We say that $\Gamma_y = (\Xi_y, \hat{r})$ is the minimal subgame that contains u if Ξ_y is the minimal subform that contains u.

In a given extensive form there are information sets that are always reached with positive probability. When this does not happen we say that the information set is avoidable, formally:

Definition 6 (Avoidable information set) An information set u is avoidable in the extensive form Ξ if $\mathbb{P}^b(u) = 0$, for some $b \in B$. Likewise, we say that the information set u is avoidable in the subform Ξ_y if $\mathbb{P}^b_y(u) = 0$, for some $b \in B$.

For reasons that will become clear in the next section, we are interested in identifying extensive games where no information set is avoidable in its minimal subform. To get an idea about the set of extensive forms that we have in mind consider Figs. 1 and 2. In the former, no information set is avoidable in the extensive form. While in the latter, no information set is avoidable in its minimal subform.

Conversely, consider Fig. 3. Player 2's information set is avoidable in the extensive form (also in its minimal subform since the entire game is the only proper subgame) because player 1 can decide not to let her move.

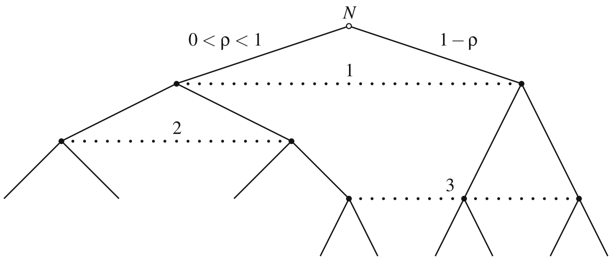


Fig. 1 Extensive form where no information set is avoidable

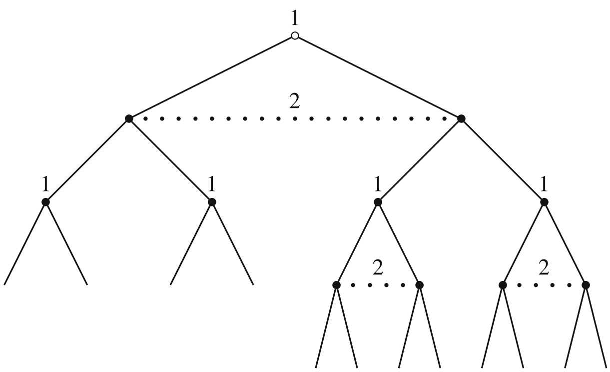
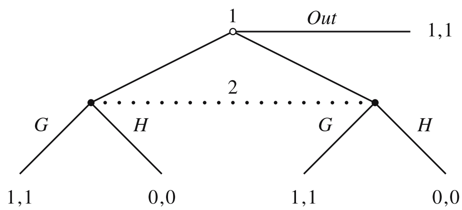


Fig. 2 Extensive form where no information set is avoidable in its minimal subform

Fig. 3 Example of the use of the algorithm contained in the proof of Proposition 1 to generate a game where SPE(Γ) ≠ SQE(Γ)



4 Results

The three "best reply" concepts introduced in Sect. 3 relate to each other, as it is shown in the first two statements of the next lemma. The third assertion of the same lemma shows that maximizing behavior at an information set is independent of the subgame of reference.

Lemma 1 *Fix a game* $\Gamma = (\Xi, r)$*. The following assertions hold*:

1. *Given a strategy profile b, if* $u \in U_i$ *is such that* $\mathbb{P}^b(u) > 0$ *and* b_i *is a best reply against b, then* b_i *is a best reply against b at the information set u.*
2. *Given a consistent assessment* (b, μ)*, if* $u \in U_i$ *is such that* $\mathbb{P}^b(u) > 0$ *and* b_i *is a best reply against b at the information set u, then* b_i *is a best reply against* (b, μ) *at the information set u.*
3. *If* Γ_y *is the minimal subgame that contains u and* (b_y, μ_y) *is the restriction of some assessment* (b, μ) *to* Γ_y*, then* b_i *is a best reply against* (b, μ) *at the information set u in the game* Γ *if and only if* $b_{y,i}$ *is a best reply against* (b_y, μ_y) *at the information set u in the game* Γ_y*.*

Proof Part 1 is known.[2] Proofs for 2 and 3 are trivial. □

In the next proposition we identify the set of extensive forms where sequential equilibrium has no additional bite over subgame perfection. The latter concept allows for the play of non-credible threats at information sets that might never be reached conditional on its minimal subgame. However, if we restrict attention to extensive form games where no information set is avoidable in its minimal subform, we can use the previous lemma to show that sequential and subgame perfect equilibrium coincide.

It turns out that not only is this particular restriction sufficient but also necessary for the equivalence, in the following sense: we can always find a payoff assignment so that the sets of subgame perfect and sequential equilibrium differ when the restriction fails to hold. The construction of such payoff assignment is based on, first, taking one information set that is avoidable in its minimal subform out of one subgame perfect equilibrium path and, second, making one of the available actions at this avoidable information set a strictly dominated action. Take for instance the game contained in Fig. 3. If player 1 moves *Out* she gives player 2 the possibility of taking the strictly dominated move *H*, which forms a subgame perfect equilibrium which is not sequential.

[2] For instance, see van Damme (1991), Theorem 6.2.1.

Proposition 1 *Let Ξ be an extensive form such that no information set u is avoidable in $\Xi(u)$. Then for any possible payoff vector r, the game $\Gamma = (\Xi, r)$ is such that* $\mathrm{SPE}(\Gamma) = \mathrm{SQE}(\Gamma)$. *Conversely, if Ξ is an extensive form with an information set u that is avoidable in $\Xi(u)$, then we can find a payoff vector r such that for the game $\Gamma = (\Xi, r)$,* $\mathrm{SPE}(\Gamma) \neq \mathrm{SQE}(\Gamma)$.

Proof Let us prove the first part of the proposition. We only have to show that $\mathrm{SPE}(\Gamma) \subseteq \mathrm{SQE}(\Gamma)$. Consider $b \in \mathrm{SPE}(\Gamma)$ and construct a consistent assessment (b, μ).[3] We have to prove that the set

$$\tilde{U}(b, \mu) = \bigcup_{i=1}^{n} \left\{ u \in U_i : b_i \notin \arg \max_{\tilde{b}_i \in B_i} R^{\mu}_{iu}(b \backslash \tilde{b}_i) \right\} \tag{1}$$

is empty. Assume to the contrary that $\tilde{U}(b, \mu) \neq \emptyset$, and consider $u \in \tilde{U}(b, \mu)$. Let Γ_y be the minimal subgame that contains u and let j be the player moving at u. By lemma 1.3, $b_{y,j}$ is not a best reply against (b_y, μ_y) at u in the game Γ_y. Part 2 implies either that $\mathbb{P}^b_y(u) = 0$ or that $b_{y,j}$ is not a best reply against b_y at u. If the latter was true, part 1 would anyway imply that $\mathbb{P}^b_y(u) = 0$. However, u is not avoidable in Ξ_y. This provides the contradiction.

Let us now prove the second part of the proposition. Suppose $u \in U_i$ is an information set that is avoidable in $\Xi(u)$ and let $c \in C_u$ be an arbitrary choice available at u. Assign the following payoffs:

$$\begin{cases} r_i(z) = 0 & \forall i \quad \text{if } z \in Z(c) \\ r_i(z) = 1 & \forall i \quad \text{elsewhere.} \end{cases} \tag{2}$$

Clearly any strategy $b_i = b_i \backslash c$ cannot be part of a sequential equilibrium since playing a different choice at u gives player i strictly higher expected payoff at that information set.

We now have to show that there exists a subgame perfect equilibrium b such that $b_i = b_i \backslash c$. By assumption there exists b' such that $\mathbb{P}^{b'}_y(u) = 0$ in the minimal subgame Γ_y that contains u. The equality $\mathbb{P}^b_y(u) = 0$ also holds for $b = b' \backslash c$. The strategy profile b_y is a Nash equilibrium of Γ_y since nobody can obtain a payoff larger than one. By the same argument, b induces a Nash equilibrium in every subgame, hence it is a subgame perfect equilibrium. This completes the proof. □

We use the extensive form of Selten's horse game (Figs. 4, 5) to show that the algorithm (used in the proof of the second part of Proposition 1) does not depend either on the particular avoidable information set, or on the particular choice that is taken to construct the payoffs. Information set u in the algorithm corresponds to player 2's

[3] A general method to define consistent assessments (b, μ) for any given $b \in B$, in an extensive form, is the following: take a sequence of completely mixed strategy profile $\{b_t\}_t \to b$ and for each t, construct $\mu^t(x) = \mathbb{P}^{b_t}(x|u) \in [0, 1], \forall x \in u$, for all information sets u. Call $k = |X \setminus P_0|$. The set $[0, 1]^k$ is compact and since $\mu^t \in [0, 1]^k, \forall t$, there exists a subsequence of $\{t\}$, call it $\{t_j\}$, such that $\{\mu^{t_j}\}_{t_j}$ converges in $[0, 1]^k$. Define beliefs as $\mu = \lim_{j \to \infty} \mu^{t_j}$.

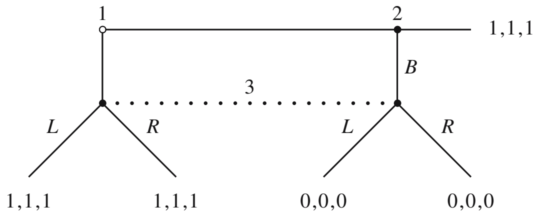


Fig. 4 Selten's horse. An example of the use of the algorithm contained in the proof of Proposition 1 to generate a game where $\mathrm{SPE}(\Gamma) \neq \mathrm{SQE}(\Gamma)$

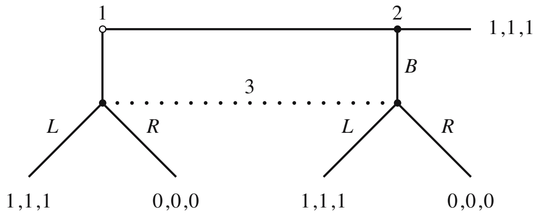


Fig. 5 Selten's horse. A different use of the algorithm contained in Proposition 1

(player 3's) information set in Fig. 4 (Fig. 5), and choice $c \in C_u$ in the algorithm corresponds to choice B (choice R) in Fig. 4 (Fig. 5).

Notice that the payoff assignment in the previous proof yields a difference in equilibrium strategies but not in equilibrium payoffs. The reason is that we cannot always achieve difference in equilibrium outcomes (therefore, neither in equilibrium payoffs). Figure 6 contains an extensive form where the second information set of player 1 is avoidable in its minimal subform, and nevertheless, the sets of sequential and subgame perfect equilibrium outcomes always coincide, regardless of what the payoffs assigned to final nodes are. Proposition 2 provides a sufficient and necessary condition for the sets of equilibrium outcomes (also, of equilibrium payoffs) to be equal for any conceivable payoff function.

Before that, we need to be able to identify which players can avoid a given information set. Let u be an information set and let $\Xi_y = \Xi(u)$. Construct the set of strategies $B(u) = \left\{b \in B : \mathbb{P}_y^b(u) > 0\right\}$.

Definition 7 We say that the information set u can be avoided in $\Xi(u)$ by player i if there exists a strategy profile $b \in B(u)$, and a choice $c \in C_v$, with $v \in U_i$, such that $\mathbb{P}_y^{b \backslash c}(u) = 0$.

Remember that for an information set u that is avoidable in $\Xi(u) = \Xi_y$ there must be a strategy profile b such that $\mathbb{P}_y^b = 0$ (Definition 6). If a player, say player i, is able to unilaterally modify a strategy profile b' for which $\mathbb{P}_y^{b'} > 0$, by changing only one of her choices, and hereby construct one b for which $\mathbb{P}_y^b = 0$, then we say that the information set u can be avoided in $\Xi(u)$ by player i. Therefore, associated with any information set, there is a (possibly empty) list of players who can avoid it in its minimal subform. Figure 6 is an example of an extensive form where for every information set such a list is either empty or contains only the owner of the information set. When

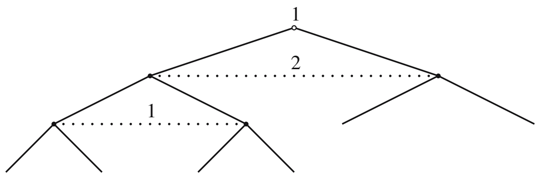


Fig. 6 The second information set of player 1 can only be avoided by player 1. Therefore, SPEP(Γ) = SQEP(Γ)

this happens, sequential equilibrium has no additional bite over subgame perfection regarding equilibrium outcomes. The reason is that subgame perfection allows a player to choose actions suboptimally, but given the particular structure of the game form, it can only happen at information sets already avoided by her own previous behavior, and choices at such information sets do not affect the outcome of the game.

This condition is also necessary for equivalence in equilibrium outcomes in the following sense: if player i can avoid the information set u in its minimal subform, and if j is the owner of the information set u, there exists a payoff assignment so that player j can "non-credibly" threaten player i (something ruled out by sequential equilibrium but not by subgame perfection) bringing about the difference in equilibrium outcomes.

The following lemma is useful for the proof of Proposition 2.

Lemma 2 *Let Ξ be an extensive form such that, whenever an information set u is avoidable in $\Xi(u)$, it can only be avoided in $\Xi(u)$ by its owner. Let (b, μ) and (b', μ') be two consistent assessments. If b and b' are such that $\mathbb{P}_y^b = \mathbb{P}_y^{b'}$ for every subform Ξ_y, then $\mu = \mu'$.*

Proof Let (b, μ) and (b', μ') be two consistent assessments such that $\mathbb{P}_y^b = \mathbb{P}_y^{b'}$ for every subform Ξ_y. Note that b' can be obtained from b by changing behavior at information sets that are reached with zero probability within their minimal subform. Hence, without loss of generality, let b and b' differ only at one such information set, say $u \in U_i$, and let $\Xi_y = \Xi(u)$. The shift from b to b' may cause a change in beliefs only at information sets that come after u and are in the same minimal subform Ξ_y. Let $v \in U_j$ be one of those information sets.

If $j = i$, perfect recall and consistency imply that there is no change in beliefs at the information set v. If $j \neq i$ there are two possible cases, either $\mathbb{P}_y^b(v) > 0$ or $\mathbb{P}_y^b(v) = 0$. In the first case the beliefs at v are uniquely defined, therefore, $\mu(x) = \mu'(x), \forall x \in v$ and moreover, $\mu(x) = \mu'(x) = 0, \forall x \in v$ such that $u < x$. In the second case, since the information set v can only be avoided by player j in $\Xi(u)$ there exists a choice $c \in C_w$ of player j such that $\mathbb{P}_y^{b\backslash c}(v) > 0$, otherwise player i would also be able to avoid the information set u in $\Xi(u)$. Let $b'' = b\backslash c$ and $b''' = b'\backslash c$, then by the discussion of the first case, $\mu''(x) = \mu'''(x), \forall x \in v$, furthermore, perfect recall and consistency imply $\mu''(x) = \mu(x)$ and $\mu'''(x) = \mu'(x), \forall x \in v$, which in turn implies $\mu(x) = \mu'(x), \forall x \in v$. □

We are now ready to state and prove our second equivalence result.

Proposition 2 *Let Ξ be an extensive form such that, whenever an information set u is avoidable in $\Xi(u)$, it can only be avoided in $\Xi(u)$ by its owner. Then for any possible payoff vector r, the game $\Gamma = (\Xi, r)$ is such that* $\mathrm{SPEO}(\Gamma) = \mathrm{SQEO}(\Gamma)$. *Conversely, if Ξ is an extensive form with an information set u that can be avoided in $\Xi(u)$ by a different player than its owner, then we can find a payoff vector r such that for the game $\Gamma = (\Xi, r)$,* $\mathrm{SPEP}(\Gamma) \neq \mathrm{SQEP}(\Gamma)$.

Proof Let us prove the first part of the proposition. We need to prove that $\forall b \in \mathrm{SPE}(\Gamma)$, $\mathbb{P}^b \in \mathrm{SQEO}(\Gamma)$. Take an arbitrary $b \in \mathrm{SPE}(\Gamma)$ and construct some consistent beliefs μ.

If the set $\tilde{U}(b, \mu) = \bigcup_{i=1}^{n} \left\{u \in U_i : b_i \notin \arg\max_{\tilde{b}_i \in B_i} R^{\mu}_{iu}(b \backslash \tilde{b}_i)\right\}$ is empty, then $b \in \mathrm{SQE}(\Gamma)$ and $\mathbb{P}^b \in \mathrm{SQEO}(\Gamma)$. Otherwise, we need to find a sequential equilibrium (b^*, μ^*) such that $\mathbb{P}^{b^*} = \mathbb{P}^b$.

Step 1: Take an information set $u \in \tilde{U}(b, \mu)$. Let i be the player that moves at this information set, and let $\Gamma_y = (\Xi(u), \hat{r})$. As in the proof of Proposition 1, notice that by Lemma 1, u should be such that $\mathbb{P}^b_y(u) = 0$, hence it is avoidable in its minimal subform. By assumption, u can only be avoided by player i.

Step 2: Let b' be the strategy profile b modified so that player i plays a best reply against (b, μ) at the information set u. Construct a consistent assessment (b', μ'). Notice that $\mathbb{P}^{b'} = \mathbb{P}^b$ and, in particular, $\mathbb{P}^{b'}_y = \mathbb{P}^b_y$. By Lemma 2, μ and μ' assign the same probability distribution on every information set.

Step 3: We now prove that $b' \in \mathrm{SPE}(\Gamma)$. For this we need $b'_y \in \mathrm{NE}(\Gamma_y)$. Given the strategy profile b'_y in the subgame Γ_y, player i cannot profitably deviate because this would mean that she was also able to profitably deviate when b_y was played in the subgame Γ_y, which contradicts $b_y \in \mathrm{NE}(\Gamma_y)$.
Suppose now that there exists a player $j \neq i$ who has a profitable deviation $b''_{y,j}$ from $b'_{y,j}$ in the subgame Γ_y. The hypothesis on the extensive form Ξ implies $\mathbb{P}_y^{b \backslash b''_{y,j}} = \mathbb{P}_y^{b' \backslash b''_{y,j}}$, which further implies that $b''_{y,j}$ should have also been a profitable deviation from b_y. However, this is impossible since $b_y \in \mathrm{NE}(\Gamma_y)$.

Step 4: By step 2, $|\tilde{U}(b', \mu')| = |\tilde{U}(b, \mu)| - 1$. If $|\tilde{U}(b', \mu')| \neq 0$, apply the same type of transformation to b'. Suppose that the cardinality of $\tilde{U}(b, \mu)$ is q, then in the qth transformation we will obtain a consistent assessment $(b^{(q)}, \mu^{(q)})$ such that $b^{(q)} \in \mathrm{SPE}(\Gamma)$, $\mathbb{P}^b = \mathbb{P}^{b^{(q)}}$, and $\tilde{U}(b^{(q)}, \mu^{(q)}) = \emptyset$. Observe that, $b^{(q)} \in \mathrm{SPE}(\Gamma)$ and $\tilde{U}(b^{(q)}, \mu^{(q)}) = \emptyset$ imply $b^{(q)} \in \mathrm{SQE}(\Gamma)$. Therefore $(b^{(q)}, \mu^{(q)})$ is the sequential equilibrium (b^*, μ^*) we were looking for.

Let us now prove the second part of the proposition. For notational convenience, it is proved for games without proper subgames, however, the argument extends immediately to the general case.

Given a node $x \in T$, the set $\mathrm{Path}(x) = \left\{c \in \bigcup_u C_u : c < x\right\}$ of choices is called path to x.

Suppose that u is an information set that can be avoided in Ξ by a player, say player j, different from the player moving at it, say player i. Note that there must exist an $x \in u$ and a choice $c \in C_v$, where $v \in U_j$, such that if $b = b \backslash \mathrm{Path}(x)$, then $\mathbb{P}^{b \backslash c}(u) = 0$ is true.

Let $f \in C_u$ be an arbitrary choice available to player i at u. Assign the following payoffs:

$$\begin{cases} r_j(z) = 0 & \text{if } z \in Z(c) \\ r_i(z) = r_j(z) = 0 & \text{if } z \in Z(f) \\ r_i(z) = r_j(z) = 1 & \text{if } z \in Z(u) \setminus Z(f). \end{cases} \tag{3}$$

Let $d \in \text{Path}(x)$ with $d \notin C_v$, assign payoffs to the terminal nodes, whenever allowed by 3, in the following fashion:

$$r_k(z) > r_k(z') \quad \text{where } z \in Z(d) \quad \text{and} \quad z' \in Z(C_w \setminus \{d\}). \tag{4}$$

Player k above is the player who has choice d available at the information set w. Give zero to every player everywhere else.

In words, player j moves with positive probability in the game. She has two choices, either moving towards the information set u and letting player i decide, or moving away from the information set u. If she moves away she gets zero for sure. If she lets player i decide, player i can either make both get zero by choosing f, or make both get one by choosing something else. Due to 4, no player will disturb this description of the playing of the game.

This game has a Nash equilibrium in which player i moves f and player j obtains a payoff equal to zero by moving c. However, in every sequential equilibrium of this game, player i does not choose f and, as a consequence, player j takes the action contained in $\text{Path}(x) \cap C_v$. Therefore, in every sequential equilibrium, players i and j obtain a payoff strictly larger than zero.[4] This completes the proof. □

For a very simple application of the previous algorithm, consider the extensive game of Fig. 3 and substitute the payoff vector following move *Out* of player 1, with the payoff vector (0, 0). Again, the first player moving *Out* and the second player taking the strictly dominated move H, is a subgame perfect equilibrium that yields an equilibrium payoff vector equal to (0, 0). However, in any sequential equilibrium, player 2 moves G and player 1 does not move *Out*, which makes (1, 1) the only sequential equilibrium payoff vector.

Remark 1 Notice that, in the set of extensive forms under study in the last proposition, beliefs are always uniquely defined for any given strategy profile (consider $b' = b$ in Lemma 2). One may incorrectly think that it is the uniqueness of the beliefs that is behind the equivalence. Consider a modification of the game form in Fig. 6 so that the second information set of player 1 is controlled by a new player 3. This modified extensive form has a unique system of consistent beliefs for any given strategy profile but, as seen in Proposition 2, the set of equilibrium outcomes is not the same for both concepts for every possible payoff vector.

[4] Equilibrium payoffs are not necessarily equal to one due to eventual moves of Nature.

5 Examples

These results can be applied to many games considered in the economic literature. It allows us to identify in a straightforward way the finite extensive form games of imperfect information for which subgame perfect equilibria are still conforming with backward induction expressed in a sequential equilibrium.

Besley and Coate (1997) proposed an economic model of representative democracy. The political process is a three-stage game. In stage 1, each citizen decides whether or not to become a candidate for public office. At the second stage, voting takes place over the list of candidates. At stage 3 the candidate with the most votes chooses the policy. Besley and Coate solved this model using subgame perfection and found multiple subgame perfect equilibria with very different outcomes in terms of number of candidates. This may suggest that some refinement might give sharper predictions. However, given the structure of the game that they considered, it follows immediately from the results of the previous section that all subgame perfect equilibria in their model are also sequential. Thus, no additional insights would be obtained by requiring this particular refinement.

The information structure of Besley and Coate's model is a particular case of the more general framework offered by Fudenberg and Levine (1983). They characterized the information structure of finite-horizon multistage games as "almost" perfect, since in each period players simultaneously choose actions, Nature never moves and there is no uncertainty at the end of each stage. As they noticed, sequential equilibrium does not refine subgame perfection in this class of games. This can also be obtained as an implication of Proposition 1 in the present paper.

In their version of the Diamond and Dybvig (1983) model, Adão and Temzelides (1998) discussed both the issue of potential banking instability as well as that of the decentralization of the optimal deposit contract. They addressed the first question in a model with a "social planner" bank. The bank offers the efficient contract as a deposit contract in the initial period. In the first stage agents sequentially choose whether to deposit in the bank or to remain in autarky. In the second stage, those agents who were selected by Nature to be patient, simultaneously choose whether to misrepresent their preferences and withdraw, or report truthfully and wait. The reduced normal form of the game has two symmetric Nash equilibria in pure strategies. The first one has all agents choosing depositing in the bank and reporting faithfully, the second one has all agents choosing autarky. The fact that both equilibria are sequential is presented in their Proposition 2. Because of the game form they used, our Proposition 1 also implies their result.

In the implementation theory framework, Moore and Repullo (1988) present the strength of subgame perfect implementation. If a choice function is implementable in subgame perfect equilibria by a given mechanism, the strategy space is finite, and no information set is avoidable in its minimal subform in the extensive form of the mechanism, then our work establishes the implementability in sequential equilibrium. (See, for instance, the example they study in Sect. 5, pp. 1213–1215.)

More examples can be found in Game Theory textbooks, like those of Fudenberg and Tirole (1996), Myerson (1997) and Osborne and Rubinstein (1994). Notice that whenever subgame perfect and sequential equilibrium differ for an extensive game, there are information sets that are avoidable in its minimal subform. As examples consider Figs. 8.4 and 8.5 in Fudenberg and Tirole (1996), Figs. from 4.8 to 4.11 in Myerson (1997) and Figs. 225.1 and 230.1 in Osborne and Rubinstein (1994).

6 Appendix: Notation and terminology

6.1 Extensive form

An n-player extensive form is a sextuple $\Xi = (T, \leq, P, U, C, p)$, where T is the finite set of nodes and $\leq$ is a partial order on T, representing precedence. We use the notation $x < y$ to say that node y comes after node x. The immediate predecessor of x is $A(x) = \max\{y : y < x\}$, and the set of immediate successors of x is $S(x) = \{y : x \in A(y)\}$. The pair $(T, \leq)$ is a tree with a unique root α: for any $x \in T$, $x \neq \alpha$, there exists a unique sequence $\alpha = x_0, x_1, \ldots, x_n = x$ with $x_i \in S(x_{i-1})$, $1 \leq i \leq n$. The set of endpoints is $Z = \{x : S(x) = \emptyset\}$ and $X = T \setminus Z$ is the set of decision points. We write $Z(x) = \{y \in Z : x < y\}$ to denote the set of terminal successors of x, and if E is an arbitrary set of nodes we write $Z(E) = \{z \in Z(x) : x \in E\}$.

6.2 Player partition

The player partition, P, is a partition of X into sets $P_0, P_1, \ldots, P_n$, where P_i is the set of decision points of player i and P_0 stands for the set of nodes where chance moves. The probability assignment p specifies for every $x \in P_0$ a completely mixed probability distribution p_x on $S(x)$.

6.3 Information partition

The information partition U is an n-tuple $(U_1, \ldots, U_n)$, where U_i is a partition of P_i into information sets of player i, such that (i) if $u \in U_i$, $x, y \in u$ and $x \leq z$ for $z \in X$, then we cannot have $z < y$, and (ii) if $u \in U_i$, $x, y \in u$, then $|S(x)| = |S(y)|$. Therefore, if u is an information set and $x \in X$, it makes sense to write $u < x$. Also, if $u \in U_i$, we often refer to player i as the owner of the information set u.

6.4 Choice partition

If $u \in U_i$, the set C_u is the set of choices available for i at u. A choice $c \in C_u$ is a collection of $|u|$ nodes with one, and only one, element of $S(x)$ for each $x \in u$. If player i chooses $c \in C_u$ at the information set $u \in U_i$ when she is actually at $x \in u$, then the next node reached by the game is the element of $S(x)$ contained in c. The entire collection $C = \{C_u : u \in \bigcup_{i=1}^{n} U_i\}$ is called the choice partition. We assume throughout that $|C_u| > 1$ for every information set u.

6.5 Extensive form game

We define a finite n-person extensive form game as a pair $\Gamma = (\Xi, r)$, where Ξ is an n-player extensive form and r, the payoff function, is an n-tuple $(r_1, \ldots, r_n)$, where r_i is a real valued function with domain Z. We assume throughout that the extensive form Ξ satisfies perfect recall, i.e. for all $i \in \{1, \ldots, n\}$, $u, v \in U_i$, $c \in C_u$ and $x, y \in v$, we have $c < x$ if and only if $c < y$. Therefore, we can say that choice c comes before the information set v (to be denoted $c < v$) and that the information set u comes before the information set v (to be denoted $u < v$).

6.6 Behavior strategies, beliefs and assessments

A behavior strategy b_i of player i is a sequence of functions $(b_i^u)_{u \in U_i}$ such that $b_i^u : C_u \to \mathbb{R}_+$ and $\sum_{c \in C_u} b_i^u(c) = 1, \forall u$. The set B_i represents the set of behavior strategies available to player i. A behavior strategy profile is an element of $B = \prod_{i=1}^n B_i$. As common in extensive form games, we restrict attention to behavior strategies.[5] Throughout, we simply refer to them as strategies. If $b_i \in B_i$ and $c \in C_u$ with $u \in U_i$, then $b_i \backslash c$ denotes the strategy b_i changed so that c is taken with probability one at u. If $b \in B$ and $b_i' \in B_i$ then $b \backslash b_i'$ is the strategy profile $(b_1, \ldots, b_{i-1}, b_i', b_{i+1}, \ldots, b_n)$. If c is a choice of player i then $b \backslash c = b \backslash b_i'$, where $b_i' = b_i \backslash c$.

A system of beliefs μ is a function $\mu : X \setminus P_0 \to [0, 1]$ with $\sum_{x \in u} \mu(x) = 1, \forall u$. An assessment (b, μ) is a strategy combination together with a system of beliefs.

6.7 Subforms and subgames

Let $\hat{T} \subset T$ be a subset of nodes such that (i) $\exists y \in \hat{T}$ with $y < x, \forall x \in \hat{T}, x \neq y$, (ii) if $x \in \hat{T}$ then $S(x) \subset \hat{T}$, and (iii) if $x \in \hat{T}$ and $x \in u$ then $u \subset \hat{T}$. Then we say that $\Xi_y = (\hat{T}, \hat{\lesssim}, \hat{P}, \hat{U}, \hat{C}, \hat{p})$ is a subform of Ξ starting at y, where $(\hat{\lesssim}, \hat{P}, \hat{U}, \hat{C}, \hat{p})$ are defined from Ξ in $\hat{T}$ by restriction. A subgame is a pair $\Gamma_y = (\Xi_y, \hat{r})$, where $\hat{r}$ is the restriction of r to the endpoints of Ξ_y. We denote by b_y the restriction of $b \in B$ to the subform Ξ_y (to the subgame Γ_y). The restriction of a system of beliefs μ to the subform Ξ_y (to the subgame Γ_y) is denoted by μ_y.

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[5] We can do this without loss of generality due to perfect recall and Kuhn's theorem, see Kuhn (1953).

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