UNIVERSIDAD CARLOS III DE MADRID

TESIS DOCTORAL

Executive Compensation in a Dynamic Model

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Getafe, marzo de 2009

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Introduction

Executive pay is a topic that has continuously interested media and academia alike. This is not surprising given its dramatic increase over the last thirty years. Jensen and Murphy (2004), for example, report a 11-fold rise in the inflation-adjusted average total remuneration of CEOs in S&P 500 firms between 1970 and 2002. The composition of executive compensation has also changed dramatically during the years. In the 1970s, it almost exclusively consisted of base salaries and performance bonuses. In 2002, salaries and bonuses had a combined share of 36 percent, while the main chunk of CEO's remuneration (47 percent) came from stock options grants.

Recently, Gabaix and Landier (2008) have related the rise in executive pay to the increase in the size of US firms. Interestingly, when Clementi and Cooley (2009) use an alternative measure of compensation defined as the annual change in the CEO's fortune that is attributable to his/her relationship with the company, they find no time trend in median compensation in the period 1993-2006. Nevertheless, they report that the median executive wealth has essentially doubled during this period.

Another issue that has been the center of controversy over the last 20 years is the sensitivity of executive pay to changes in shareholder value. Jensen and Murphy (1990) estimated that on average CEO wealth changes by only $3.25 for every $1,000 change in shareholder wealth. Later research has documented higher pay-performance sensitivity [see, for example Hall and Liebman (1998), Schaefer (1998), Aggarwal and Samwick (1999), Clementi and Cooley (2009)], but the question remains open especially given the lack of agreement on a proper theoretical benchmark.

Even the current financial crisis did not manage to turn the attention away from managerial pay. In March this year, Treasury Secretary Tim Geithner and general public alike were outraged at the plan of the bailed-out AIG to spend $165 mln of taxpayer money on executive bonuses. As a matter of fact, as reported by Attorney General Andrew Cuomo, 73 out of 400 executives received more than $1mln in bonuses with top 10 taking $42 mln in total. While the AIG CEO Edward Liddy referred to the role of compensation in attracting and retaining the "best and brightest talent", the House of Representatives passed a bill introducing a 90 percent tax on AIG bonuses. Following discussions on the G20 summit in London related to curbing executive bonuses, Russia took rather tough stance on CEO pay. The State Duma, the Russian parliament's lower house, is currently considering imposing a cap of 4 mln rubles (about $120,000) on annual executive compensation for all state companies and for private companies that receive government aid.

Indeed, since 1990s there has been a big debate about the effectiveness of the observed compensation schemes in creating the proper incentives while providing insurance to risk-averse managers. While there are a lot of empirical recipes and surveys, the most important question remains as to how the optimal contract should actually look like. Aseff and Santos (2005) consider a static setup and show that a simple quasilinear compensation scheme comes close to inducing high effort. However, given the inability of static models to effectively capture the role of deferred compensation as an incentive mechanism, the focus should really fall on dynamic contracting. Moreover, the agency literature has mainly regarded contracts inducing optimal effort while the participation constraints have largely been ignored. Indeed, we do not know what would happen if the outside options of the manager actually change with the history of observables. For example, it is unrealistic to treat the reservation utility of a CEO as fixed regardless of the situation in his/her firm, industry, or the economy as a whole. The dependence could come through many channels- externalities, different types of agents, a certain structure of beliefs, but more importantly, it can significantly influence the nature of the relationship and the form of the optimal contract. Indeed, risk-averse managers would like to be insured against these fluctuations when signing a contract with a company. How could such an insurance be provided? What would be the resulting contract and how successful would it be in creating managerial incentives? Would executive compensation and wealth increase in the long run and would the resulting distribution depend on the initial shock to the reservation utilities. Would both current and deferred compensation be used for incentive purposes and would the result change if we consider a simple stock-option scheme? Would there be a significant change if we consider contract terminations and who will be the most affected?

All these questions fall into the scope of the current dissertation. The thesis generalizes the notion of commitment by defining the outside options on the history observed in a dynamic contractual setting. It studies how history dependence through reservation utilities affects the provision of insurance and incentives and the dynamics of the contract with a special focus on executive compensation. Stock options and contract terminations are also analyzed.

The first chapter considers a moral hazard problem in an infinitely repeated principal-agent interaction where both parties can only commit to a short-term contract. Unlike previous literature, their outside options are not necessarily fixed across the history of outcomes. More precisely, to keep the model tractable, the reservation utilities are defined on a finite truncation of the publicly observed history. The framework is as follows. A principal contracts an agent to implement a sequence of actions where the choice of an action each period is effectively a choice of an end-of-period probability distribution over outcomes. Since the exercise of an action brings disutility to the agent, he/she should be compensated by a monetary transfer from the principal. Everything is common knowledge except for the particular action implemented which is only observed by the agent. The contract will, therefore, need to induce the agent to implement a particular action sequence recommended by the principal. The incentives, however, are restricted by the inability of the parties to commit to a long-term relationship and need to be consistent with their history-dependent participation constraints.

I prove existence of an optimal self-enforcing, incentive-compatible contract and obtain the first in the literature characterization of such an environment. The characterization is very general in terms of assumptions and, more importantly, is fully recursive. Its convergence properties make it perfect for computing the optimal contract for a general class of dynamic hidden action models. The idea is to reduce incentive compatibility to Green (1987)'s temporary incentive compatibility and treating the expected utility of the agent as a state variable to construct the optimal contract recursively through a series of single-period static contracts in the spirit of Spear and Srivastava (1987). Unlike Aseff (2004), I do not pre-suppose the optimality of a particular action sequence and, unlike Wang (1997), I focus on limited commitment which is introduced in the sense of Phelan (1995), but on both contractual parties. Note, however, that we need to enlarge the state space in order to obtain stationarity. Indeed, since reservation utilities are history-dependent, so would be the contract satisfying any party's participation constraints. The solution is to expand the state space by including the relevant truncated histories.

Consider an auxiliary problem where the principal can fully commit to a long-term contract. The solution to this problem can be recursively characterized on a state space of the type discussed above. This state space will indeed be two-dimensional because of the history-dependence of the agent's participation constraints and will contain the state space of the original problem. It will also be endogenous, but in the spirit of Abreu, Pearce and Stachetti (1990) I show that it is a fixed point of a set operator and can be obtained by successively iterating on this operator until convergence. Once, the state space is known, the solution to the problem with one-sided commitment can be obtained by dynamic programming routines. Given the value function of this auxiliary problem, I resort to a procedure outlined by Rustichini (1998) in order to solve for the optimal incentive-compatible, two-side participation guaranteed super-contract. This is achieved by severely punishing the principal for any violation of his/her participation constraint. The procedure allows of recovering the sub-space of agent's expected discounted utilities supportable by a self-enforcing incentive-compatible contract.

The general framework discussed above is given more structure in the second chapter. There, I consider a long term contract between a risk-neutral principal proxying for firm's shareholders and a risk-averse CEO, where both parties are unable to commit in the long run and face history-dependent reservation utilities. I treat the variable of interest to the principal as profit, the monetary transfer as managerial compensation, the action as an effort level, and assume that CEO's period utility is separable in money and effort. Since I am interested in the long term dynamics of the contract and the resulting wealth distribution, I focus on long-term self-enforcing schemes that are incentive-compatible. As a matter of fact, I am the first to study how shocks on the reservation utilities may affect the parties to a dynamic contractual relationship. In particular, I investigate whether the optimal contract insures the manager against variability in the value of his/her outside options. I build up the intuition behind the possible effect of such an insurance on the manager's utility in the short and the long run and relate it to the properties of the limiting distribution.

I start by deriving the state space when reservation utilities are constant. Then, I consider a single-period history dependence and show that if the manager's reservation utilities are sufficiently dispersed, his/her participation constraint does not bind under the worst case scenario, which is also observed when the manager can essentially commit when his/her outside option is at its lowest value. In other words, the minimum utility the CEO can be promised for initial histories characterized by lower reservation utility is generally boosted by higher reservation utilities for other states. Alternatively put, the optimal contract provides the CEO with some insurance against fluctuations in the value of his/her outside options, which ultimately smooths his/her consumption across (initial history) states. In case of positive correlation between firm's profits and manager's reservation utilities, this translates into the participation constraint of the manager being non-binding in states characterized by low profits. Computing the model actually shows that utility promises close to the reservation level are possible only under the manager's best-case scenarios when his/her reservation utility is the highest (i.e., when the highest profit has been observed).

In order to compute the dynamically optimal executive pay, I parameterize the model following the calibration of Aseff (2004) and Aseff and Santos (2005) based on the results of Hall and Liebman (1998) and Margiotta and Miller (2000). I focus on the more interesting case when the value of the outside offer to the manager is positively correlated with current profit (different types of agents whose ability may be considered positively related to firm's performance by outside potential employers; different economic environments: harder to find a job in a through than in a boom, etc.). In such a setting, we may expect that the manager would be motivated to increase the probability of high profits in the future (by choosing a higher level of effort). At the same time, risk-averse managers would like to smooth consumption across states, which may require that their participation constraint does not bind for lower profit realizations. Moreover, it may become increasingly more difficult to motivate richer CEOs, especially when the shareholders face some borrowing constraints, which may lead to the suboptimality of inducing high effort for such CEOs.

Regarding the numerical computation, I use an algorithm that is quite general and allows for non-convexities of the underlying state space. The main idea is to discretize the guess for the equilibrium set elementwise, extract small open balls around the gridpoints unfeasible with respect to the (non-updated) guess and use the remaining set, i.e., the guess less the extracted intervals, as a new guess for the equilibrium set. The procedure stops if the representations of two successive guesses have the same number of closed sets element by element and the suitably defined difference between the representations is less than some pre-specified tolerance level.

The numerical results suggest that inducing high effort is the predominant strategy of the principal, but shirking may still be optimal for sufficiently rich (in expected utility terms) managers. The optimal wage scheme and the future utility of the manager tend to grow in both his/her current utility and in the end-of-period profit realization. Intuitively, both current and future compensation are used to induce poor and mid-range managers to work hard, while rich managers prove too difficult to motivate. The latter shirk and while they may face some fluctuations in their current income stream in case of binding credit constraints on part of the firm, their lifetime utility remains relatively flat. The manager's utility tends to increase weakly in the long run. This increase is most pronounced for managers with initial utilities below the highest reservation utility. These managers first have their utilities pushed well above their reservation level based on the insurance effect against fluctuations in the value of their outside options. Then, the principal motivates them to work hard by rewarding success through continuation utilities while providing them with insurance through flatter wages. In this way, the probability of a higher profit and, therefore, higher reservation utility tomorrow increases which rises the manager's expected continuation utility. Since wage is increasing in initial utility, the executive pay has a similar dynamic behavior. Therefore, in the long run, both consumption (wage) and wealth (utility) are smoother across initial history states. The result can also be interpreted as a decreasing (wage- and utility-) inequality (as far as the poorest managers are concerned). More interestingly, due to the insurance effect of the contract, the fluctuations in the CEO's reservation utilities tend to lose importance in the long run. The long term distribution of manager's utility is non-degenerate and depends on the initial utility promise but not directly on the relevant initial history at least as far as short initial histories are concerned.

The third chapter is motivated by the frequent use of a quasilinear compensation schemes in the real world. While there is a growing body of literature searching for possible reasons to account for the abundance of such contracts, very little has been done in terms of computing the optimal stock option contract and comparing it with the optimal contract per se. Clementi, Cooley and Wang (2006) consider a two-period principal-agent model of hidden action and show that under severe commitment problems shareholders' value can potentially be improved by including stock options in the agent's compensation package. Aseff and Santos (2005) analyze the properties of the optimal stock option contract obtained in a principal-agent framework by restricting the set of admissible compensation schemes to one consisting of a fixed component and a grant of stock options with a particular strike price. They calibrate the model and find that the cost of implementing the optimal stock contract vs. the optimal contract is negligible. However, their model is static and therefore fails to address issues such as smoothing consumption and incentives over time. Indeed, stock option grants are a purely dynamic phenomenon.

In this chapter I extend and generalize the analysis of Aseff and Santos (2005) in a dynamic framework. The setting is an infinite-horizon hidden-action problem marked by two-sided limited commitment and history-dependent reservation utilities. The history dependence of reservation utilities may in fact be particularly relevant for the case of stock options as indicated in Oyer (2004) who analyzes the use of a broad stock option plans in a static setting. I demonstrate that restricting the space of admissible compensation schemes to canonical stock options does not affect the theoretical results developed in the first chapter. Therefore, the problem of finding the optimal stock option contract can be recursively characterized and numerically computed in a three-step procedure. I analyze the properties of the endogenous state space to obtain results similar but weaker than the ones derived in the second chapter. Interestingly, the manager is still provided with some insurance against non-negligible fluctuations in the value of his/her outside options. I follow the calibration of Aseff and Santos (2005) and compute the optimal stock option contract. Its estimated value function is very flat for lower utility promises and very steep for higher utility promises. As under the optimal contract per se, less wealthy managers tend to work hard, but high effort proves suboptimal for the richest CEOs. There is a notable difference, however, in the way incentives are provided. Under the optimal contract both current and deferred compensation are used while under the stock option contract future utility promise appears to be a more powerful incentive device. It tends to increase with the initial utility promise and, on average, grows with the stock price realization. The compensation package, on the other hand, shows little dynamics and only gains significance for high utility promises where the resulting compensation jumps due to an increased fixed salary and a big stock option grant with a low strike price.

The fourth chapter extends the analysis to contracts that allow for permanent separations. So far, I have considered dynamic contracts which are self-enforcing, i.e., contracts referring to long-term relationships which, by construction, neither party has an incentive to renege on. Spear and Wang (2005) focus on contract terminations instead by allowing for replacements of the contracted agent with a new one from a labor market pool and for golden parachutes at termination. Sleet and Yeltekin (2001) consider a dynamic model of temporary layoffs and permanent separations where the period profit of the firm is subject to publicly observable random shocks.

In my environment, there is no profit perturbation, but the outside options are allowed to vary across the history of observables. In particular, I consider a potentially repeated hidden-action problem where both the principal and the agent can only commit to single-period contracts while facing finite-history-dependent reservation utilities. Each party can walk out of the relationship if his/her utility is below the respective reservation level in which case the relationship terminates and the parties receive their outside options. This possibility explicitly enters the strategy space of the contract signed at some initial period of contracting. Therefore, the continuation utilities for both parties should be no less than their respective reservation utilities at all contingencies that are actually reached.

My main interest is to compare the behavior of this contract in providing incentives and insurance with the self-enforcing contract analyzed in the first chapter. In particular, I seek to answer the following questions. Does the option to terminate the relationship have a global effect on incentives, or does it only affect the poorest managers? In fact, when do separations occur and what continuation utilities can be supported by the threat of a future separation?

As before, I characterize the contract recursively on the space of truncated outcome histories matched with agent's expected discounted utilities. I parameterize and numerically compute the model in view of top executive compensation. The results show similarities to the optimal self-enforcing contract. High effort appears optimal for most but the richest (in expected utility terms) managers. Executive compensation and continuation utility tend to increase in both current utility and firm's profit. The differences are related to the ability of the contract to support much lower utility promises, which significantly decreases their smoothing across initial profits. The most affected are the poorest (in expected utility terms) managers who are motivated to work hard by much lower continuation utilities under the threat of a future separation.

For future research, I plan to calibrate the model dynamically to recent US data by matching Compustat's ExecuComp with CRSP database on stock prices. I consider investigating the effects of shocks (to the primitives of the model) on the optimal provision of insurance and incentives in the short and the long run, including the resulting dynamics of executive pay and wealth. In this sense, it is essential to differentiate between idiosyncratic and aggregate shocks. The model can be used for estimating pay-performance sensitivity whether measured in the tradition of Jensen and Murphy (1990) or as an elasticity following Clementi and Cooley (2009). An extended version of the model can also be used to address the relative performance evaluation puzzle. The framework can be enriched in two other directions. First, we can allow the manager to borrow and save and study how this affects the provision of incentives under the optimal contract. Second, we can endogenize the outside options available to the parties to the contract and motivate the fluctuations in their reservation utility values.

Dynamic Moral Hazard with History-Dependent Participation Constraints*

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March, 2009

Abstract

This paper considers a moral hazard problem in an infinite-horizon, principal-agent framework. In the model, both the principal and the agent can commit only to short-term (single-period) contracts and their reservation utilities are allowed to depend on some finite truncation of the history of observables. After existence is proved, the original problem of obtaining the optimal incentive-compatible self-enforcing contract is given an equivalent recursive representation on a properly defined state space. I construct an auxiliary version of the problem where the participation of the principal is not guaranteed. The endogenous state space of agent's expected discounted utilities which on a different dimension includes the set of truncated initial histories in order to account for their influence on the reservation utilities is proven to be the largest fixed point of a set operator. Then, the self-enforcing contract is shown to be recursively obtainable from the solution of the auxiliary problem by severely punishing any violation of the principal's participation constraint.

Keywords: principal-agent problem, moral hazard, dynamic contracts, limited commitment

Journal of Economic Literature Classification Numbers: C61, C63, D82

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*I am particularly grateful to Manuel Santos for his guidance and support. I also wish to thank the members of my thesis committee, participants in the 2008 Meetings of the Econometric Society in Pittsburgh, Wellington, and Singapore, in the Ph.D. and Macro seminars at Universidad Carlos III de Madrid and the ICEF seminar at the State University - Higher School of Economics for useful suggestions and comments.

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1 Introduction

During the last years, there has been a revived interest in the theory of dynamic contracting[1]. However, although most of the research incorporates some form of limited commitment/enforcement, little has been done in terms of extending the notion of commitment per se. In particular, there is no reason to believe that (the value of) the outside option is constant across the history of observables. For example, it is unrealistic to treat the reservation utility of a CEO as fixed regardless of the situation in his/her firm, industry, or the economy as a whole. The dependence could come through many channels- externalities, different types of agents, a certain structure of beliefs, but more importantly, it can significantly influence the nature of the relationship and the form of the optimal contract. It would be interesting to see how the agent is actually compensated for variability in the value of his/her outside options. When would his/her participation constraint bind? How is the agent's wealth affected in the short and the long run? In fact, would there be a limiting distribution and how would it depend on initial conditions? Such questions can only be analyzed in a generalized framework allowing for history-dependent reservation utilities. Moreover, extending the notion of commitment can bring some important insights into various contractual problems. For example, in order to address the wide use of broad-based stock option plans, Oyer (2004) builds a simple two-period model where adjusting compensation is costly and employee's outside opportunities are correlated with the firm's performance.

The current paper generalizes the notion of commitment by defining the outside options on the history observed in a dynamic contractual setting. I prove existence and obtain the first in the literature characterization of such an environment. The characterization is very general in terms of assumptions and, more importantly, is fully recursive. Its convergence properties make it perfect for computing the optimal contract for a general class of dynamic hidden action models.

I consider a moral hazard problem in an infinitely repeated principal-agent interaction while allowing the reservation utilities of both parties to vary across the history of observables. More precisely, to keep the model tractable, the reservation utilities are assumed to depend on some finite truncation of the publicly observed history. The rest of the model is standard in the sense that the principal wants to implement some sequence of actions which stochastically affect a variable of his/her interest, but suffers from the fact that the actions are unobservable. For this purpose, the optimal contract needs to provide the proper incentives for the agent to exercise the sequence of actions suggested by the principal. The incentives, however, are restricted by the inability of the parties to commit to a long-term relationship. It is here where the dynamics of

[1] See, for example, Fernandes and Phelan (2000), Ligon, Thomas and Worrall (2000), Wang (2000), Phelan and Stacchetti (2001), Sleet and Yeltekin (2001), Ligon, Thomas and Worrall (2002), Ray (2002), Thomas and Worrall (2002), Doepke and Townsend (2004), Jarque (2005), Abraham and Pavoni (2008).

the reservation utilities enters the relationship by reshaping the set of possible self-enforcing, incentive-compatible contracts.

In order to be able to characterize the optimal contract in such a setting, I construct a reduced stationary representation of the model in line with the dynamic insurance literature. The representation benefits from Green (1987)- the notion of temporary incentive compatibility, Spear and Srivastava (1987)- the recursive formulation of the problem with the agent's expected discounted utility taken as the state variable, and Phelan (1995)- the recursive structure with limited commitment, but is closest to Wang (1997) as far as the recursive form is concerned. Unlike Wang (1997), however, I formally introduce limited commitment on both sides and provide a rigorous treatment of its effect on the structure of the reduced computable version of the model. A parallel research by Aseff (2004) uses a similar general formulation[2], but via a transformation due to Grossman and Hart (1983) constructs a dual, cost-minimizing recursive form closer to Phelan (1995) in order to solve for the optimal contract. Such a procedure, however, exogenously imposes the optimality of a certain action on every possible contingency.

After existence is proved, the general form of the model is reduced to a more tractable, recursive form where the state is given by the agent's (promised) expected discounted utility. On a different dimension, the state space includes the set of possible truncated histories in order to account for their influence[3] on the reservation utilities. This recursive formulation does not rely on the first-order approach and is not based on Lagrange multipliers [cf. Marcet and Marimon, (1998)]. In fact, all I need is continuity of the momentary utilities. I first consider an auxiliary version where the participation of the principal is not guaranteed. The solution of this problem can be computed through standard dynamic programming methods once the state space is determined. Following the approach of Abreu, Pearce and Stacchetti (1990), the state space is shown to be the fixed point of a set operator and can be obtained through successive iteration on this operator until convergence. Given the solution of the auxiliary problem, I resort to a procedure outlined by Rustichini (1998) in order to solve for the optimal incentive compatible, two-side participation guaranteed supercontract. This is achieved by severely punishing the principal for any violation of his/her participation constraint. The procedure allows of recovering the subspace of agent's expected discounted utilities supportable by a self-enforcing incentive-compatible contract.

The rest of the paper is structured as follows. Section 2 presents the dynamic model. Section 3 derives the reduced recursive formulation. Section 4 concludes. Appendix 1 contains all the proofs.

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[2] His benchmark model is a full-commitment one, but he considers limited commitment on part of the agent as an extension.

[3] The relationship between the history of observables and the reservation utilities is predetermined since the reservation utilities are exogenous to the problem.

2 Dynamic model

The model considers a moral hazard problem in an infinite horizon principal-agent framework with limited commitment on both sides. Each period, the principal needs the agent to implement some action that stochastically affects a variable of principal's interest, but suffers from the fact that the action is observable only by the agent. Given that the variable of interest to the principal is publicly observable, the principal may want to condition the wage of the agent on the realization of this variable instead. However, the issue of inducing the proper incentives is further complicated by the lack of commitment to a long-term relationship. The commitment problem is structured very generally in the sense that the reservation utilities are allowed to depend on some truncation of the publicly observed history.

Consider, for example, the interaction between the firm's shareholders (the principal) and its CEO (the agent). The CEO may exert a different amount of effort which on its turn randomly affects the success of the corporation illustrated by its observed gross profit. Both the principal and the agent have some outside options: the firm may close, while the agent may quit and start working for another employer. These options are represented by reservation utilities which may vary on the history of observables (in this case, the history of firm's realized gross profits).

First, I will introduce some notation. Let $\mathbb{Z}$ be the set of integers with $\mathbb{Z}_{++}$ and $\mathbb{Z}_{+}$ denoting the sets of positive and respectively nonnegative integers. Time is discrete and indexed by $t \in \mathbb{Z}$. Let y_t denote a particular realization of the variable of interest to the principal in period t. This outcome is realized and observed by both the principal and the agent at the end of the period. As a matter of fact, at the beginning of period t there is a stream of previously realized outcomes which we denote by y^{t-1}. Given that the end-of-period-t realization is y_t, the history of outcomes at the beginning of period $t+1$ is simply $y^t = \left(y^{t-1}, y_t\right)$. The set of possible outcomes is assumed a time- and history-invariant, finite set of real numbers which is denoted by Y. For concreteness, we assume that it consists of $n > 1$ distinct elements.

There is an initial period of contracting which we normalize to 0. At the beginning of this initial period, an outcome history of length $\theta \in \mathbb{Z}_{+}$ is observed. Therefore, a period-0 contracting problem should be defined on n^{θ} initial history nodes.[4] Both the principal and the agent can only commit to short term contracts, therefore it is natural to start with a series of single-period contracts defined on all possible contingencies stemming from some initial node. Each such

[4] As it will become clear afterwards, history will not matter at the initial period of contracting unless the reservation utility of either the principal or the agent is history dependent. Since in order to keep the problem tractable, I allow the reservation utilities to vary across a finite truncation of the observed history with length θ (Assumption 2), it would be natural to consider the contracting problem as defined on n^{θ} initial nodes. As for the existence of an initial period of contracting, note that we can modify the period-0 contracting problem [PP] so that the principal should provide the agent with a given initial (expected discounted) utility level resulting from a previous round of long-term contracting.

contract is history dependent and specifies an action and a monetary transfer from the principal to the agent contingent on the particular outcome observed at the end of the period. The timing is as follows. A short-term contract is signed at the beginning of the period. Then, the agent implements some action which is unobserved by the principal and may not be the one specified in the contract. Nature observes the action and draws a particular element of the set of possible outcomes according to some probability distribution. The outcome is observed by both parties and the agent receives the transfer corresponding to this particular outcome.[5] Then, a new period starts, a new short-term contract is signed, and so on.

Formally, at the beginning of each period $t \in \mathbb{Z}_+$, after a particular history y^{t-1} has been publicly observed,[6] a single-period contract $c_t\left(y^{t-1}\right) := \left\{a_t\left(y^{t-1}\right), w_t\left(y^{t-1}, y\right) : y \in Y\right\}$ is signed between the principal and the agent. Hereafter, for the sake of simplicity, I will often denote such a contract by c_t with the clear understanding that it is defined on a particular history y^{t-1}. The contract specifies an action a_t to be implemented by the agent. To make the analysis tractable, the action is assumed one-dimensional and the action space is taken compact, time- and history-invariant. Formally, $a_t \in A$, where $A \subset \mathbb{R}$ compact. The contract also specifies a compensation scheme $\{w_t(., y) : y \in Y\}$ under which the agent will receive a monetary payoff $w_t(., y)$ in the end of the period if the (end-of-period) outcome is y for any $y \in Y$. The space of possible wages, W, is assumed a compact, time- and history-invariant subset of $\mathbb{R}$.[7] After the contract is signed, the agent exercises action $a_t' \in A$ which is not necessarily the one prescribed by the contract. Then, outcome y_t is realized and the agent receives $w_t(., y_t)$. At the beginning of period $t+1$, contract $c_{t+1}(y^t)$ is signed and so on.

Hereafter, I will refer to any sequence of outcomes, actions, or wages as admissible if all their elements belong to Y, A, or W, respectively.

[5] Given the above setup, the principal's ability to commit to a short-term contract should be understood as an ability to commit to providing the agent with the promised monetary transfer. Indeed, the transfer specified in the short-term contract signed at the beginning of the period occurs at the end of the same period.

[6] You may note that an outcome history y^{t-1} consists of θ elements corresponding to the initial history observed at the beginning of period 0 and t elements from period 0 to period $t-1$.

[7] The compactness assumption can easily be defended by economic considerations. Consider $W = [\underline{w}, \overline{w}] \subset \mathbb{R}$, where $\underline{w}$ may either be zero or a higher number that corresponds to the legally established minimum wage, while $\overline{w}$ is some finite number reflecting the boundedness of the principal's total wealth (the discounted sum of maximum possible income flows). For example, if we treat y as profits, then $\underline{w}$ may be taken equal to $\frac{\max Y}{1-\beta_P}$, where β_P is the relevant discount factor, or to a lower number reflecting restrictions on the principal's ability to borrow against future profits. In Morfov (2009a), a minimum wage level is assumed and from there a theoretical upper bound on the wage is derived in Proposition 1. In the same paper, two other possibilities are considered. The first deals with the case where the principal can borrow up to $\max Y - y$ units of consumption, where y is current gross profit. Then, we can take $\overline{w} = \max Y$. The second case assumes that the principal is prohibited from borrowing, so the wage cannot exceed the current gross profit realization. Note that we can easily extend this case to the environment described here, by taking $\overline{w} = \max Y$ and additionally requiring $w_t(., y) \leq y$, $\forall y \in Y$.

In order to simplify the analysis, I assume that the probability distribution of the variable of interest to the principal depends only on the action taken (earlier) in the same period[8] and that each value in the admissible set Y is reached with a strictly positive probability.

Assumption 1 *For any period $t \in \mathbb{Z}$, any admissible outcome history $y^t = \left(y^{t-1}, y_t\right)$, and any admissible action sequence $a^t = \left(a^{t-1}, a_t\right)$, the probability that y_t is realized given y^{t-1} has been observed and a^t has been implemented equals $\pi\left(y_t, a_t\right)$ where $\pi : Y \times A \to (0,1)$ such that $\forall a \in A$, $\sum\limits_{y \in Y} \pi\left(y, a\right) = 1$ and $\forall y \in Y$, $\pi\left(y, .\right)$ continuous on A.*

The continuity of π in its second argument is a regularity condition which is trivially satisfied if A is finite.

The principal's (end-of-)period-t utility is denoted by $u\left(w_t, y_t\right)$, where $u : W \times Y \longrightarrow \mathbb{R}$ is assumed continuous, decreasing in the agent's wage, and increasing in the outcome. The principal discounts the future by a factor $\beta_P \in (0,1)$. The agent's (end-of-)period-t utility is given by $\nu\left(w_t, a_t\right)$ with $\nu : W \times A \longrightarrow \mathbb{R}$ continuous, increasing in wage, and decreasing in the implemented action.[9] The agent discounts the future by a factor $\beta_A \in (0,1)$. Note that given our assumptions, the expected discounted utilities of both the agent and the principal are bounded at any node.

As already mentioned, the agent need not necessarily implement the action specified in the contract. Indeed, if another action brings the agent strictly higher utility, he/she will find it profitable to deviate. Therefore, the contract should provide the proper incentives to the agent in order for him/her to exercise exactly the action recommended by the principal.

Limited commitment is assumed on both parts in the sense that both the principal and the agent can commit only to short-term (single-period) contracts. This assumption is intended to reflect legal issues on the enforcement of long-term contracts. However, at the initial period the principal can offer a long term contract (a supercontract) that neither he/she, nor the agent would like to renege on,[10] and that would provide the necessary incentives for the agent

[8] While the framework can be modified to include some form of "action" persistence [see, for example, Fernandes and Phelan (2000) and Jarque (2005)], such an extension will be of little value here since the current paper aims to characterize the effect of a generalized form of limited commitment on the optimal dynamic contract. Given that the reservation utilities are allowed to vary across the history of observables, we have another form of persistence which should be analyzed in isolation from potential long-term effects coming from agent's action choice.

[9] Note that we effectively prohibit the agent from borrowing or saving. While extending the model in that direction is possible, introducing such a behavior would shift the focus to incentive-compatibility, while in the current research I seek to analyze the role of the participation constraints in the optimal contract. Moreover, without a set of strong assumptions justifying the first-order approach, such an extension would be very hard to deal with on a practical level given the increase in the dimensionality of the state space of the recursive form.

[10] That is, a self-enforcing contract extending the definition of Phelan (1995) to my generalized notion of limited commitment.

to exercise the sequence of actions proposed by the principal. We will refer to this supercontract as a self-enforcing, incentive-compatible contract and would concentrate on the one maximizing the utility of the principal.

Regarding the issue of commitment, the reservation utilities take values in $\mathbb{R}$ and are allowed to vary across the history of observables. Since it is not practical to define reservation utilities on infinite histories, I make the following assumption.

Assumption 2 *The reservation utilities of both the principal and the agent exogenously depend on the previous θ outcomes.*

The assumption says that the reservation utilities are finite-history dependent, but time independent. Note that the history dependence is truncated to the realizations in the previous θ periods. This is no coincidence. Analogously, we could have started with potentially infinite histories in period 0, introduced finite-history dependence of different length: θ_P for the principal and θ_A for the agent, and then considered finite truncations with length $\theta := \max\{\theta_P, \theta_A\}$ of the infinite histories observed in period 0.

Let Y^θ denote the set of possible outcome streams of length θ periods, or, alternatively, the set of possible initial histories observed at the beginning of period 0. For concreteness, let us enumerate this set using some bijective function $l : Y^\theta \to L$, where $L := \{1, ..., n^\theta\}$. Hereafter, all functions and correspondences with domain Y^θ will be considered as vectors or Cartesian products of sets indexed by L. Moreover, we will often abuse the notation and use l as its inverse, namely, as the initial history to which the particular index corresponds.

Given this indexing, we will denote the reservation utilities of the principal and the agent at node $y^{t-1} \in Y^t \times l$ as $\underline{U}_l$ and $\underline{V}_l$ respectively, $\forall t \in \mathbb{Z}_+$, $\forall l \in L$. For example, if the history observed in the previous θ periods has been $(y_{t-\theta}, ..., y_{t-1})$, the principal's reservation utility in the current period will be $\underline{U}_l$, where $l = l\,(y_{t-\theta}, ..., y_{t-1})$ is the index of the particular outcome stream.

For any $l \in L$, we will define a long-term contract (a supercontract), $c := (a, w)$, where $a := \{a_t\,(y^{t-1}) : y^{t-1} \in l \times Y^t\}_{t=0}^{\infty}$ and $w := \{w_t\,(y^{t-1}, y_t) : (y^{t-1}, y_t) \in l \times Y^t \times Y\}_{t=0}^{\infty}$ are the plan of actions and respectively the sequence of wages defined the whole tree of contingencies stemming from an initial history l.[11] The supercontract prescribes a single action at every node, but specifies the agent's compensation as further dependent on the end-of-period outcome, i.e., as a function with domain Y, or alternatively, as a vector of n elements.[12] Let $V_\tau\,(c, y^{\tau-1})$ and $U_\tau\,(c, y^{\tau-1})$ be the expected discounted utilities of the agent and respectively the principal at node $y^{\tau-1}$ given a supercontract c, i.e.:

[11] Note that the supercontract depends on the initial history, but to ease up the exposition, I suppress this dependence notationally.

[12] Remember that Y is finite with cardinality n.

$$V_\tau\left(c,y^{\tau-1}\right):=\sum_{t=\tau}^{\infty}\beta_A^{t-\tau}\sum_{y_t\in Y}...\sum_{y_\tau\in Y}\nu\left(w_t,a_t\right)\prod_{i=\tau}^{t}\pi\left(y_i,a_i\left(y^{i-1}\right)\right),$$
$$U_\tau\left(c,y^{\tau-1}\right):=\sum_{t=\tau}^{\infty}\beta_P^{t-\tau}\sum_{y_t\in Y}...\sum_{y_\tau\in Y}u\left(w_t,y_t\right)\prod_{i=\tau}^{t}\pi\left(y_i,a_i\left(y^{i-1}\right)\right).$$

At time 0, after a truncated history l has been observed, the principal is solving the following problem:

[PP]

$\sup_c U_0\left(c,l\right)$ s.t.:

$$a_\tau\in A,\ \forall nai(l) \tag{1}$$

$$w_\tau\left(.,y\right)\in W,\ \forall y\in\ Y,\ \forall nai(l) \tag{2}$$

$$V_\tau\left(a,w,y^{\tau-1}\right)\geq V_\tau\left(a',w,y^{\tau-1}\right),\ \forall(a':\forall nai(y^{\tau-1}),\ a'_t\in A),\ \forall nai(l) \tag{3}$$

$$V_\tau\left(c,y^{\tau-\theta-1},\widetilde{l}\right)\geq \underline{V}_{\widetilde{l}},\ \forall nai(l) \tag{4}$$

$$U_\tau\left(c,y^{\tau-\theta-1},\widetilde{l}\right)\geq \underline{U}_{\widetilde{l}},\ \forall nai(l) \tag{5}$$

where "$\forall nai(l)$" should be understood as "for any *node after* and *including* l", that is $\forall y^{\tau-1}\in l\times Y^\tau$, $\forall\tau\in\mathbb{Z}_+$.

This is a time-0, history-l contracting problem that mimics dynamic contracting from this node on. That is, at l, the principal solves for a sequence of future strategies on all possible contingencies, so at each node the continuation strategy needs to be self-enforcing and incentive compatible. As in the standard model of dynamic contracting, these strategies are history-dependent. Here, we additionally have that each decision node is characterized by a specific pair of reservation utilities which depend on the history of observables. Nevertheless, as the next section shows, the problem does possess a recursive representation in the spirit of Spear and Srivastava (1987).

Constraints (1) and (2) guarantee that the action plan and respectively the wage scheme are admissible. That is, at any node of the tree stemming from l, the supercontract prescribes an action from A and specifies a compensation scheme that maps Y to W. (3) guarantees that the contract is incentive compatible on any node. For example, at the initial node l, it requires that the action plan of the principal should make the agent weakly better off in terms of period-0 expected discounted utility than any other sequence of admissible actions.[13]

[13] In our framework, we actually have that incentive compatibility on any node is equivalent to initial (time-0) incentive compatibility (see Lemma 1 in the Appendix).

(4) and (5) are the participation constraints of the agent and respectively the principal which due to limited commitment should hold at any node. These constraints guarantee the participation of both parties at each contingency. For example, at node $y^{\tau-1} = \left(y^{\tau-\theta-1}, \widetilde{l}\right)$, the expected discounted utility of the agent should be no less than his/her respective reservation utility at this node, $\underline{V}_{\widetilde{l}}$, and the expected discounted utility of the principal should be greater or equal to $\underline{U}_{\widetilde{l}}$.

For future reference, we denote the problem above as [PP] and its supremum as U_l^{**}.

The solution of [PP], if such a solution exists, would be the self-enforcing, incentive-compatible contract that maximizes the utility of the principal at the initial period of contracting.

Let $\Gamma_{y^{\tau-1}} := \left\{c : (1) - (5) \text{ hold after } y^{\tau-1}\right\}$. This is the set of admissible, incentive-compatible, self enforcing contracts that can be signed at node $y^{\tau-1}$. In particular, consider Γ_l, the set of such contracts available at an initial history l. We shall assume that this set is non-empty for any $l \in L$.[14]

Assumption 3 *$\forall l \in L$, $\Gamma_l \neq \emptyset$.*

3 Recursive Form

In this section, we will prove existence and construct an equivalent recursive representation of [PP]. We start by establishing the equivalence of incentive compatibility at all contingencies to Green (1987)'s temporary incentive compatibility at all contingencies.

Proposition 1 *Let (1) and (2) hold after $l \in L$. Then, (3) $\Leftrightarrow$*

$$\forall nai\,(l)\,,\ V_\tau\left(a, w, y^{\tau-1}\right) \geq V_\tau\left(a', w, y^{\tau-1}\right),$$
$$\forall a' : a'_\tau\left(y^{\tau-1}\right) \in A, \text{ and } \forall y \in Y,\ \forall nai\left(y^{\tau-1}, y\right),\ a'_t\,(.) = a_t\,(.) \qquad (6)$$

[14] If the set is empty for some initial history, then there does not exists an incentive-compatible, self-enforcing supercontract at this node. As our numerical estimates in Morfov (2009a) demonstrate, this is hardly the case: in fact there is a wide interval of possible utility promises to the agent that can be supported by a contract of such type for any initial history node. Also note that for suitably chosen reservation utility values, the incentive compatible contract will behave as a full-commitment one, so any violation of Assumption 3 will directly imply the non-existence of the latter. Therefore, it is more a problem of choosing the "proper" (not too high) reservation utilities than anything else. Nevertheless, Morfov (2009b) considers an extension that allows for permanent separations and does not require an assumption of this sort.

The proposition says that constraint (3) is equivalent to requiring that at any date τ, after any history $y^{\tau-1}$, there is no profitable deviation in the current period which will make the agent strictly better off (in expected utility terms) given that he/she fully complies to the plan in the future. The proposition allows us to focus on single-period deviations, which is the first step towards a recursive structure.

Consider two types of supercontracts. The first, hereafter referred to as a 2P contract, is an incentive-compatible supercontract which is self-enforcing, i.e., guarantees the participation of both the agent and the principal. The second, hereafter referred to as an AP contract, is an incentive-compatible supercontract which guarantees the participation of the agent, but not necessarily the one of the principal.[15] Note that the set of possible 2P contracts that can be signed at node $y^{\tau-1}$ was already denoted by $\Gamma_{y^{\tau-1}}$. Let $\Gamma^{AP}_{y^{\tau-1}}$ be the set of possible AP contracts that can be signed at node $y^{\tau-1}$. Formally, $\Gamma^{AP}_{y^{\tau-1}} := \left\{c : (1) - (4) \text{ hold after } y^{\tau-1}\right\}$. Now, we are going to consider the sets of agent's initial utilities that can be guaranteed/supported by a 2P and respectively an AP contract.

Let l be some initial history node. Take an arbitrary period τ and a history $y^{\tau-1}$ stemming from l, i.e., $y^{\tau-1} \in l \times Y^{\tau}$. Let $V^{2P}_{\tau}\left(y^{\tau-1}\right)$ be the set of admissible values for the expected discounted utility of the agent signing at date τ after a history $y^{\tau-1}$ a 2P contract with the principal. Formally, $V^{2P}_{\tau}\left(y^{\tau-1}\right) := \{V \in \mathbb{R} : \exists c \in \Gamma_{y^{\tau-1}} \text{ such that } V_{\tau}\left(c, y^{\tau-1}\right) = V\}$. Let us also introduce another set, $V^{AP}_{\tau}\left(y^{\tau-1}\right)$, which gives us the possible discounted utilities of the agent signing at date τ after a history $y^{\tau-1}$ an AP contract with the principal. Formally, $V^{AP}_{\tau}\left(y^{\tau-1}\right) := \{V \in \mathbb{R} : \exists c \in \Gamma^{AP}_{y^{\tau-1}} \text{ such that } V_{\tau}\left(c, y^{\tau-1}\right) = V\}$. Since every 2P contract is an AP contract, the agent's utilities supportable by a 2P contract will be a subset of the agent's utilities supportable by an AP contract. Formally, $V^{2P}_{\tau}\left(y^{\tau-1}\right) \subset V^{AP}_{\tau}\left(y^{\tau-1}\right)$ for any $l \in L$, $\tau \in \mathbb{Z}_{+}$, and $y^{\tau-1} \in l \times Y^{\tau}$. Now, we are ready to introduce the sets of principal's initial utilities that can be supported by a 2P and respectively an AP contract promising a certain initial utility to the agent.

For any $V \in V^{2P}_{\tau}\left(y^{\tau-1}\right)$, let $U^{2P}_{\tau}\left(V, y^{\tau-1}\right)$ be the set of possible values for the expected discounted utility of the principal signing at node $y^{\tau-1}$ at time τ a 2P contract that would give the agent an initial expected discounted utility of V, i.e., $U^{2P}_{\tau}\left(V, y^{\tau-1}\right) := \{U \in \mathbb{R} : \exists c \in \Gamma_{y^{\tau-1}} \text{ such that } V_{\tau}\left(c, y^{\tau-1}\right) = V$ and $U_{\tau}\left(c, y^{\tau-1}\right) = U\}$. For any $V \in V^{AP}_{\tau}\left(y^{\tau-1}\right)$, let $U^{AP}_{\tau}\left(V, y^{\tau-1}\right)$ be the corresponding set (defined accordingly) in case the principal is signing an AP contract instead. Then, for any $V \in V^{2P}_{\tau}\left(y^{\tau-1}\right)$, we have $U^{2P}_{\tau}\left(V, y^{\tau-1}\right) \subset U^{AP}_{\tau}\left(V, y^{\tau-1}\right)$, while for $V \in V^{AP}_{\tau}\left(y^{\tau-1}\right) \setminus V^{2P}_{\tau}\left(y^{\tau-1}\right)$, $U^{2P}_{\tau}\left(V, y^{\tau-1}\right)$ is not defined.

[15] It may be easier to remember the abbreviations in the following way: AP="agent participates"; 2P= "two [...] participate", i.e., both the agent and the principal participate.

Proposition 2 *Let $l \in L$ and $i \in \{2P, AP\}$. Then, for any $\tau \in \mathbb{Z}_+$ and $y^{\tau-1} \in Y^\tau \times l$: (a) $V_\tau^i\left(y^{\tau-1}\right) = V_0^i(l)$ compact; (b) $\forall V \in V_\tau^i\left(y^{\tau-1}\right)$, $U_\tau^i\left(V, y^{\tau-1}\right) = U_0^i(V,l)$ compact.*

Part (a) of the proposition says that the sets of possible expected discounted utility values for the agent signing a 2P or AP contract are time invariant and compact. Furthermore, the history dependence of these sets is restricted only to the previous (as of signing) θ realizations. As part (b) indicates, the result also applies to the set of possible expected discounted utilities of the principal signing a 2P or AP contract guaranteeing a particular initial utility to the agent.

To ease up the notation, we will hereafter refer to these sets as $V^{2P}(l)$, $V^{AP}(l)$, $U^{2P}(V,l)$, and $U^{AP}(V,l)$.

Remember that U_l^{**} is the supremum of the principal's problem [PP]. We state the following result.

Proposition 3 *(Existence of an optimal contract): For any $l \in L$, $\exists c_l \in \Gamma_l$ s.t. $U_l^{**} = U_0(c_l, l)$.*

The proposition establishes the existence of an optimal 2P contract. However, due to the complexity of the problem, the optimal contract cannot be derived analytically. Nevertheless, I show that it can be characterized and given a computable representation. In the spirit of Spear and Srivastava (1987), this is done by constructing a recursive version of [PP] taking the agent's expected discounted utility as a state variable. Up to certain qualifications, this new formulation of the problem can be addressed by dynamic programming routines.

I will first establish a useful result which is related to the transformation of the dynamic principal's problem to a series of static problems defined on an endogenously obtained state space.

Fix $l \in L$. By Proposition 2 (b) for any $V \in V^{2P}(l)$, $U^{2P}(V,l)$ is compact and therefore, we can define $U^*(V,l) := \max U^{2P}(V,l)$ as the maximum utility the principal can get by signing a 2P supercontract offering V to the agent. Furthermore, let $U_l^* := \sup\limits_{V \in V^{2P}(l)} U^*(V,l)$.

Proposition 4 *$\forall l \in L$, $U_l^{**} = U_l^* = \max\limits_{V \in V^{2P}(l)} U^*(V,l)$.*

This proposition shows that the principal is indifferent between directly maximizing his/her utility given l, or first finding the maximum utility he/she can obtain by guaranteeing the agent a certain initial level of utility and then maximizing over the resulting set.[16]

Let $l_+ : L \times Y \to L$ give the index of the initial history tomorrow given the index of the initial history today and the new realization of the variable. Consider, for example, that we are in period t. At the beginning of t, an initial history $(y_{t-\theta}, ..., y_{t-1})$ with index l has been observed. At the end of the period, an outcome y_{t-1} is realized. Then, at the beginning of period $t+1$, the observed initial history will be $(y_{t-\theta+1}, ..., y_t)$ and will have an index $l_+ (l, y_t)$. We will often abuse the notation and use l_+ instead of $l^{-1}(l_+)$, i.e., replace the initial history tomorrow by its index.

Consider the state space $\left\{(V, l) : V \in V^{AP}(l)\right\}_{l\in L}$ that matches initial histories of outcomes with initial utility promises supportable by an AP contract.[17] Let $c_R(V, l) = \{(a_-(V, l), w_+(V, l, y), V_+(V, l, y)) : y \in Y\}$ be a stationary contract defined on a point (V, l) of the state space, where $a_-(.)$ is the agent's action in the beginning of the period, $w_+(., y)$ is the wage the agent will receive in the end of the period if the realization of the variable of interest to the principal is y, $\forall y \in Y$, and $V_+(., y)$ is the end-of-period expected discounted utility of the agent in case of realization y, $\forall y \in Y$. Since the realization of the variable in question is not known when this contract is signed, the wage and the end-of-period utility of the agent are specified for all possible outcomes, Y. Although the stationary contract depends on the initial history and the particular expected discounted utility of the agent in the beginning of the period, I will often suppress this dependence notationally and refer to the contract simply as $c_R = \{(a_-, w_+(y), V_+(y)) : y \in Y\}$. Let $USCB_l$ denote the space of bounded upper semicontinuous (usc) functions from $V^{AP}(l)$ to $\mathbb{R}$ endowed with the sup metric. Define $V^{AP} := \left\{V^{AP}(l)\right\}$ as the set of possible initial discounted utilities of the agent signing an AP contract ordered by initial history. Since L is finite, this set inherits the properties of $V^{AP}(l)$ established in Proposition 2 (a). Then, for any $U = \{U_l\}$ with $U_l \in USCB_l$, $\forall l \in L$, define the operator T as follows. For any $V = \{V_l\} \in \left\{V^{AP}\right\}$, $T(U)_{(V)} := \left\{T_l(U)_{(V_l)}\right\}$, where:

$$T_l(U)_{(V_l)} := \max_{c_R}\{\sum_{y\in Y}[u(w_+(y), y) + \beta_P U_{l_+(l,y)}(V_+(y))]\pi(y, a_-)\} \text{ s.t.:}$$

[16] Note that the original problem can be set as the principal maximizing expected discounted utility given an initial truncated history l at period 0, where the maximum is taken over a set of 2P supercontracts promising the agent an initial expected discounted utility of V_l for any $l \in L$ and $V_l \in V^{2P}(l)$. The promise should be consistent (in a sense that will soon become clear; see (9)) and can be considered a leftover from a (remote) previous round of contracting. Then, the original problem is defined on $\{V^{2P}(l) : l \in L\}$ and the recursive representation will be equivalent to the one obtained here without the need to maximize $U^*(., l)$ over $V^{2P}(l)$. Namely, we would have $U^{**}(., l) = U^*(., l)$ over $V^{2P}(l)$. Since the static form characterizing both [PP] and the problem described here is the same, I choose to present the former because of the more involved description and notation of the latter.

[17] The possible initial histories (of length θ) enter the picture because they could potentially affect the reservation utility values.

$$a_{-} \in A \tag{7}$$

$$w_{+}(y) \in W,\ \forall y \in Y \tag{8}$$

$$\sum_{y \in Y} [\nu (w_{+}(y), a_{-}) + \beta_A V_{+}(y)] \pi (y, a_{-}) = V_l \tag{9}$$

$$\sum_{y \in Y} [\nu \left(w_{+}(y), a'_{-}\right) + \beta_A V_{+}(y)] \pi \left(y, a'_{-}\right) \leq V_l,\ \forall a'_{-} \in A \tag{10}$$

$$V_{+}(y) \in V^{AP}(l_{+}(l, y)),\ \forall y \in Y \tag{11}$$

Notice that the maximization above is over a set of static contracts at a particular point (V_l, l) of the state space.[18] Also note that if the initial history today is l and the end-of-the-period realization is y, then the initial history tomorrow will be l_{+} which in general will be different from l. Therefore, it is important that we keep track of the initial history update and so each T_l is applied to U, not just to U_l.[19] The use of max instead of sup in the definition of T is justified by the fact that we are maximizing an usc function over a compact set. Constraints (7), (8), and (10) are the stationary versions of (1), (2), and (6) respectively. In particular, (7) guarantees that the action is admissible (i.e., an element of A), (8) guarantees that the compensation scheme is admissible (i.e., mapping Y to W), and (10) is temporary incentive compatibility.[20] (9) is a promise keeping constraint[21] which guarantees the agent an expected discounted utility of V_l today. It is a requirement on the static contract that makes the principal's initial utility promise to the agent at node l, V_l, consistent with the future promise given the proposed action and compensation scheme. (11) is another consistency constraint requiring that the discounted expected utility that the agent will get next period can be supported by an AP supercontract. Note that (11) implies that the agent's continuation utility should not fall below

[18] This may not show up directly since I have simplified the notation by suppressing the dependence of c_R on the initial history l and the particular initial utility V_l promised to the agent.

Notice also that $\{(V_l, l) : V_l \in V^{AP}(l)\}$ is endogenous to the model, so one may doubt the usefulness of defining the operator T on an unknown state space as well as the practical benefit of constraint (11). Further in this section, however, we will demonstrate that V^{AP} can be recovered from the primitives by a recursive procedure in the spirit of Abreu, Pearce and Stacchetti (1990).

[19] Of course, if $\theta = 0$, the reservation utilities will be constant at all nodes, so the initial history will be immaterial for the static contract. The state space will shrink to a single dimension; namely, the set of expected discounted utilities that can be promised to the agent will be the same at every node. Then, the history update will prove irrelevant since U will be defined on the one-dimensional V^{AP}.

[20] Note that we have made use of (9) when stating temporary incentive compatibility as (10). Namely, given (9) holds, temporary incentive compatibility is equivalent to (10).

[21] It is referred to as a re-generation constraint in Spear and Srivastava (1987).

the reservation level on any respectively updated initial history, i.e., $V_+(.,y) \geq \underline{V}_{l_+(l,y)}$, $\forall y \in Y$. In fact, constraints (9) and (11) guarantee the dynamic consistency of the series of static contracts generated by iterating on the operator T.

For any $l \in L$ and $V \in V^{AP}(l)$, we have that $U^{AP}(V,l)$ is compact by Proposition 3 (b). Then, we can define $U^{AP^*}(V,l) := \max U^{AP}(V,l)$ as the maximum utility the principal can get by signing an AP supercontract offering V to the agent. For any $V \in V^{AP}$, let $U^{AP^*}(V) = \left\{U^{AP^*}(V_l,l)\right\}$ be the vector of these maximum utilities indexed by initial history. Next, I will show that $U^{AP^*} : V^{AP} \to \mathbb{R}^{n^\theta}$ is the unique fixed point of the operator T and can be obtained as the limit of successively iterating on T. I start with a proposition that establishes some useful properties of U^{AP^*}.

Proposition 5 *For any $l \in L$, $U^{AP^*}(.,l)$ is usc and bounded on $V^{AP}(l)$.*

Note that these properties can directly be translated to U^{AP^*}, say with the sup metric over Y^θ.

Proposition 6 $T\left(U^{AP^*}\right) = U^{AP^*}$.

The proposition says that U^{AP^*} is a fixed point of the operator T.

For the purposes of the next proposition, I introduce some additional notation. Let $ß_l$ denote the space of bounded functions from $V^{AP}(l)$ to $\mathbb{R}$ endowed with the sup metric. For any $U', U'' \in \{USCB_l\}$, define the metric $\mu(U',U'') := \sup_{l\in L} \mu_l(U',U'')$, where $\mu_l(U',U'') := \sup_{V_l \in V^{AP}(l)} |U'_l(V_l) - U''_l(V_l)|$, $\forall l \in L$. Note that both suprema in the above definition are achieved.

Proposition 7 *(a) T maps $(\{USCB_l\},\mu)$ into itself; (b) T is a contraction mapping with modulus β_P in terms of the metric μ; (c) Let $\widetilde{U} \in (\{ß_l\},\mu)$: $T\left(\widetilde{U}\right) = \widetilde{U}$. Then, $\widetilde{U} = U^{AP^*}$; (d) $\forall U \in (\{USCB_l\},\mu)$, $\mu\left(T^n(U), U^{AP^*}\right) \underset{n\to\infty}{\to} 0$, where $T^n(U) := T(T^{n-1}(U))$ for any $n \in \mathbb{Z}_{++}$ with $T^0(U) := U$.*

This proposition shows that the fixed point of T is unique and can be obtained as a limit of successive iterations on T. Consequently, we can use standard dynamic programming techniques in order to solve for the optimal AP contract.

However, what we are ultimately interested in is solving for the optimal 2P contract. For this purpose, I resort again to dynamic programming using a method outlined by Rustichini (1998).

First, I will introduce some notation. For any $l \in L$ and $V_l \in V^{AP}(l)$, let $\Gamma_R(V_l, U, l) := \{c_R : (7) - (11)$ hold at (V_l, l) and $U_{l_+(l,y)}(V_+(y)) \geq \underline{U}_{l_+(l,y)}$, $\forall y \in Y\}$ for some function $U : V^{AP} \to (\mathbb{R} \cup \{-\infty\})^{n^\theta}$. Additionally, let

$$\Lambda_R(V_l, U, l) := \begin{cases} \Gamma_R(V_l, U, l) & \text{if } U_l(V_l) \geq \underline{U}_l \\ \Lambda_R(V_l, U, l) := \emptyset & \text{otherwise.} \end{cases}$$

Denote by $USCBA_l$ the space of usc, bounded from above functions from $V^{AP}(l)$ to $\mathbb{R} \cup \{-\infty\}$. Then, for any $U = \{U_l\}$ with $U_l \in USCBA_l$, $\forall l \in L$, define the operator $\underline{T}$ as follows. For any $V \in V^{AP}$, $\underline{T}(U)_{(V)} := \left\{\underline{T}_l(U)_{(V_l)}\right\}$, where

$$\underline{T}_l(U)_{(V_l)} := \max_{\substack{c_R \in \\ \Lambda_R(V_l, U, l)}} \{\sum_{y \in Y} [u(w_+(y), y) + \beta_P U_{l_+(l,y)}(V_+(y))] \pi(y, a_-)\}$$

following the convention that $\underline{T}_l(U)_{(V_l)} = -\infty$ if $\Lambda_R(V_l, U, l) = \emptyset$.

This operator encompasses the lower bounds on the utility of the principal in the form of additional constraints. The only difference with T is that in case U is lower than the reservation utility of the principal today or at any possible contingency tomorrow, $\underline{T}$ becomes $-\infty$. The idea is that any violation of the constraints in this stationary framework is punished severely making the contract in question non-optimal. What remains to be shown is that iterating on this operator will indeed lead us to the optimal dynamic contract.

Proposition 8 $\underline{T}$ *maps* $\{USCBA_l\}$ *into itself.*

For any $V \in V^{AP}$, let $D_0(V) := U^{AP^*}(V)$ and $D_{i+1}(V) := \underline{T}(D_i)$, $\forall i \in \mathbb{Z}_+$. Note that by Proposition 8 and the fact that $\Gamma_R(V_l, U, l)$ is compact if non-empty for any $V_l \in V^{AP}(l)$, $U \in \{USCBA_l\}$, and $l \in L$ (trivial), D_i is well defined on V^{AP} for any $i \in \mathbb{Z}_+$.

Proposition 9 *(a)* $\{D_i\}_{i=1}^{\infty}$ *is a weakly decreasing sequence and* $\exists D_\infty \in \{USCBA_l\} : D_i(V_l, l) \underset{i \to \infty}{\to} D_\infty(V_l, l)$, $\forall V_l \in V^{AP}(l)$, $\forall l \in L$; *(b)* $\underline{T}(D_\infty) = D_\infty$; *and (c) if* $\exists D' \in \{USCBA_l\} : \underline{T}(D') = D'$, *then* $D' \leq D_\infty$.

This proposition says that if we start iterating on the operator $\underline{T}$ taking U^{AP^*} as an initial guess, we will ultimately converge (pointwise) to D_∞, the largest fixed point of $\underline{T}$. Next, I establish the relationship between U^* and D_∞.

In the subsequent analysis it will be useful to extend U^* on V^{AP}. For any $V \in V^{AP}$, let $\widehat{U}^*(V) := \left\{\widehat{U}^*(V_l, l)\right\}$ with $\widehat{U}^*(V_l, l) := U^*(V_l, l)$ if $V_l \in V^{2P}(l)$ and $\widehat{U}^*(V_l, l) := -\infty$ otherwise.

Proposition 10 $\underline{T}\left(\widehat{U}^*\right) = \widehat{U}^*$.

This proposition establishes that the extension of U^* on V^{AP} is a fixed point of $\underline{T}$. What remains to be shown is how to recover U^* from D_∞. The next proposition gives the answer.

Proposition 11 *For any* $V \in V^{AP}$, $\widehat{U}^*(V) = D_\infty(V)$.

The proposition provides a straight-forward method of solving for the optimal 2P supercontract. After we have found the optimal AP contract we take it as an initial guess and start iterating on the operator $\underline{T}$ until convergence is reached. Note that convergence here is pointwise and is meant to be on $\mathbb{R} \cup \{-\infty\}$. After we have obtained the limit function D_∞, we can recover the set of possible values for the expected discounted utility of the agent signing a 2P contract by taking the subset of the domain of D_∞ on which the limit function takes finite values. More precisely, for any $l \in L$ we can restrict ourselves only to values of $D_\infty(., l)$ above $\underline{U}_l$. Formally, $V^{2P}(l) := \left\{V \in V^{AP}(l) : D_\infty(V, l) \geq \underline{U}_{-\theta_y}\right\}$. Then, for any$V \in V^{2P}(l)$, we have $U^*(V; l) = D_\infty(V, l)$.

However, note that the state space of the recursive problem constructed for computing the optimal AP contract, V^{AP}, is endogenous. Nevertheless, it is the largest fixed point of a set operator and can be obtained through successive iterations in a procedure introduced by Abreu, Pearce and Stacchetti (1990).

Choose some $\widehat{V} \in \mathbb{R} : \widehat{V} \geq \max_{l \in L}\left\{\max V^{AP}(l)\right\}$, where the right-hand side of the inequality is well defined given $V^{AP}(l)$ compact, $\forall l \in L$ and L finite. Note that given Assumption *3*, $\left[\underline{V}_l, \widehat{V}\right] \neq \emptyset$, $\forall l \in L$. Then, for any $X = \{X_l\} : X_l \in \mathbb{R}$, $\forall l \in L$ let $B(X) := \{B_l(X)\}$ with

$$B_l(X) := \{V_l \in \left[\underline{V}_l, \widehat{V}\right] : \exists c_R : (7) - (10) \text{ and } (12) \text{ hold at } (V_l, l)\},$$

where (12) is defined as:

$$V_+(V_l, y) \in X_{l_+(l,y)} \cap \left[\underline{V}_{l_+(l,y)}, +\infty\right), \forall y \in Y \tag{12}$$

Note that $B_l(X)$ gives the set of agent's initial utilities that are not below the reservation level $\left(\text{which follows from } V_l \in \left[\underline{V}_l, \widehat{V}\right]\right)$ and that can be supported by a single-round (stationary) contract at l that is admissible [i.e., (7) and (8) hold], consistent [i.e., satisfies (9)], temporary incentive-compatible [i.e., satisfies (10)] and has continuation utilities which are taken from X and are not below the relevant reservation level [i.e., (12) holds]. In short, B maps continuation utilities to relevant initial utilities. It is this operator that will help us recover the endogenous state space of T, V^{AP}.

Proposition 12 *(a) $B\left(V^{AP}\right) = V^{AP}$; and (b) if $\exists X \subset \mathbb{R}^{n^\theta} : B(X) = X$, then $X \subset V^{AP}$.*

This proposition establishes that the set of agent's expected discounted utilities supportable by an AP supercontract, V^{AP}, is the largest fixed point of B.

Proposition 13 *Let X_0 compact : $V^{AP} \subset X_0 \subset \mathbb{R}^{n^\theta}$ and $B(X_0) \subset X_0$. Define $X_{i+1} := B(X_i)$, $\forall i \in \mathbb{Z}_+$. Then, $X_{i+1} \subset X_i$, $\forall i \in \mathbb{Z}_+$ and $X_\infty := \lim_{i\to\infty} X_i = V^{AP}$.*

The proposition says that if we start iterating on B taking as an initial guess some compact set X_0 that contains both $B(X_0)$ and V^{AP}, we will ultimately converge to the largest fixed point of the operator, V^{AP}. This is sufficient for obtaining V^{AP} since we can always take $X_0 = \{X_{0,l}\} : \left[\underline{V}_l, \widehat{V}\right] \subset X_{0,l} \subset \mathbb{R}$ with $X_{0,l}$ compact, $\forall l \in L$. However, an even more computationally efficient result exists.

Let us modify the operator B as follows. For any $X = \{X_l\} : X_l \in \mathbb{R}, \forall l \in L$ let $\widetilde{B}(X) := \left\{\widetilde{B}_l(X)\right\}$ with

$$\widetilde{B}_l(X) := \{V_l \in X_l : \exists c_R : \text{(7)-(10) and (13) hold at } (V_l, l)\},$$

where (13) is defined as:

$$V_+(y) \in X_{l_+(l,y)}, \forall y \in Y \tag{13}$$

Note that the operator $\widetilde{B}$ does not require that the agent should commit to the contract. Namely, we do not impose a constraint keeping the continuation values for the utility of the agent above the lower bound given by the reservation utility. From a computational point of view, we are increasing the efficiency since we are relaxing the set of constraints.

Proposition 14 *(a) Take* $\widetilde{X}_0 := \left\{\widetilde{X}_{0,l}\right\}$ *with* $\widetilde{X}_{0,l} = \left[\underline{V}_l, \widehat{V}\right]$, $\forall l \in L$ *and let* $\widetilde{X}_{i+1} := \widetilde{B}\left(\widetilde{X}_i\right)$, $\forall i \in \mathbb{Z}_+$. *Then,* $\widetilde{X}_{i+1} \subset \widetilde{X}_i$, $\forall i \in \mathbb{Z}_+$ *and* $\widetilde{X}_\infty := \lim_{i\to\infty} \widetilde{X}_i = V^{AP}$. *(b)* $\widetilde{B}\left(V^{AP}\right) = V^{AP}$*; and (c) if* $\exists X : \emptyset \neq X \subset \widetilde{X}_0$ *and* $\widetilde{B}(X) = X$, *then* $X \subset V^{AP}$.

This proposition outlines a practical way of obtaining V^{AP}. Namely, we start with the set $\left\{\left[\underline{V}_l, \widehat{V}\right]\right\}$ and iterate on the set operator $\widetilde{B}$ until convergence in a properly defined sense is attained. Note that we can always take $\widehat{V} = \frac{\nu(\max\{W\},\min\{A\})}{1-\beta_A}$.

4 Conclusion

This paper builds a framework for analyzing dynamic moral hazard problems characterized by limited commitment and history-dependent reservation utilities. This is achieved by constructing an equivalent recursive representation that is stationary on a properly defined state space. The state space which contains the expected discounted utilities of the agent on one dimension and the initial histories on the other is characterized by a generalized Bellman equation. Given the state space, the optimal AP contract is recursively characterized by standard dynamic programming routines on bounded usc functions and in the same time is used as an initial guess for the optimal 2P in a procedure severely punishing any violation of the principal's participation constraint.

This general setting can be used to address multiple dynamic problems including but not limited to executive compensation, stock option packages, tenure decisions, optimal insurance, and investment. It would also be interesting to try to endogenize the external options in a model directly providing the link between fundamentals/beliefs and reservation utilities.

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APPENDIX

Lemma 1 *Let (1) and (2) hold after $l \in Y^{\theta}$. Then, (3) $\Leftrightarrow$*

$$V_0\left(a, w, l\right) \geq V_0\left(a', w, l\right), \forall (a' : \forall nai\left(l\right), \, a'_t \in A) \tag{14}$$

Proof. It is trivial to show (3)$\Rightarrow$(14). Just take $\tau = 0$. In the other direction, let (14) hold, but assume that (3) is not satisfied, i.e., there is a node $y^{\tau-1}$ s.t. $\exists a'$ admissible $\forall \text{nai}\left(y^{\tau-1}\right)$ and $V_\tau\left(a', w, y^{\tau-1}\right) > V_\tau\left(a, w, y^{\tau-1}\right)$. Let a'' : $\forall \text{nai}\left(y^{\tau-1}\right)$, $a''_t = a'_t$, with $a''_t = a_t$ elsewhere. Given (1) and (2), ν is continuous on a compact set and, therefore bounded. Then, we obtain:

$$\begin{gathered} V_0\left(a'', w, l\right) = \\ \sum_{t=0}^{\tau-1} \beta_A^t \sum_{y_t \in Y} ... \sum_{y_0 \in Y} \nu\left(w_t\left(y^t\right), a''_t\left(y^{t-1}\right)\right) \prod_{i=0}^{t} \pi\left(y_i, a''_i\left(y^{i-1}\right)\right) + \\ \beta_A^\tau \sum_{y_{t-1} \in Y} ... \sum_{y_0 \in Y} V_\tau\left(a'', w, y^{\tau-1}\right) \prod_{i=0}^{t-1} \pi\left(y_i, a''_i\left(y^{i-1}\right)\right) \\ > V_0\left(a, w, l\right), \end{gathered} \tag{A1}$$

where the inequality follows from the construction of a'' since $V_\tau\left(a', w, y^{\tau-1}\right) > V_\tau\left(a, w, y^{\tau-1}\right)$ and $\pi > 0$ by Assumption *1*. Given that a'' is admissible after l by construction, (A1) contradicts (14). ■

This proposition shows that incentive compatibility at an initial node ${}^{-\theta}y$ is equivalent to incentive compatibility at all the nodes following l.

Proof of Proposition 1. It is trivial that (3) implies (6). In the other direction, assume (6) holds at every node, but $\exists$ an admissible plan $a' : V_0\left(a', w, l\right) > V_0\left(a', w, l\right)$. We have:

$$\begin{gathered} V_0\left(a', w, l\right) = \\ \sum_{t=0}^{T} \beta_A^t \sum_{y_t \in Y} ... \sum_{y_0 \in Y} \nu\left(w_t\left(y^t\right), a'_t\left(y^{t-1}\right)\right) \prod_{i=0}^{t} \pi\left(y_i, a'_i\left(y^{i-1}\right)\right) + \\ \beta_A^{T+1} \sum_{y_T \in Y} ... \sum_{y_0 \in Y} V_{T+1}\left(a', w, y^T\right) \prod_{i=0}^{T} \pi\left(y_i, a'_i\left(y^{i-1}\right)\right), \end{gathered}$$

where the second term on the right-hand side can be made arbitrarily small by choosing T big enough given (1), (2) and the assumptions on β_A, ν, A, W. Therefore, $\exists T \in \mathbb{Z}_+$ and an admissible plan $a'' : a''_t\left(y^{t-1}\right) = a'_t\left(y^{t-1}\right)$, $\forall y^{t-1} \in l \times Y^t$, $\forall t \leq T$, and $a''_t = a_t$ elsewhere, s.t. $V_0\left(a'', w, l\right) > V_0\left(a, w, l\right)$ Then, take $\tau \in \mathbb{Z}_+ : \tau \leq T$ s.t. $\exists y^{\tau-1} : a''_\tau\left(y^{\tau-1}\right) \neq a_\tau\left(y^{\tau-1}\right)$ and $\nexists \tau' \in \mathbb{Z}_{++} : \tau < \tau' \leq T : a''_{\tau'}\left(y^{\tau'-1}\right) \neq a_{\tau'}\left(y^{\tau'-1}\right)$ for some $y^{\tau'-1} \in l \times Y^{\tau'}$. If we define an admissible plan $a''' : a'''_\tau\left(y^{\tau-1}\right) = a_\tau\left(y^{\tau-1}\right), \forall y^{\tau-1} \in l \times Y^\tau$ and $a'''_t = a''_t$ elsewhere, by (6) at $\forall y^{\tau-1} \in l \times Y^\tau$, we have that $V_\tau\left(a''', w, y^{\tau-1}\right) \geq V_\tau\left(a'', w, y^{\tau-1}\right)$, from where $V_0\left(a''', w, l\right) \geq V_0\left(a'', w, l\right)$. Proceeding in this way we can eliminate all the deviations (note that $\tau \in \mathbb{Z}_+ : \tau \leq T$) to obtain $V_0\left(a, w, l\right) \geq V_0\left(a'', w, l\right)$, i.e., a contradiction. Therefore, we obtain (6)$\Rightarrow$(14), which by Lemma 1 results in (6)$\Rightarrow$(3). ■

For any $l \in L$, let $C_l := \{c : \text{(1) and (2) hold after } l\}$.

Proof of Proposition 2. (a) Fix $l \in L$. Take $\tau', \tau'' \in \mathbb{Z}_+ : \tau' \leq \tau''$ and arbitrary $y^{\tau'-1} \in Y^{\tau'} \times l$ and $y^{\tau''-1} \in Y^{\tau''} \times l$. Take an arbitrary $V' \in V^{2P}_{\tau'}\left(y^{\tau'-1}\right)$. Then, there exists a contract $c' = (a', w') \in \Gamma_{y^{\tau'-1}} : V_{\tau'}\left(c', y^{\tau'-1}\right) = V'$. Define $c'' = (a'', w'')$ such that for any $\widetilde{y}^t \in y^{\tau''-1} \times Y^{t-\tau''+1}$ with $t \geq \tau''$, $a''_t\left(\widetilde{y}^{t-1}\right) := a'_{\tau'+t-\tau''}\left(y^{\tau'-1}, \widetilde{y}_{\tau''}, ..., \widetilde{y}_{t-1}\right)$, $w''_t\left(\widetilde{y}^t\right) := w'_{\tau'+t-\tau''}\left({}^{-\theta}y^{\tau'-1}, \widetilde{y}_{\tau''}, ..., \widetilde{y}_t\right)$. It is straight-forward that $V_{\tau''}\left(c'', y^{\tau''-1}\right) = V'$ and $c'' \in \Gamma_{y^{\tau''-1}}$. Therefore, we have that $V' \in V^{2P}_{\tau''}\left(y^{\tau''-1}\right)$. The same argument holds in the other direction, so we have proven that $V^{2P}_{\tau''}\left(y^{\tau'-1}\right) = V^{2P}_{\tau''}\left(y^{\tau''-1}\right)$.

Fix $l \in L$. $V^{2P}(l)$ is bounded given (1) and (2). Regarding the compactness of $V^{2P}(l)$, we should also prove that it is closed. Take an arbitrary convergent sequence $\{V_i\}_{i=1}^\infty : V_i \in V^{2P}(l)$, $\forall i \in \mathbb{Z}_{++}$ with limit V_∞. We need to show that $V_\infty \in V^{2P}(l)$. By the construction of the sequence, for any $i \in \mathbb{Z}_{++}$, $\exists c_i \in \Gamma_l$ such that $V_0(c_i, l) = V_i$. Then, for any $i \in \mathbb{Z}_{++}$, $c_i \in C_l$. Let us endow C_l with the product topology. C_l is compact as a product of compact spaces. Consequently, there exists a convergent subsequence $\{c_{i_k}\}_{k=1}^\infty$ of $\{c_i\}_{i=1}^\infty$ such that $c_\infty := \lim_{k\to\infty}(c_{i_k}) \in C_l$, from where c_∞ satisfies (1) and (2) after l. For any $T \in \mathbb{Z}_+ : T \geq \tau$, let $V^T_\tau(c, y^{\tau-1}) := \sum_{t=\tau}^{T} \beta_A^{t-\tau} \sum_{y_t \in Y} ... \sum_{y_\tau \in Y} [\nu\left(w_t\left(y^t\right), a_t\left(y^{t-1}\right)\right)] \prod_{i=\tau}^{t} \pi\left(y_i, a_i\left(y^{i-1}\right)\right)$. Notice that $V_\tau(c, y^{\tau-1}) - V^T_\tau(c, y^{\tau-1}) = \beta_A^{T+1} \sum_{y_T \in Y} ... \sum_{y_\tau \in Y} V_{T+1}\left(c, y^T\right) \prod_{i=\tau}^{T} \pi\left(y_i, a_i\left(y^{i-1}\right)\right) \in [\beta_A^{T+1} \frac{\upsilon(\min W, \max A)}{1-\beta_P}, \beta_A^{T+1} \frac{\upsilon(\max W, \min A)}{1-\beta_P}]$, $\forall T \in \mathbb{Z}_+ : T \geq \tau$, $\forall c \in C_l$, $\forall y^{\tau-1} \in$

$l\times Y^{\tau}$, $\forall\tau\in\mathbb{Z}_{+}$. Moreover, $V_{\tau}^{T}(.,y^{\tau-1})$ is continuous on C_l. Then, $V_{\tau}(.,y^{\tau-1})$ is continuous on C_l. Analogously, we can show that $U_{\tau}(.,y^{\tau-1})$ is continuous on C_l. As a result, we have that c_{∞} satisfies (4), (5), (6) after l and $V_0\left(c_{\infty},l\right)=V_{\infty}$.

Following the same logic, we can show that $V_{\tau}^{AP}\left(y^{\tau-1}\right)$ is time invariant and compact and depends only on the last θ observations prior to signing.

(b) Analogous to the proof of (a). ■

Proof of Proposition 3. Fix $l\in L$. We have $\Gamma_l\subset C_l$. Let's endow C_l with a metric inducing the product topology. Then, following the argument of the proof of Proposition 2, we obtain that Γ_l is compact and $U_0\left(.,l\right)$ is continuous on C_l. ■

Proof of Proposition 4. Fix $l\in L$. By Proposition 3, we have that $\exists c\in\Gamma_l$ and $U_0\left(c,l\right)=U_l^{**}$. Let $V^{**}:=V_0\left(c,l\right)$. By Proposition 1, $V^{**}\in V^{2P}\left(l\right)$ and $U_l^{**}\in U\left(V^{**},l\right)$. Therefore, $U_l^{*}\geq U_l^{**}$. Suppose $U_l^{*}>U_l^{**}$. Then, $\exists V^{*}\in V^{2P}\left(l\right):U_l^{**}<U^{*}\left(V^{*},l\right)\leq U_l^{*}$. Since $U^{*}\left(V^{*},l\right)\in U\left(V^{*},l\right)$, $\exists c^{*}\in\Gamma_l$, $V_0\left(c^{*},l\right)=V^{*}$ and $U_0\left(c^{*},l\right)=U^{*}\left(V^{*},l\right)$. Then, by the definition of U_l^{**} and Proposition 1 we have that $U_l^{**}\geq U^{*}\left(V^{*},l\right)$, i.e., a contradiction is reached. Consequently, $U_l^{*}=U_l^{**}$ and the supremum in the definition of U_l^{*} is achieved. ■

For any $l\in L$ and $\forall V\in V^{AP}\left(l\right)$, define $\Gamma_l^{AP}\left(V\right):=\left\{c:(1),(2),(4),(6)\text{ hold after }l\text{ and }V_0\left(c,l\right)=V\right\}$ and $G_l^{AP}\left(V\right):=\left\{c\in\Gamma_l^{AP}\left(V\right):U_0\left(c,l\right)=U^{AP^{*}}\left(V,l\right)\right\}$.

Lemma 2 *For any $l\in L$, $\Gamma_l^{AP}\left(.\right)$ is upper hemi-continuous (uhc) on $V^{AP}\left(l\right)$.*

Proof. Fix $l\in L$ and $V\in V^{AP}\left(l\right)$ and note that $\Gamma_l^{AP}\left(V\right)$ is non-empty and compact. Take a sequence $\left\{V_i\right\}_{i=1}^{\infty}$ s.t. $V_i\in V^{AP}\left(l\right),\forall i\in\mathbb{Z}_{++}$ and $V_i\underset{i\to\infty}{\to}V$. Let $c_i\in\Gamma_l^{AP}\left(V_i\right)$, $\forall i\in\mathbb{Z}_{++}$. Note that $\Gamma_l^{AP}\left(V_i\right)\subset C_l$, $\forall i\in\mathbb{Z}_{++}$ with C_l compact . Then, $\exists$ a subsequence $\left\{c_{i_j}\right\}_{j=1}^{\infty}$ of $\left\{c_i\right\}_{i=1}^{\infty}:c_{i_j}\underset{j\to\infty}{\to}c\in C_l$. Since $V_{\tau}\left(.,^{-\theta}y^{\tau-1}\right)$ is continuous on C_l, c satisfies (4) and (6) after l and $V_0\left(c,l\right)=V$. Therefore, $c\in\Gamma_l^{AP}\left(V\right)$. ■

Proof of Proposition 5. Fix $l\in L$ and $V\in V^{AP}\left(l\right)$. Take a sequence $\left\{V_i\right\}_{i=1}^{\infty}$ s.t. $V_i\in V^{AP}\left(l\right),\forall i\in\mathbb{Z}_{++}$ and $V_i\underset{i\to\infty}{\to}V$. Let $c_i\in G_l^{AP}\left(V_i\right)$, $\forall i\in\mathbb{Z}_{++}$. Define $\overline{U_l^{AP^{*}}}:=\overline{\lim_{i\to\infty}}U^{AP^{*}}\left(V_i,l\right)$. $\exists$ a subsequence $\left\{c_{i_j}\right\}_{j=1}^{\infty}$ of $\left\{c_i\right\}_{i=1}^{\infty}:\lim_{j\to\infty}U_0\left(c_{i_j},l\right)=\overline{U_l^{AP^{*}}}$. Since $G_l^{AP}\left(.\right)\subset\Gamma_l^{AP}\left(.\right)$ and $\Gamma_l^{AP}\left(.\right)$ is uhc from Lemma 2, $\exists$ a subsequence $\left\{c_{i_{j_n}}\right\}_{n=1}^{\infty}$ of $\left\{c_{i_j}\right\}_{j=1}^{\infty}:c_{i_{j_n}}\underset{n\to\infty}{\to}c$ with $c\in\Gamma_l^{AP}\left(V\right)$. Then, $\overline{U_l^{AP^{*}}}=\lim_{n\to\infty}U_0\left(c_{i_{j_n}},l\right)=U_0\left(c,l\right)\leq U^{AP^{*}}\left(V,l\right)$ where the first equality comes from the fact that $\{c_{i_{j_n}}\}_{n=1}^{\infty}$ is a subsequence of $\{c_{i_j}\}_{j=1}^{\infty}$ and $\lim_{j\to\infty} U_0\left(c_{i_j}, l\right) = \overline{U_l^{AP^*}}$, the second follows from the continuity of $U_0\,(.,l)$ and the third obtains directly from $c \in \Gamma_l^{AP}(V)$ and the definition of $U^{AP^*}(V,l)$. Therefore, $U^{AP^*}(.,l)$ is usc on $V^{AP}(l)$.

Regarding the boundedness of $U^{AP^*}(.,l)$, note that for any $V \in V^{AP}(l)$, $U^{AP^*}(V,l) = U_0(c_V, l)$ for some $c_V \in \Gamma_l^{AP}(V) \subset C_l$ with C_l non-empty and compact. Since $U_0(.,l) : C_l \to \mathbb{R}$ is continuous on a compact set, it is also bounded. Consequently, $U^{AP^*}(.,l)$ is bounded on $V^{AP}(l)$. ■

Lemma 3 *Fix arbitrary $l \in L$ and $V \in V^{AP}(l)$, and let $c \in G_l^{AP}(V)$. Then,* $U_\tau\left(c, ., \widetilde{l}^{-1}\right) = U^{AP^*}\left(V_\tau\left(c^*, ., \widetilde{l}^{-1}\right), \widetilde{l}^{-1}\right)$, *$\forall nai(l)$.*

Proof. Note that *$\forall nai(l)$,* $V_\tau\left(c, ., \widetilde{l}^{-1}\right) \in V^{AP}\left(\widetilde{l}^{-1}\right)$ and, therefore, $U^{AP^*}\left(V_\tau\left(c, ., \widetilde{l}^{-1}\right), \widetilde{l}^{-1}\right)$ is well defined. Since for $\tau = 0$, the result is trivial, take arbitrary $\tau \in \mathbb{Z}_{++}$ and $y^{\tau-1} = \left(y^{\tau-\theta-1}, \widetilde{l}^{-1}\right) \in l \times Y^\tau$, and assume that the lemma does not hold. Then, there exists a supercontract $c' \in \Gamma_{\widetilde{l}^{-1}}^{AP}\left(V_\tau\left(c, y^{\tau-1}\right)\right) : U_0\left(c', \widetilde{l}^{-1}\right) > U_\tau\left(c, y^{\tau-1}\right)$. Let us construct a supercontract c'' after l s.t. $\left(a_t''\left(y^{t-1}\right), w_t''\left(y^{t-1}, y_t\right)\right) = (a_{t-\tau}'(\widetilde{l}^{-1}, y_\tau, ..., y_{t-1}),$ $w_{t-\tau}'(\widetilde{l}^{-1}, y_\tau, ..., y_{t-1}, y_t))$, $\forall \text{nai}\left(y^{\tau-1}\right)$, with $\left(a_t''\left(y^{t-1}\right), w_t''\left(y^{t-1}, y_t\right)\right) = \left(a_t\left(y^{t-1}\right), w_t\left(y^{t-1}, y_t\right)\right)$ elsewhere. By the definition of c and the construction of c'' we have that c'' satisfies (1), (2), (4), (6) after l and $V_0(c'', l) = V_0(c, l) = V$. Then, $U_0(c'', l) \in U^{AP}(V, l)$. However, since $U_\tau\left(c'', y^{\tau-1}\right) > U_\tau\left(c, y^{\tau-1}\right)$, we have that $U_0(c'', l) > U_0(c, l)$, which contradicts the fact that $U_0(c, l) = U^{AP^*}(V, l)$. ■

The lemma says that at any contingency, the expected discounted utility of the principal who has signed the AP supercontract maximizing his/her utility at period 0 while guaranteeing the agent particular initial expected discounted utility also gives the maximum initial utility the principal can get by signing a new AP supercontract guaranteeing the agent an initial utility equal to the utility the agent would receive in that contingency under the previous contract. In other words, at the optimum the principal can neither lose nor gain by breaching the original contract and signing a new one guaranteeing the same utility stream to the agent.

For any $l \in L$ and $V_l \in V^{AP}(l)$, define $\Gamma_R^{AP}(V_l, l) := \{c_R : (7) - (11) \text{ hold at } (V_l, l)\}$.

Proof of Proposition 6. Take an arbitrary $V = \{V_l\} \in V^{AP}$. Fix $l \in L$. Given the existence of $U^{AP^*}(V_l, l)$, $\exists\ c \in \Gamma_l : V_0(c, l) = V_l$ and $U_0(c, l) =$

$U^{AP^*}(V_l, l)$. For any $y \in Y$, let $a_- := a_0(l)$, $w_+(y) := w_0(l, y)$, and $V_+(y) := V_1(c, l, y)$. Then, we immediately have that (9) holds. Moreover, (1) $\Rightarrow$ (7), (2) $\Rightarrow$ (8), (6) $\Rightarrow$ (10). As in the proof of Proposition 2 (a), for any $y \in Y$, we can construct $c'_y \in \Gamma_{l_+(l,y)} : V_0\left(c'_y, l_+(l, y)\right) = V_1(c, l, y)$, from where (11) also holds. By Lemma 3, for any $y \in Y$, we have $U_1(c, l, y) = U^{AP^*}(V_1(c, l, y), l_+(l, y))$ $= U^{AP^*}(V_+(y), l_+(l, y))$. Consequently, $U^{AP^*}(V_l, l) = U_0(c, l) = \sum_{y\in Y}[u(w_0(l, y), y) + \beta_P U_1(c, l, y)]\pi(y, a_0(l)) = \sum_{y\in Y}[u(w_+(y), y) + \beta_P U^{AP^*}(V_+(y), l_+(l, y))]\pi(y, a_-)$, where U^{AP^*} is usc and bounded from Proposition 5. Then, by the definition of $T(.)$, we have that $T_l\left(U^{AP^*}\right)_{(V_l)} \geq U^{AP^*}(V_l, l)$. Since V and l were chosen randomly, the result generalizes to $T\left(U^{AP^*}\right) \geq U^{AP^*}$.

Fix arbitrary $V = \{V_l\} \in V^{AP}$ and $l \in L$. We have demonstrated above that $\Gamma_R^{AP}(V_l, l) \neq \emptyset$. Then, since $\Gamma_R^{AP}(V_l, l)$ can be shown to be compact and U^{AP^*} is usc and bounded, there exists $c_R^* \in \Gamma_R^{AP}(V_l, l) : T_l\left(U^{AP^*}\right)_{(V_l)} = \sum_{y\in Y}[u\left(w_+^*(y), y\right) + \beta_P U^{AP^*}\left(V_+^*(y), l_+(l, y)\right)]\pi\left(y, a_-^*\right)\}$. By (11), for any $y \in Y$, $V_+^*(y) \in V^{AP}(l_+(l, y))$, from where there exists $c_y^* \in \Gamma_{l_+(l,y)}^{AP}\left(V_+^*(y)\right) :$ $U_0\left(c_y^*, l_+(l, y)\right) = U^{AP^*}\left(V_+^*(y), l_+(l, y)\right)$. Then, let c^{**} be a supercontract s.t.: $\left(a_0^{**}(l), w_0^{**}(l, y)\right) = \left(a_-^*, w_+^*(y)\right)$ and $\forall$nai(l, y), $\left(a_t^{**}(l, y, .), w_t^{**}(l, y, .)\right) = (a_{y,t-1}^*(l_+(l, y), .), w_{y,t-1}^*(l_+(l, y), .))$, $\forall y \in Y$. It is immediate that c^{**} satisfies (1), (2), (4), (6) after (l, y), $\forall y \in Y$. Moreover, (7)$\Rightarrow$ $a_0^{**}(l) \in A$, (8)$\Rightarrow$ $w_0^{**}(l, y) \in W$, $\forall y \in Y$. By construction and (10), we have that (6) holds at l. By (9), we obtain that $V_0(c^{**}, l) = V_l \in V^{AP}(l)$, from where (4) is satisfied at l. Finally, we have that $T_l\left(U^{AP^*}\right)_{(V_l)} = U_0(c^{**}, l) \in U^{AP}(V_l, l)$, from where $U^{AP^*}(V_l, l) \geq T_l\left(U^{AP^*}\right)_{(V_l)}$. As before, this immediately generalizes to $T\left(U^{AP^*}\right) \geq U^{AP^*}$. ■

Proof of Proposition 7. (a) Analogously to the proof of Lemma 2, we can show that for any $l \in L$, $\Gamma_R^{AP}(., l)$ is uhc on $V^{AP}(l)$. Then, following an argument similar to the proof of Proposition 5, we conclude that $T(U)_{(.)}$ is usc on V^{AP}. It is trivial to show that $T(U)_{(.)}$ is also bounded.

(b) The result follows by the argument of Theorem 3.3 in Stokey and Lucas (1989) since it is trivial that T satisfies the Blackwell's sufficient conditions.

(c) Assume on the contrary that $\mu\left(\widetilde{U}, U^{AP^*}\right) > 0$. We have that $\mu\left(\widetilde{U}, U^{AP^*}\right) = \mu\left(T\left(\widetilde{U}\right), T\left(U^{AP^*}\right)\right) \leq \beta_P \mu\left(\widetilde{U}, U^{AP^*}\right)$, where the equality follows from the fact that both $\widetilde{U}$ and U^{AP^*} are fixed points of T (the first - by assumption, the second - by Proposition 6) and the inequality obtains by (b). However, this contradicts $\beta_P \in (0,1)$. Consequently, $\mu\left(\widetilde{U}, U^{AP^*}\right) = 0$.

(d) Since by (a) T maps $(\{USCB_l\}, \mu)$ into itself, the existence of $T^n(U)$ is guaranteed for any $n \in \mathbb{Z}_+$. Using Proposition 6 and successively applying (b), we obtain $\mu\left(T^n(U), U^{AP^*}\right) \leq \beta_P^n \mu\left(U, U^{AP^*}\right)$. Note that $\mu\left(U, U^{AP^*}\right) < \infty$ since U is bounded by assumption and U^{AP^*} is bounded by Proposition 5. Therefore, given $\beta_P \in (0,1)$, the result follows. ■

Proof of Proposition 8. Take arbitrary $U \in \{USCBA_l\}$, $l \in L$, $V_\infty \in V^{AP}(l)$ and $\{V_i\}_{i=1}^\infty$ s.t. $V_i \in V^{AP}(l)$, for any $i \in \mathbb{Z}_{++}$ and $V_i \underset{i\to\infty}{\to} V_\infty$. If $\overline{\lim_{i\to\infty}} \underline{T}_l(U)_{(V_i)} = -\infty$, the result is trivial. If $\overline{\lim_{i\to\infty}} \underline{T}_l(U)_{(V_i)} > -\infty$, we can always extract a subsequence $\{V_{i_k}\}_{k=1}^\infty$ of $\{V_i\}_{i=1}^\infty$ s.t. $\underline{T}_l(U)_{\left(V_{i_k}\right)} > -\infty$, $\forall k \in \mathbb{Z}_{++}$ and $\lim_{k\to\infty} \underline{T}_l(U)_{\left(V_{i_k}\right)} = \overline{\lim_{i\to\infty}} \underline{T}_l(U)_{(V_i)}$. Since $\Lambda_R(V_{i_k}, U, l) \neq \emptyset$, $\forall k \in \mathbb{Z}_{++}$, we can apply the argument used in the proof of Proposition 5 to obtain $\overline{\lim_{i\to\infty}} \underline{T}_l(U)_{(V_i)} \leq \underline{T}_l(U)_{(V_\infty)}$. ■

Proof of Proposition 9. (a) Notice that $U^{AP^*} \in \{USCB_l\} \subset \{USCBA_l\}$. Then, directly from the definition of T and $\underline{T}$, we have $\underline{T}\left(U^{AP^*}\right) \leq T\left(U^{AP^*}\right) = U^{AP^*}$, where the equality follows from Proposition 6. Since $\underline{T}_l$ is monotonic for any $l \in L$, $\{D_i\}_{i\in Z_+}$ is a weakly decreasing sequence of bounded from above usc functions, therefore $\exists D_\infty \in \{USCBA_l\} : D_i(V_l, l) \underset{i\to\infty}{\to} D_\infty(V_l, l)$, $\forall V_l \in V^{AP}(l)$, $\forall l \in L$.

(b) First we are going to prove $\underline{T}(D_\infty) \geq D_\infty$. Fix $l \in L$ and $V_l \in V^{AP}(l)$. Let us assume that $D_\infty(V_l, l) > -\infty$ because otherwise the result is trivial. Since $D_\infty(V_l, l)$ is a limit of a weakly decreasing sequence, we have that $D_i(V_l, l) > -\infty$, $\forall i \in \mathbb{Z}_+$. Consequently, $D_i(V_l, l) \geq \underline{U}_l$, $\forall i \in \mathbb{Z}_+$ since $D_i(V_l, l) < \underline{U}_l \Rightarrow \Lambda_R(V_l, D_i, l) = \emptyset \Rightarrow D_{i+1}(V_l, l) = -\infty$. This immediately implies that $D_\infty(V_l, l) \geq \underline{U}_l$. Moreover, $\Gamma_R(V_l, D_{i-1}, l) \neq \emptyset$, $\forall i \in \mathbb{Z}_{++}$ since if $\Gamma_R(V_l, D_{i-1}, l) = \emptyset$, we would have $D_i(V_l, l) = -\infty$. Then, for any $i \in \mathbb{Z}_{++}$, since D_{i-1} is usc and $\Gamma_R(V_l, D_{i-1}, l)$ is compact (trivial given D_{i-1} is usc), there exists a contract $c_{R,i} \in \Gamma_R(V_l, D_{i-1}, l)$ such that $D_i(V_l, l) = \sum_{y\in Y} [u(w_{+,i}(y), y) + \beta_P D_{i-1}(V_{+,i}(y), l_+(l, y))]\pi(y, a_{-,i}) \geq \underline{U}_l$. Since for any $i \in \mathbb{Z}_{++}$, $\Gamma_R(V_l, D_{i-1}, l) \subset \Gamma_R^{AP}(V_l, l)$ and $\Gamma_R^{AP}(V_l, l)$ is compact, $\exists$ a convergent subsequence of $\{c_{R,i}\}_{i=1}^\infty$, $\{c_{R,i_k}\}_{k=1}^\infty$, s.t. $c_{R,\infty} := \lim_{k\to\infty} c_{R,i_k} \in \Gamma_R^{AP}(V_l, l)$. Fix an arbitrary $y \in Y$. Then, we have:

$$D_\infty(V_{+,\infty}(y), l_+(l, y)) =$$

$$\lim_{j\to\infty} D_{i_j-1}(V_{+,\infty}(y), l_+(l, y)) \geq$$

$$\lim_{j\to\infty} \overline{\lim_{k\to\infty}} D_{i_j-1}(V_{+,i_k}(y), l_+(l, y)) \geq$$

$$\lim_{j\to\infty}\overline{\lim_{k\to\infty}} D_{i_k-1}\left(V_{+,i_k}(y), l_+(l,y)\right) =$$
$$\overline{\lim_{k\to\infty}} D_{i_k-1}\left(V_{+,i_k}(y), l_+(l,y)\right),$$

where the first equality follows from $\left\{D_{i_j-1}\right\}_{j=1}^{\infty}$ being a subsequence of a sequence converging to D_∞ by (a), the first inequality results from the upper semicontinuity of D_{i_j-1}, the second inequality derives from the fact that $\{D_i\}_{i=0}^{\infty}$ is weakly decreasing, hence $D_{i_k-1}\left(V_{+,i_k}(y), l_+(l,y)\right) \leq D_{i_j-1}\left(V_{+,i_k}(y), l_+(l,y)\right)$, $\forall k \geq j$, and the last equality is trivial. Notice that $D_{i_k-1}\left(V_{+,i_k}(y), l_+(l,y)\right) \geq \underline{U}_{l_+(l,y)}$, $\forall k \in \mathbb{Z}_{++}$ since by construction $c_{R,i_k} \in \Gamma_R\left(V_l, D_{i_k-1}, l\right) \neq \emptyset$. Then, $D_\infty\left(V_{+,\infty}(y), l_+(l,y)\right) \geq \underline{U}_{l_+(l,y)}$, from where $c_{R,\infty}(V_l) \in \Gamma_R\left(V_l, D_\infty, l\right)$. Finally,

$$\underline{T}_l\left(D_\infty\right)_{(V_l)} =$$
$$\max_{\substack{c_R\in\\ \Gamma_R(V_l,D_\infty,l)}} \sum_{y\in Y}[u\left(w_+(y), y\right) + \beta_P D_\infty\left(V_+(y), l_+(l,y)\right)]\pi\left(y, a_-\right) \geq$$
$$\sum_{y\in Y}[u\left(w_{+,\infty}(y), y\right) + \beta_P D_\infty\left(V_{+,\infty}(y), l_+(l,y)\right)]\pi\left(y, a_{-,\infty}\right) \geq$$
$$\overline{\lim_{k\to\infty}} \sum_{y\in Y}[u\left(w_{+,i_k}(y), y\right) + \beta_P D_{i_k-1}\left(V_{+,i_k}(y), l_+(l,y)\right)]\pi\left(y, a_{-,i_k}\right) =$$
$$\overline{\lim_{k\to\infty}} D_{i_k}\left(V_l, l\right) =$$
$$D_\infty\left(V_l, l\right),$$

where the first equality follows from the fact that $D_\infty(V_l, l) \geq \underline{U}_l$, D_∞ is usc, $\Gamma_R\left(V_l, D_\infty, l\right)$ is non-empty and compact, the first inequality - from $c_{R,\infty}(V_l) \in \Gamma_R\left(V_l, D_\infty, l\right)$, the second inequality - by using the result obtained earlier by developing for $D_\infty\left(V_{+,\infty}(y), l_+(l,y)\right)$, the following equality - by construction, and the last equality - by construction and (a).

To conclude the proof, we need to show that $\underline{T}\left(D_\infty\right) \leq D_\infty$. Fix $l \in L$ and $V_l \in V^{AP}(l)$. If $\underline{T}_l\left(D_\infty\right)_{(V_l)} = -\infty$, the result is trivial, so assume $\underline{T}_l\left(D_\infty\right)_{(V_l)} > -\infty \Rightarrow \Lambda_R\left(V_l, D_\infty, l\right) \neq \emptyset$. From (a), we have that for any $i \in \mathbb{Z}_+$, $D_\infty \leq D_i$, from where $\Lambda_R\left(V_l, D_\infty, l\right) \subset \Lambda_R\left(V_l, D_i, l\right)$, $\forall i \in \mathbb{Z}_+$. Then, for any $i \in \mathbb{Z}_+$:

$$\underline{T}_l\left(D_\infty\right)_{(V_l)} =$$
$$\max_{\substack{c_R\in\\ \Lambda_R(V_l,D_\infty,l)}} \sum_{y\in Y}[u\left(w_+(y), y\right) + \beta_P D_\infty\left(V_+(y), l_+(l,y)\right)]\pi\left(y, a_-\right) \leq$$

$$\max_{\substack{c_R \in \\ \Lambda_R(V_l, D_i, l)}} \sum_{y \in Y} [u(w_+(y), y) + \beta_P D_i(V_+(y), l_+(l,y))] \pi(y, a_-) =$$

$$D_{i+1}(V_l, l)$$

Consequently, $\underline{T}_l(D_\infty)_{(V_l)} \leq \lim_{i \to \infty} D_{i+1}(V_l, l) = D_\infty(V_l, l)$.

(c) Let $D' \in \{USCBA_l\} : \underline{T}(D') = D'$. Note that $\exists \overline{D} \in \{USCB_l\} : \overline{D} \geq D'$. Consequently, $T(\overline{D}) \geq T(D') \geq D'$, where the first inequality follows from the monotonicity of T, while the second comes from $T \geq \underline{T}$ and the fact that $\underline{T}(D') = D'$. Repeating the argument, we obtain $T^n(\overline{D}) \geq D'$ for any $n \in \mathbb{Z}_+$. Then, by Proposition 7 (d) we have that $U^{AP^*} = \lim_{n \to \infty} T_U^n(\overline{D}) \geq D'$, where the convergence is in terms of μ. Fix $l \in L$ and $V_l \in V^{AP}(l)$. By the monotonicity of $\underline{T}$, we have $D_i = \underline{T}^i(U^{AP^*}) \geq \underline{T}^i(D') = D'$, $\forall i \in \mathbb{Z}_+$. Therefore, $D_\infty \geq D'$. ■

Lemma 4 $\underline{T}\left(\widehat{U}^*\right) \geq \widehat{U}^*$*.*

Proof. [Adapted from the first part of the proof of Proposition 6] Fix an arbitrary $l \in L$. If $V_l \in V^{AP}(l) \backslash V^{2P}(l)$, $\widehat{U}^*(V_l, l) = -\infty$ and the result is trivial. Therefore, take $V_l \in V^{2P}(l)$. Then, $\widehat{U}^*(V_l, l) = U^*(V_l, l)$. Given the existence of $U^*(V_l, l)$, we have that $\exists c \in \Gamma_l : V_0(c, l) = V_l$ and $U_0(c, l) = U^*(V_l, l)$. For any $y \in Y$, let $a_- := a_0(l)$, $w_+(y) := w_0(l, y)$, $V_+(y) := V_1(c, (l, y))$. Then we immediately have that (9) holds. Moreover, (1) $\Rightarrow$ (7), (2) $\Rightarrow$ (8), (6) $\Rightarrow$ (10). For any $y \in Y$, we can construct $c' \in \Gamma_{l_+(l,y)}$: $V_0(c', l_+(l, y)) = V_1(c, l, y)$, from where we have that $V_+(y) \in V^{2P}(l_+(l, y)) \subset V^{AP}(l_+(l, y))$, i.e., (11) holds. From (5), we have $U^*(V_l, l) \geq \underline{U}_l$. Furthermore, by slightly modifying the argument of Lemma 3, we have that for any $y \in Y$, $U^*(V_+(y), l_+(l, y)) = U^*(V_1(c, l, y), l_+(l, y)) = U_1(c, l, y)$. Then, from (5) we obtain that $U^*(V_+(y), l_+(l, y)) \geq \underline{U}_{l_+(l,y)}$, $\forall y \in Y$. Finally, $\widehat{U}^*$ is usc (by the argument used in the proof of Proposition 5 given the qualifications stated in the proof of Proposition 8) and bounded from above. Then, by the definition of $\underline{T}$, we have that $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} \geq \widehat{U}^*(V_l, l)$. ■

Lemma 5 $\underline{T}\left(\widehat{U}^*\right) \leq \widehat{U}^*$.

Proof. [Adapted from the second part of the proof of Proposition 6] Take $l \in L$ and $V_l \in V^{AP}(l)$. If $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} = -\infty$, the result is trivial; therefore, assume that $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} > -\infty$. This implies that $\widehat{U}^*(V_l, l) \geq \underline{U}_l$, from where we immediately have that $V_l \in V^{2P}(l)$ and $\widehat{U}^*(V_l, l) = U^*(V_l, l)$. Since we would trivially obtain $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} \le \widehat{U}^*(V_l, l)$ if $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} \le \underline{U}_l$, assume $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} > \underline{U}_l$. Also note that $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} = \max_{\substack{c_R \in \\ \Gamma_R(V_l, \widehat{U}^*, l)}} \sum_{y \in Y} [u(w_+(y), y) + \beta_P \widehat{U}^*(V_+(y), l_+(l, y))] \pi(y, a_-)$ with $\Gamma_R\left(V_l, \widehat{U}^*, l\right) \neq \emptyset$. Given that $\widehat{U}^*$ is usc and $\Gamma_R\left(V_l, \widehat{U}^*, l\right)$ is compact, there exists a contract c_R^* such that $(7)-(11)$ hold at (V_l, l), $\widehat{U}^*\left(V_+^*(y), l_+(l, y)\right) \ge \underline{U}_{l_+(l,y)}$, $\forall y \in Y$ and $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} = \sum_{y \in Y} [u\left(w_+^*(y), y\right) + \beta_P \widehat{U}^*\left(V_+^*(y), l_+(l, y)\right)] \pi\left(y, a_-^*\right)$. By (11), $V_+^*(y) \in V^{AP}(l_+(l, y))$, which together with $\widehat{U}^*\left(V_+^*(y), l_+(l, y)\right) \ge \underline{U}_{l_+(l,y)}$ implies $V_+^*(y) \in V^{2P}(l_+(l, y))$. Since $\widehat{U}^*\left(V_+^*(y), l_+(l, y)\right) = U^*\left(V_+^*(y), l_+(l, y)\right)$, $\exists c_y^* \in \Gamma_{l_+(l,y)} : V_0\left(c_y^*, l_+(l, y)\right) = V_+^*(y)$ and $U_0\left(c_y^*, l_+(l, y)\right) = \widehat{U}^*\left(V_+^*(y), l_+(l, y)\right)$. Note that this is true for any $y \in Y$. Then, let c^{**} be defined as follows: $(a_0^{**}(l), w_0^{**}(l, y)) = \left(a_-^*, w_+^*(y)\right)$ and $(a_t^{**}(l, y, .), w_t^{**}(l, y, .)) = (a_{y,t-1}^*(l_+(l, y), .)), w_{y,t-1}^*(l_+(l, y), .))$, $\forall \mathrm{nai}(l, y)$, $\forall y \in Y$. It is immediate that $c^{**} \in \Gamma_{l,y}$, $\forall y \in Y$. Moreover, $(7) \Rightarrow a_0^{**}(l) \in A$, $(8) \Rightarrow w_0^{**}(l, y) \in W$, $\forall y \in Y$. By construction and (10), we have that (6) holds at l. By (9), we obtain that $V_0(c^{**}, l) = V_l \in V^{2P}(l)$, from where (4) is satisfied at l. Furthermore, we have that $U_0(c^{**}, l) = T_l\left(\widehat{U}^*\right)_{(V_l)} > \underline{U}_l$. Therefore, $T_l\left(\widehat{U}^*\right)_{(V_l)} \in U^{2P}(V_l, l)$, from where $\widehat{U}^*(V_l, l) = U^*(V_l, l) \ge T_l\left(\widehat{U}^*\right)_{(V_l)}$. ■

Proof of Proposition 10. From Lemmas 4 and 5. ■

Proof of Proposition 11. Since $\widehat{U}^* \in \{USCBA_l\}$, by Propositions 9 (c) and 10 we obtain $\widehat{U}^* \le D_\infty$. What remains to be shown is that $\widehat{U}^* \ge D_\infty$. Fix $l \in L$ and $V_l \in V^{AP}(l)\}$. If $D_\infty(V_l, l) = -\infty$, the result is trivial; therefore, assume $D_\infty(V_l, l) > -\infty$. Then, $D_\infty(V_l, l) = \underline{T}_l(D_\infty)_{(V_l)} = \max_{\substack{c_{R,l}(V_l) \in \\ \Gamma_R(V_l, D_\infty, l)}} \sum_{y \in Y} [u(w_+(V_l, y), y) + \beta_P D_\infty(V_+(V_l, y), l_+(l, y))] \pi(y, a_-(V_l))$ with $\Gamma_R(V_l, D_\infty, l)$ nonempty and $D_\infty(V_l, l) \ge \underline{U}_l$ since otherwise we would have $D_\infty(V_l, l) = -\infty$. Since D_∞ is usc and $\Gamma_R(V_l, D_\infty, l)$ is compact, we have that $\exists c_R^*(V_l, l) \in \Gamma_R(V_l, D_\infty, l)$ s.t. $\underline{T}_l(D_\infty)_{(V_l)} = \sum_{y \in Y} [u\left(w_+^*(V_l, y), y\right) + \beta_P D_\infty\left(V_+^*(V_l, y), l_+(l, y)\right)] \pi\left(y, a_-^*(V_l)\right)$. Since $D_\infty\left(V_+^*(V_l, y), l_+(l, y)\right) \ge \underline{U}_{l_+(l,y)}$, $\forall y \in Y$, we have:

$$D_\infty\left(V_+^*(V_l, y), l_+(l, y)\right) = \underline{T}_{l_+(l,y)}(D_\infty)_{V_+^*(V_l,y)} =$$

$$\max_{\substack{c_R\left(V_+^*(V_l,y), l_+(l,y)\right) \in \\ \Gamma_R\left(V_+^*(V_l,y), D_\infty, l_+(l,y)\right)}} \sum_{y' \in Y} [u\left(w_+\left(V_+^*(V_l, y), y'\right), y'\right) +$$

$$\beta_P D_\infty \left(V_+ \left(V_+^* \left(V_l, y\right), y'\right), l_+ \left(l_+ \left(l, y\right), y'\right)\right)]\pi \left(y', a_- \left(V_+^* \left(V_l, y\right)\right)\right)$$

with $\Gamma_R \left(V_+^* \left(V_l, y\right), D_\infty, l_+ \left(l, y\right)\right)$ nonempty, so the previous analysis applies. Proceeding in this way, we can construct a supercontract c such that $\forall \text{nai}(l)$, $a_t \left(y^{t-1}\right) := a_-^* \left(V_+^{*\ t} \left(V_l, y^{t-1}\right), y^{t-1}\right)$, $w_t \left(y^{t-1}, y_t\right) := w_+^* \left(V_+^{*\ t} \left(V_l, y^{t-1}\right), y^{t-1}, y_t\right)$, where $V_+^{*\ t} \left(V_l, l, y_0, ..., y_{t-1}\right) := V^*_{+\ \langle y_{t-1} \rangle} \circ ... \circ V^*_{+\ \langle y_0 \rangle} \left(V_l, l\right)$ for any $t \in \mathbb{Z}_{++}$ and $V_+^{*\ 0} \left(V_l, l\right) := V_l$ with $V^*_{+\ \langle y_\tau \rangle} \left(V, y^{\tau-1}\right) := V_+^* \left(V, y^{\tau-1}, y_\tau\right)$, $\forall y_\tau \in Y$, $V \in V^{AP} \left(l \left(y_{\tau-\theta}, ..., y_{\tau-1}\right)\right)$, $y^{\tau-1} \in l \times Y^\tau$, $\tau \in \mathbb{Z}_+$. We immediately have (7) $\Rightarrow$ (1) and (8) $\Rightarrow$ (2). Moreover, by construction and successively applying (9), we obtain that $\forall \text{nai}(l)$:

$$V_t \left(c, y^{t-1}\right) - V_+^{*\ t} \left(V_l, y^{t-1}\right) =$$

$$\lim_{T \to \infty} \beta_A{}^T \sum_{y_{t+T-1} \in Y} ... \sum_{y_t \in Y} [V_{t+T} \left(c, y^{t+T-1}\right) -$$

$$V_+^{*\ t+T} \left(V_l, y^{t+T-1}\right)] \prod_{i=t}^{t+T-1} \pi \left(y_i, a_i \left(y^{i-1}\right)\right)$$

Consequently, $\forall \text{nai}(l)$, $V_t \left(c, y^{t-1}\right) = V_+^{*\ t} \left(V_l, y^{t-1}\right)$ since by (11) $V_+^{*\ t} \left(V_l, y^{t-1}\right) \in V^{AP} \left(l \left(y_{\tau-\theta}, ..., y_{\tau-1}\right)\right)$ and is therefore bounded, while $V_t \left(c, y^{t-1}\right)$ is bounded given (7) and (8). In particular, $V_0 \left(c, y^{\tau-1}\right) = V_l$. Then (10) $\Rightarrow$ (6). Furthermore, $V_t \left(c, y^{t-1}\right) \in V^{AP} \left(l \left(y_{\tau-\theta}, ..., y_{\tau-1}\right)\right)$, implies that (4) holds after l. Since $U_\tau \left(c, y^{t-1}\right)$ is bounded given (7) and (8) and $D_\infty \left(V_+^{*\ t} \left(V_l, y^{t-1}\right), l \left(y_{\tau-\theta}, ..., y_{\tau-1}\right)\right)$ is bounded from above by Proposition 9 (a).and from below by $\min_{l \in L} \underline{U}_l$ (well defined given L finite), we also have that $U_t \left(c, y^{t-1}\right) = D_\infty \left(V_+^{*\ t} \left(V_l, y^{t-1}\right), l \left(y_{\tau-\theta}, ..., y_{\tau-1}\right)\right)$, $\forall \text{nai}(l)$. In particular, $U_0 \left(c, l\right) = D_\infty \left(V_l, l\right)$. Then, (5) is satisfied at any node. Therefore, $V_l \in V^{2P} \left(l\right)$ and $D_\infty \left(V_l, l\right) \in U^{2P} \left(V_l, l\right)$. Then, $\widehat{U}^* \left(V_l, l\right) = U^* \left(V_l, l\right) \geq D_\infty \left(V_l, l\right)$. ■

Lemma 6 *$V^{AP} \subset B\left(V^{AP}\right)$.*

Proof. Let $V \in V^{AP}$ and fix an arbitrary $l \in L$. From $V_l \in V^{AP} \left(l\right)$, $\exists c \in \Gamma_l^{AP} \left(V_l\right)$. By construction, $V_l \in \left[\underline{V}_l, \widehat{V}\right]$. For any $y \in Y$, let $a_- := a_0 \left(l\right)$, $w_+ \left(y\right) := w_0 \left(l, y\right)$, and $V_+ \left(y\right) := V_1 \left(c, l, y\right)$. Given these choices, we immediately have that (9) holds. Moreover, (1) $\Rightarrow$ (7), (2) $\Rightarrow$ (8), (6) $\Rightarrow$ (10). Note that for any $y \in Y$, $V^{AP} \left(l_+ \left(l, y\right)\right) \cap \left[\underline{V}_{l_+ (l,y)}, +\infty\right) = V^{AP} \left(l_+ \left(l, y\right)\right)$. Since for any $y \in Y$ we can construct a supercontract $c'_y \in \Gamma^{AP}_{l_+(l,y)}(V_1 \left(c, (l, y)\right))$, we have that (12) is satisfied. Therefore, $V_l \in B_l\left(V^{AP}\right)$. Since $l \in L$ was chosen randomly, this generalizes to $V \in B\left(V^{AP}\right)$. ■

The lemma establishes that V^{AP} is self-generating in the terminology of Abreu, Pearce and Stacchetti (1990).

Lemma 7 *Assume $X = \{X_l\} : \emptyset \neq X_l \subset B_l(X)$, $\forall l \in L$. Then, $B(X) \subset V^{AP}$.*

Proof. Let the condition of the lemma hold and take $V \in B(X)$. Fix an arbitrary $l \in L$. Since $V_l \in B_l(X)$, $\exists c_{R,l}(V_l)$: (7)-(10) and (12) hold at l. By (12) and $X_{l_+(l,y)} \subset B_{l_+(l,y)}(X)$, we obtain that $V_{+,l}(V_l, y) \in B_{l_+(l,y)}(X)$. Then, $\forall y \in Y$, $\exists c_R$: (7)-(10) and (12) hold at $(V_{+,l}(V_l, y), l_+(l, y))$. Proceeding this way, as in the proof of Proposition 11, we can consecutively construct a supercontract c after l s.t. $c \in \Gamma_l^{AP}(V_l)$. Here, it deserves noting that while (12) implies (4) on every node but the first, $V_l \in B_l(X) \subset \left[\underline{V}_l, \widehat{V}\right]$, from where (4) is also satisfied at l. Therefore, $V_l \in V^{AP}(l)$, which generalizes to $V \in V^{AP}$. ■

The lemma says that the image of every nonempty, self-generating set is a subset of V^{AP}.

Proof of Proposition 12. (a) By Assumption *3* and Lemma 6, V^{AP} satisfies the condition of Lemma 7. Therefore, $B\left(V^{AP}\right) \subset V^{AP}$, which together with Lemma 6 implies the result.

(b) It follows by Lemma 7. ■

Lemma 8 *Assume $X' = \{X'_l\}$ and $X'' = \{X''_l\} : X'_l \subset X''_l \subset \mathbb{R}$, $\forall l \in L$. Then, $B_l(X') \neq \emptyset \Rightarrow B_l(X') \subset B_l(X'')$, $\forall l \in L$.*

Proof. Trivial. ■

Lemma 9 *Assume $X = \{X_l\} : X_l \subset \mathbb{R}$ compact, $\forall l \in L$. Then, $B_l(X) \neq \emptyset \Rightarrow B_l(X)$ compact, $\forall l \in L$.*

Proof. Let the condition of the lemma hold and assume $B_l(X) \neq \emptyset$ for some $l \in L$. Note that $B_l(X) \subset \left[\underline{V}_l, \widehat{V}\right] \subset \mathbb{R}$ is bounded by definition. We should also show that it is closed. Take an arbitrary convergent sequence $\{V_i\}_{i=1}^{\infty} : V_i \in B_l(X)$, $\forall i \in \mathbb{Z}_{++}$ with $V_i \underset{i \to \infty}{\to} V_\infty$. We need to prove that $V_\infty \in B_l(X)$. By construction, we have that for any $i \in \mathbb{Z}_{++}$, $V_i \in \left[\underline{V}_l, \widehat{V}\right]$ and $\exists c_{R,i}$:(7)-(10),

(12) hold at (V_i, l). By $V_i \in \left[\underline{V}_l, \widehat{V}\right]$, $\forall i \in \mathbb{Z}_{++}$, we obtain $V_\infty \in \left[\underline{V}_l, \widehat{V}\right]$. By (7), (8), (12), Assumption 2, L finite, and $X_l \subset \mathbb{R}$ compact for any $l \in L$, we have that $\{c_{R,i}\}_{i=1}^{\infty}$ is uniformly bounded, therefore $\exists$ a subsequence $\{c_{R,i_k}\}_{k=1}^{\infty}$ of $\{c_{R,i}\}_{i=1}^{\infty} : c_{R,i_k} \underset{k\to\infty}{\to} c_{R,\infty}$. It is immediate that $c_{R,\infty}$ satisfies (7)-(10), (12) at (V_∞, l). ■

Proof of Proposition 13. For any $l \in L$ and $i \in \mathbb{Z}_+$, denote by $X_{i,l}$ the element of X_i corresponding to initial history l. By the condition of the Proposition and Assumption *3*, we have that $\emptyset \neq V^{AP}(l) \subset X_{0,l} \subset \mathbb{R}$, $\forall l \in L$. Since by Proposition 12 (a) $B_l\left(V^{AP}\right) = V^{AP}(l)$, we can apply Lemma 8 to obtain $\emptyset \neq V^{AP}(l) \subset X_{1,l} \subset \mathbb{R}$, $\forall l \in L$. Using $X_1 \subset X_0$ and repeating the argument, we reach $V^{AP} \subset X_{i+1} \subset X_i$, $\forall i \in \mathbb{Z}_+$ Then, $\{X_i\}_{i=0}^{\infty}$ is a sequence of non-empty, compact (by Lemma 9 since X_0 compact), monotonically decreasing (nested) sets; therefore it converges to $X_\infty = \overset{\infty}{\underset{i=0}{\cap}} X_i \supset V^{AP}$ with X_∞ compact.

What remains to be shown is that $X_\infty \subset V^{AP}$. By Lemma 7, it is enough to show that $X_\infty \subset B(X_\infty)$. Let $V \in X_\infty$. This implies that $V \in X_i$, $\forall i \in \mathbb{Z}_+$. Fix an arbitrary $l \in L$. We have that $\exists c_{R,i}$: (7)-(10), (12) hold at (V_l, l). By (7), (8), (12), Assumption 2, L finite, and $X_i \subset X_0 \subset \mathbb{R}^{n^\theta}$ compact, $\forall i \in \mathbb{Z}_+$, we have that $\{c_{R,i}\}_{i\in\mathbb{Z}_+}$ is uniformly bounded; therefore, $\exists$ a subsequence $\{c_{R,i_k}\}_{k=1}^{\infty}$ of $\{c_{R,i}\}_{i=1}^{\infty} : c_{R,i_k} \underset{k\to\infty}{\to} c_{R,\infty}$. It is immediate that $c_{R,\infty}$ satisfies (7)-(10) at (V_l, l). Moreover, $V_{+,\infty}(y) \geq \underline{V}_{l_+(l,y)}$, $\forall y \in Y$. We also need to show that for any $y \in Y$, $V_{+,\infty}(y) \in X_{\infty,l_+(l,y)}$. Fix an arbitrary $y \in Y$ and assume, on the contrary, that $V_{+,\infty}(y) \notin X_{\infty,l_+(l,y)}$. Since $X_{\infty,l_+(l,y)} = \overset{\infty}{\underset{i=0}{\cap}} X_{i,l_+(l,y)} = \overset{\infty}{\underset{k=0}{\cap}} X_{i_k,l_+(l,y)}$, we have that $\exists k' \in \mathbb{Z}_+ : V_{+,\infty}(y) \notin X_{i_{k'},l_+(l,y)}$. Furthermore, $\left\{X_{i_{k'}}\right\}_{k=0}^{\infty}$ was shown to be a monotonically decreasing (nested) sequence, from where $V_{+,i_k}(y) \in X_{i_k,l_+(l,y)} \subset X_{i_{k'},l_+(l,y)}$, $\forall k \in \mathbb{Z}_+ : k \geq k'$. Since $X_{i_{k'},l_+(l,y)}$ is closed and $V_{+,i_k}(y) \underset{k\to\infty}{\to} V_{+,\infty}(y)$, we obtain that $V_{+,\infty}(y) \in X_{i_{k'},l_+(l,y)}$, i.e., a contradiction is reached. This proves $V_{+,\infty}(y) \in X_{\infty,l_+(l,y)}$, $\forall y \in Y$. Consequently, (12) holds for $c_{R,\infty}$. Finally, note that $V_l \in \left[\underline{V}_l, \widehat{V}\right]$ follows immediately from $V_l \in X_{1,l}$. Therefore, $V_l \in B_l(X_\infty)$, which generalizes to $V \in B(X_\infty)$. ■

For any $X = \{X_l\} : X_l \in \mathbb{R}, \forall l \in L$ let $B'(X) := \{B'_l(X)\}$ with

$$B'_l(X) := \{V_l \in \left[\underline{V}_l, \widehat{V}\right] : \exists c_R : (7) - (10) \text{ and } (13) \text{ hold at } (V_l, l)\}.$$

Note that the only difference between this operator and operator $\widetilde{B}$ defined in Section 3 is that $B'_l(X) \subset \left[\underline{V}_l, \widehat{V}\right]$, while $\widetilde{B}_l(X) \subset X_l$.

Lemma 10 *Take* $X_0' := \left\{X_{0,l}'\right\}$ *with* $X_{0,l}' := \left[\underline{V}_l, \widehat{V}\right]$, $\forall l \in L$ *and let* $X_{i+1}' := B'\left(X_i'\right)$, $\forall i \in \mathbb{Z}_+$. *Then,* $X_{i+1}' \subset X_i'$, $\forall i \in \mathbb{Z}_+$ *and* $X_\infty' := \lim_{i\to\infty} X_i' = V^{AP}$.

Proof. We have that X_0' is compact and $V^{AP} \subset X_0' \subset \mathbb{R}^{n^\theta}$. Note that for any $X \subset \mathbb{R}^{n^\theta} : B_l(X) \neq \emptyset$, we have $B_l(X) \subset B_l'(X)$. Then, by Lemma 8 and Proposition 12 (a), we obtain $V^{AP} \subset B\left(X_0'\right) \subset B'\left(X_0'\right)$. Using the same arguments plus the monotonicity of B' (trivial), we have $V^{AP} \subset X_i'$, $\forall i \in \mathbb{Z}_+$. Moreover, by construction $B'\left(X_0'\right) \subset X_0'$. Then, the condition $B\left(X_0'\right) \subset X_0'$ is satisfied. Observe that for any $l \in L$, $X_{1,l}' = \{V_l \in \left[\underline{V}_l, \widehat{V}\right] : \exists c_R$ s.t. (7) $-$ (10), (13) hold at $(V_l, l)\} = \{V_l \in \left[\underline{V}_l, \widehat{V}\right] : \exists c_R$ s.t. (7) $-$ (10), (12) hold at $(V_l, l)\} = B_l\left(X_0'\right)$ since, by construction, we have that $X_{0,l_+(l,y)}' \cap \left[\underline{V}_{l_+(l,y)}, +\infty\right) = X_{0,l_+(l,y)}'$, $\forall y \in Y$. Furthermore, by $X_1' \subset X_0'$ and the monotonicity of B', we obtain $X_{i+1}' \subset X_i'$, $\forall i \in \mathbb{Z}_+$. Then, it is trivial that $X_{i+1}' = B\left(X_i'\right)$, $\forall i \in \mathbb{Z}_+$ Therefore, Proposition 13 applies to $\{X_i'\}_{i=1}^\infty$. ■

Lemma 11 *Let* $\{X_i'\}_{i=1}^\infty$ *be defined as in Lemma 10. Take* $\widetilde{X}_0 := X_0'$ *and let* $\widetilde{X}_{i+1} := \widetilde{B}\left(\widetilde{X}_i\right)$, $\forall i \in \mathbb{Z}_+$. *Then,* $\widetilde{X}_i = X_i'$, $\forall i \in \mathbb{Z}_+$.

Proof. Assume $\widetilde{X}_{i-1} = X_{i-1}'$ for some $i \in \mathbb{Z}_{++}$. By Lemma 10, $\emptyset \neq X_i' \subset X_{i-1}'$. Fix $l \in L$ and let $V \in X_{i,l}'$ Then, we have $V \in \widetilde{X}_{i-1,l}$, which together with $V \in B_l'\left(\widetilde{X}_{i-1}\right)$ implies that $V \in \widetilde{B}_l\left(\widetilde{X}_{i-1}\right)$. Since l and V were chosen randomly, this generalizes to $X_i' \subset \widetilde{X}_i$. Then, $\widetilde{X}_i$ is non-empty. Note that $\widetilde{X}_{i-1} = X_{i-1}' \subset X_0'$ by Lemma 10. Consequently, $\emptyset \neq \widetilde{B}_l\left(\widetilde{X}_{i-1}\right) \subset B_l'\left(\widetilde{X}_{i-1}\right)$, i.e., $\widetilde{X}_i \subset X_i'$.

We have that $\widetilde{X}_0 = X_0'$ by definition and have just shown that $\widetilde{X}_{i-1} = X_{i-1}'$ would imply $\widetilde{X}_i = X_i'$; therefore, by induction we obtain that $\widetilde{X}_i = X_i'$ for any $i \in \mathbb{Z}_+$. ■

Proof of Proposition 14. (a) From Lemmas 10 and 11.

(b) Similarly to the proof of Lemma 6, we can show that $V^{AP} \subset \widetilde{B}\left(V^{AP}\right)$. Since $\widetilde{B}\left(V^{AP}\right)$ is nonempty, it can easily be obtained that $\widetilde{B}\left(V^{AP}\right) \subset V^{AP}$.

(c) Since $\emptyset \neq X \subset \widetilde{X}_0$, we can use the monotonicity of $\widetilde{B}$ and $\widetilde{B}(X) = X$ to obtain $X \subset \widetilde{X}_i$, $\forall i \in \mathbb{Z}_+$. Then, by (a), we have $X \subset \widetilde{X}_\infty = V^{AP}$. ■

Dynamically Optimal Executive Compensation when Reservation Utilities Are History-Dependent*

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March, 2009

Abstract

This paper computes the optimal executive compensation in an infinite-horizon moral hazard framework characterized by limited commitment and history-dependent reservation utilities. The model is given a recursive form and some properties of the state space are established. I derive a sufficient condition for the optimal contract to provide the CEO with insurance against fluctuations in the value of his/her outside options under a short-term history dependence. In the numerical computation of the endogenous state space, I use an innovative algorithm which does not rely on the convexity of the underlying set. Exerting effort appears to be the predominant strategy for the principal, but shirking may still be optimal when the manager is rich enough. The optimal wage scheme and the future utility of the CEO tend to grow in both his/her current utility and in the firm's future profit. The manager's utility tends to increase weakly in the long run and appears to have a non-degenerate long-term distribution depending on the initial utility promise but not on the initial history.

Keywords: principal-agent problem, moral hazard, dynamic contracts, limited commitment, executive compensation

Journal of Economic Literature Classification Numbers: C63, D82, G30

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*I am particularly grateful to Manuel Santos for his guidance and support. I also wish to thank the members of my thesis committee and participants in the ICEF seminar at the State University - Higher School of Economics for useful suggestions and comments.

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1 Introduction

Executive pay is a topic that has continuously interested media and academia alike. Since Jensen and Murphy (1990)'s seminal paper, there has a big debate about the effectiveness of the observed compensation schemes in inducing the proper incentives while providing insurance to risk-averse managers. Empirical surveys and recipes abound.[1] The most important question, however, is how the optimal compensation scheme should actually look like. Estimating the top-management pay is not a trivial. task. It relates to the vast literature on dynamic contracts pioneered by Green (1987) and Spear and Srivastava (1987). In a dynamic model of adverse selection, Thomas and Worrall (1990) demonstrated that a legally enforceable contract would have the borrower's utility converging to minus infinity with probability one. Phelan (1995) showed that in a dynamic insurance setting characterized by one-sided commitment, there exists a non-degenerate long-run distribution of consumption. While the agency literature has mainly focused on deriving contracts inducing optimal effort, the participation constraints have largely been ignored. Some notable exceptions are Sleet and Yeltekin (2001) and Spear and Wang (2005) who concentrate on contract terminations and Cao and Wang (2008) who endogenize agent's reservation utility.

In the current paper, I compute the dynamically optimal executive compensation. Since I am interested in the long term dynamics of the contract and the resulting wealth distribution, I focus on long-term self-enforcing schemes that are incentive compatible. The setting is an infinite-horizon moral hazard problem characterized by limited commitment and history-dependent reservation utilities. Each period, the firm's shareholders (treated as a risk-neutral principal) and the CEO (a risk-averse agent) sign a contract which specifies a recommended level of effort to be exercised by the agent this period and the compensation the agent will receive in the end of the period. The effort exerted by the agent is not observed by the principal and influences the firm's (gross) profit in a non-deterministic fashion. Therefore, the compensation of the agent cannot be based on the specific level of effort exercised. However, it can be made contingent on firm's realized profit. More generally, since the firm's profit is publicly observable and no amnesia is introduced in the model, the contract can be based on the whole history of profit realizations and the compensation can additionally be made contingent on the profit to be realized in the end of the period. Moreover, the contract should provide the proper incentives to the manager in order for him/her to exert exactly the level of effort recommended by the firm's shareholders. Limited commitment is assumed on both parts in the sense that both the shareholders and the CEO can commit only to short-term (single-period) contracts. This assumption is intended to reflect legal issues on the enforcement of long-term contracts. However, at the initial period the shareholders can offer a long-term contract (a supercontract) that neither they,

[1] See Murphy (1999) and Jensen and Murphy (2004) for a review.

nor the manager would like to renege on, and that would provide the necessary incentives for the manager to exercise the sequence of effort levels suggested by the principal.

Wang (1997) and Aseff (2004) use a similar framework in order to analyze the optimal contract. The former, however, does not rigorously analyze the effects of limited commitment on both parts of the relationship while the latter restrictively pre-supposes the optimality of high effort and effectively estimates the optimal compensation scheme that induces it.

Furthermore, in my treatment of limited commitment, I allow for correlation between the reservation utilities of the agent and the principal and the (finitely truncated) history of profits. This extension directly affects the set of possible endogenous utilities, but also permits the analysis of some interesting dynamic effects. For example, if the outside offer for the manager is positively correlated with current profit (due to, say, a belief on part of the outside employers that the firm's performance reveals information about the quality/type of the manager), we may expect that he/she would be motivated to increase the probability of high profits in the future (by choosing a higher level of effort). At the same time, the risk-averse managers would like to smooth consumption across states, which may require that their participation constraint does not bind for lower profit realizations. Moreover, it may become increasingly more difficult to motivate richer CEOs, especially when the shareholders face some borrowing constraints, which may lead to the suboptimality of inducing high effort for such CEOs.

The current paper is the first to look at how shocks on the reservation utilities may affect the parties to a dynamic contractual relationship. In particular, we investigate whether the optimal contract insures the manager against variability in the value of his/her outside options. We build up the intuition behind the possible effect of such an insurance on the manager's utility in the short and the long run and relate it to the properties of the limiting distribution.

The framework falls into the scope of Morfov (2009); therefore, an optimal contract exists and the problem can be characterized recursively and addressed by dynamic programming techniques.

The estimation is conducted in three steps. First, the state space of an auxiliary problem that does not require the participation of the principal but binds the wage from above is recovered as the limit of a generalized Bellman equation. Second, the aforementioned auxiliary problem is solved by a standard recursive procedure. Third, the optimal recursive contract and its state space are recovered by severely punishing the principal for each violation of his/her participation constraint.

In order to estimate the model, I parameterize it following the calibration of Aseff (2004) and Aseff and Santos (2005) based on the results of Hall and Liebman (1998) and Margiotta and Miller (2000).

Regarding the numerical computation, one point deserves special attention. In computing the endogenous state space we are iterating on sets and therefore need to represent them efficiently. For the class of infinitely repeated games with perfect monitoring, Judd, Yeltekin and Conklin (2003) are able to construct inner and outer convex polytope approximations based on the convexification of the equilibrium value set through a public randomization device. The algorithm I use may be of independent interest since it does not rely on the convexity of the underlying set. The main idea is to discretize the guess for the equilibrium set elementwise, extract small open balls around the gridpoints unfeasible with respect to the (non-updated) guess and use the remaining set, i.e., the guess less the extracted intervals, as a new guess for the equilibrium set. The procedure stops if the structure of the representations of two successive guesses coincides[2] and the suitably defined difference between the representations is less than some prespecified tolerance level.

I derive the state space under constant reservation utilities. Then, I consider a single-period history dependence and show theoretically that if the manager's reservation utilities are sufficiently dispersed, his/her participation constraint does not bind under the worst case scenario, which is also observed when the manager can essentially commit when his/her outside option is at its lowest value. In other words, the minimum utility the CEO can be promised for initial histories characterized by lower reservation utility is generally boosted by higher reservation utilities for other states. Alternatively put, the optimal contract provides the CEO with some insurance against fluctuations in the value of his/her outside options, which ultimately smooths his/her consumption across (initial history) states. In case of positive correlation between firm's profits and manager's reservation utilities, this translates into the participation constraint of the manager being non-binding in states characterized by low profits. Computing the model actually shows that utility promises close to the reservation level are possible only under the manager's best-case scenarios when his/her reservation utility is the highest (i.e., when the highest profit has been observed).

The numerical results suggest that with a loose upper bound on wages, the optimal contract can support extremely high values for the expected discounted utility of the CEO when the participation of the principal is not guaranteed. However, when solving for the self-enforcing contract, these values naturally disappear since they violate principal's participation constraint. Exerting effort appears to be the predominant strategy for the principal, but shirking may still be optimal when the agent is rich enough. The optimal wage scheme and the future utility of the manager tend to grow in both current utility and future profit. Intuitively, both current and future compensation are used to induce poor and mid-range managers to work hard, while rich managers prove too difficult to motivate. The latter shirk and while they may face some fluctuations in their current income stream in case of binding credit constraints on part of the firm, their lifetime utility remains relatively flat.

Simulations suggest that CEO's utility weakly increases in the long run. In particular, agents who start rich tend to keep their utility level while those who start poor get richer in time. The increase is most pronounced for managers with initial utilities below the highest reservation utility. These managers first have their utilities pushed well above their reservation level. Then, the principal motivates them to work hard by rewarding success through continuation utilities

[2] Namely, if the representations have the same number of closed sets element by element.

while providing insurance through flatter wages. In this way, the probability of success and, therefore, of a higher reservation utility tomorrow increases which rises the manager's expected continuation utility. The long term distribution of manager's utility is non-degenerate and depends on the initial utility promise but not directly on the relevant initial history at least as far as short initial histories are concerned.

The rest of the paper is structured as follows. Section 2 presents the model in a general and recursive form. Section 3 explains the numerical algorithm at a practical level and discusses the results. Section 4 concludes. Appendix 1 contains all the proofs. Appendix 2 presents the results.

2 Model

The setting describes a dynamic interaction between the shareholders of a corporation and its chief executive officer (CEO). The shareholders are exclusively interested in the profit realized by the corporation. They need the CEO to run the company but cannot observe the level of effort he/she exerts on the job. If offered a fixed compensation, the manager will naturally prefer to shirk rather than work hard, so such a scheme would have no incentive impact whatsoever. On the other hand, since the shareholders know the distribution of firm's (gross) profits conditional on executive's effort, they can offer a wage scheme contingent on the future profit realization in order to invoke the manager to adhere to a certain type of behavior. While the shareholders would prefer the manager to work hard every period, it may be costly to induce such a behavior. The CEO who is risk averse in the money he/she receives, would require a higher average remuneration in order to compensate him/her for the increased volatility of his/her current income. Since the manager is free to walk out of the relationship, incentive compatibility may go against individual rationality, namely, it may become difficult to induce the CEO to work hard and keep him/her in the company. The situation may further be complicated by the shareholders' own limited commitment. While it is very interesting to see how the optimal contractual agreement would look like in terms of incentives, insurance, compensation, induced behavior, and wealth distribution, characterizing it may prove quite involved given the parties' inability to commit and the realistic possibility that the manager's outside job offers/opportunities may vary with firm's realized profit (different types of agents whose ability may be considered related to firm's performance by outside potential employers; different economic environments: harder to find a job in a through than in a boom, etc.)[3]. Albeit the technical difficulties, analyzing this problem increases our understanding of the mechanics of incentive compatibility and self-enforcement in a dynamic setting.

[3]For example, in order to address the wide use of broad-based stock option plans, Oyer (2004) builds a simple two-period model where adjusting compensation is costly and employee's outside opportunities are correlated with the firm's performance.

Would the manager require some form of insurance against fluctuations in the value of his/her outside options? How would this affect the CEO's utility in the short and the long run? Would shocks to reservation utilities have an impact on the long term distribution of executive's wealth? All these questions fall into the scope of the current paper which brings more structure to the model presented in Morfov (2009), establishes some interesting properties of the state space, computes the model numerically and provides intuition for the results.

Let us formally introduce the environment. Time is discrete and the set of firm's possible profits, Y, is a time- and history-invariant set of $n > 1$ distinct real numbers. For concreteness, we will index the set of possible profits of length $\theta \geq 0$, Y^θ, by $L := \{1, ..., n^\theta\}$. Hereafter, we will refer to a particular element of Y^θ as an initial history and will frequently denote it by its index $l \in L$.[4] Moreover, all functions and correspondences with domain Y^θ will be considered as vectors or Cartesian products of sets indexed by L. At the beginning of period 0, the firm's shareholders and the manager sign an incentive-compatible, self-enforcing supercontract. The wage received by the CEO has a uniform lower bound $\underline{w}$ which can be considered a minimum wage level. The level of effort exerted by the manager belongs to the compact, time- and history-invariant set A. Additionally, we make the following assumptions.

Assumption 1 *The profit realization at any period of time depends only on the effort exerted by the CEO in the beginning of the same period and is characterized by the probability function* $\pi(., a) : Y \to (0, 1)$, $\forall a \in A$, *where* $\pi(y, .)$ *is continuous on* A *for any* $y \in Y$.

Assumption 2 *The shareholders of the corporation are proxied by a "principal" with period utility* $y - w$ *for any (gross) profit realization* y *and wage* w. *They discount the future by a factor* $\beta_P \in (0, 1)$.

Assumption 3 *The CEO's period-utility is specified as* $\upsilon(w) - a$ *for any wage* w *and level of effort* a, *where* $\upsilon(.)$ *is assumed continuous, strictly increasing and concave.*[5] *He/she discounts the future by a factor* $\beta_A \in (0, 1)$.

[4] Occasionally, we will treat l as a bijective function mapping Y^θ to L.

[5] This specification actually requires that the manager should consume his/her exact wage at each contingency thus preventing him/her from smoothing his/her consumption stream through borrowing and/or saving. Ceteris paribus, the principal will find it cheaper to motivate the CEO. Note, however, that in our framework problems with commitment are likely to have an adverse effect on the provision of incentives, so by ignoring possible readjustments in the manager's consumption stream, we will be able to study the role of limited commitment in isolation. On a practical level, without imposing a very strong set of assumptions on the primitives of the model in order to justify the use of the first-order approach, allowing the agent to save will significantly complicate the numerical estimation of the model.

Assumption 4 *At any time t, the reservation utilities map Y^θ to $\mathbb{R}$. Given a particular initial history l observed at the beginning of period t, the reservation utilities of the principal and the CEO are respectively $\underline{U}_l$ and $\underline{V}_l$ (denoted as $\underline{U}$ and $\underline{V}$ if $\theta = 0$).*

Given a profit history $y^{t-1} \in l \times Y^t$ observed in the beginning of period $t \geq 0$, and an admissible supercontract $c = (a, w)$ signed at node l at the beginning of period 0,[6] define the expected discounted utility of the principal at node y^{t-1} as $U_t\left(c, y^{t-1}\right)$. Analogously, define $V_t\left(c, y^{t-1}\right)$ as the expected discounted utility of the manager at that node. The supercontract specifies a recommended level of effort and a contingent compensation scheme on all possible contingencies after signing. The admissibility of the contract refers to the effort belonging to A and the wage being greater or equal to its minimum level $\underline{w}$ at any contingency (after signing).

Then, at period 0 at node l, the principal will be solving the following problem:

[PPx]

$\sup_c U_0(c, l)$ s.t.:

$$c \text{ admissible} \tag{1}$$

$$V_t\left(c, y^{t-1}\right) \geq V_t\left(c', y^{t-1}\right), \forall c' = (a', w) \text{ admissible}, \forall y^{t-1}, \forall t \tag{2}$$

$$V_t\left(c, ., \widetilde{l}\right) \geq \underline{V}_{\widetilde{l}}, \forall t, \forall \widetilde{l} \in L \tag{3}$$

$$U_t\left(c, ., \widetilde{l}\right) \geq \underline{U}_{\widetilde{l}}, \forall t, \forall \widetilde{l} \in L \tag{4}$$

where (1) is an admissibility constraint, (2) requires that the recommended plan of effort levels is incentive compatible at every node, while (3) and (4) are participation constraints for the manager and respectively the principal which are required to hold at any node after (and including) l.[7]

Having defined the problem, we will assume that the set of constraints forms a non-empty set.[8]

[6] Note that the history y^{t-1} consists of θ initial outcomes observed before period 0 and t profit realizations from time 0 to time $t-1$.

[7] In the current paper, the environment, the principal's problem and the recursive form are only presented schematically. For a more detailed and motivated exposition, refer to the more general framework of Morfov (2009).

[8] This assumption is the equivalent of Assumption 3 in Morfov (2009) [for details, see the comments in Footnote 14 in the aforementioned paper].

Assumption 5 *$\forall l \in L$, $\{c : (1)–(4)$ hold after $l\} \neq \emptyset$.*

Proposition 1 *Let (1) and (4) hold after some l. Then at any node after (and including) l we have $w_t(.) \leq \overline{w}$, where $\overline{w} := \underline{w} + \frac{1}{\underline{\pi}}\left(\frac{\overline{y}-\underline{w}}{1-\beta_P} - \underline{\underline{U}}\right)$ with $\underline{\pi} := \min_{(y,a)\in Y\times A} \pi(y,a)$, $\overline{y} := \max Y$, and $\underline{\underline{U}} := \min_{\widetilde{l}\in L} \underline{U}_l$.*

The proposition says that an admissible contract that guarantees the commitment of the principal effectively binds the wage from above. Note that the upper bound $\overline{w}$ does not depend on the initial history l. Therefore, for any contract in the constrained set of the problem [PPx], we have that $w_t(.) \in W := [\underline{w}, \overline{w}]$ which is a compact subset of $\mathbb{R}$. Consequently, all the results of Morfov (2009) are valid for such a contract. In particular, there exists an equivalent recursive representation of [PPx] which is stationary upon a properly defined state space. A brief outline of the characterization follows.

Let AP denote an admissible, incentive-compatible supercontract that only guarantees the participation of the agent, while 2P stays for an admissible, incentive-compatible supercontract that guarantees the participation of both parties. Denote by $V^{AP}(l)$ the set of expected discounted utilities for the manager signing an AP contract at l with $\overline{w}$ imposed as a uniform upper bound for the wage.[9] Let $V^{AP} := \left\{V^{AP}(l)\right\}$ be the Cartesian product of such sets indexed by L. Let V^{2P} be the corresponding product of sets of expected discounted utilities for the CEO signing a 2P contract. For any $\forall V = \{V_l\} \in V^{AP}$, define $U^{AP^*}(V)$ as a vector with a general element $U^{AP^*}(V_l, l)$ that stays for the maximum utility the principal can get by signing an AP contract offering V_l to the manager. Respectively, for any $V \in V^{2P}$, $U^*(V)$ is a vector with a general element $U^*(V_l, l)$ that denotes the maximum utility the principal can get by signing a 2P supercontract offering V_l to the manager. Let $\widehat{U}^*$ be the extension of U^* on V^{AP} s.t. for any $V \in V^{AP}$, $\widehat{U}^*(V)$ is a vector with a general element

$$\widehat{U}^*(V_l, l) = \begin{cases} U^*(V_l, l) \text{ if } V_l \in V^{2P}(l) \\ -\infty \text{ otherwise} \end{cases}$$

Let $l_+ : L \times Y \to L$ map today's initial histories and current profit realizations to tomorrow's initial histories. Finally, three important operators are defined.

[9]Note that the "true" AP contract does not require that $w_t(.) \leq \overline{w}$. This condition comes from the participation constraints of the principal which only hold for the 2P contract. Therefore, we will actually characterize the AP contract that allows for wages not higher than $\overline{w}$. Imposing this additional condition to the AP contract, however, have no impact on the 2P contract since by Proposition 1, the original problem [PPx] is equivalent to one where wages are bounded from above by $\overline{w}$. Also note that working with explicit bounds for the wage will be an advantage in the forthcoming numerical computation.

For any $X = \{X_l\} \in \mathbb{R}^{n^{\theta}}$, $\widetilde{B}(X) = \left\{\widetilde{B}_l(X)\right\}$ with $\widetilde{B}_l(X) := \{V \in X_l : \exists$ a (single-round) contract $c_R(V, l) = \{a_-, w_+(y), V_+(y)\}_{y \in Y}$ s.t.:

$$a_- \in A \tag{5}$$

$$w_+(y) \in W, \forall y \in Y \tag{6}$$

$$\sum_{y \in Y} [\upsilon(w_+(y)) - a'_- + \beta_A V_+(y)] \pi\left(y, a'_-\right) \leq V, \forall a'_- \in A \tag{7}$$

$$\sum_{y \in Y} [\upsilon(w_+(y)) - a_- + \beta_A V_+(y)] \pi(y, a_-) = V \tag{8}$$

$$V_+(y) \in X_{l_+(l,y)}, \forall y \in Y \tag{9}$$

hold$\}$.

For any $U = \{U_l\}$ with $U_l : V^{AP}(l) \to \mathbb{R}$ upper semi-continuous (usc) and bounded with respect to the sup metric, and any $V \in \{V_l\} \in V^{AP}$, $T(U)_{(V)}$ is a vector with a general element defined as follows:

$$T_l(U)_{(V_l)} := \max_{c_R} \sum_{y \in Y} [y - w_+(y) + \beta_P U_{l_+(l,y)}(V_+(y))] \pi(y, a_-) \text{ s.t.:}$$

$$(5) - (8) \text{ hold, and}$$

$$V_+(y) \in V^{AP}(l_+(l, y)), \ \forall y \in Y \tag{10}$$

For any $l \in L$ and $V_l \in V^{AP}(l)$, let $\Gamma_R(V_l, U, l) := \{c_R : (5){-}(8), (10)$ hold at (V_l, l) and $U_{l_+(l,y)}(V_+(y)) \geq \underline{U}_{l_+(l,y)}, \forall y \in Y\}$ for some function $U : V^{AP} \to (\mathbb{R} \cup \{-\infty\})^{n^{\theta}}$. Additionally, let

$$\Lambda_R(V_l, U, l) := \begin{cases} \Gamma_R(V_l, U, l) \text{ if } U_l(V_l) \geq \underline{U}_l \\ \emptyset \text{ otherwise} \end{cases}$$

For any $U = \{U_l\}$ with $U_l : V^{AP}(l) \to \mathbb{R} \cup \{-\infty\}$ usc and bounded from above, and any $V \in \{V_l\} \in V^{AP}$, $\underline{T}(U)_{(V)}$ is a vector with a general element:

$$\underline{T}_l(U)_{(V_l)} := \begin{cases} -\infty \text{ if } \Lambda_R(V_l, U, l) = \emptyset \\ \max\limits_{\substack{c_R \in \\ \Lambda_R(V_l, U, l)}} \sum\limits_{y \in Y} [y - w_+(y) + \beta_P U_{l_+(l,y)}(V_+(y))] \pi(y, a_-) \text{ otherwise} \end{cases}$$

Following the results of Morfov (2009), the optimal 2P contract is recursively characterized in three steps[10]:

Step 1. Start with the set $\widetilde{X}_0 := \left\{ \left[\underline{V}_l, \widehat{V} \right] \right\}$ where $\widehat{V} = \frac{\upsilon(\overline{w})-\underline{a}}{1-\beta_A}$ with $\underline{a} := \min\{A\}$ and iterate on the set operator $\widetilde{B}$ until convergence. The limit is V^{AP}.

Step 2. Take a function $U = \{U_l\}$ with $U_l : V^{AP}(l) \to \mathbb{R}$ usc and bounded with respect to the sup metric, $\forall l \in L$. Iterate on $T(.)$ until convergence. The limit is $U^{AP^*}(.)$.

Step 3. Take $U^{AP^*}(.)$ as an initial guess and iterate on $\underline{T}(.)$ until convergence. The limit is $\widehat{U}^*(.)$. Moreover, $V^{2P}(l) = \{V \in V^{AP}(l) : \widehat{U}^*(V, l) \geq \underline{U}_l\}$. Then, for any $V \in V^{2P}(l)$, we have $U^*(V, l) = \widehat{U}^*(V, l)$.

Although we cannot solve the model analytically, we have constructed an equivalent recursive representation that can be addressed by numerical techniques in a three-step procedure as outlined above. Now, we are ready to parameterize the model and compute the optimal solutions. Before that, I will provide some intuition for the results to follow.

Proposition 2 *If $\theta = 0$, we have* $V^{AP} = \left[\max\left\{ \underline{V}, \frac{\upsilon(\underline{w})-\underline{a}}{1-\beta_A} \right\}, \widehat{V} \right]$.

This proposition derives the state space of the optimal AP contract when manager's reservation utility is constant across profit histories. V^{AP} is an interval and its lower limit is either the CEO's reservation utility or his/her discounted utility under a supercontract paying the minimum wage and inducing the lowest level of effort at every single node whichever is bigger. Indeed, when in the computation, I consider $\underline{V} = \frac{\upsilon(\underline{w})-\overline{a}}{1-\beta_A}$, where $\overline{a} := \max A$, the lowest possible utility supportable by an AP contract is exactly $\frac{\upsilon(\underline{w})-\underline{a}}{1-\beta_A}$ (cf. Table 1 in Appendix 2, LLL). The other possible values for $\underline{V}$ are chosen to be greater than $\frac{\upsilon(\underline{w})-\overline{a}}{1-\beta_A}$, so they immediately become the lower limit of the respective state spaces (cf. Table 1 in Appendix 2, MMM and HHH). Regarding the upper limit of the interval, it is given by $\frac{\upsilon(\overline{w})-\underline{a}}{1-\beta_A}$, i.e., the discounted utility of the manager under a contract that pays him/her the highest possible wage $\overline{w}$ and induces the lowest possible effort at every node. Note that $\overline{w}$ was obtained in Proposition 1 as a theoretical bound on wages under the AP contract that would not affect the subsequent derivation of the optimal 2P contract. In practice, we can improve on this bound using economic considerations (see next section). Proposition 2

[10] Step 1 generalizes on Abreu, Pearce and Stacchetti (1990). Step 2 is standard dynamic programming over upper semi-continuous, bounded functions. Step 3 is based on Rustichini (1998).

will not be affected by any uniform bound on wages. We simply need to redefine $\underline{w}$. One economically interesting case, however, requires that wage does not exceed the future (gross) profit realization reflecting the inability of firm's shareholders in raising additional funds to support higher compensation values for the manager. In this case, the upper limit of the state space is, indeed, affected.

For the purposes of the next proposition, let $E_a(.)$ denote the mathematical expectation conditional on a current effort level a. For example, $E_{\underline{a}}(y) = \sum_{y \in Y} y\pi(y, \underline{a})$.

Proposition 3 *If $\theta = 0$ and $w_t(., y) \leq y$, $\forall y \in Y$, we have that* $\min V^{AP} = \max\left\{\underline{V}, \frac{\upsilon(\underline{w})-a}{1-\beta_A}\right\}$ *and* $\max V^{AP} = \frac{\max_{a \in A}\{E_a \upsilon(\min\{y, \overline{w}\})-a\}}{1-\beta_A}$*. Moreover, if $\underline{a} \in \arg\max_{a \in A}\{E_a \upsilon(\min\{y, \overline{w}\}) - a\}$, then V^{AP} is convex.*

This proposition establishes that when the shareholders are effectively prohibited from borrowing, the maximum of the state space of the AP contract is simply the expected discounted utility of the manager under a supercontract that maximizes his/her period utility across the set of admissible actions and wages. For example, if $A = \{\underline{a}, \overline{a}\}$, $\overline{y} \leq \overline{w}$ and $E_{\overline{a}}\upsilon(y) - E_{\underline{a}}\upsilon(y) < \overline{a} - \underline{a}$, then $V^{AP} = \left[\max\left\{\underline{V}, \frac{\upsilon(\underline{w})-a}{1-\beta_A}\right\}, \frac{E_a\upsilon(y)-a}{1-\beta_A}\right]$ (cf. Table 1 in Appendix 2, LLL, MMM, and HHH).

So far, we have established the limits of the state space V^{AP} for the case where the reservation utility of the manager remains constant across profit realizations. Since the focus of the paper is history dependent participation constraints, it would be interesting to see if we can say something about the case where the outside options vary with the history of observables. In what follows, I will concentrate on a one-period dependence.

Proposition 4 *Let $\theta = 1$ and $\underline{V}_{\widehat{l}} = \min_{l \in L} \underline{V}_l$. Then,* $\max V^{AP}(l) = \widehat{V}$, $\forall l \in L$*. Moreover, if* $\max_{a \in A}\{\beta_A E_a \underline{V} - a\} > \underline{V}_{\widehat{l}} - \upsilon(\underline{w})$*, then* $\min V^{AP}\left(\widehat{l}\right) > \underline{V}_{\widehat{l}}$*; otherwise,* $\min V^{AP}(l) = \underline{V}_l$, $\forall l \in L$.

Here, $E_a\underline{V}$ is the expected reservation utility of the agent tomorrow conditional on a current effort level a. Formally, $E_a\underline{V} = \sum_{y \in Y} \underline{V}_{l(y)}\pi(y, a)$.

Before commenting on the proposition, I will introduce some more structure. Let us order the elements of Y ascendingly and index them accordingly such that the lowest element corresponds to an index 1 and the highest to an index n. We also let the reservation utilities of the manager be positively correlated with the firm's realized profit. This last assumption is made solely for the purpose of illustration; it is not necessary for establishing the result of the proposition.

Notice that the manager's reservation utility, the minimum wage, and the probability distribution of the firm's profit conditional on manager's effort are all exogenous to the model. Therefore, the proposition relates the slackness of the manager's participation constraint to the values of the exogenous parameters. Indeed, it is just a restatement of the fact that if the firm's profit is at its lowest level today and a temporary incentive-compatible contract providing the manager with the minimum wage and a continuation utility equal to the reservation value at each contingency tomorrow guarantees him/her today a utility strictly higher than his/her outside option, then the manager's participation constraint under the optimal contract will not bind at initial state y_1. Consider, for example, the case of two possible actions, $\underline{a} < \overline{a}$. If $E_{\overline{a}}\underline{V} - E_{\underline{a}}\underline{V} > \frac{\overline{a}-\underline{a}}{\beta_A}$, then $\{\overline{a}, \underline{w}, \underline{V}_l\}_{l=1}^{n}$ is the temporary incentive-compatible contract that minimizes the manager's current level of utility. The worst-case scenario (from the point of view of the manager) is when the firm's profit is lowest since then his/her reservation utility is at its minimum level. If, in such a case, inducing high effort by promising the minimum salary and the respective reservation utility on any node tomorrow guarantees utility of at most the reservation level today, then manager's participation constraint binds under the optimal contract irrespective of the history of profits. If, however, the manager can only be promised a current utility higher than his/her reservation one, then his/her participation constraint will not bind and the shrinking of the set of possible continuation utilities from below will eventually lead to increasing the lower limit of the state space for low enough profits. Note that the shrinking of $\widetilde{B}_1\left(\widetilde{X}_0\right)$ may lead to shrinking in $\widetilde{B}_l^i\left(\widetilde{X}_0\right)$, $l = 2, ..., n-1$, $i = 2, ...$ even if $\underline{V}_l > \underline{V}_1$ and $\max_{a\in A}\{\beta_A E_a \underline{V} - a\} > \underline{V}_l - \upsilon(\underline{w})$. The reason is that raising $\min \widetilde{B}_1^i\left(\widetilde{X}_0\right)$ increases $E_a \min \widetilde{B}^i\left(\widetilde{X}_0\right)$ relative to $\min \widetilde{B}_l^i\left(\widetilde{X}_0\right) = \underline{V}_l$ for any $a \in A$.[11] If $E_{\overline{a}}\underline{V} - E_{\underline{a}}\underline{V} < \frac{\overline{a}-\underline{a}}{\beta_A}$, then letting the manager shirk by paying him/her the minimum wage and promising him/her the reservation utility at any continuation node is temporary incentive-compatible and minimizes the agent's current level of utility. The same analysis as before applies.

Proposition 4 indicates that, ceteris paribus, decreasing (increasing) the variance of the manager's reservation utility, his/her patience, utility of effort, or utility of consuming the legally-established minimum wage level will increase (decrease) the number of scenarios under which the manager's participation constraint would actually be binding. In the extreme case where the manager's reservation utility, $\underline{V}$, is constant across the history of observables (i.e., $\theta = 0$) and is (reasonably assumed) higher or equal to the lowest utility level supportable by an admissible incentive-compatible contract ignoring the issue of manager's commitment, $\frac{\upsilon(\underline{w})-\underline{a}}{1-\beta_A}$, then the result of Proposition 4 reduces to $\min V^{AP} = \underline{V}$ as also implied by Proposition 2, i.e., the poorest (in initial

[11] Notice that $E_a \min \widetilde{B}^i\left(\widetilde{X}_0\right) = \sum_{y\in Y} \min\left(\widetilde{B}_{l(y)}^i\left(\widetilde{X}_0\right)\right)\pi(y,a)$.

expected discounted utility terms) manager is guaranteed exactly his/her reservation utility level under the optimal contract. What would happen, however, if the manager's reservation utility actually varies across the observed profit histories?

Corollary 1 *If* $\theta = 1$, $\underline{V}_{\widehat{l}} = \min_{l \in L} \underline{V}_l \leq \frac{\upsilon(\underline{w})-a}{1-\beta_A}$ *and* $\exists l \in L : \underline{V}_l > \underline{V}_{\widehat{l}}$, *then* $\min V^{AP}\left(\widehat{l}\right) > \frac{\upsilon(\underline{w})-a}{1-\beta_A}$.

The corollary says that if we have a (non-reducible)[12] one-period dependence (i.e., reservation utilities at the beginning of each period do vary only with the profit realized at the end of the previous period) and we consider a manager who can essentially commit in the worst-case scenario (i.e., $\underline{V}_{\widehat{l}} \leq \frac{\upsilon(\underline{w})-a}{1-\beta_A}$), there will be cases under the optimal contract where the manager would receive utility strictly higher than the respective value of his/her outside option.

For higher values of $\underline{V}_{\widehat{l}}$, whether participation will bind or not depends on the specific parameter values. Nevertheless, if the manager's reservation utilities are not bunched on a very tiny interval, we would expect some gain above reservation utility levels for the least wealthy of the managers with worse performance records.[13]

To summarize, if the manager's reservation utilities are sufficiently dispersed, his/her participation constraint will not bind under the worst case scenario, which is also observed if the manager can essentially commit when his/her outside option is at its lowest value. In other words, the minimum utility the CEO can be promised for initial histories characterized by lower reservation utility is generally boosted by higher reservation utilities for other states. Alternatively put, the optimal contract provides the CEO with some insurance against fluctuations in the value of his/her outside options. In case of positive correlation between firm's profit and manager's reservation utility, this translates into the participation constraint of the manager being non-binding in states characterized by low profits.

Another point that deserves attention is whether V^{AP} is convex. We have seen that when the reservation utility of the principal is constant across profit histories, the state space is indeed an interval. This result, however, cannot be easily generalized for the case of varying reservation utilities. Indeed, if

[12] Note that assuming $\theta = 1$ and $\underline{V}_l = \underline{V}$, $\forall l \in L$, is equivalent (or, alternatively put, is reducible) to $\theta = 0$ with a manager's reservation utility of $\underline{V}$.

[13] In the next section, I consider positive correlation between yesterday's profit and manager's current reservation utility (i.e., a one-period positive dependence). The reservation utility values generally allow for the more interesting case of non-binding participation constraints. The state space V^{AP} is estimated numerically (for different combinations of reservation utility values and different borrowing arrangements) and the results are presented in Table 1 in Appendix 2. They indicate that if $\max_{a \in A}\{\beta_A E_a \underline{V} - a\} > \underline{V}_{\widehat{l}} - \upsilon(\underline{w})$, there is some utility gain on the lower limit of the state space for all but the best-record managers. Another observation is that the worse the record, the higher the gain.

for some $i = 1, \dots$ the set $\widetilde{X}_i := \widetilde{B}^i \left(\widetilde{X}_0\right)$ exhibits a hole, then this hole can potentially persist into V^{AP}. Let us assume that $\theta = 1$, $A = \{\underline{a}, \overline{a}\}$, $\underline{a} < \overline{a}$, $\exists l \in L : \underline{V}_l > \min_{l \in L} \underline{V}_l$, i.e., we have a non-reducible one-period dependence and two possible levels of effort. Consider $\widetilde{B}\left(\widetilde{X}_0\right)$. Is it convex given that $\widetilde{X}_0$ is? Let $V_{\underline{a}}$ and $V_{\overline{a}}$ be the sets of initial utility values that are supportable by admissible incentive-compatible contracts guaranteeing continuation utilities in $\widetilde{X}_0$ and inducing low and, respectively, high effort. Note that both these sets are compact and convex. Then, $\widetilde{B}\left(\widetilde{X}_0\right)$ is convex if and only if $V_{\underline{a}} \cap V_{\overline{a}} \neq \emptyset$. From the proof of Proposition 4, we know that $\max V_{\underline{a}} > \max V_{\overline{a}}$, so the necessary and sufficient condition for the convexity of $\widetilde{B}\left(\widetilde{X}_0\right)$ is equivalent to $\max V_{\overline{a}} \geq \min V_{\underline{a}}$. It is not straight-forward, however, to derive this condition in terms of parameters. We can certainly derive sufficient conditions, but they need not be necessary. For example, let $\widetilde{y} \in \arg\max_{y \in Y} \{\pi(y, \overline{a}) - \pi(y, \underline{a})\}$ and $\underline{V} = \max_{l \in L} \underline{V}_l$. Take the contract $\{\underline{a}, \upsilon(\underline{w}), \underline{V}\}$ which is clearly incentive-compatible and guarantees the manager an initial utility of $\upsilon(\underline{w}) + \beta_A \underline{V} - \underline{a}$. Now, consider the contract recommending high effort while promising wage $\overline{w}$ and a continuation utility $\widehat{V}$ if the profit realization is $\widetilde{y}$ and, respectively, $\underline{w}$ and $\underline{V}$ otherwise. This contract would be incentive compatible and would guarantee the manager an initial utility of at least $\upsilon(\underline{w}) + \beta_A \underline{V} - \underline{a}$ if $\overline{a} - \underline{a} \leq \min\{(\pi(\widetilde{y}, \overline{a}) - \pi(\widetilde{y}, \underline{a}))\left(\upsilon(\overline{w}) + \beta_A \widehat{V} - \upsilon(\underline{w}) - \beta_A \underline{V}\right), \pi(\widetilde{y}, \overline{a})(\upsilon(\overline{w}) + \beta_A \widehat{V}) - \pi(\widetilde{y}, \underline{a})\left(\upsilon(\underline{w}) + \beta_A \underline{V}\right)\}$. This inequality is basically satisfied if $\underline{V}$ is not too high. Notice, however, that by Proposition 4 $\underline{V}$ will be higher the next iteration if $\max_{a \in A} \{\beta_A E_a \underline{V} - a\} > \underline{V}_{\widehat{l}} - \upsilon(\underline{w})$, i.e., if the guess for the state space shrinks. Also note that while we have constructed a sufficient condition for the convexity of $\widetilde{B}\left(\widetilde{X}_0\right)$, this condition is far from necessary.

Given the previous discussion, can we say anything more about the properties of the value function U^{AP^*} and its associated policies? We already know by Proposition 5 in Morfov (2009) that U^{AP^*} is upper semi-continuous (usc) and bounded. Is it continuous? For any $l \in L$ and $V \in V^{AP}(l)$, define $\Gamma_R^{AP}(V, l) := \{c_R : \text{(5)-(8), (10) hold at } (V, l)\}$ and $G_R^{AP}(V, l) := \{c_R \in \Gamma_R^{AP}(V, l) : U^{AP^*}(V, l) = E_a(y - w_+(y) + \beta_P U^{AP^*}_{l_+(l,y)}(V_+(y)))\}$. Namely, $\Gamma_R^{AP}(V, l)$ is the set of admissible, incentive-compatible, one-period contracts guaranteeing the manager an initial utility V at an initial history l, while $G_R^{AP}(V, l)$ is the subset of optimal (from the point of view of the principal) contracts.

Proposition 5 *For any $l \in L$, $\Gamma_R^{AP}(., l)$ is upper hemi-continuous on $V^{AP}(l)$.*

To show that the value function U^{AP^*} is continuous on V^{AP}, we also need $\Gamma_R^{AP}(., l)$ to be lower hemi-continuous on V^{AP}. This is where the problem stems from. For example, consider two possible effort levels $\underline{a} < \overline{a}$ and let $V_{\underline{a}}^{AP}$

and $V_{\bar{a}}^{AP}$ be the sets of initial utility values that are supportable by admissible incentive-compatible contracts guaranteeing continuation utilities in V^{AP} and inducing low and, respectively, high effort. Fix $l \in L$. By Proposition 4, $\max V_{\underline{a}}^{AP}(l) > \max V_{\bar{a}}^{AP}(l)$; therefore, if V^{AP} is convex, $\Gamma_R^{AP}(., l)$ may violate lower hemi-continuity at $\max V_{\bar{a}}^{AP}(l)$ and/or $\max\left\{\min V_{\underline{a}}^{AP}(l), \min V_{\bar{a}}^{AP}(l)\right\}$. Call these points V_1 and V_2, respectively. Then, by the theorem of the maximum[14] $U^{AP^*}(., l)$ will be continuous and $G_R^{AP}(., l)$ will be upper hemi-continuous on $V^{AP}(l) \setminus \{V_1, V_2\}$. If $\theta = 0$, we know that V^{AP} is convex, so the previous analysis applies.

Notice that the problems surrounding the potential discontinuities of Γ_R^{AP} may be related to the possible non-convexity of the set of effort levels, A.[15] However, in view of the numerical estimation, working with an interval of efforts is unfeasible. Moreover, multiple actions may require ranking conditions and the calibration of such a model may prove a difficult task. Therefore, in the next section, I will concentrate on the case of only two possible levels of managerial effort: high (working hard) and low (shirking).[16]

3 Computation and Results

The computation of the model starts with solving for V^{AP}, the set of manager's expected discounted utilities supportable by an AP contract. While Proposition 14 from Morfov (2009) gives the theoretical background for the estimation of V^{AP}, some caveats remain. In particular, $\widetilde{B}$ is a set operator and in order to apply the iterative procedure in practice we need an efficient representation of the sequence of sets $\left\{\widetilde{X}_i\right\}_{i \in Z_+}$. For the class of infinitely repeated games with perfect monitoring, Judd, Yeltekin and Conklin (2003) are able to construct inner and outer convex polytope approximations based on the convexification of the equilibrium value set through a public randomization device. Here, I follow a more general approach which does not rely on assuming that V^{AP} is convex or convexifying it by introducing public randomization.[17] The main idea is to discretize the elements of the initial guess $\widetilde{X}_0$ and start extracting small open

[14] See, for example, Stokey and Lucas (1989).

[15] Indeed, the problem may be attenuated if we assume A convex [cf. Phelan and Townsend (1991)].

[16] Note that if we presuppose the optimality of a certain level of effort, say high effort [see, for example, Aseff (2004)], we will have $V^{AP} = V_{\underline{a}}^{AP}$ convex, Γ_R^{AP} lower hemi-continuous and, therefore (given Proposition 5), continuous, so by the theorem of the maximum U^{AP^*} will be continuous and G_R^{AP} will be upper hemi-continuous. Such an assumption, however, is not as innocuous as it may seem since it appears that shirking (low effort) is optimal for a wide interval of initial utility values in the upper region of the state space (see Figure 10).

[17] Such a general approach is particularly useful in addressing extensions as for example estimating the endogenous state space of agent's expected discounted utilities supportable by an AP stock option contract, because of the non-convexities inherent to the stock option contract.

intervals, the midpoints of which are unfeasible with respect to $\widetilde{X}_0$. The extraction is done elementwise without updating the previous elements. In particular, I start from the discretization of the first[18] element of $\widetilde{X}_0$, find the points that cannot be supported by a one-period AP contract with a continuation utility profile contained in $\widetilde{X}_0$, i.e., the points of the discretization which are not in the first element of $\widetilde{B}\left(\widetilde{X}_0\right)$, and extract small open balls around these points. Next, I find the gridpoints in the second element of $\widetilde{X}_0$ which are unfeasible with respect to $\widetilde{X}_0$, extract their small open neighborhoods and proceed in a similar fashion until I cover all the elements of $\widetilde{X}_0$. The remaining set, i.e., $\widetilde{X}_0$ less the extracted intervals, becomes $\widetilde{X}_1$, our new guess for V^{AP}. Given that $\widetilde{X}_0$ is a vector of n^θ closed intervals in $\mathbb{R}$, each of the n^θ elements of $\widetilde{X}_1$ will be a finite union of closed intervals in $\mathbb{R}$. In order to increase efficiency, intervals with length less than some prespecified level are reduced to their midpoints. The procedure stops if for each element of $\widetilde{X}_i$ the number of closed intervals representing it equals the respective number for the same[19] element in $\widetilde{X}_{i-1}$ and, in addition, the representation of $\widetilde{X}_i$ differs from the representation of $\widetilde{X}_{i-1}$ by less than some prespecified tolerance level. In order to apply this stopping criterion, one still needs to construct a measure for the difference between representations. For this purpose, I find the difference in absolute terms between each endpoint (minimum or maximum point) of each interval of each element of $\widetilde{X}_i$ and $\widetilde{X}_{i-1}$ respectively and take the maximum one to be the difference between the representations of $\widetilde{X}_i$ and $\widetilde{X}_{i-1}$. This difference is well defined given that the two representations share the same structure, which is actually the first condition of the stopping criterion.

Once V^{AP} is obtained, it is elementwise discretized and used as a state space in the dynamic program for obtaining U^{AP^*}. At each iteration, the guess for U^{AP^*} being defined only on the discretization needs to be interpolated over the state space. Interpolation is also required in the subsequent iterative procedure which uses U^{AP^*} as an initial guess for $\widehat{U}^*$, the extension of U^* on V^{AP}.

It should be noted that for computational purposes, I do not work with w directly, but use $\upsilon := \upsilon(w)$ instead. This simple change of variables makes the set of constraints linear in a, υ, and V_+, which significantly improves the numerical optimization. We can always recover the optimal wage by inverting the optimal υ.

Table 1 in Appendix 2 contains V^{AP}, the state space of the optimal AP contract. The results are obtained by parameterizing the model in line with the calibration of Aseff and Santos (2005) based on the results of Hall and Liebman (1998) and Margiotta and Miller (2000). Namely, the set of possible profit realizations which are interpreted as stock price returns $Y = \left\{y_{(1)}, y_{(2)}, y_{(3)}\right\} = \{0.55, 1.125, 1.7\}$, the space of effort levels $A = \{\underline{a}, \overline{a}\} = \{0.1253, 0.1469\}$, the conditional probabilities $\pi\left(y_{(1)}, \underline{a}\right) = 0.1508$, $\pi\left(y_{(2)}, \underline{a}\right) = 0.8121$, $\pi\left(y_{(3)}, \underline{a}\right) =$

[18] Note that $\widetilde{X}_0$ is a Cartesian product of n^θ sets.

[19] Here, 'same' refers to the index of the element, i.e. to the initial history to which it corresponds.

0.0371, $\pi\left(y_{(1)},\overline{a}\right)=0.1268$, $\pi\left(y_{(2)},\overline{a}\right)=0.8082$, $\pi\left(y_{(3)},\overline{a}\right)=0.065$.[20] I fix $\underline{w}=0$ and equalize the discount factors for the agent and the principal $\beta_A=\beta_P=0.96$. The period utility with no effort, $v(.)=\sqrt{(.)}$, is as in Aseff (2004)[21]. The reservation utility of the principal is assumed constant across initial histories with a value $\underline{U}=0$. As regards the upper bound of the manager's compensation, I consider three different cases. Case 1 uses Proposition 1 to derive the uniform upper bound for the wage $\overline{w}$ given the minimum reservation utility of the principal $\underline{U}$. Cases 2 and 3 still honor the upper bound $\overline{w}$, but impose further restrictions on the manager's period compensation[22] Case 2 bounds the wage by $\overline{y}$ at each contingency.[23] It implicitly allows the shareholders to borrow up to $\overline{y}-y$ every period given a realized profit y. Case 3 implicitly prevents the shareholders from borrowing. At each possible contingency, they can pay the CEO no more than the realized profit. For case 1, I take the upper bound for the initial guess $\widehat{V}=\frac{v(\overline{w})-\underline{a}}{1-\beta_A}$, while for cases 2 and 3, I use $\widehat{V}=\frac{v(\min\{\overline{w},\overline{y}\})-\underline{a}}{1-\beta_A}$. I analyze the case of $\theta=1$, which encompasses $\theta=0$ as a subcase. Then, I have to deal with $n^{\theta}=3$ (initial history) states. I use the natural notation l for the state with initial history $y_{(l)}$, $l\in\{1,2,3\}$. I consider three possible values for the reservation utility of the CEO: L= $\frac{v(\underline{w})-\overline{a}}{1-\beta_A}$ = -3.6725, M = 0, H = -L. Then, I analyze the more interesting case of nonnegative correlation between initial histories and manager's reservation utilities. This limits the number of possible combinations of reservation utility values across initial histories to 10. For example, LMH, which stays for $\underline{V}_1$ = L, $\underline{V}_2$ = M, $\underline{V}_3$ = H, is allowed, while LHM is not. Note that KKK is equivalent to the case of $\theta=0$ and $\underline{V}$ =K, where K$\in${L,M,H}. Each cell of Table 1, contains V^{AP} for a particular combination of reservation utility values (table rows) and a particular case (table columns). In each cell, the left subcolumn corresponds to the intervals' minimum points and the right - to the maximum points, while each subrow corresponds to a particular initial history. For example, for LMH, (case) 1, $V^{AP}(1)=[0.8275, 843.0178]$, $V^{AP}(2)=[0.8200, 843.0178]$, $V^{AP}(3)=[3.6725, 843.0178]$.

The results suggest that for any $l\in\{1,2,3\}$, $V^{AP}(l)$ is convex from where come the single intervals in Table 1. Note that at least for cases 1 and 2 the upper bound of $V^{AP}(.)$ remains constant across initial histories and reservation utility combinations. In fact, it equals the theoretical bound given the case: $\frac{v(\overline{w})-\underline{a}}{1-\beta_A}$ for case 1 and $\frac{v(\overline{y})-\underline{a}}{1-\beta_A}$ for case 2. This means that wages can be high enough to support high expected discounted utilities for the manager. Note, however, that $V^{2P}\subset V^{AP}$ and we lose high utility values in solving for U^* as Figure 1 in Appendix 2 indicates. The reason is that the value function is decreasing in the upper region of V^{AP}, which results in violations of the principal's participation constraint for high utility values of the manager.

[20] Aseff and Santos (2005) actually consider two conditional distributions over an interval of possible stock price returns [0.55, 1.7]. In this numerical experiment, I concentrate the mass of each distribution on 3 points of this interval: the minimum, middle, and maximum point.

[21] Running the algorithm with $v(w)=log(1+w)$ as in Aseff and Santos (2005) showed no qualitative changes in the results.

[22] Cf. Wang (1997).

[23] Remember that $\overline{y}$ is the highest possible profit realization, i.e. $y_{(3)}$ in our setting.

Since the results are similar across cases, we concentrate on the economically motivated cases 2 and 3 with a special focus on case 3.[24] Figures 1 and 2 plot U^* and U^{AP^*} over V^{AP} for cases 2 and 3 respectively. In each graph, the left panel corresponds to an initial history 1, the middle - to 2, and the right - to 3. Note that although similar, the value functions are not the same across initial profit histories for both the auxiliary and the original problem. The main difference comes from the substantial shrinking of the state space from the left when the initial history is the one characterized by the highest reservation utility (i.e., 3). Given an initial history 3, the maximum utility the principal can get by signing an AP or 2P contract with the CEO is less than what he/she can obtain under 1 or 2 since the contract should guarantee a higher initial utility to the manager. Note that U^* and U^{AP^*} are almost identical for case 3, while U^* does not cover the uppermost part of the domain of U^{AP^*} in case 2. The reason is that very high initial utility promises should be supported with sufficiently high wages, which would eventually decrease the expected discounted utility of the principal below its reservation value at some node. Therefore, in case 2, the minimum utility the principal can obtain by signing a 2P contract with the manager is higher than the minimum under an AP contract. This is not observed (or, in general, less pronounced) for case 3 since then the principal is essentially prevented from borrowing, so he/she cannot offer the manager wages that are sufficiently high to violate his/her own participation constraint under the 2P contract. The graph also suggests that the value functions are concave and monotonically decreasing, properties which, however, are not so easy to generalize.

Regarding the characteristics of the optimal contract, the recommended effort level is predominantly the high one. However, low effort appears to be optimal in some utility regions. Since the results are similar across cases, I only report the relationship for LMH, case 3. As Figure 3) indicates, shirking is optimal for sufficiently high initial utility values. Intuitively, the manager is so rich (in expected utility terms) that the firm cannot effectively reward or punish him/her and, therefore, finds motivating him/her to exert high effort suboptimal. The CEO's utility tomorrow increases in both the end-of-period profit and the initial utility promise as illustrated in Figures 4 and 5 respectively. Let us focus on Figure 4 which plots the relation for each possible initial utility. While the future utility promise is basically flat for high initial utilities, for low utility values the increase is driven by the participation constraint of the agent which is binding tomorrow at a profit realization $y_{(3)}$. This is also reflected on the left and the middle panel of Figure 5 as the kink of the graph of $V_+(., y_3)$. Note that this is not the case for initial history 3 which requires higher future utility promises. In general, the manager's wage increases in both

[24] As regards the numerical computation, case 3 is the clearest case followed by case 2. Case 1 is the noisiest case since the state space of the auxiliary problem, V^{AP}, is the largest due to the higher upper bound of the manager's utility, $\widehat{V}$. This requires a coarser grid and also introduces numerical mistakes due to the high absolute values of the negative numbers the guess for (and the actual) U^{AP^*} takes in the upper regions of the state space, regions which we in fact lose when estimating U^* since they violate principal's participation.

the end-of-period profit and the initial utility promise. Notice that the compensation scheme is much flatter across profit realizations for case 2 than for case 3 (Figures 6 and 7 respectively). This is because current consumption smoothing (across profit realizations) which is achieved by a flat wage scheme for the initially poor (in terms of utility promises) managers, is no longer possible for richer CEOs because the credit constraint imposed in case 3 starts to bind. This is particularly relevant for the lowest profit realization $y_{(1)}$. The same point can be illustrated by Figures 8 and 9. Note that wage contingent on a low profit tomorrow is strictly increasing on the whole domain of initial utilities for case 2, while in case 3 it steadily increases until $y_{(1)}$ is reached and then with the credit constraint binding stays constant at that level.

The results suggests that both current and future compensation are used to induce poor and mid-range managers to work hard, while rich managers prove too difficult to motivate. The latter shirk and while they may face some fluctuations in their current income stream due to binding credit constraints on part of the firm, their lifetime utility remains relatively flat.

Since there is a sufficient dispersion in agent's reservation utility values,[25] the minimum utility supportable by an AP/2P contract for initial histories characterized by lower reservation utility is boosted by higher reservation utilities for other states. More specifically, in the presence of positive correlation between profits and reservation utilities, the participation constraint of the agent does not bind in states characterized by low profits. In other words, the AP/2P contract provides the manager with some insurance against fluctuations in the value of his/her outside options, which ultimately smooths his/her consumption across (initial history) states. Interestingly, while the theoretical result of Proposition 1 only refers to initial history 1, we observe a cascade effect which leads to a significant rise in the lower limits of the possible utility promises for both 1 and 2. Finally, note that if the reservation utility remains the same across some, but not all of the truncated initial histories, $V^{AP}(.)$ is identical for the initial histories with the same reservation utility. While this seems obvious for $\theta \leq 1$, longer history dependence will potentially break the relation since the set of possible tomorrow's histories will depend on the history today.

Table 2 in Appendix 2 shows the effect of changing the value of the minimum reservation utility of the principal for LLL, case 1. Theoretically, we have that increasing $\underline{U}$ decreases $\overline{w}$, which in turn causes $\widehat{V}$ to fall. Since the analysis so far suggests that the theoretical upper bounds for agent's utility can be supported by an AP contract, the only effect of changing $\underline{U}$ comes from the resulting change in the theoretical bound. Moreover, since the 2P contract cannot support manager's utilities in the upper region of $V^{AP}(.)$, the optimal self-enforcing contract is not affected.

I also use Monte Carlo simulations to investigate the dynamic behavior of the optimal contract. Namely, I construct "typical" time paths of length T for the manager's effort, wage, and expected discounted utility, the firm's profits,

[25] The only exception observed is when the reservation utility remains flat across past outcomes, so in fact we are in the case of $\theta = 0$.

and the principal's expected discounted utility. Each such path is taken to be the mean of I independently generated paths which are constructed following the transition and the policies (and if relevant, the value function) of the 2P contract. The "typical" path is well defined given an initial condition (V_0, l), where $V_0 \in V^{2P}(l)$ and $l \in \{1, 2, 3\}$. Figures 10-17 present the results for LMH, case 3 where I take $T = 50$ and $I = 450$.[26]

Figure 10 illustrates how the manager's effort optimally develops in time. Each curve on the l'th panel of the graph represents a time path conditional on a particular expected discounted utility being promised to the manager in the beginning of period 0 given an initial history $l \in \{1, 2, 3\}$. Relating each curve to its corresponding initial utility indicates that initial effort persists for sufficiently low or sufficiently high initial utilities (high effort for low initial utilities, and low effort for high initial utilities), there is some dynamics in the middle-utility range, mostly expressed in diminishing effort.

Figures 11 and 14 suggest that manager's compensation and, respectively, his/her expected discounted utility grow weakly in the long run where the increase is pronounced for sufficiently low initial utilities, while the mid-range and high initial utility paths tend to be relatively stable at their initial levels. In other words, CEOs who start rich (in expected utility terms) tend to keep their utility level while those who start poor get richer in time. Note that the increase is most pronounced for managers with initial utilities below the highest reservation utility, i.e., the poorest managers in 1 and 2.[27] These managers first have their utilities pushed well above their reservation level based on the insurance effect outlined in Corollary 1. Then, the principal motivates them to work hard by rewarding success through continuation utilities while providing them with insurance through flatter wages. In this way, the probability of a higher profit and, therefore, higher reservation utility tomorrow increases, which rises the manager's expected continuation utility. Since wage is increasing in initial utility, the resulting pattern is observed. Therefore, in the long run, both consumption (wage) and wealth (utility) are smoother across initial history states. The result can also be interpreted as a decreasing (wage- and utility-) inequality (as far as the poorest managers are concerned).

Figure 12 shows the profit fluctuations under the optimal contract. The average profit realization is substantially higher for lower than for higher initial utilities with some sudden drop at the mid range. This is understandable given that high effort is optimal for lower utility values while low effort is optimal for high utility values.

As Figure 13 indicates, the principal's expected discounted utility tends to decrease weakly in the long run where the decrease is more pronounced when lower initial utility is promised to the agent. For higher utility promises, the principal tends to keep his/her initial utility value. This is easily explained by

[26] Longer paths were also simulated but the results did not show significant difference from the ones presented here while memory limitations progressively restricted the precision of the estimates.

[27] Remember, that 3 is the manager's best initial history since it is associated with his/her highest reservation utility $\underline{U}_3 = \mathrm{H}$.

the dynamics of the manager's utility given that the value function is decreasing.

In a setting of dynamic risk sharing, Green (1987) and Thomas and Worrall (1990) demonstrate that the agent becomes infinitely poor in the long run. Phelan (1995) shows that this result does not hold if limited commitment is introduced on part of the agent, namely that there exists a non-degenerate limiting distribution of agent's expected discounted utility and consumption. In a CARA setup with unobservable actions, Wang (1997) shows numerically that agent's wealth and consumption tend to fluctuates over time. Aseff (2004) numerically demonstrates that in a contract that optimally induces high effort, the agent's expected discounted utility increases in the long run and has a non-degenerate limiting distribution. In a more general setup characterized by limited commitment on both parts and history-dependent reservation utilities, I obtain a similar result as indicated in Figures 15-17. Each of these graphs considers an initial state l and plots the empirical distributions of manager's expected discounted utility after 50 periods conditional on 12 different initial utility promises. Since the lower bound of the set of possible initial utility promises for 3 is greater than those for the other two initial history states and I use an equidistant grid of 100 points ($V_{(1)}$,...,$V_{(100)}$), the i-th point of the grid for 3 will generally larger than the i-th point of the grids for 1 and 2 respectively. Having this in mind, we see that the limiting distribution does not vary considerably across initial history states, i.e., in the long run it would not matter what the initial profit was as far as the initial utility promise was the same (at least for a single-period history dependence). Note however that since the curves on each panel of Figure 14 generally do not cross, it still matters where (in terms of utility promise) you start - the poor get rich but it is still better to start richer.

4 Conclusion

This paper considers the dynamic principal-agent interaction between firm's shareholders and a CEO in a setting characterized by limited commitment and history-dependent reservation utilities. I analyze the state space of the recursive form of the problem under a short-term history dependence and derive conditions under which the optimal contract offers the manager a utility strictly higher than the reservation level. The model is parameterized and computed under different structural arrangements. I find evidence that the optimal contract provides the manager with insurance against (non-negligible) fluctuations in the value of his/her outside options, which ultimately smooths his/her consumption across (initial history) states. Exerting effort appears to be the predominant strategy for the principal, but shirking may still be optimal when the CEO is rich enough. The optimal wage scheme and the future utility of the manager tend to grow in both his/her current utility and in the future profit realization.

In the long run, the CEO does not get poorer in utility terms. In particular, managers who start rich tend to keep their utility level while those who start poor get richer in time. The manager's utility tends to increase weakly in the long run and appears to have a non-degenerate long-term distribution depending on the initial utility promise.

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APPENDIX 1

Proof of Proposition 1. At any node $y^{\tau-1}$ after (and including) l, we have $\underline{\underline{U}} \leq U_\tau\left(.,y^{\tau-1}\right) \leq \frac{\overline{y}-\underline{w}}{1-\beta_P}$, where the first inequality follows from (3) and Assumption 4, and the second from (1), Assumptions 1, 2, and the properties of A and Y. If we define $\underline{y} := \min Y$, it is straight-forward that $\sum_{t=\tau}^{\infty} \beta_P^{t-\tau} \sum_{y_t\in Y} ... \sum_{y_\tau\in Y} y_t \prod_{i=\tau}^{t} \pi\left(y_i, a_i\left(y^{i-1}\right)\right) \in \left[\frac{\underline{y}}{1-\beta_P}, \frac{\overline{y}}{1-\beta_P}\right]$. Consequently, we have that $\widehat{W}\left(c, y^{\tau-1}\right) := \sum_{t=\tau}^{\infty} \beta_P^{t-\tau} \sum_{y_t\in Y} ... \sum_{y_\tau\in Y} w_t\left(y^t\right) \prod_{i=\tau}^{t} \pi\left(y_i, a_i\left(y^{i-1}\right)\right) \in \left[\frac{\underline{w}}{1-\beta_P}, \frac{\overline{y}}{1-\beta_P} - \underline{\underline{U}}\right]$. Let us take some admissible a. Since $\widehat{W}\left(a, w, y^{\tau-1}\right) = \sum_{y_\tau\in Y} w_\tau\left(y^\tau\right)\pi\left(y_\tau, a\left(y^{\tau-1}\right)\right) + \beta \sum_{y_\tau\in Y} \widehat{W}\left(c, y^\tau\right)\pi\left(y_\tau, a\left(y^{\tau-1}\right)\right)$, we obtain $\sum_{y_\tau\in Y} w_\tau\left(y^\tau\right)\pi\left(y_\tau, a\left(y^{\tau-1}\right)\right) \in \left[\underline{w}, \frac{\overline{y}-\beta_P\underline{w}}{1-\beta_P} - \underline{\underline{U}}\right]$. Now, consider $w_\tau\left(y^{\tau-1}, y\right)$ for some $y \in Y$. Note that by Assumption 1 and the properties of A and Y, $\underline{\pi}$ is well defined and $\pi\left(y, a\left(y^{\tau-1}\right)\right) > \underline{\pi} > 0$. Then, we have:

$$
\begin{gathered}
w_\tau\left(y^{\tau-1}, y\right) \leq \\
\frac{1}{\pi\left(y, a\left(y^{\tau-1}\right)\right)}\left(\frac{\overline{y}-\beta_P\underline{w}}{1-\beta_P} - \underline{\underline{U}} - \sum_{y_\tau\in Y\setminus\{y\}} w_\tau\left(y^\tau\right)\pi\left(y_\tau, a\left(y^{\tau-1}\right)\right)\right) \leq \\
\frac{1}{\pi\left(y, a\left(y^{\tau-1}\right)\right)}\left(\frac{\overline{y}-\beta_P\underline{w}}{1-\beta_P} - \underline{\underline{U}} - \sum_{y_\tau\in Y\setminus\{y\}} \underline{w}\pi\left(y_\tau, a\left(y^{\tau-1}\right)\right)\right) = \\
\frac{1}{\pi\left(y, a\left(y^{\tau-1}\right)\right)}\left(\frac{\overline{y}-\underline{w}}{1-\beta_P} - \underline{\underline{U}}\right) + \underline{w} \leq \\
\frac{1}{\underline{\pi}}\left(\frac{\overline{y}-\underline{w}}{1-\beta_P} - \underline{\underline{U}}\right) + \underline{w},
\end{gathered}
$$

where the last inequality follows from $\left(\frac{\overline{y}-\underline{w}}{1-\beta_P} - \underline{\underline{U}}\right)$ being nonnegative by Assumption 5. Since $\left(y^{\tau-1}, y\right)$ was taken randomly, we are done. ■

Proof of Proposition 2. Given that $\theta = 0$, the initial guess for V^{AP} in step 1 will be $\widetilde{X}_0 := \left[\underline{V}, \widehat{V}\right]$. Then, $\max \widetilde{B}\left(\widetilde{X}_0\right) = \min\left\{\upsilon\left(\overline{w}\right) + \beta_A\widehat{V} - \underline{a}, \widehat{V}\right\}$ $= \min\left\{\widehat{V}, \widehat{V}\right\} = \widehat{V}$ since $\widehat{V} = \frac{\upsilon(\overline{w})-\underline{a}}{1-\beta_A}$. Consequently, by Proposition 14 from Morfov (2009), we have $\max V^{AP} = \widehat{V}$.

The stationary contract $\{\underline{a}, \underline{w}, \underline{V}\}$ promises the same wage and the same continuation utility for any profit realization. It is temporary incentive compatible and guarantees a current expected discounted utility of $\upsilon(\underline{w}) + \beta_A \underline{V} - \underline{a}$ to the manager. Can we find a contract that guarantees a current utility strictly lower than that? Assume such a contract exists, i.e., $\exists \{a_-, w_+(y), V_+(y)\}_{y\in Y}$ admissible, such that $E_a\{\upsilon(w_+) + \beta_A V_+\} - a < \upsilon(\underline{w}) + \beta_A \underline{V} - \underline{a}$, where E_a is the expectation over the profit realization y conditional on the current action being a. However, this contract will fail to satisfy temporary incentive compatibility. Indeed, (7) requires that $E_a\{\upsilon(w_+) + \beta_A V_+\} - a_- \geq E_{\underline{a}}\{\upsilon(w_+) + \beta_A V_+\} - \underline{a} \geq \upsilon(\underline{w}) + \beta_A \underline{V} - \underline{a}$ which contradicts our assumption that $\{a_-, w_+(y), V_+(y)\}_{y\in Y}$ guarantees a strictly lower current utility than $\{\underline{a}, \underline{w}, \underline{V}\}$ does. Therefore, $\min \widetilde{B}\left(\widetilde{X}_0\right) = \max\{\upsilon(\underline{w}) + \beta_A \underline{V} - \underline{a}, \underline{V}\}$. Note that $\upsilon(\underline{w}) + \beta_A \underline{V} - \underline{a} \geq \underline{V}$ is equivalent to $\underline{V} \leq \frac{\upsilon(\underline{w}) - \underline{a}}{1-\beta_A}$. Then, by Proposition 14 from Morfov (2009), it is trivial that $\min V^{AP} = \max\left\{\underline{V}, \frac{\upsilon(\underline{w}) - \underline{a}}{1-\beta_A}\right\}$.

Finally, we will show that V^{AP} is an interval. $\widetilde{X}_0$. Let $\upsilon_+(.) := \upsilon(w_+(.))$. Given that $\upsilon(.)$ is strictly increasing by Assumption 3, the inverse function of $\upsilon(.)$ is well defined and we have $w_+ = \upsilon^{-1}(\upsilon(w_+))$. Then, we can effectively work with υ_+ instead of w_+. Indeed, (6) simply becomes $\upsilon_+ \in [\upsilon(\underline{w}), \upsilon(\overline{w})]$. Now, let us concentrate on stationary contracts of the form $\{a_-, \upsilon_+(y), V_+(y)\}_{y\in Y}$. Note that $\widetilde{B}\left(\widetilde{X}_0\right)$ is a compact set and its lower and upper limits are utilities supportable by stationary contracts inducing the lowest possible level of effort. Then, any utility between $\min\left\{\widetilde{B}\left(\widetilde{X}_0\right)\right\}$ and $\max\left\{\widetilde{B}\left(\widetilde{X}_0\right)\right\}$ can be obtained as a linear combination of the respective stationary contracts that support them. The linear combination will satisfy (5)-(8). (9) will also hold since $\widetilde{X}_0$ is an interval. In that way, we can show that $\widetilde{X}_i$ is a convex set for any $i = 0, 1, ...$ Since $\widetilde{X}_i$ is a sequence of decreasing (nested), compact, convex sets, we have that their limit is also convex. ■

Proof of Proposition 3. The minimum of the state space is obtained as in the proof of Proposition 2. Note that $\max_{a\in A}\{E_a \upsilon(\min\{y, \overline{w}\}) - a\}$ is well defined given that A is compact and $\pi(y, .)$ is continuous on A for any $y \in Y$ by Assumption 1. Let $\widetilde{X}_0 := \left[\underline{V}, \widetilde{V}\right]$, where $\widetilde{V} = \frac{\upsilon(\min\{\overline{y}, \overline{w}\}) - \underline{a}}{1-\beta_A}$. Here, $\widetilde{V}$ is chosen so that $V^{AP} \subset \widetilde{X}_0$. Let $\widehat{A} = \arg\max_{a\in A}\{E_a \upsilon(\min\{y, \overline{w}\}) - a\}$. Choose $\widehat{a} \in \widehat{A}$. Then, the stationary contract $\left\{\widehat{a}, \min\{y, \overline{w}\}, \widetilde{V}\right\}_{y\in Y}$ satisfies (5)-(7), (9) and guarantees a current utility of $E_{\widehat{a}} \upsilon(\min\{y, \overline{w}\}) + \beta_A \widetilde{V} - \widehat{a} \leq \widetilde{V}$. Assume a contract $\{a_-, w_+(y), V_+(y)\}_{y\in Y}$ that has $w_+(y) \leq y$, $\forall y \in Y$ and satisfies (5)-(7), (9) can guarantee a strictly higher current utility to the manager, i.e., $E_{a_-}\{\upsilon(w_+) + \beta_A V_+\} - a_- > E_{\widehat{a}} \upsilon(\min\{y, \overline{w}\}) + \beta_A \widetilde{V} - \widehat{a}$. However, we have that $E_{\widehat{a}} \upsilon(\min\{y, \overline{w}\}) + \beta_A \widetilde{V} - \widehat{a} \geq E_{a_-} \upsilon(\min\{y, \overline{w}\}) + \beta_A \widetilde{V} - a_- \geq E_{a_-}\{\upsilon(w_+) + \beta_A V_+\} - a_-$, so a contradiction is reached. By Proposition 14 from Morfov (2009), we obtain $\max V^{AP} = \frac{\max_{a\in A}\{E_a \upsilon(\min\{y,\overline{w}\})-a\}}{1-\beta_A}$. In case $\underline{a} \in \widehat{A}$, the convexity of V^{AP} is established as in the proof of Proposition 2. ■

Proof of Proposition 4. The maximum is obtained as in the proof of Proposition 2. Let $\widetilde{X}_0 := \left\{\left[\underline{V}_l, \widehat{V}\right]\right\}$. Take $\widehat{a} \in \arg\max_{a\in A}\{E_a\underline{V} - a\}$. Then, the stationary contract $\{\widehat{a}, \underline{w}, \underline{V}_l\}$ satisfies (5)-(7), (9) and guarantees the manager a current utility of $\upsilon(\underline{w}) + \beta_A E_{\widehat{a}}\underline{V} - \widehat{a}$. Assume that there exists another contract that satisfies (5)-(7), (9) and guarantees a strictly lower level of current utility to the agent. Let $\{a_-, w_+(y), V_+(y)\}$ be such a contract, i.e., $E_{a_-}\{\upsilon(w_+) + \beta_A V_+\} - a_- < \upsilon(\underline{w}) + \beta_A E_{\widehat{a}}\underline{V} - \widehat{a}$. Then, $E_{a_-}\{\upsilon(w_+) + \beta_A V_+\} - a_- \geq E_{\widehat{a}}\{\upsilon(w_+) + \beta_A V_+\} - \widehat{a} \geq \upsilon(\underline{w}) + \beta_A E_{\widehat{a}}\underline{V} - \widehat{a}$. where the first inequality follows from incentive compatibility and the second from (6) and (9). A contradiction is reached, so $\{\widehat{a}, \underline{w}, \underline{V}_l\}$ must bring minimum utility to the manager today. If $\upsilon(\underline{w}) + \beta_A E_{\widehat{a}}\underline{V} - \widehat{a} \leq \underline{V}_{\widehat{l}}$, we have that $\min \widetilde{B}_l\left(\widetilde{X}_0\right) = \underline{V}_l$, $\forall l \in L$ since $\underline{V}_{\widehat{l}} = \min_{l\in L}\{\underline{V}_l\}$. If $\upsilon(\underline{w}) + \beta_A E_{\widehat{a}}\underline{V} - \widehat{a} > \underline{V}_{\widehat{l}}$, $\min \widetilde{B}_{\widehat{l}}\left(\widetilde{X}_0\right) = \upsilon(\underline{w}) + \beta_A E_{\widehat{a}}\underline{V} - \widehat{a}$. Since applying $\widetilde{B}$ successively on $\widetilde{X}_0$ leads to a sequence of decreasing (nested) compact sets that converges to V^{AP}, we obtain that $\min V^{AP}\left(\widehat{l}\right) \geq \min \widetilde{B}_{\widehat{l}}\left(\widetilde{X}_0\right) > \underline{V}_{\widehat{l}}$. ■

Proof of Corollary 1. From Proposition 4, it is enough to show that $\max_{a\in A}\{\beta_A E_a\underline{V} - a\} > \underline{V}_{\widehat{l}} - \upsilon(\underline{w})$. We have that $\max_{a\in A}\{\beta_A E_a\underline{V} - a\} > \max_{a\in A}\left\{\beta_A \underline{V}_{\widehat{l}} - a\right\} = \beta_A \underline{V}_{\widehat{l}} - \underline{a} \geq \underline{V}_{\widehat{l}} - \upsilon(\underline{w})$, where the first inequality follows from the definition of $\underline{V}_{\widehat{l}}$, the assumption that for at least one $l \in L$, $\underline{V}_{\widehat{l}} = \min_{l\in L}\{\underline{V}_l\} < \underline{V}_l$, and $\pi(y,a) > 0$ from Assumption 1, the equality is trivial, and the last inequality results directly from $\underline{V}_{\widehat{l}} \leq \frac{\upsilon(\underline{w})-\underline{a}}{1-\beta_A}$. ■

Proof of Proposition 5. Analogous to the proof of Lemma 2 in the Appendix of Morfov (2009). ■

APPENDIX 2

Table 1
State Space of the Optimal AP Contract

Case		1		2		3	
		[	]	[	]	[	]
LLL		-3.1325	843.0178	-3.1325	29.4635	-3.1325	22.4035
	$y_{(1)}$	-1.4325	843.0178	-1.4325	29.4635	-1.4325	22.4035
LLM	$y_{(2)}$	-1.4325	843.0178	-1.4325	29.4635	-1.4325	22.4035
	$y_{(3)}$	0.0000	843.0178	0.0000	29.4635	0.0000	22.4035
	$y_{(1)}$	0.8075	843.0178	0.8056	29.4635	0.8050	22.4035
LLH	$y_{(2)}$	0.8075	843.0178	0.8056	29.4635	0.8050	22.4035
	$y_{(3)}$	3.6725	843.0178	3.6725	29.4635	3.6725	22.4030
	$y_{(1)}$	-0.1425	843.0178	-0.1460	29.4635	-0.1461	22.4035
LMM	$y_{(2)}$	0.0000	843.0178	0.0000	29.4635	0.0000	22.4035
	$y_{(3)}$	0.0000	843.0178	0.0000	29.4635	0.0000	22.4035
	$y_{(1)}$	0.8275	843.0178	0.8182	29.4635	0.8280	22.4035
LMH	$y_{(2)}$	0.8200	843.0178	0.8200	29.4635	0.8200	22.4035
	$y_{(3)}$	3.6725	843.0178	3.6725	29.4635	3.6725	22.4030
	$y_{(1)}$	3.3575	843.0178	3.3635	29.4635	3.3632	22.3724
LHH	$y_{(2)}$	3.6725	843.0178	3.6725	29.4635	3.6725	22.3783
	$y_{(3)}$	3.6725	843.0178	3.6725	29.4635	3.6725	22.3783
MMM		0.0000	843.0178	0.0000	29.4635	0.0000	22.4035
	$y_{(1)}$	0.8100	843.0178	0.8100	29.4635	0.8100	22.4035
MMH	$y_{(2)}$	0.8100	843.0178	0.8100	29.4635	0.8100	22.4035
	$y_{(3)}$	3.6725	843.0178	3.6725	29.4635	3.6725	22.4030
	$y_{(1)}$	3.3600	843.0178	3.3594	29.4635	3.3592	22.3623
MHH	$y_{(2)}$	3.6725	843.0178	3.6725	29.4635	3.6725	22.3703
	$y_{(3)}$	3.6725	843.0178	3.6725	29.4635	3.6725	22.3703
HHH		3.6725	843.0178	3.6725	29.4635	3.6725	22.4001

Table 2
Effects of Changing the Minimum Reservation Utility of the Principal
(LLL, case 1)

$\underline{U}$	0	5	10
$\overline{w}$	1145.5526	1010.7817	876.0108
$\widehat{V}$	843.0178	791.6873	736.8045
V^{AP}	[-3.1325, 843.0178]	[-3.1325, 791.6873]	[-3.1325, 736.8045]

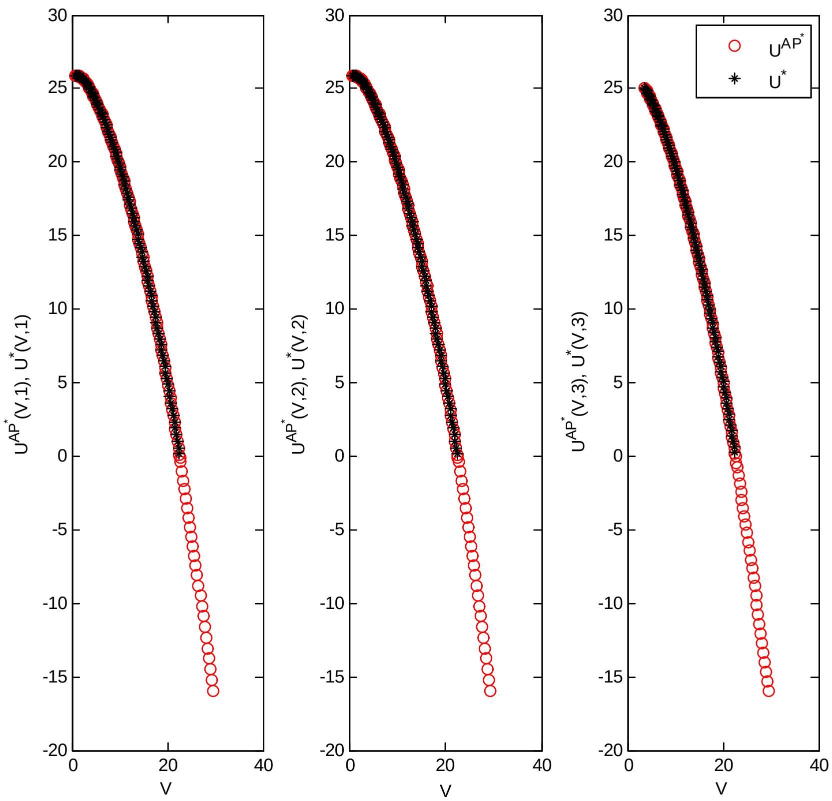

Figure 1: Value functions for the AP and 2P contracts ordered by initial history: $U^{AP^*}(.,l)$, $U^*(.,l)$, $l \in \{1,2,3\}$ (LMH, case 2)

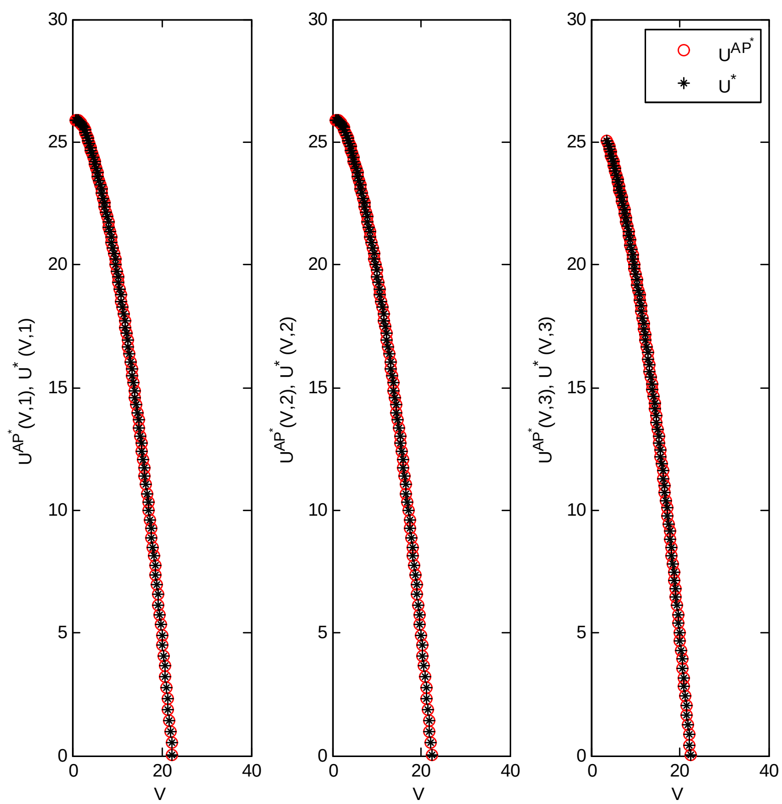

Figure 2: Value functions for the AP and 2P contracts ordered by initial history: $U^{AP^*}(.,l)\,,\, U^*(.,l),\, l \in \{1,2,3\}$ (LMH, case 3)

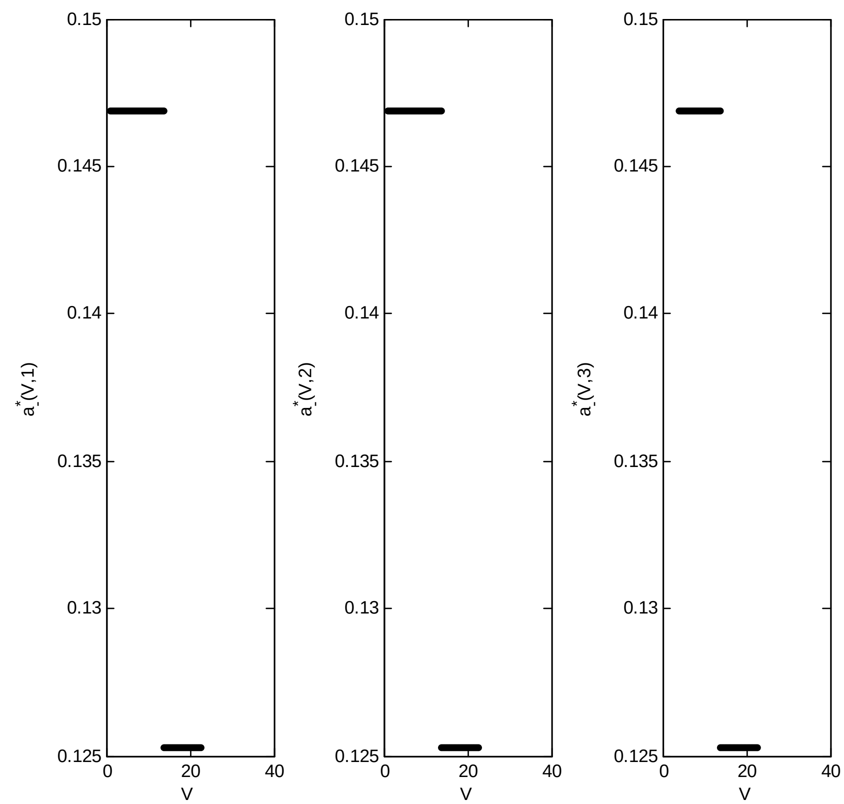

Figure 3: Optimal effort as a function of initial utility promise: $a_{-}^{*}(.,l)$, $l \in \{1,2,3\}$ (LMH, case 3)

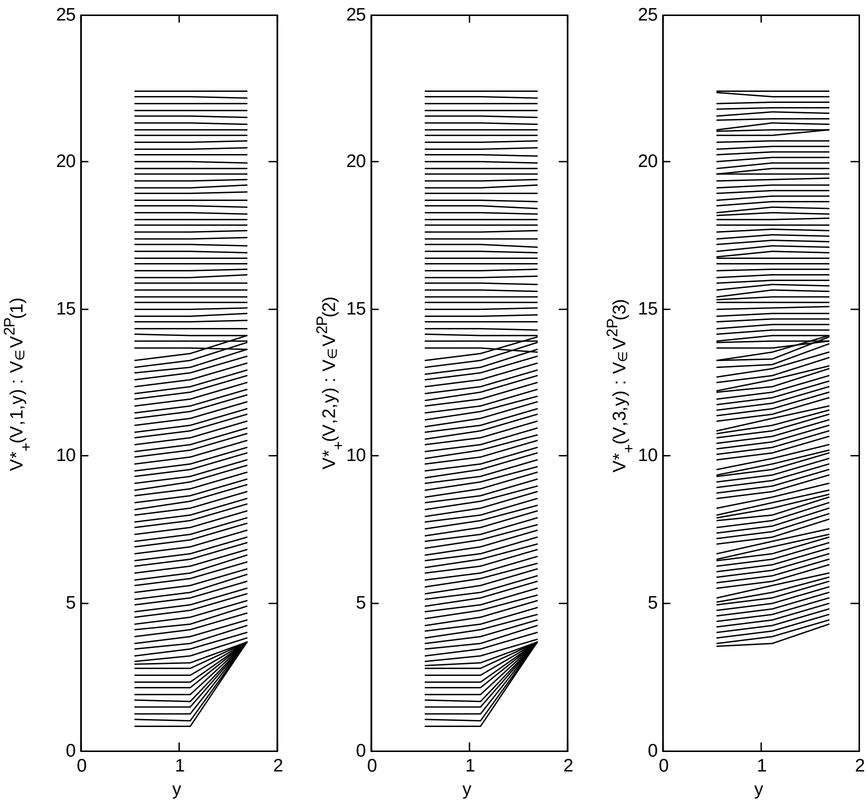

Figure 4: Optimal future utility promise as a function of future profit: $V_{+}^{*}(V, l, .)$: $V \in V^{2P}(l)$, $l \in \{1, 2, 3\}$ (LMH, case 3)

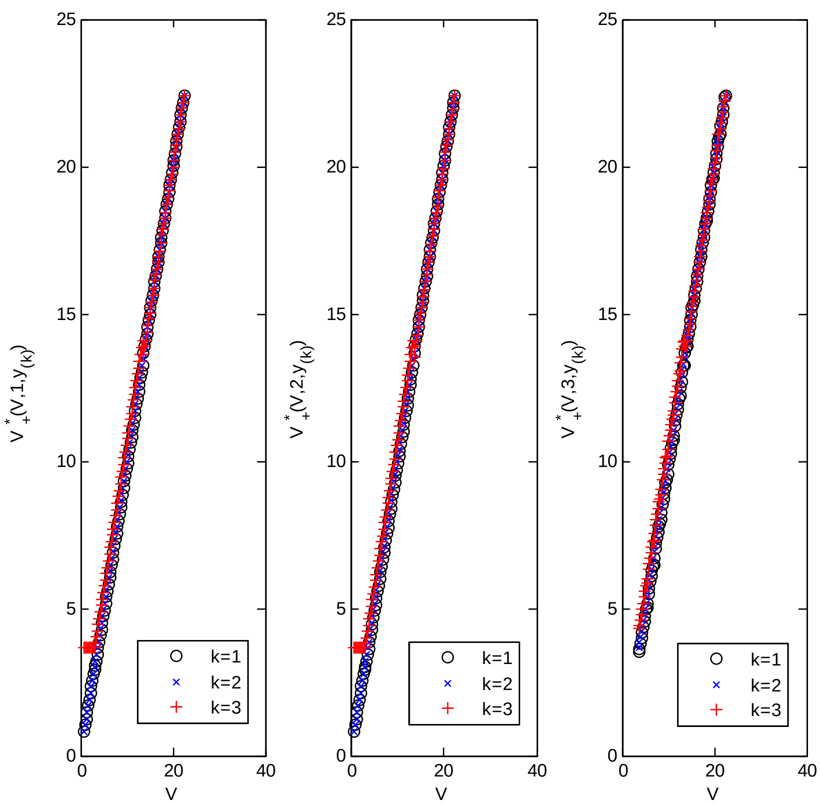

Figure 5: Optimal future utility promise as a function of initial utility promise: $V_+^*\left(.,l,y_{(k)}\right)$, $l,k \in \{1,2,3\}$ (LMH, case 3)

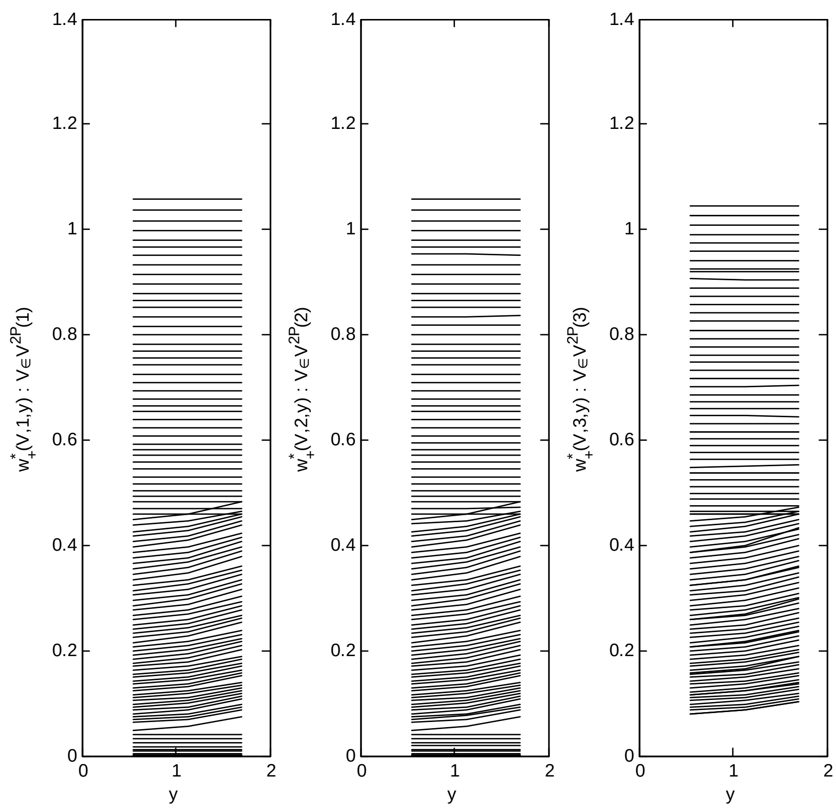

Figure 6: Optimal wage as a function of future profit: $w_+^*(V, l, .)$: $V \in V^{2P}(l)$, $l \in \{1, 2, 3\}$ (LMH, case 2)

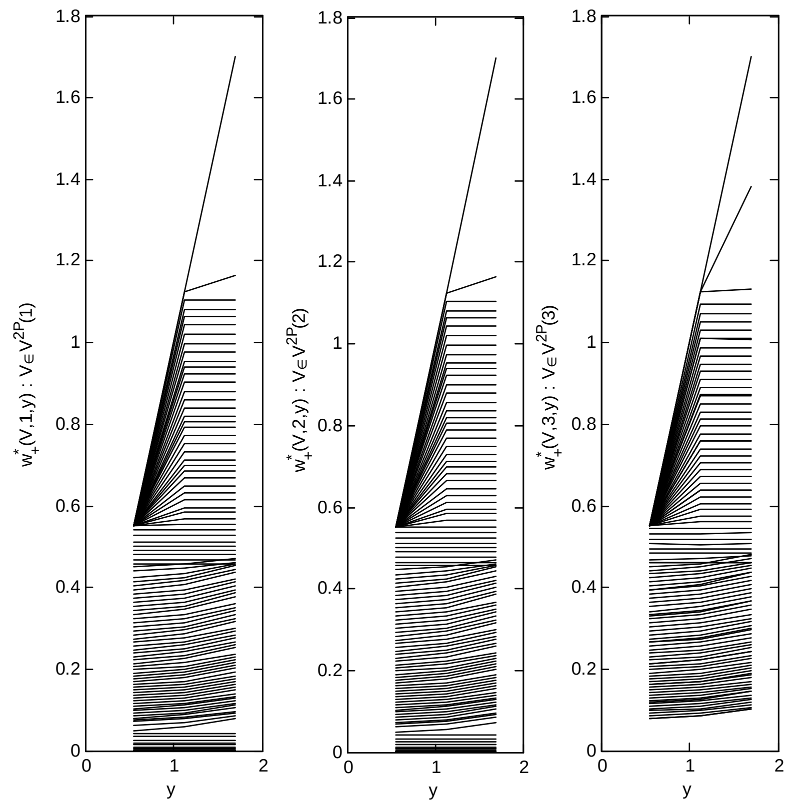

Figure 7: Optimal wage as a function of future profit: $w_+^* \left(V, l, .\right)$: $V \in V^{2P} \left(l\right)$, $l \in \{1, 2, 3\}$ (LMH, case 3)

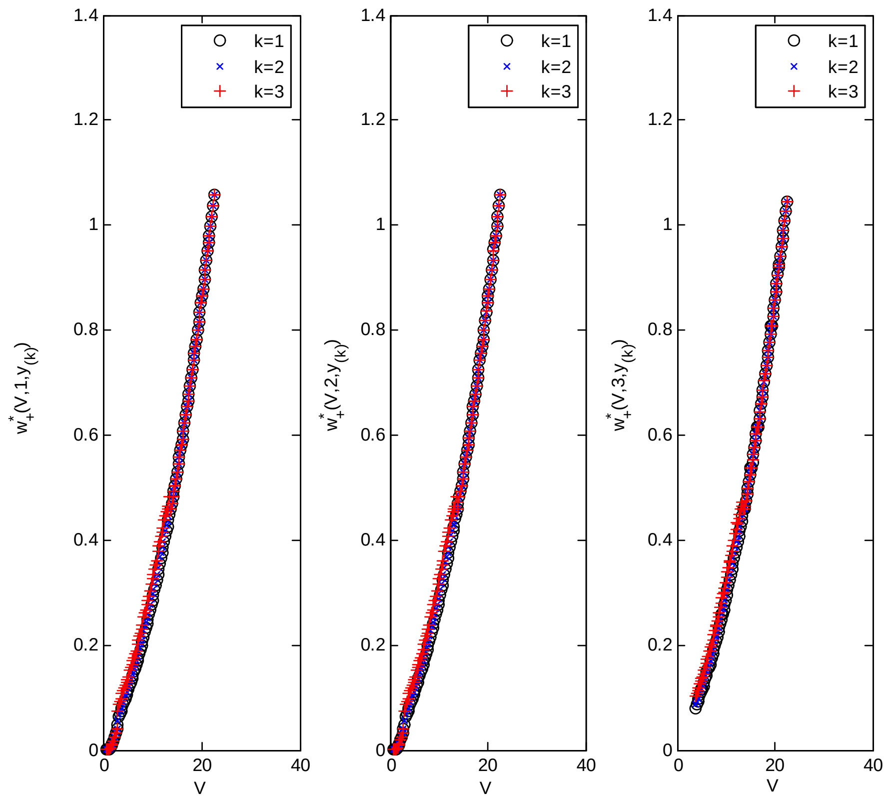

Figure 8: Optimal wage as a function of initial utility promise: $w_+^*\left(.,l,y_{(k)}\right)$, $l,k \in \{1,2,3\}$ (LMH, case 2)

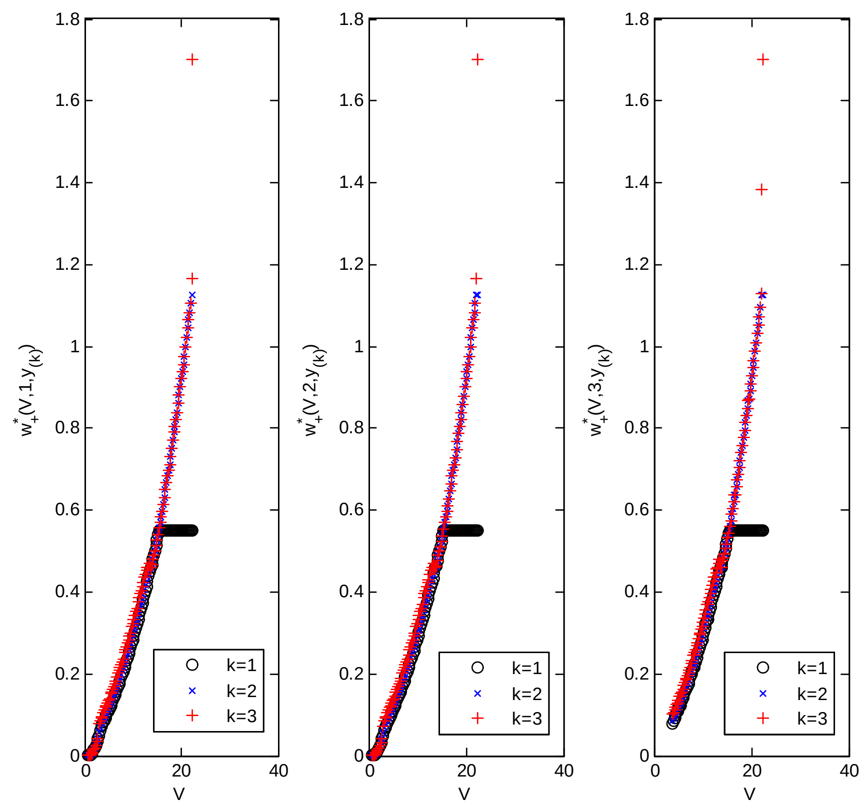

Figure 9: Optimal wage as a function of initial utility promise: $w_+^*\left(.,l,y_{(k)}\right)$, $l,k \in \{1,2,3\}$ (LMH, case 3)

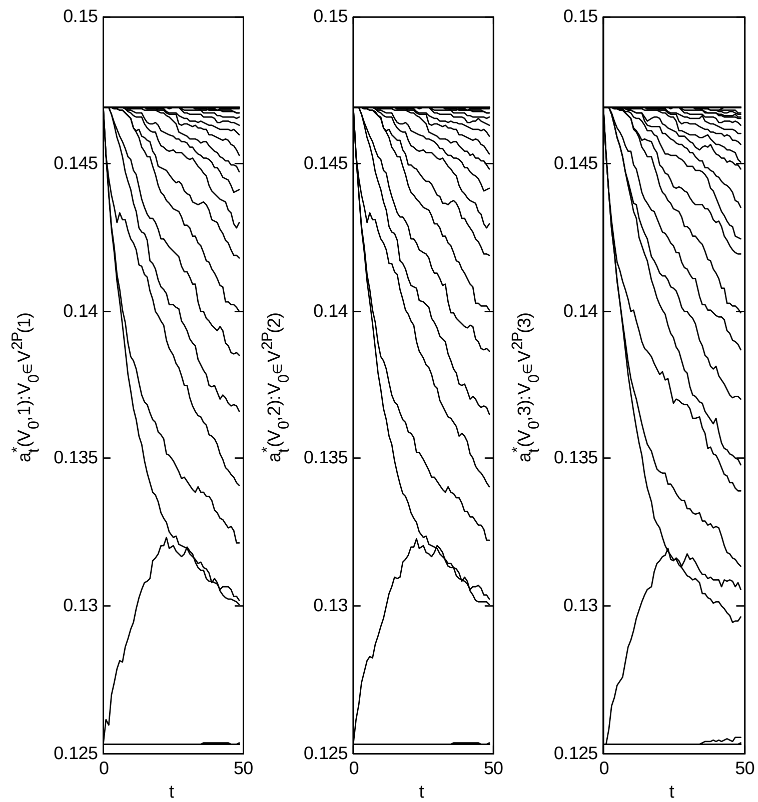

Figure 10: Optimal effort in time: $a_t\left(V_0, l\right)$: $V_0 \in V^{2P}\left(l\right)$, $l \in \{1, 2, 3\}$, LMH, case 3

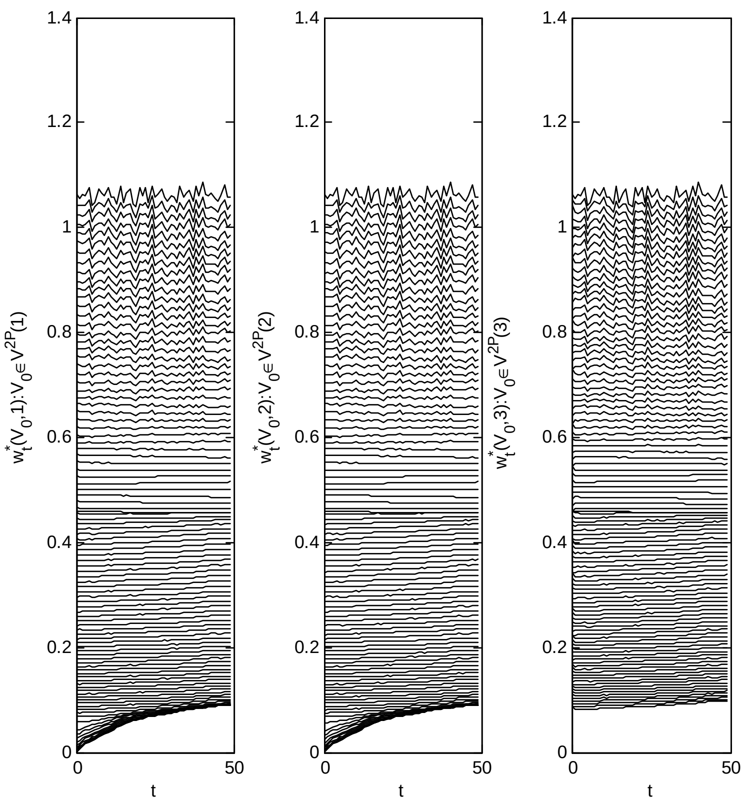

Figure 11: Optimal wage in time: $w_t\,(V_0, l)$: $V_0 \in V^{2P}\,(l)$, $l \in \{1, 2, 3\}$, LMH, case 3

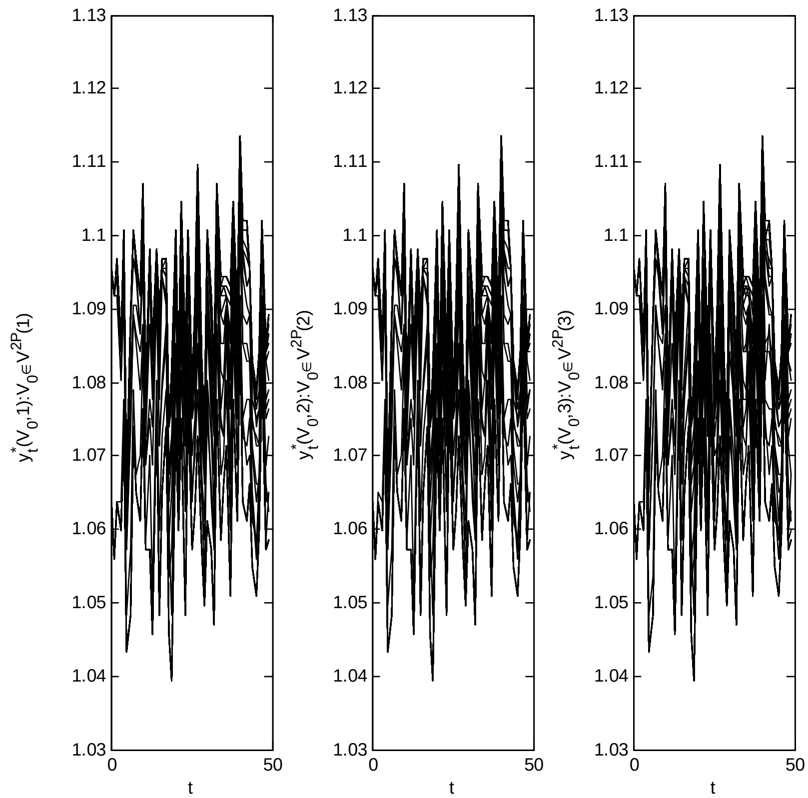

Figure 12: Firm's profit in time: $y_t\,(V_0, l)$: $V_0 \in V^{2P}\,(l)$, $l \in \{1, 2, 3\}$, LMH, case 3

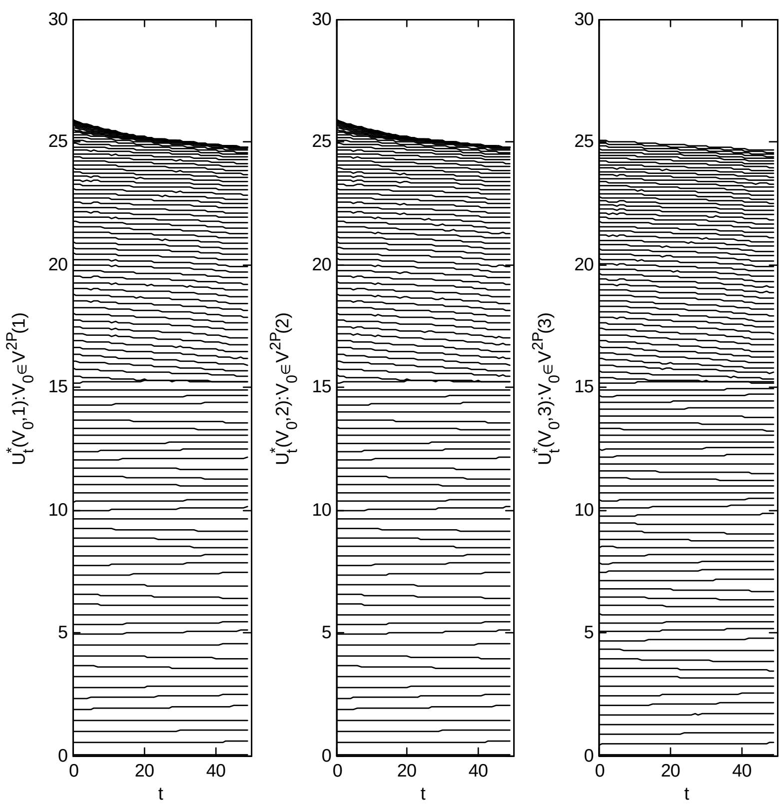

Figure 13: Principal's utility in time: $U_t\,(V_0, l)$: $V_0 \in V^{2P}\,(l)$, $l \in \{1, 2, 3\}$, LMH, case 3

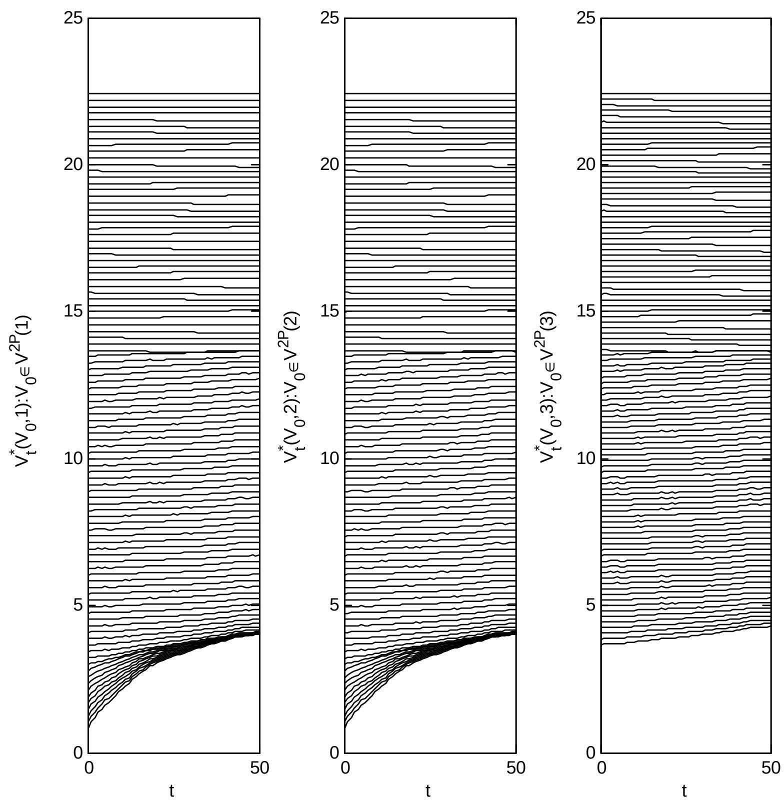

Figure 14: Manager's utility in time: $V_t\,(V_0, l)$: $V_0 \in V^{2P}\,(l)$, $l \in \{1, 2, 3\}$, LMH, case 3

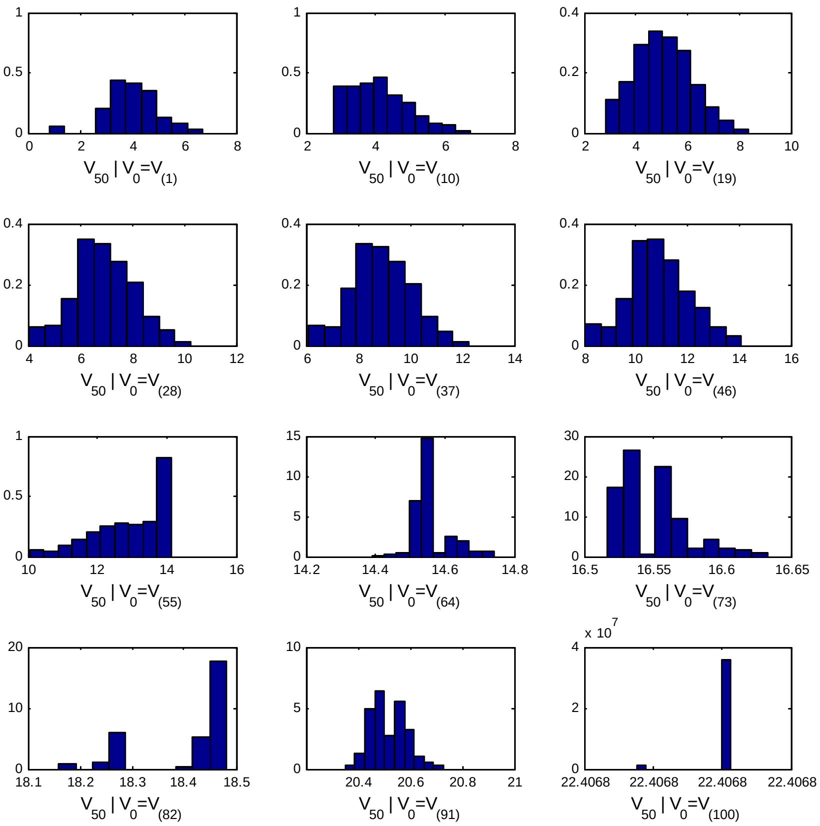

Figure 15: Empirical distribution of manager's utility after 50 periods, V_{50}, conditional on initial history $y_0 = y_{(1)}$ and initial utility promise $V_0 \in V^{2P}(y_0)$, LMH, case 3

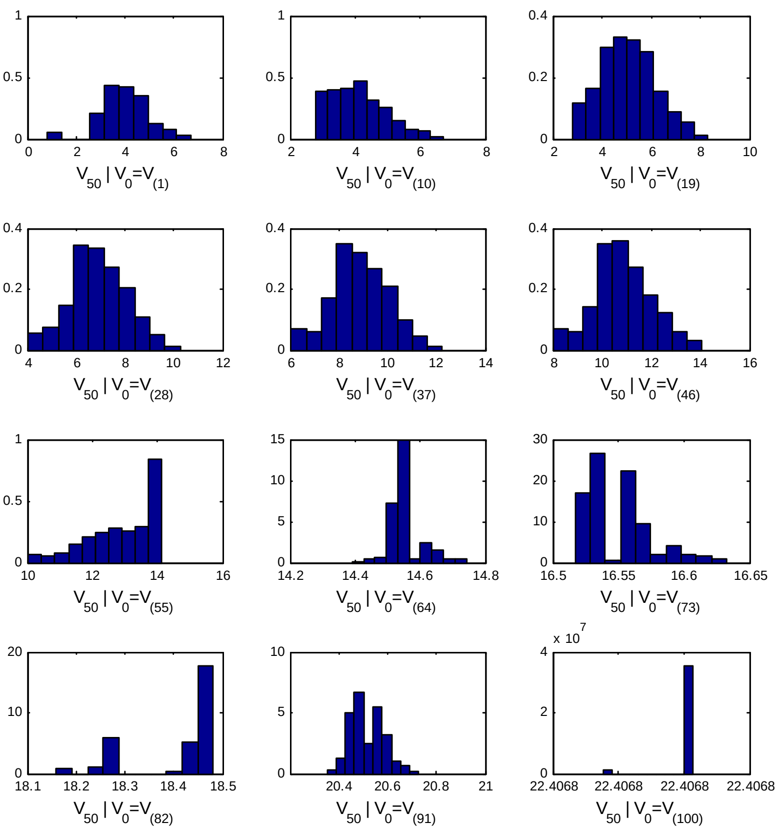

Figure 16: Empirical distribution of manager's utility after 50 periods, V_{50}, conditional on initial history $y_0 = y_{(2)}$ and initial utility promise $V_0 \in V^{2P}(y_0)$, LMH, case 3

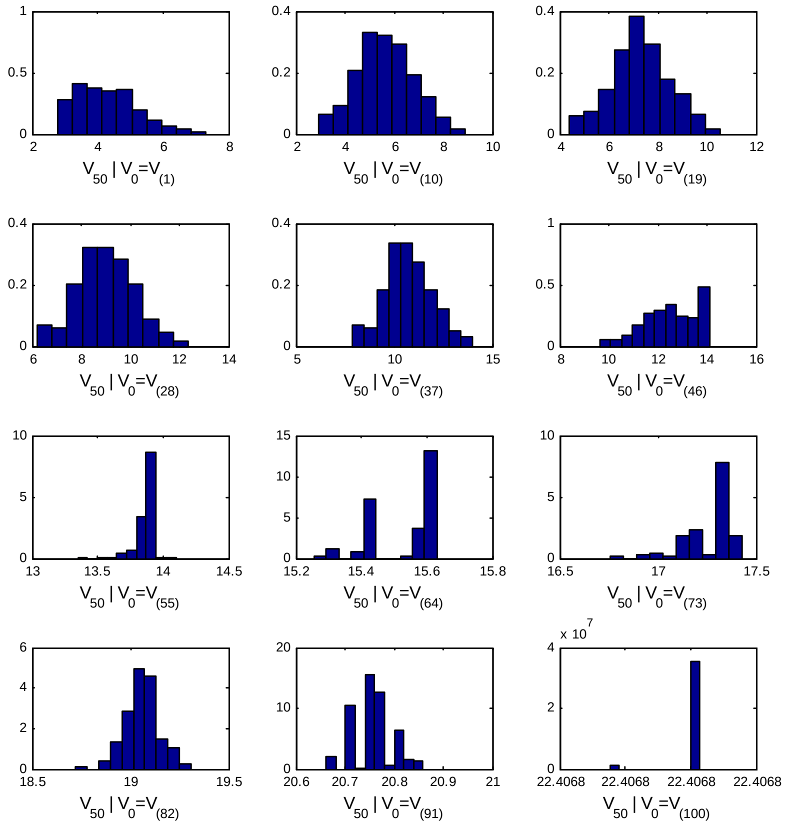

Figure 17: Empirical distribution of manager's utility after 50 periods, V_{50}, conditional on initial history $y_0 = y_{(3)}$ and initial utility promise $V_0 \in V^{2P}(y_0)$, LMH, case 3

Stock Options in the Manager's Compensation Package - A Dynamic View*

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March, 2009

Abstract

The paper uses a dynamic hidden-action framework marked by limited commitment and history-dependent reservation utilities to characterize the optimal incentive-compatible, self-enforcing contract that offers the agent a fixed salary and a stock option grant with a particular strike price. The model is parameterized and estimated in view of top executive compensation. The optimal stock option contract seems to motivate less wealthy managers to work hard, but high effort proves suboptimal for the richest CEOs. The compensation package shows little dynamics and only gains significance for high utility promises where the resulting compensation jumps due to an increased fixed salary and a big stock option grant with a low strike price. The future utility promise appears to be a more powerful incentive device. It tends to increase in the initial utility promise and, on average, grows with the stock price realization. The contract also offers some partial insurance against non-negligible fluctuations in the manager's outside options.

Keywords: principal-agent problem, moral hazard, dynamic contracts, executive compensation, stock options

Journal of Economic Literature Classification Numbers: C63, D82, G30

*I am particularly grateful to Manuel Santos for his guidance and support. I also wish to thank the members of my thesis committee and participants in the IX Annual Scientific Conference of the State University - Higher School of Economics for useful suggestions and comments.

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1 Introduction

Recently, there has been a widespread use of stock option contracts with respect to both firms' top managers[1] and rank-and-file workers[2]. With their simple, piecewise-linear payoff structure, these schemes are difficult to identify as the optimal incentive schemes predicted by economic theory.[3] While there is a growing body of literature searching for possible reasons[4] to account for the abundance of such contracts, very little has been done in terms of computing the optimal stock option contract and comparing it with the optimal contract per se.

Clementi, Cooley and Wang (2006) consider a two-period principal-agent model of hidden action and show that under severe commitment problems shareholders' value can potentially be improved by including stock options in the agent's compensation package. Aseff and Santos (2005) (hereafter, AS) analyze the properties of the optimal stock option contract obtained in a principal-agent framework by restricting the set of admissible contracts to canonical stock option contracts[5]. They calibrate the model and find that the cost of implementing the optimal stock contract vs. the optimal contract is negligible. However, their model is static and therefore fails to address issues such as smoothing consumption and incentives over time. Indeed, stock option grants are a purely dynamic phenomenon.

The current paper extends and generalizes the analysis of AS in a dynamic framework. The set-up is an infinite-horizon hidden-action problem characterized by two-sided limited commitment (a natural generalization of Phelan (1995)'s one-sided commitment) and history-dependent reservation utilities. The history dependence of the agent's outside options can be defended by economic considerations and may also be particularly relevant for the case of stock options as indicated in Oyer (2004). The space of admissible compensation schemes is exogenously restricted to the family of canonical stock option contracts; i.e., each period, the firm's shareholders can offer the manager only a package consisting of a fixed salary component and a stock option grant with a particular strike price. The framework can be effectively described by a model of the class analyzed in Morfov (2009a) and so can fully benefit from the recursive form constructed there. Namely, we can start from an auxiliary environment where the principal can commit to a long-term contract and consider him/her maximizing expected discounted utility over all incentive-compatible contracts that recommend an admissible plan of actions and offer a series of single-round stock option contracts. The auxiliary problem can be recursively characterized

[1] See, for example, Jensen and Murphy (2004).

[2] See, for example, Hall and Murphy (2003).

[3] See, for example, Stiglitz (1991).

[4] Reasons proposed include attraction, retention, motivation, sorting ("options-as-finance"), accounting and tax advantages [see Hall and Murphy (2003), Oyer and Schaefer (2005) for a review].

[5] A canonical stock option contract is fully characterized by a fixed salary, stock option grant and its strike price.

on an endogenous state space that matches price histories with utility promises to the agent. This state space is the largest fixed point of a set operator and is also the limit of this operator given a sufficiently big initial set. With the state space obtained, we can solve the recursive representation of the auxiliary problem by dynamic programming. The original problem where neither the principal nor the agent can commit to a long-term relationship is also recursively characterizable on a subset of the aforementioned state space. Both its state space and value function are recoverable by a modified dynamic programming routine using the value function of the auxiliary problem as an initial guess and severely punishing any violation of the principal's participation constraint.

I show theoretically that if the manager's reservation utilities are sufficiently dispersed, his/her participation constraint does not bind under the worst case scenario, which is also observed when the manager can essentially commit when his/her outside option is at its lowest value. In other words, the minimum utility the CEO can be promised for initial histories characterized by lower reservation utility is generally boosted by higher reservation utilities for other states.

Regarding the numerical computation, one point deserves special attention. In computing the endogenous state space we are iterating on sets and therefore need to represent them efficiently. For the class of infinitely repeated games with perfect monitoring, Judd, Yeltekin and Conklin (2003) are able to construct inner and outer convex polytope approximations based on the convexification of the equilibrium value set through a public randomization device. The algorithm I use may be of independent interest since it does not rely on the convexity of the underlying set. The main idea is to discretize the guess for the equilibrium set elementwise, extract small open balls around the gridpoints unfeasible with respect to the (non-updated) guess and use the remaining set, i.e. the guess less the extracted intervals, as a new guess for the equilibrium set. The procedure stops if the structure of the representations of two successive guesses coincides[6] and the suitably defined difference between the representations is less than some prespecified tolerance level.

The model is parameterized in line with the calibration of AS who derive the stock price distribution conditional on manager's effort based on the results of Hall and Liebman (1998) and take the value of low effort from Margiotta and Miller (2000).

The estimated value function is very flat for lower utility promises and very steep for higher utility promises. As under the optimal contract per se, the stock option contract induces less wealthy managers to work hard. High effort, however, proves suboptimal for the richest CEOs. The stock option package shows very little dynamics, it only plays a role for high utility promises where the resulting compensation jumps due to increased fixed salary and a big stock option grant with a low strike price. The future utility promise appears to be a more powerful incentive device. Manager's utility tomorrow tends to increase with the initial utility promise and, on average, grows with the stock price realization. The contract also appears to partly insure the manager against

[6] Namely, if the representations have the same number of closed sets element by element.

fluctuations in his/her outside options.

The rest of the paper is structured as follows. Section 2 presents the model in a general and recursive form. Section 3 parameterizes the model, comments on the numerical algorithm and discusses the results. Section 4 concludes. Appendix 1 contains the proofs. All the graphs are presented in Appendix 2.

2 Model

Consider the general framework described in Morfov (2009a). Namely, each period, a principal needs an agent to operate some stochastic technology transforming an action space A to a set of possible outcomes Y. While the particular realization $y \in Y$ is observed by both individuals who also know the history of y's so far, the particular action $a \in A$ exercised by the agent is his/her private knowledge. $A \subset \mathbb{R}$ and $Y \subset \mathbb{R}_+$ are compact, time- and history-invariant and Y is also assumed finite, say with n distinct elements. The distribution of outcomes conditional on a particular action choice is assumed independent and identical across past actions and outcomes. It is described by a commonly known probability function $\pi(., a) : Y \to [\underline{\pi}, 1]$ for any $a \in A$ with $\underline{\pi} > 0$, where $\pi(y, .)$ is continuous on A for every y. The principal is trying to design a long-term contract to optimally induce the agent to follow some (endogenous) action plan. The contract consists of a monetary transfer $w \in W \subset \mathbb{R}$ from the principal to the agent and a recommended action a on each contingency. Principal's utility at every node is state and time independent and is described by $u : W \times Y \to \mathbb{R}$ continuous, decreasing in its first argument and increasing in the second. At every node, the agent's utility is given by $\nu : W \times A \to \mathbb{R}$ continuous, increasing in its first argument and decreasing in the second. The principal and the agent discount the future by factors β_P and β_A respectively, where β_P, $\beta_A \in (0, 1)$. The principal and the agent also have some outside opportunities available at each contingency which are time-independent, but history-dependent. In particular, I restrict the history dependence to the previous θ outcomes, where θ is a non-negative integer. The limited commitment of both parties motivates the principal to offer a self-enforcing contract in the sense of Phelan (1995), i.e. a contract with continuation utilities that are weakly higher than the relevant reservation utilities at every node. Note that since actions are unobservable and non-verifiable, the agent may deviate from the recommended action plan if that brings him/her strictly higher utility. Therefore, the contract needs to be incentive-compatible, that is, to induce the proper incentives for the agent to comply with the suggested plan of actions.

The specifics are that at each contingency, the principal wants to use a monetary transfer consisting of a fixed and a state-contingent component with the latter having a fixed fraction of the payoff of a call option on y. More precisely, the transfer is described by a triple $\sigma_t := (\omega_t, b_t, p_t) \in \Omega \times [0, 1] \times Y$, where ω_t is the fixed component, p_t is the strike price of the call on y_t, and b_t

is the fraction of the payoff of the call. The the set Ω is assumed a compact subset of $\mathbb{R}$. The end-of-period transfer would then equal $w(\sigma_t, y_t) = \omega_t + b_t \max\{y_t - p_t, 0\}$ if the realization y_t is observed.[7]

Given that history dependence is only relevant through its impact on reservation utilities, we can normalize the initial time of contracting to equal 0 and concentrate on all the possible histories of length θ that could have been previously observed. Let this set of initial histories Y^θ be indexed using the bijective function $l : Y^\theta \to L := \{1, .., n^\theta\}$. Hereafter, we will commonly refer to a particular element of Y^θ by its index l. Let $l_+ : L \times Y \to L$ map today's initial histories of length θ and current outcomes to tomorrow's initial histories (of length θ). It gives the index corresponding to the stream of θ outcomes obtained from the current initial history by deleting the oldest (most leftward) outcome and adding the current outcome (by concatenating it to the right of the stream). Let U_l and V_l denote the reservation utilities of the principal and respectively the agent on any node $y^{t-1} \in Y^t \times l$, for any $l \in L$. A supercontract is defined as $c := (a, \sigma) = (a, \omega, b, p)$ where each element of c denotes a plan across all contingencies after a particular initial history. For example, at node y^{t-1}, the plan recommends an action $a_t(y^{t-1})$ and offers a compensation package described by $\sigma_t(y^{t-1})$. While it should be understood that the supercontract is defined on a particular initial history, i.e., we should actually write c_l instead of only c, I usually omit the subscript to ease up the notation. Then, given a long-term contract c, let $V_\tau(c, y^{\tau-1})$ and $U_\tau(c, y^{\tau-1})$ be the expected discounted utilities of the agent and respectively the principal at node $y^{\tau-1}$. Formally,

$$V_\tau\left(c, y^{\tau-1}\right) := \sum_{t=\tau}^{\infty} \beta_A^{t-\tau} \sum_{y_t \in Y} ... \sum_{y_\tau \in Y} \nu\left(w\left(\sigma_t, y_t\right), a_t\right) \prod_{i=\tau}^{t} \pi\left(y_i, a_i\left(y^{i-1}\right)\right)$$
$$U_\tau\left(c, y^{\tau-1}\right) := \sum_{t=\tau}^{\infty} \beta_P^{t-\tau} \sum_{y_t \in Y} ... \sum_{y_\tau \in Y} u\left(w\left(\sigma_t, y_t\right), y_t\right) \prod_{i=\tau}^{t} \pi\left(y_i, a_i\left(y^{i-1}\right)\right)$$

We will refer to a supercontract c as admissible if at any contingency y^{t-1} after the initial history l observed at the time of signing we have that $(a, \omega, b, p)\left(y^{t-1}\right) \in A \times \Omega \times [0, 1] \times Y$. Finally, let $\forall nai(l)$ denote "at any *node after* and *including* l". Now, we are ready to formulate the principal's problem.

[7] Note that we implicitly assume that at the beginning of each period the principal may only grant European-style options that expire at the end of the same period, which, of course, is quite simplistic given the vesting arrangements accompanying stock option grants in reality. As a matter of fact, this eliminates the inter-period retention effect of stock options and restricts the ability of the principal to substitute them for current cash. Note, however, that we would also prohibit the agent from borrowing and saving, which would decrease his/her ability to insure against the variance introduced in his/her income by the principal for incentive purposes. Although these two effects are far from cancelling each other, they do work in opposite directions, which increases the applicability of the current analysis. Moreover, even such a stylized view of stock-option grants does bring some important insights into how quasilinear compensation schemes may provide incentives and insurance in a dynamic setting.

At initial node $l \in L$ the principal is solving:

[PPso]

$\sup_c U_0(c, l)$ s.t.:

$$c \text{ admissible} \tag{1}$$

$$V_t(c, .) \geq V_t(c', .), \forall (c' = (a', \sigma) \text{ admissible}), \forall nai(l) \tag{2}$$

$$V_t\left(c, ., \widetilde{l}\right) \geq \underline{V_{\widetilde{l}}}, \forall nai(l) \tag{3}$$

$$U_t\left(c, ., \widetilde{l}\right) \geq \underline{U_{\widetilde{l}}}, \forall nai(l) \tag{4}$$

Constraint (1) guarantees that the contract specifies admissible actions and transfers at every node. (2) guarantees incentive compatibility at any node. Constraints (3) and (4) are the participation constraints for the agent and, respectively, the principal.

Assumption 1 $\exists c$: *(1) - (4) hold.*

This is a standard assumption requiring that the set of constraints is non-empty.[8]

Note that the only difference between the above problem (hereafter, referred to as [PPso]) and the problem [PP] defined in Morfov (2009a) is that the former imposes additional constraints on the contract and more specifically on the monetary transfers. By (1), $c(.) \in A \times \Omega \times [0,1] \times Y$ which is compact in the product topology. Moreover, the transfer $w(\sigma_t, y_t) = \omega_t + b_t \max\{y_t - p_t, 0\}$ is continuous in σ_t. Then, the analysis of Morfov (2009a) applies to the case considered here. Namely, we can show existence and characterize the problem recursively by three Bellman equations. The first characterizes the set of possible utility promises to the agent (matched with initial histories of length θ) that are supportable by an admissible, temporary incentive-compatible [after Green (1987)] supercontract that guarantees the participation of the agent, but not necessarily the one of the principal. This is in fact the state space of a second Bellman equation which recursively characterizes the optimal contract of the type described above. The third characterization deals with the optimal temporary incentive-compatible, self-enforcing contract, i.e., recursively characterizes the solution of [PPso].

[8] For a discussion of this assumption and a more detailed description of the environment, the reader is referred to the more general framework of Morfov (2009a).

In view of the future estimation of the model, I will give it some more structure.

Hereafter, I will treat the principal as a proxy for firm's shareholders and the agent as the firm's CEO. The stochastic technology operated by the agent will be considered a black box with manager's effort as an input and firm's stock price as an output. The monetary transfer from the principal to the agent will be the manager's compensation which will take the form of a pre-determined salary ω_t and a stock option grant b_t with a strike price p_t. Assume that the principal is risk neutral with end-of-period utility $y_t - w(\sigma_t, y_t)$. The manager is risk-averse in monetary compensation and experiences disutility of effort. His/her end-of-period utility is given by $\upsilon(w(\sigma_t, y_t)) - a_t$, where $\upsilon(.)$ is continuous, increasing and concave. Similarly to Morfov (2009b), relax the assumption that $\omega_t \in \Omega$, compact, to $\omega_t \geq \underline{\omega}$, where $\underline{\omega} \in \mathbb{R}$ can be considered a legal lower bound on wage (a minimum wage or some subsistence level of income). Then, it is trivial that $w(\sigma_t, y_t) \geq \underline{\omega}$ as well. Now, using the result of Proposition 1 in Morfov (2009b), we have that if $(a, \omega, b, p)(.) \in A \times [\underline{\omega}, \infty) \times [0,1] \times Y$ and (2)-(4) hold, then $w(\sigma_t, y_t) \leq \overline{\omega} := \underline{\omega} + \frac{1}{\pi}\left(\frac{\overline{y}-\omega}{1-\beta_P} - \underline{\underline{U}}\right)$ with $\overline{y} := \max Y$, and $\underline{\underline{U}} := \min_{l \in L} \underline{U}_l$. Consequently, $\omega_t \leq \overline{\omega}$, and so we can re-define $\Omega := [\underline{\omega}, \overline{\omega}]$. So, under these functional forms and a uniform lower bound on the fixed salary, we will have that the wage is also uniformly bounded from above. Therefore, I will not differentiate between $\omega \in [\underline{\omega}, \overline{\omega}]$ and $\omega \in [\underline{\omega}, \infty)$ with the clear understanding that imposing $\overline{\omega}$ as an upper bound on wages in the problem with one-sided commitment will result in a different problem, but the solution to the problem where nobody can commit will not change.[9]

Denote by 2Pso an incentive-compatible, self-enforcing supercontract in stock options, and by APso - an incentive-compatible supercontract under one-sided commitment in the sense of Phelan (1995) where the total payoff of the stock option package is uniformly bounded from above by $\overline{\omega}$.[10] Let $V^{APso} = \{V^{APso}(l)\}$ where $V^{APso}(l)$ is the set of possible utility promises to a manager supportable by an APso contract at l. Let V^{2Pso} be the corresponding Cartesian product of sets of initial expected discounted utilities of the CEOs signing a 2Pso contract with the shareholders of the firm. For every $V \in V^{APso}$, define $U^{APso^*}(V)$ as the vector of principal's maximum utilities $U^{APso^*}(V_l, l)$ achievable through

[9] In fact, I will also impose $w(\sigma_t, y_t) \leq \overline{\omega}$, $\forall y_t \in Y$ as it was originally derived. This does not affect the optimal self-enforcing contract and proves quite useful in applications.

[10] Note that we could have alternatively defined the APso contract as an incentive-compatible supercontract guaranteeing the participation of the CEO, but not necessarily the one of the principal, where the fixed salary component and not the total compensation (which also includes the realized payoff of the stock option grant) is uniformly bounded from above by $\overline{\omega}$. Note that $w(\sigma_t, y_t) \leq \overline{\omega}$ implies $\omega_t \leq \overline{\omega}$, so given that we are interested in the optimal 2Pso contract, using this alternative definition for APso would make no difference. Indeed, we have derived $\overline{\omega}$ in order to work with an explicit bound on wages without affecting the optimal solution. In fact, I care to bound the total compensation explicitly as well only since is makes the framework convenient to deal with some economically motivated extensions imposing additional restrictions on managerial compensation. Such restrictions may include $w(\sigma, y) \leq \overline{y}$, $\forall y \in Y$ (restricted borrowing) or $w(\sigma, y) \leq y$, $\forall y \in Y$ (no borrowing on part of the firm's shareholders).

signing an APso contract with an initial utility promise V_l at l. Analogously, for any $V \in V^{2Pso}$, $U^{so^*}(V)$ is the vector of the maximum utilities $U^{so^*}(V_l, l)$ the principal can get by signing a 2Pso contract offering V_l to the manager at l. Let $\widehat{U}^{so^*}$ be the extension of U^{so^*} on V^{APso} such that for any $V \in V^{APso}$, $\widehat{U}^{so^*}(V)$ is a vector with a general element $\widehat{U}^{so^*}(V_l, l) = U^{so^*}(V_l, l)$ if $V_l \in V^{2Pso}(l)$ and $\widehat{U}^{so^*}(V_l, l) := -\infty$ otherwise. Now, we are ready to define three important operators: $\widetilde{B}^{so}$, T^{so}, and $\underline{T}^{so}$.

For any $X \subset \mathbb{R}^{n^\theta}$, $\widetilde{B}^{so}(X)$ is a vector with a general element $\widetilde{B}^{so}_l(X) := \{V \in X_l : \exists$ a (single-round) stock option contract $c_R = \{a_-, \sigma_-, V_+(y) : y \in Y\}$ s.t.:

$$a_- \in A \tag{5}$$

$$\sigma_- = (\omega_-, b_-, p_-) \in \Omega \times [0,1] \times Y \tag{6}$$

$$w(\sigma_-, y) \leq \overline{\omega},\ \forall y \in Y \tag{7}$$

$$\sum_{y \in Y} [\upsilon(w(\sigma_-, y)) - a_- + \beta_A V_+(y)]\pi(y, a_-) = V \tag{8}$$

$$\sum_{y \in Y} [\upsilon(w(\sigma_-, y)) - a'_- + \beta_A V_+(y)]\pi(y, a'_-) \leq V,\ \forall a'_- \in A \tag{9}$$

$$V_+(y) \in X_{l_+(l,y)} \tag{10}$$

hold}.

For any function $U = \{U_l\}$ with $U_l : V^{APso}(l) \to \mathbb{R}$ upper semicontinuous (usc) and bounded with respect to the sup metric, and any $V \in \{V_l\} \in V^{APso}$, $T^{so}(U)_{(V)}$ is a vector with a general element defined as follows:

$$T^{so}_l(U)_{(V_l)} := \max_{c_R}\{\sum_{y \in Y}[y - w(\sigma_-, y) + \beta_P U_{l_+(l,y)}(V_+(y))]\pi(y, a_-)\} \text{ s.t.:}$$

$$(5) - (9) \text{ hold, and}$$

$$V_+(y) \in V^{APso}(l_+(l,y)),\ \forall y \in Y \tag{11}$$

For any $l \in L$ and $V \in V^{APso}(l)$, define $\Gamma_R(V, U, l) := \{c_R : (5)\text{–}(9), (11)$ hold at (V, l) and $U_{l_+(l,y)}(V_+(y)) \geq \underline{U}_{l_+(l,y)}, \forall y \in Y\}$ for some function $U : V^{APso} \to (\mathbb{R} \cup \{-\infty\})^{n^\theta}$. Let

$$\Lambda_R(V,U,l) := \begin{cases} \Gamma_R(V,U,l) & \text{if } U_l(V) \geq \underline{U}_l \\ \emptyset \text{ otherwise} \end{cases}$$

For any function $U = \{U_l\}$with $U_l : V^{APso}(l) \to \mathbb{R}\cup\{-\infty\}$ usc and bounded from above, and any $V = \{V_l\} \in V^{APso}$, $\underline{T}^{so}(U)_{(V)}$ is a vector with a general element

$$\underline{T}_l^{so}(U)_{(V_l)} :=$$
$$\begin{cases} -\infty \text{ if } \Lambda_R(V_l,U,l) = \emptyset \\ \max\limits_{\substack{c_R\in \\ \Lambda_R(V_l,U,l)}} \sum\limits_{y\in Y} [y - w(\sigma_-,y) + \beta_P U_{l_+(l,y)}(V_+(y))]\pi(y,a_-)\} \text{ otherwise} \end{cases}$$

Following the results of Morfov (2009a), the optimal 2Pso contract is recursively characterized in three steps[11]:

Step 1. Start with the set $\widetilde{X}_0 := \left\{\left[\underline{V}_l, \widehat{V}\right]\right\}$ where $\widehat{V} = \frac{\upsilon(\overline{\omega})-\underline{a}}{1-\beta_A}$ with $\underline{a} := \min A$ and iterate on the set operator $\widetilde{B}^{so}$ until convergence. The limit is V^{APso}.

Step 2. Take a function $U = \{U_l\}$ with $U_l : V^{APso}(l) \to \mathbb{R}$ usc and bounded with respect to the sup metric, $\forall l \in L$. Iterate on $T^{so}(.)$ until convergence. The limit is $U^{APso^*}(.)$.

Step 3. Take $U^{APso^*}(.)$ as an initial guess and iterate on $\underline{T}^{so}(.)$ until convergence. The limit is $\widehat{U}^{so^*}(.)$. Moreover, $V^{2Pso}(l) = \{V \in V^{APso}(l) : \widehat{U}^{so^*}(V,l) \geq \underline{U}_l\}$. Then, for all $V \in V^{2Pso}(l)$, we have $U^{so^*}(V,l) = \widehat{U}^{so^*}(V,l)$.

Next, we will briefly discuss some of the characteristics of the state space of the optimal APso contract. We start by analyzing the case where the manager's reservation utility is not history-dependent.

Proposition 1 *If $\theta = 0$, we have* $V^{APso} = \left[\max\left\{\underline{V}, \frac{\upsilon(\underline{\omega})-\underline{a}}{1-\beta_A}\right\}, \frac{\upsilon(\overline{\omega})-\underline{a}}{1-\beta_A}\right]$.

This proposition shows that the result established in Proposition 2 in Morfov (2009b) is not affected by restricting the space of admissible compensation schemes to canonical stock options. Indeed, abstracting from manager's participation, both the limits of V^{APso} can be achieved by setting the stock option grant b equal to zero and using only the fixed wage component at any contingency. Moreover, taking as initial guess $\left[\underline{V}, \frac{\upsilon(\overline{\omega})-\underline{a}}{1-\beta_A}\right] \supset V^{APso}$, at any iteration

[11] Step 1 generalizes on Abreu, Pearce and Stacchetti (1990). Step 2 is standard dynamic programming over upper semi-continuous, bounded functions. Step 3 is based on Rustichini (1998).

of $\widetilde{B}^{so}$ we can obtain the minimum and maximum points of the resulting set by one-period contracts satisfying (5)-(10) and inducing the lowest level of effort by promising a fixed salary and no stock option grant on any continuation node. Then, we can support any linear combination of the utility bounds by a linear combination of these contracts[12], and in the limit we would have a convex state space.

What would happen if we consider another, potentially tighter, bound on the compensation? Take, for example, the case when the firm can pay the manager no more than the discounted present value of its highest possible earnings stream, i.e. $w_t(., y_t) \leq \frac{\overline{y}}{1-\beta_P}$. The above bound is still quite loose, but consider a firm that can borrow up to $\overline{y} - y$ to pay the manager, i.e. $w_t(., y_t) \leq \overline{y}$. If the firm's shareholders are effectively prevented from borrowing, then the CEO's compensation cannot exceed the stock price realization, i.e. $w_t(., y_t) \leq y_t$. In case the bound is uniform on Y, the result of Proposition 1 remains valid. We can simply take $\overline{\omega}' = \min\left\{\overline{\omega}, \frac{\overline{y}}{1-\beta_P}\right\}$ or, respectively, $\overline{\omega}' = \min\{\overline{\omega}, \overline{y}\}$ and use it instead of $\overline{\omega}$ as the upper bound of Ω and on the left-hand side of (7). In the case of no borrowing, however, the bound varies on Y. Then, the following result proves useful.

Proposition 2 *If* $\theta = 0$, $w(., y) \leq y$, $\forall y \in Y$, *we have that* $\max V^{APso} = \frac{\max_{a \in A}\{E_a v(\min\{y, \overline{\omega}\}) - a\}}{1-\beta_A}$ *and if* $\frac{v(\underline{\omega})-a}{1-\beta_A} \geq \underline{V}$, *then* $\min V^{APso} = \frac{v(\underline{\omega})-a}{1-\beta_A}$.

This proposition is an analog of Proposition 3 of Morfov (2009b). This version, however, is quite weaker. The reason is that we may not be able to cover the interval between two initial utility promises if they are supported by contracts with a non-zero stock option component. In other words, a linear combination of the momentary utilities of total compensation resulting from two admissible stock option contracts is not necessarily attainable as the momentary utility of the total payoff of an admissible stock option contract. Nevertheless, the proposition shows that the upper bound of the state space, which was observed under the optimal AP contract can also be attained under the optimal APso contract.[13] Indeed, $w(\omega, b, p, y) = y$ for any $y \in Y$ if we take $\omega = \underline{y}$, $b = 1$, and $p = \underline{y}$, where $\underline{y} := \min\{Y\}$. That is, we achieve the compensation bound by fixing the manager's salary to the lowest stock price realization and providing him/her with the maximum stock option grant with a strike price equal to the firm's lowest stock price.

[12] More precisely, the combination is linear in the modified contracts with components: effort level, momentary utility of total compensation, and continuation utility. Given that the boundary contracts do not grant any stock options, the resulting contract will offer a compensation that only consists of a fixed salary which in turn can be easily recovered from the respective momentary utility.

[13] An AP contract [cf. Morfov (2009b)] is an incentive-compatible supercontract that guarantees the participation of the agent, but not necessarily the one of the principal. In fact, the APso contract is an AP contract restricting the managerial pay to a canonical stock-option package.

The following proposition characterizes the state space of the optimal APso contract when the CEO's reservation utilities depend only on the past period's outcomes (one-period dependence).

Proposition 3 *Let* $\theta = 1$ *and* $\underline{V}_{\widehat{l}} = \min_{l\in L}\{\underline{V}_l\}$. *Then,* $\max V^{APso}(l) = \frac{\upsilon(\overline{\omega})-a}{1-\beta_A}$, $\forall y \in Y$. *Moreover, if* $\max_{a\in A}\{\beta_A E_a \underline{V} - a\} > \underline{V}_{\widehat{l}} - \upsilon(\underline{\omega})$, *then* $\min V^{APso}\left(\widehat{l}\right) > \underline{V}_{\widehat{l}}$; *otherwise,* $\min V^{APso}(l) = \underline{V}_l$, $\forall l \in L$.

The result establishing when the manager will receive just his/her reservation utility under the optimal contract remains valid for stock options. Indeed, if we ignore manager's participation, the minimum and maximum utilities supportable by an admissible and temporary incentive compatible single-round contract, $\upsilon(\underline{\omega}) + \max_{a\in A}\{\beta_A E_a \underline{V} - a\}$ and $\upsilon(\overline{\omega}) + \beta_A \frac{\upsilon(\overline{\omega})-a}{1-\beta_A} - \underline{a}$, respectively, can be achieved by setting the fixed wage to $\underline{\omega}$ and $\overline{\omega}$, respectively, and granting no stock options at all. Therefore, if $\max_{a\in A}\{\beta_A E_a \underline{V} - a\} \leq \underline{V}_{\widehat{l}} - \upsilon(\underline{\omega})$, i.e., if the aforementioned minimum utility does not exceed any possible reservation utility of the manager, then the CEO's reservation utility associated to any past period outcome will be supportable by a contract which effort level and continuation utilities are a linear combination of the respective values for the boundary contracts described above, and which fixed wage component gives the manager a momentary utility equal to the linear combination of the momentary utilities of consumption in the boundary contracts. Then, CEO's participation will bind at the minimum of the state space for any initial history. If, on the contrary, $\max_{a\in A}\{\beta_A E_a \underline{V} - a\} > \underline{V}_{\widehat{l}} - \upsilon(\underline{\omega})$, participation will not bind at the lower bound of the set associated with a past outcome $\widehat{l}$ and resulting from any iteration of $\widetilde{B}^{so}$ (given an initial guess $\left\{\left[\underline{V}_l, \frac{\upsilon(\overline{\omega})-a}{1-\beta_A}\right]\right\} \supset V^{APso}$), and, therefore, will neither bind at the minimum of $V^{APso}\left(\widehat{l}\right)$. Which case is observed depends on the parameter values of the model, but in general, the smaller the variation of agent's reservation utilities across past outcomes, the higher the chance that his/her participation constraint binds under the optimal contract.

Corollary 1 *If* $\theta = 1$, $\underline{V}_{\widehat{l}} = \min_{l\in L}\{\underline{V}_l\} \leq \frac{\upsilon(\omega)-a}{1-\beta_A}$ *and* $\exists l \in L : \underline{V}_l > \underline{V}_{\widehat{l}}$, *then* $\min V^{APso}\left(\widehat{l}\right) > \frac{\upsilon(\omega)-a}{1-\beta_A}$.

This corollary says that if the manager does not have problems committing for some, but not all past outcomes, then at these natural commitment states he/she is sure to receive more than the value of his/her outside options.

Establishing the convexity of the state space is even more problematic under canonical stock options than under the optimal contract per se.[14] The

[14] Cf. Morfov (2009b).

additional complication comes from the non-convexity of total compensation under a stock option contract. When the reservation utility of the manager is constant across past outcomes, the problem disappears since both the minimum and the maximum of the state space can be guaranteed by contracts promising a fixed wage and no stock option grant in all contingencies. In this particular case, we can easily span all the intermediate utility values by combining the boundary contracts. However, if the agent's reservation utilities do vary across initial histories, this would generally be impossible. Even if the initial utility bounds are supportable by contracts recommending the same level of effort, we may not be able to combine them appropriately if the compensation offered contains a stock-option grant. We can certainly come up with sufficient conditions for convexity, but they would be stricter with stock options than under the optimal contract per se.

Next, we will define the constraint set of the stationary APso version of the [PPso] and the associated policy correspondence. For any $l \in L$ and $V \in V^{APso}(l)$, let $\Gamma_R^{APso}(V,l)$ be the set of stationary APso contracts guarantying the manager an initial utility of V at an initial history l, i.e. satisfying (5)-(9) and $V_+(y) \in V^{APso}(l_+(l,y))$, $\forall y \in Y$. Also define $G_R^{APso}(V,l)$ as the set of optimal stationary APso contracts, i.e. the subset of $\Gamma_R^{APso}(V,l)$ such that $U^{APso^*}(V,l) = E_{a_-}(y - w(\sigma_-, y) + \beta_P U_{l_+(l,y)}^{APso^*}(V_+(y)))$.

Proposition 4 *For any $l \in L$, $\Gamma_R^{APso}(.,l)$ is upper hemi-continuous on $V^{APso}(l)$.*

By Proposition 5 in Morfov (2009a), U^{APso^*} is upper semi-continuous (usc) and bounded.[15] If we could prove that Γ_R^{APso} is also lower hemi-continuous, then it could be shown that the value function U^{APso^*} is continuous on V^{APso} and that the policy correspondence G_R^{APso} is upper hemi-continuous on V^{APso}. However, even establishing sufficient conditions for the lower hemi-continuity of Γ_R^{APso} proves quite involved. Note that if assume $\underline{a} < \overline{a}$ and let $V_{\underline{a}}^{APso}$ and $V_{\overline{a}}^{APso}$ be the sets of managerial utilities supportable by stationary APso contracts that have continuation utilities in V^{APso} and implement low and, respectively, high effort, we cannot directly continue along the lines of Morfov (2009b) to establish some sufficient conditions for lower hemi-continuity on a subset of Γ_R^{APso}. The reason is that a convexity of V^{APso}, will not imply in general that $V_{\underline{a}}^{APso}$ and $V_{\overline{a}}^{APso}$ are convex because of possible stock option grants. Therefore, it should be noted that the discontinuity problem will not necessarily be attenuated by assuming a convex set A.

[15] By Propositions 5, 8 and 11 in Morfov (2009a), we also have that the extension of U^{so^*} on V^{APso}, $\widehat{U}^{so^*}$, is upper semi-continuous and bounded from above.

3 Computation and Results

The numerical estimation follows the algorithms described in Morfov (2009b). The computation of the model starts with solving for V^{APso}, the set of manager's expected discounted utilities supportable by an APso contract. While Proposition 14 from Morfov (2009a) gives the theoretical background for the estimation of V^{APso}, some caveats remain. In particular, $\widetilde{B}^{so}$ is a set operator and in order to apply the iterative procedure in practice we need an efficient representation of the sequence of sets $\left\{\widetilde{X}_i\right\}_{i=0}^{\infty}$. For the class of infinitely repeated games with perfect monitoring, Judd, Yeltekin and Conklin (2003) are able to construct inner and outer convex polytope approximations based on the convexification of the equilibrium value set through a public randomization device. Here, I follow a more general approach which does not rely on assuming that V^{APso} is convex or convexifying it by introducing public randomization. The main idea is to discretize the elements of the initial guess $\widetilde{X}_0$ and start extracting small open intervals, the midpoints of which are unfeasible with respect to $\widetilde{X}_0$. The extraction is done elementwise without updating the previous elements. In particular, I start from the discretization of the first[16] element of $\widetilde{X}_0$, find the points that cannot be supported by a one-period APso contract with a continuation utility profile contained in $\widetilde{X}_0$, i.e. the points of the discretization which are not in the first element of $\widetilde{B}^{so}\left(\widetilde{X}_0\right)$, and extract small open balls around these points. Next, I find the gridpoints in the second element of $\widetilde{X}_0$ which are unfeasible with respect to $\widetilde{X}_0$, extract their small open neighborhoods and proceed in a similar fashion until I cover all the elements of $\widetilde{X}_0$. The remaining set, i.e. $\widetilde{X}_0$ less the extracted intervals, becomes $\widetilde{X}_1$, our new guess for V^{APso}. Given that $\widetilde{X}_0$ is a vector of n^θ closed intervals in $\mathbb{R}$, each of the n^θ elements of $\widetilde{X}_1$ will be a finite union of closed intervals in $\mathbb{R}$. In order to increase efficiency, intervals with length less than some prespecified level are reduced to their midpoints. The procedure stops if for each element of $\widetilde{X}_i$ the number of closed intervals representing it equals the respective number for the same[17] element in $\widetilde{X}_{i-1}$ and, in addition, the representation of $\widetilde{X}_i$ differs from the representation of $\widetilde{X}_{i-1}$ by less than some prespecified tolerance level. In order to apply this stopping criterion, one still needs to construct a measure for the difference between representations. For this purpose, I find the difference in absolute terms between each endpoint (minimum or maximum point) of each interval of each element of $\widetilde{X}_i$ and $\widetilde{X}_{i-1}$ respectively and take the maximum one to be the difference between the representations of $\widetilde{X}_i$ and $\widetilde{X}_{i-1}$. This difference is well defined given that the two representations share the same structure, which is actually the first condition of the stopping criterion.

Once V^{APso} is obtained, it is elementwise discretized and used as a state

[16] Note that $\widetilde{X}_0$ is a Cartesian product of sets indexed by L.

[17] Here, 'same' refers to the index of the element, i.e. to the initial history to which it corresponds.

space in the dynamic program for obtaining $U^{AP so^*}$ as outlined in Proposition 7 in Morfov (2009a). At each iteration, the guess for $U^{AP so^*}$ being defined only on the discretization needs to be interpolated over the state space. Interpolation is also required in the subsequent iterative procedure which uses $U^{AP so^*}$ as an initial guess for $\widehat{U}^{so^*}$, the extension of U^{so^*} on $V^{AP so}$.

Compared to the estimation of the optimal contract per se [cf. Morfov (2009b)], the procedure for computing the optimal stock option contract proves much more demanding and computationally intensive. The difficulty comes from the fact that the manager is risk averse and the stock option contract is characterized by three different components. Therefore, we cannot use a change of variables to arrive to a set of linear constraints. Then, at each iteration, for each initial price history, and each feasible initial utility promise, we should maximize a non-linear function over a set of non-linear constraints.

The model is parameterized in line with the calibration of AS who obtain the stock price distribution conditional on high effort from the results of Hall and Liebman (1998), estimate the stock price distribution conditional on low effort, derive the value of high effort, and take the value of low effort from Margiotta and Miller (2000). The exact parameter values I use are shown below. The set of possible stock prices $Y = \{y_{(1)}, y_{(2)}, y_{(3)}\} = \{0.55,\ 1.125,\ 1.7\}$, the action space $A = \{\underline{a}, \overline{a}\} = \{0.1253,\ 0.1469\}$, the conditional probabilities $\pi\left(y_{(1)}, \underline{a}\right) = 0.1891$, $\pi\left(y_{(2)}, \underline{a}\right) = 0.7687$, $\pi\left(y_{(3)}, \underline{a}\right) = 0.0421$, $\pi\left(y_{(1)}, \overline{a}\right) = 0.1555$, $\pi\left(y_{(2)}, \overline{a}\right) = 0.7654$, $\pi\left(y_{(3)}, \overline{a}\right) = 0.0791$.[18, 19] I take $\underline{\omega} = 0$ and consider a common discount factor for both the manager and the principal $\beta_A = \beta_P = 0.96$. The period utility of monetary payoff w is $\upsilon(w) = \log(1 + w)$, where w can be considered in relative terms. Although I investigate the case of shareholders who are effectively unconstrained in their borrowing ($w(.) \leq \overline{\omega}$, or similarly, $w(.) \leq \frac{\overline{y}}{1-\beta_P}$) or face loose borrowing constraints ($w(.) \leq \overline{y}$), I concentrate on the computationally cleanest[20] case where the shareholders are prevented from borrowing ($w(.) \leq y$). The reservation utility of the principal is taken constant across stock price histories at $\underline{U} = 0$. For the no-borrowing case, a loose upper bound for the initial utility promise is $\widehat{V} = \frac{\upsilon(\overline{\omega}) - \underline{a}}{1-\beta_A}$. I take $\theta = 1$, which may also cover the case of $\theta = 0$. I consider three possible values for the reservation utility of the agent: L= $\frac{\upsilon(\underline{\omega}) - \max A}{1-\beta_A}$ = -3.6725, M = 0, H = -L.[21] Then, I focus on the case of

[18] AS consider continuous distributions over an interval of possible stock option prices [0.55, 1.7]. Here, I choose 3 points of this interval: the minimum, middle, and maximum point and construct discrete probabilities that retain the first two moments of each distribution.

[19] The probabilities conditional on $\underline{a}$ may seem to add to 0.9999 instead of 1. The reason is that here I report the numbers at a much lower precision than the one used in the numerical computation. For example, if we round the probabilities to the fifth digit after the decimal point rather than to the fourth, the reported numbers would be 0.18915, 0.76871, and 0.04214 respectively.

[20] Here, the initial guess for the state space is smallest, so the grid is not so coarse as in the other cases which reduces the numerical mistakes related to working with approximations and very large absolute values for the second-stage value function in the uppermost region of its domain, i.e. at high utility promises which will later be impossible to support by an 2Pso contract.

[21] Note that L is actually the theoretical lowest bound of manager's utility.

nonnegative correlation between initial histories and agent's reservation utilities. I use the above letters to describe the manager reservation utility values across initial histories. For example, LMH stays for $\underline{V}_1 = $ L, $\underline{V}_2 = $ M, $\underline{V}_3 = $ H.

The results suggest that, as was the case for the optimal contract, there is little difference between the value function of the auxiliary and the original problem, U^{APso^*} and U^{so^*} respectively (see Figure 1). A reasonable explanation is that when shareholders are prohibited from borrowing, they have only a limited amount to spare for manager's compensation and so the chance that they violate their own participation constraint (under an APso contract) is much smaller than in the cases where they have easy access to credit. As suggested by Proposition 3, the insurance across initial history states is still present; namely, the minimum utility supportable by an APso/2Pso contract for initial histories characterized by lower reservation utility is boosted by higher reservation utilities for other states. As under the optimal contract, the participation constraint of the manager does not bind in states characterized by low stock prices (and, therefore, with lower reservation utilities). In other words, the APso/2Pso contract still allows the risk averse manager to smooth his/her consumption across (initial history) states. The estimated value function is very flat for lower utility promises and very steep for higher utility promises.

As under the optimal contract per se, the stock option contract induces less wealthy managers to work hard, while high effort proves suboptimal for the richest CEOs (Figure 2). As expected, the wealthiest (in utility terms) individuals prove the most difficult to reward or punish, but the stock option contract still manages to motivate a big bulk of CEOs, including the mid-range. Inspecting the optimal values for the elements of the stock option compensation package: the fixed wage (Figure 3), the stock option grant (Figure 4), and the strike price (Figure 5) proves unrewarding. The results seem very noisy with constant jumps of all the values except for the fixed component ω which remains at zero for most of the domain and jumps up at its uppermost region (highest utility promises).One should have in mind, however, that if the stock option grant is 0, the strike price is without any significance. Also, if the strike price is set equal to the highest stock price realization, the stock option grant has no importance whatsoever. Finally, unlike the optimal contract case where compensation is described by a single variable, here we deal with three[22], so an inspection of the resulting payoff conditional on the stock price realization makes much more sense. The payoff of the stock option package is depicted on Figure 6 as a function of the initial utility promise and the realized stock price. The resulting compensation shows very little dynamics: it only has a role for high utility promises where the resulting compensation jumps (Figure 7) due to

[22] Still, they are predetermined with respect to the stock price realization and cannot span the whole space of wages achievable when solving for the optimal contract. Moreover, the strike prices are restricted to take values only in Y. This is done in order to simplify the heavy computation algorithm, but one should also have in mind that in reality most stock options are issued at the money (AS show that the strike price has an intermediate role in the provision of insurance and incentives and that deviations from its optimal value can be partially compensated by realigning both the stock option grant and the fixed salary).

an increased fixed salary and a big stock option grant with a low strike price. For this uppermost part of the utility domain, the stock option grant is substantial and the pronounced increase of the payoff in the stock price realization (Figure 8) is a natural consequence of the profitable exercise of stock options. For the rest of the utility domain, the compensation is constant at zero and all the action comes from the future utility promise. Figure 9 shows the expected discounted utility of the manager tomorrow as a function of today's utility promise and the stock price realization. Compared to the optimal contract the results are very noisy. Manager's utility tomorrow tends to increase in the initial utility promise (Figure 10) and on average grows with the stock price realization (Figure 11).

4 Conclusion

The paper uses a dynamic hidden-action framework marked by limited commitment and history dependent reservation utilities to characterize the optimal incentive-compatible, self-enforcing contract that offers the agent a fixed monetary transfer and a fraction of the payoff of a call option on an outcome of interest to the principal. The model is shown to be of the class described in Morfov (2009a) and is given a recursive representation. Then, some more structure is added and the attention shifts to executive compensation in the form of canonical stock option contracts. The model is parameterized and computed numerically. The results indicate that while the optimal stock option contract induces most of the managers to exert high effort, shirking may still be optimal for the richest CEOs. In fact, the compensation package shows very little dynamics, it only has a role for high utility promises where the resulting compensation jumps due to an increased fixed salary and a big stock option grant with a low strike price. Given the restrictions imposed on the current payoff, the future utility promise appears to be a more powerful incentive device: it tends to grow with current utility and, on average, increases with the stock price. The contract also offers some partial insurance (in terms of promised initial utility) against (non-negligible) fluctuations in the manager's outside options.

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APPENDIX 1

Proof of Proposition 1. Note that $\min \widetilde{B}^{so}\left(\left[\underline{V}, \frac{\upsilon(\overline{\omega})-a}{1-\beta_A}\right]\right) = \max\{\underline{V}, \upsilon(\underline{\omega}) + \beta_A \underline{V} - \underline{a}\}$ and $\max \widetilde{B}^{so}\left(\left[\underline{V}, \frac{\upsilon(\overline{\omega})-a}{1-\beta_A}\right]\right) = \upsilon(\underline{\omega}) + \beta_A \frac{\upsilon(\overline{\omega})-a}{1-\beta_A} - \underline{a}$. Note that an initial utility of $V_L = \upsilon(\underline{\omega}) + \beta_A \underline{V} - \underline{a}$ can be supported by an admissible, temporary incentive-compatible one-period contract $\{\underline{a}, \underline{\omega}, 0, p, \underline{V}\}$ for any $p \in Y$. On the other hand, $\left\{\underline{a}, \overline{\omega}, 0, p, \frac{\upsilon(\overline{\omega})-a}{1-\beta_A}\right\}$, $\forall p \in Y$, is a one-period contract satisfying (5)-(10) for $V_H = \upsilon(\overline{\omega}) + \beta_A \frac{\upsilon(\overline{\omega})-a}{1-\beta_A} - \underline{a}$. Then, we can support any utility value between V_L and V_H by taking linear combinations of $\{\underline{a}, \upsilon(w(\underline{\omega}, 0, p)), \underline{V}\}$ and $\left\{\underline{a}, \upsilon(w(\overline{\omega}, 0, p)), \frac{\upsilon(\overline{\omega})-a}{1-\beta_A}\right\}$. Note that this is possible since the contracts supporting the bounds have $b = 0$, so p is immaterial. Then, we have that $\widetilde{B}^{so}\left(\left[\underline{V}, \frac{\upsilon(\overline{\omega})-a}{1-\beta_A}\right]\right)$ which is a compact set is in fact a closed interval. Since $V^{APso} \subset \left[\underline{V}, \frac{\upsilon(\overline{\omega})-a}{1-\beta_A}\right]$, by Proposition 14 from Morfov (2009a), we have that V^{APso} is also a closed interval. With the above qualifications regarding the admissibility of taking linear combinations, we can apply the argument of the proof of Proposition 2 in Morfov (2009b) to show that $V^{APso} = \left[\max\left\{\underline{V}, \frac{\upsilon(\underline{\omega})-a}{1-\beta_A}\right\}, \frac{\upsilon(\overline{\omega})-a}{1-\beta_A}\right]$. ■

Proof of Proposition 2. Once we notice that $w\left(\underline{y}, 1, \underline{y}, y\right) = y$, $\forall y \in Y$, the results are obtained along the lines of the proof of Proposition 3 in Morfov (2009b). ■

Proof of Proposition 3. Note that for any $p \in Y$, $w(\underline{\omega}, 0, p, .) = \underline{\omega}$ on Y. Then, the results follow from the proof of Proposition 4 in Morfov (2009b). ■

Proof of Corollary 1. Analogous to the proof of Corollary 1 in Morfov (2009b). ■

Proof of Proposition 4. Analogous to the proof of Lemma 2 in the Appendix of Morfov (2009a). ■

APPENDIX 2

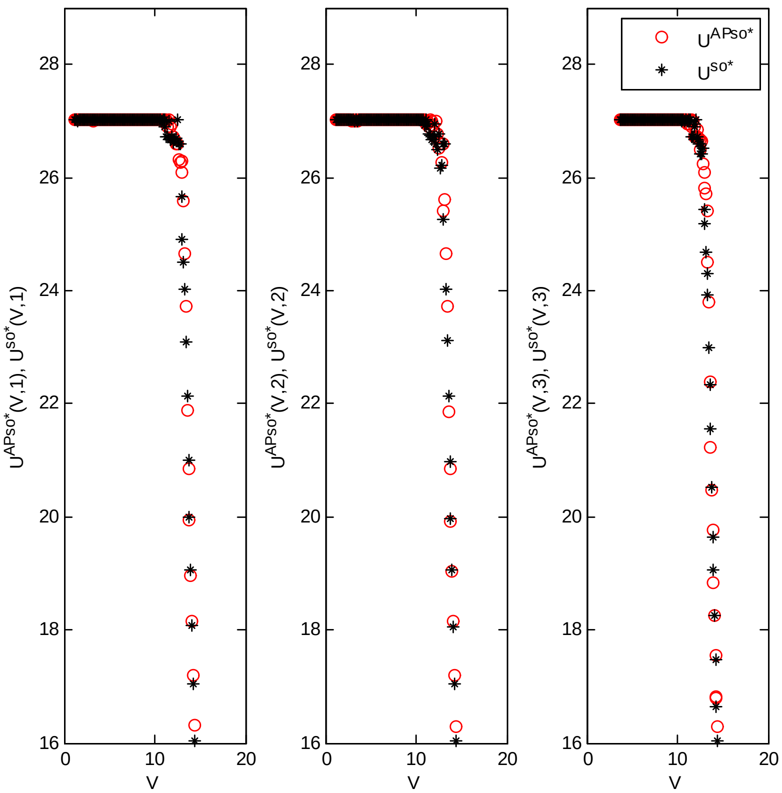

Figure 1: Value functions for the APso and 2Pso contracts ordered by initial stock-price history: $U^{APso^*}(.,l)$, $U^{so^*}(.,l)$, $l \in \{1,2,3\}$ (LMH, case 3)

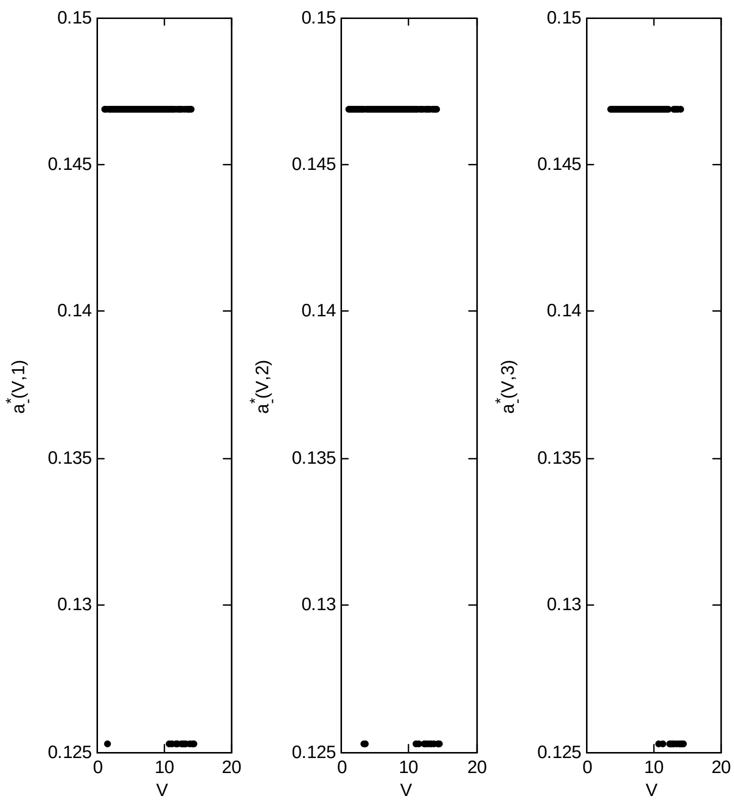

Figure 2: Optimal effort under the 2Pso contract as a function of initial utility promise: $a^*_-(.,l)$, $l \in \{1,2,3\}$ (LMH, case 3)

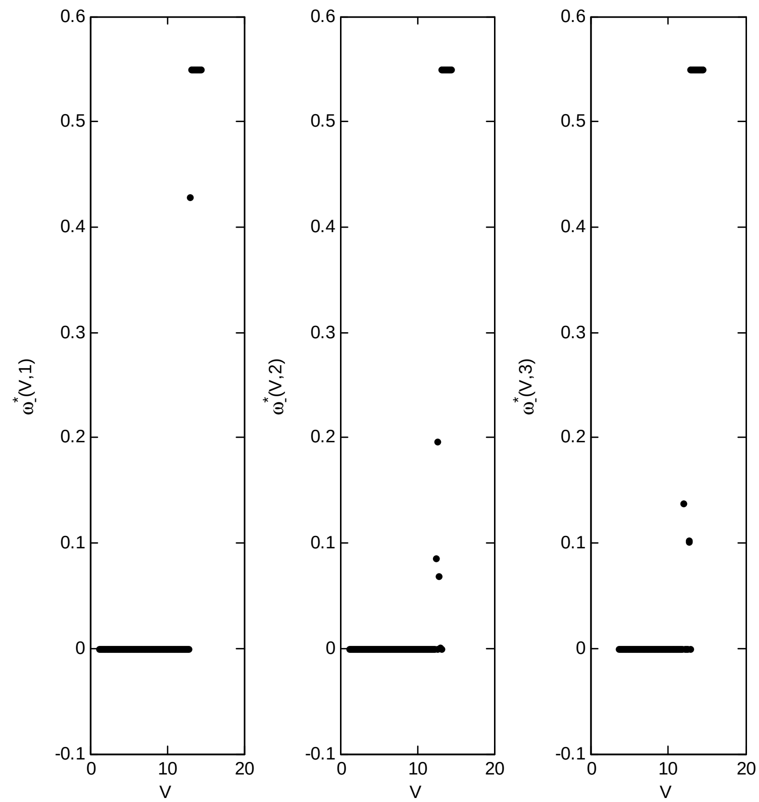

Figure 3: Optimal fixed wage component under the 2Pso contract as a function of initial utility promise: $\omega^*_-(.,l)$, $l \in \{1,2,3\}$ (LMH, case 3)

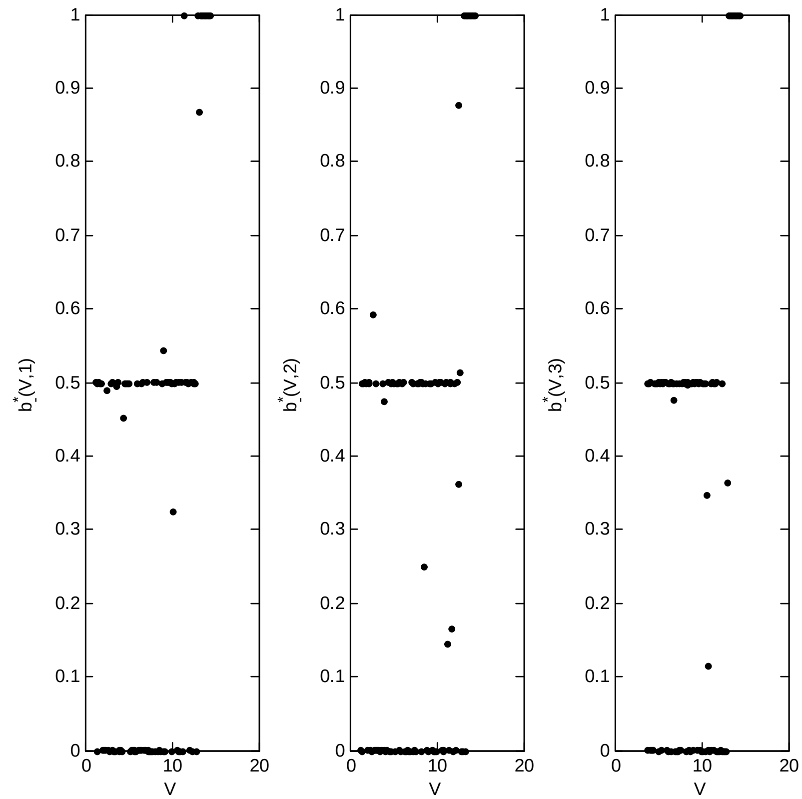

Figure 4: Optimal stock option grant under the 2Pso contract as a function of initial utility promise: $b^*_-(.,l)$, $l \in \{1,2,3\}$ (LMH, case 3)

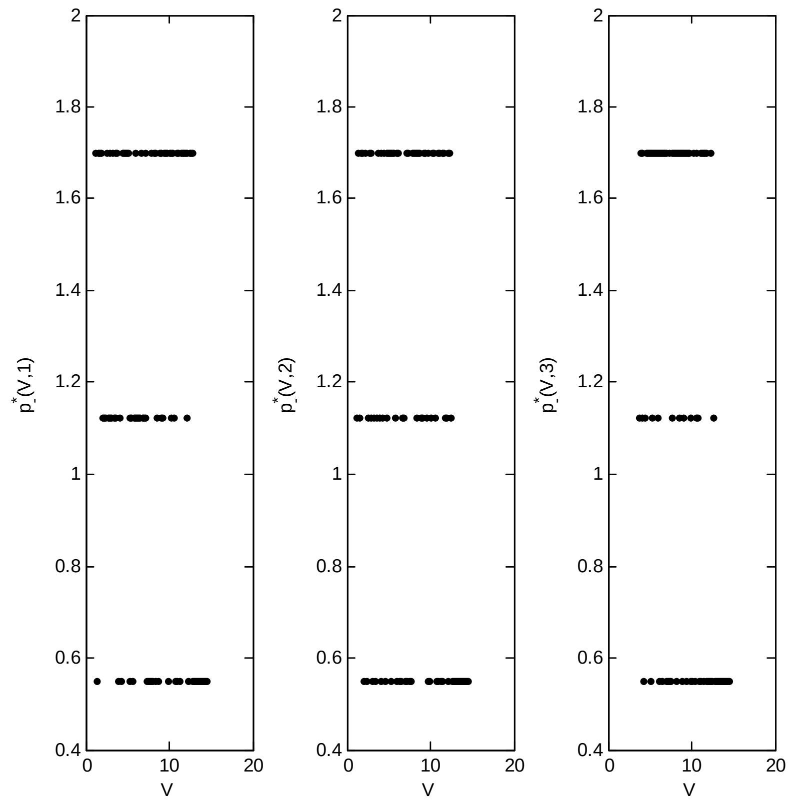

Figure 5: Optimal strike price under the 2Pso contract as a function of initial utility promise: $p^*_-(,l)$, $l \in \{1, 2, 3\}$ (LMH, case 3)

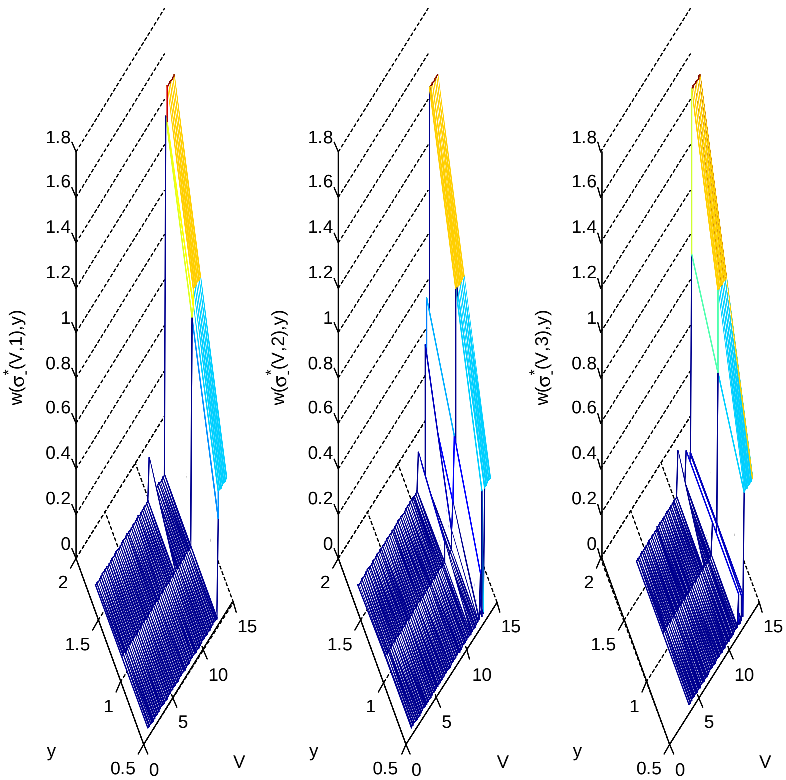

Figure 6: Optimal total compensation under the 2Pso contract as a function of initial utility promise and future stock price: $w\left(\sigma_{-}^{*}\left(.,l\right),.\right)$, $l\in\{1,2,3\}$ (LMH, case 3)

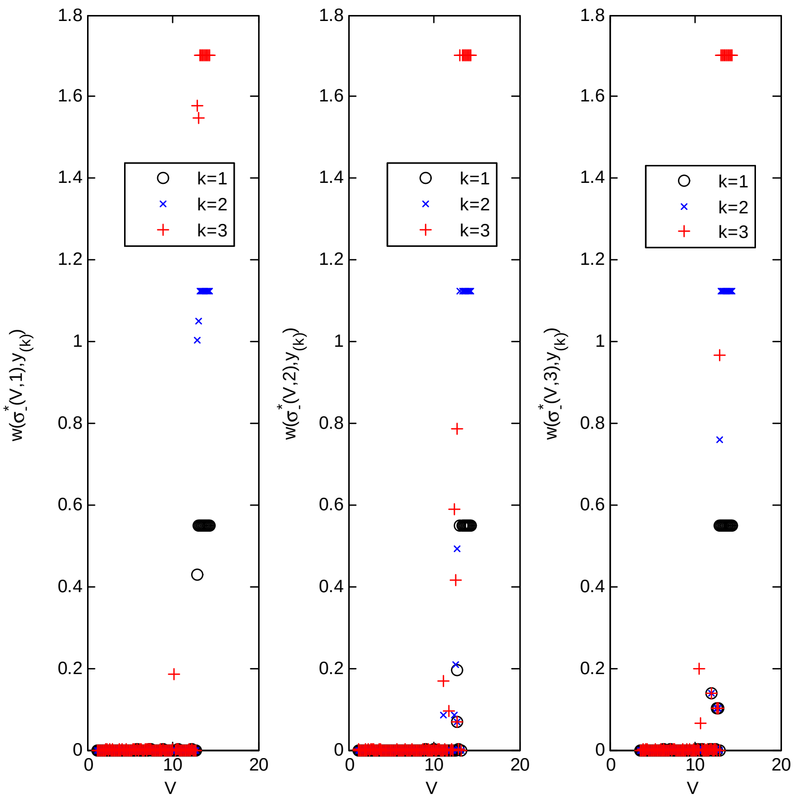

Figure 7: Optimal total compensation under the 2Pso contract as a function of initial utility promise: $w\left(\sigma^*_-\left(.,l\right),y_{(k)}\right)$, $l,k \in \{1,2,3\}$ (LMH, case 3)

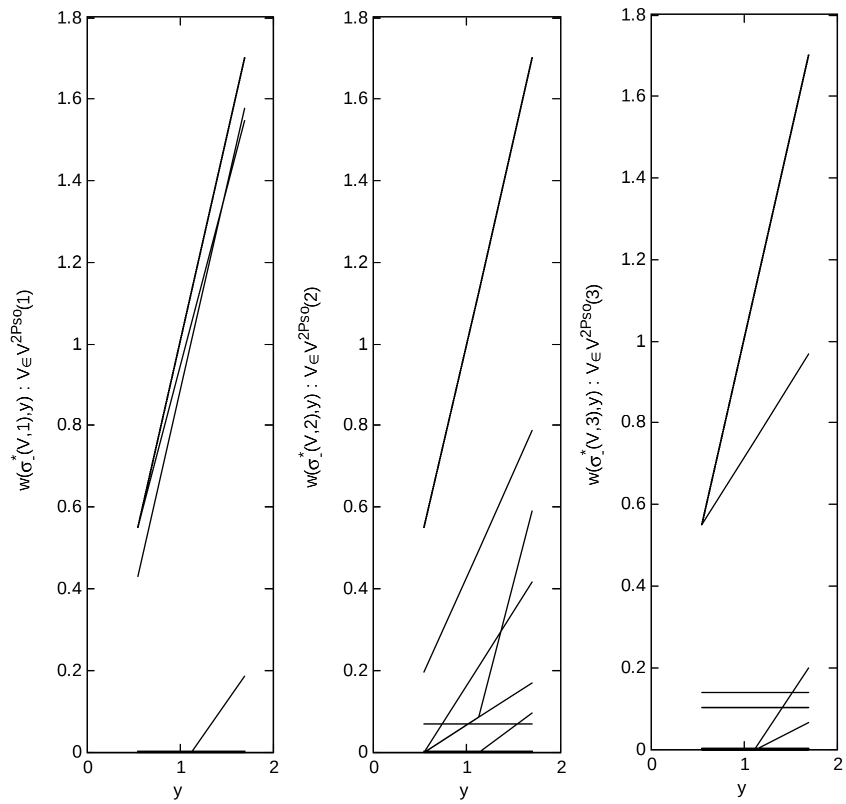

Figure 8: Optimal total compensation under the 2Pso contract as a function of future stock price: $w\left(\sigma_{-}^{*}\left(V,l\right),.\right)$: $V \in V^{2Pso}\left(l\right)$, $l \in \{1,2,3\}$ (LMH, case 3)

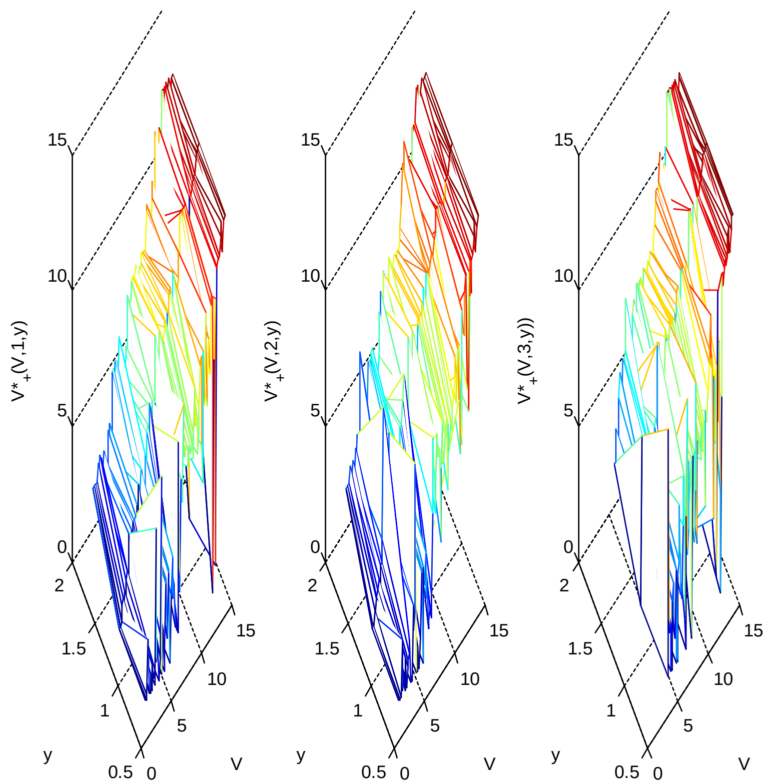

Figure 9: Optimal future utility promise under the 2Pso contract as a function of initial utility promise and future stock price: $V_{+}^{*}(., l, y)$, $l \in \{1, 2, 3\}$ (LMH, case 3)

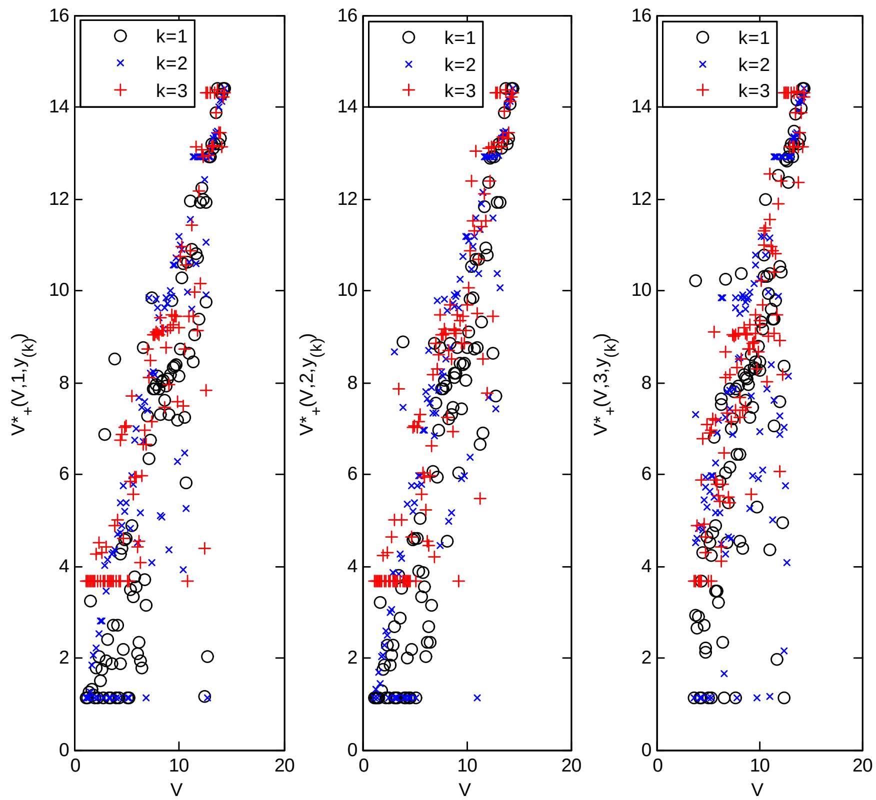

Figure 10: Optimal future utility promise under the 2Pso contract as a function of initial utility promise: $V_+^*\left(.,l,y_{(k)}\right)$, $l,k \in \{1,2,3\}$ (LMH, case 3)

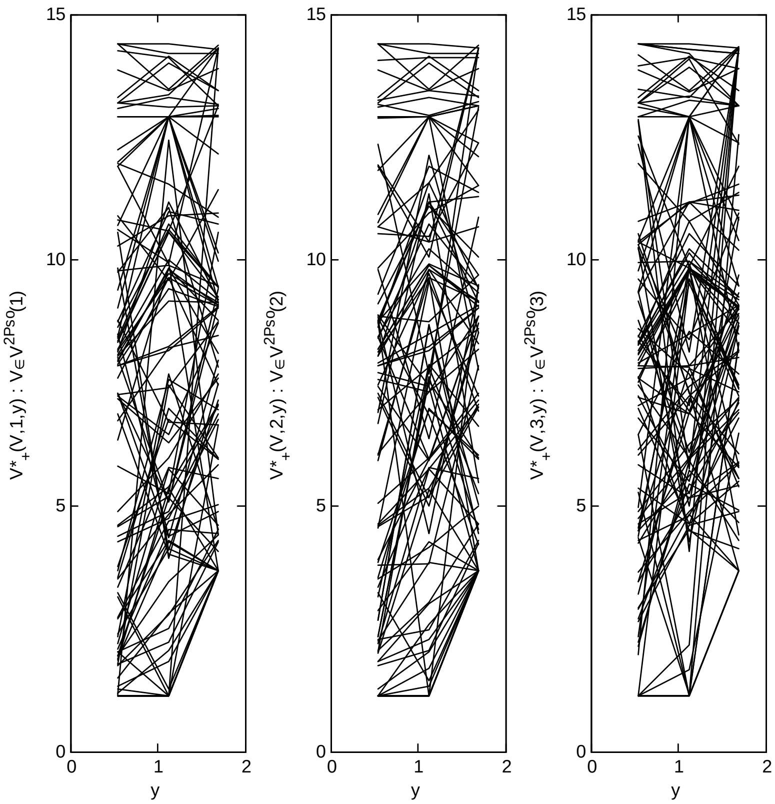

Figure 11: Optimal future utility promise under the 2Pso contract as a function of future stock price: $V_{+}^{*}(V,l,.)$: $V \in V^{2Pso}(l)$, $l \in \{1,2,3\}$ (LMH, case 3)

Permanent Separations and Optimal Compensation with History-Dependent Reservation Utilities*

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March, 2009

Abstract

This paper considers an infinite-horizon, hidden action model characterized by limited commitment and history-dependent reservation utilities. I focus on the optimal contract allowing for permanent separations which can be triggered by any party. I prove existence and characterize the solution recursively. I compute the optimal contract for top executives and find little difference with the optimal self-enforcing contract. Namely, high effort is optimal for most but the richest managers. Compensation and future utility promise increase in both current utility and firm's future profit. The contract is less successful, however, in smoothing initial utility promises across observable histories. The most affected are the poorest managers who are motivated to work hard by much lower continuation utilities under the threat of a future separation.

Keywords: principal-agent problem, moral hazard, dynamic contracts, executive compensation, limited commitment

Journal of Economic Literature Classification Numbers: C63, D82, G30

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*I am particularly grateful to Manuel Santos for his guidance and support. I also wish to thank the members of my thesis committee and participants in the Xth International Academic Conference on Economic and Social Development in Moscow for useful suggestions and comments.

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1 Introduction

An important strand of the dynamic agency literature has investigated the long run properties of optimal contracts. Green (1987), Thomas and Worrall (1990) and Phelan and Townsend (2001) have shown that legally enforceable contracts lead to degenerate long-term wealth distributions. Atkenson and Lucas (1995) demonstrate that if the agent's expected discounted utility is bounded from below, the model induces a non-degenerate invariant distribution. Phelan (1995) achieves a similar result by assuming limited commitment on part of the agent. In such a setting, a self-enforcing contract induces participation at every contingency, so the agent has no incentive to leave the relationship.

The techniques developed by Abreu, Pearce and Stacchetti (1990) prove useful in characterizing the space of agent's feasible continuation utilities. The optimal dynamic contract can then be represented recursively by a series of static contracts defined on this space in line with Spear and Srivastava (1987). Introducing limited commitment on part of the principal poses a different problem related to the fact that the value function of the recursive problem enters the set of constraints and standard dynamic programming fails since the operator used is no longer a contraction. Nevertheless, Rustichini (1998) based on some earlier results by Streufert (1992) shows that a simple modification of the operator allows a recursive representation of the dynamic problem.

The dynamic contracts of the form described above refer to long-term relationships which (by construction) neither party has an incentive to renege on. Spear and Wang (2005) focus on contract terminations instead and achieve stationary and ergodic dynamics by allowing for replacements of the contracted agent with a new one from a labor market pool and for golden parachutes at termination. Sleet and Yeltekin (2001) consider a dynamic model of temporary layoffs and permanent separations where their limited commitment case allows for both the firm and the employee to dissolve the relationship and pursue a fixed outside option. They introduce publicly observable shocks on firm's period profits, enlarge the state space of continuation utilities, and use a monotonic operator to recover it and the respective value function.

In this paper, I consider an environment similar to the one described in Sleet and Yeltekin (2001) where there is no profit perturbation, but the outside options are allowed to vary across (finite truncations of) the history of observables. The model is of the hidden action variety. A principal (firm) seeks a potentially infinite relationship with an agent (CEO) where the agent operates a stochastic technology transforming actions (effort) into outcomes (firm's output, revenues, (gross) profit, or stock price (return)). The outcomes are publicly observable, but the actions are not, so the contract should induce the proper incentives for the agent to exercise some desired sequence of actions. Both the principal and the agent cannot commit to a long term relationship, so the contract should implicitly offer them continuation utilities weakly above the value of their outside options at any contingency that is actually reached. The contract may be optimally terminated at any contingency in which case both parties receive their respective reservation utilities. This possibility effectively enters the set of possible strategies when signing the contract.

My main interest is to compare the behavior of this contract in providing incentives and insurance with the self-enforcing contract analyzed in Morfov (2009). Does the option to terminate the relationship have any global effect on incentives, or does it only affect the insurance of the poorest managers? In fact, when do separations occur and what continuation utilities can be supported by the threat of a future separation?

I represent the problem recursively on the space of truncated histories of outcomes matched with agent's expected discounted utilities. The state space which is endogenous is further characterized in line with Abreu, Pearce and Stacchetti (1990). I parameterize and numerically compute the model in view of top executive compensation. The results show that the optimal contract estimated here is similar to the optimal incentive-compatible contract which is self-enforcing in the sense of Phelan (1995). The firm finds it optimal to induce high effort for most but the richest (in expected utility terms) managers. Executive compensation and continuation utility increase in both current utility and firm's profit. The possibility of separation, however, significantly decreases the smoothing of the manager's lower utility bound across initial histories. As a matter of fact, the contract supports much lower managerial utilities than the self-enforcing contract. Indeed, the most affected are the poorest (in expected utility terms) managers who are motivated to work hard by much lower continuation utilities under the threat of a future separation.

The rest of the paper is structured as follows. Section 2 presents the dynamic model. Section 3 recursively characterizes the optimal contract. Section 4 computes the contract numerically and discusses the results. Section 5 concludes. The theory behind the characterization is developed in Appendix 1. The numerical results are presented in Appendix 2.

2 Dynamic model

The framework is as in Morfov (2009). A principal contracts an agent to implement a sequence of actions where the choice of an action each period is, in fact, a choice of an end-of-period probability distribution over outcomes. Since the exercise of an action brings disutility to the agent, he/she should be compensated by a monetary transfer from the principal. Everything is common knowledge but the particular action implemented which is only observed by the agent. The contract will, therefore, need to be incentive-compatible, i.e., to induce the agent to implement a particular action (or action sequence) recommended by the principal. Since both the principal and the agent are unable to commit to long term relationships, a long-term contract would require their (finite or infinite) participation. Unlike the analysis in Morfov (2009), here either party is allowed to dissolve a long-term relationship at contingencies where the contract fails to provide a continuation utility that is at least as high as this party's respective reservation utility.[1] At such a node, the relationship is terminated and both the principal and the agent "consume" their outside option, i.e., receive their reservation utilities. Notice the difference: before, we had the parties signing a self-enforcing contract which guaranteed their participation at every node, while here the contract ex-ante allows for termination at any node. As before, the reservation utilities are allowed to vary across some finite truncation of the history of past outcomes.

Time is discrete and there is an initial period of contracting normalized to 0. Let Y, A, and W be the sets of possible outcomes, actions, and monetary transfers which are all assumed compact subspaces of $\mathbb{R}$. In particular, Y is assumed finite with n distinct elements, a minimum element $\underline{y}$, and a maximum element $\overline{y}$. Let $\pi : Y \times A \to [\underline{\pi}, 1]$ describe the probability distribution of outcomes conditional on actions, where $\underline{\pi} \in (0,1)$ and $\pi(y,.)$ is continuous on A, $\forall y \in Y$. Denote by $u : W \times Y \to \mathbb{R}$ the (end-of-period) utility function of the principal which is assumed continuous, decreasing in the monetary transfer and increasing in the outcome. The (end-of-period) utility function of the agent $\nu : W \times A \to \mathbb{R}$ is continuous, increasing in the monetary transfer and decreasing in the action. The principal and the agent discount future utility by discount factors β_P and β_A respectively, where $\beta_P, \beta_A \in (0,1)$. The reservation utilities of both parties depend on the previous θ outcomes, where θ is a nonnegative integer. Consider the set of possible sequences of outcomes of length θ, Y^θ. For concreteness, index this set by the bijective function $l : Y^\theta \to L := \left\{1, ..., n^\theta\right\}$. Hereafter, we will often denote an element of Y^θ by its respective index $l = l(.)$. Given this indexing, the reservation utility of the principal and the agent at any node $y^{t-1} \in Y^t \times l$ are denoted by $\underline{U}_l$ and $\underline{V}_l$ respectively.[2] Let a supercontract

[1] We are going to consider supercontracts signed at some initial period that allow for a termination at any future contingency. That is, the possibility of a future contract termination explicitly enters the strategies considered at the period of signing. The supercontracts are defined on all future contingencies (i.e., they may prescribe actions and compensation schemes even on contingencies that would not be reached; this is convenient since the terminations are actually decision variables and what happens on nodes that are not reached given the termination decision plan is immaterial). In our setting, the principal maximizes his/her utility over the set of such incentive-compatible and individually rational supercontracts. Then, the prescription of the optimal contract is followed by both parties and if a node is reached where the contract prescribes termination, a separation occurs and both parties receive their relevant outside options. So, since the principal proposes the contract to the agent, it is the principal's decision how the contract would look like, including whether or when it may terminate. The agent's is passive in the sense that he/she influences the contract only indirectly by his/her incentive compatibility and individual rationality constraints. Indeed, he/she may accept or reject the proposed contract but given rationality, it is in fact the principal that takes this decision by offering a contract that satisfies or violates agent's individual rationality. So, when I say that both parties are allowed to dissolve the relationship, I simply mean that any party can walk away if optimal, i.e., if the contract offers him/her a expected discounted utility lower than the reservation value. In other words, individual rationality matters for both the principal and the agent.

[2] In order to have the reservation utilities well defined in period 0, we assume that at the initial period of contracting a history of length θ has already been observed. Then, any particular history of outcomes available at the beginning of period t, y^{t-1}, consists of θ elements referring to the outcomes observed before period 0 and t elements referring to the

$c := (h, a, w)$ be a plan of termination (or more properly, continuation) decisions, actions and compensation schemes defined on all possible contingencies. For example, at some history y^{t-1} observed in the beginning of period t, the contract terminates if $h_t\left(y^{t-1}\right) = 0$ or proceeds if $h_t\left(y^{t-1}\right) = 1$, recommends an action $a_t\left(y^{t-1}\right)$, and specifies a monetary transfer $w_t\left(y^{t-1}, y\right)$ contingent on an (end-of-period-t) outcome y, $\forall y \in Y$. Note that the supercontract is defined on every node of the tree of possible outcome histories, independent of whether the contract has been terminated before (including at) that node or not.[3] We will refer to the supercontract as admissible if at every node y^{t-1}, we have $h_t\left(y^{t-1}\right) \in \{0, 1\}$, $a_t\left(y^{t-1}\right) \in A$, and $w_t\left(y^{t-1}, y\right) \in W$, $\forall y \in Y$. Then, we can define the expected discounted utility of the principal at some node $y^{\tau-1}$ given an admissible supercontract c as

$$U_\tau\left(c, y^{\tau-1}\right) := \sum_{t=\tau}^{\infty} \beta_P^{t-\tau} \sum_{y_\tau \in Y} \dots \sum_{y_{t-1} \in Y} [\left(1 - h_t\left(y^{t-1}\right)\right) \underline{U}_{l(y_{t-\theta}, \dots, y_{t-1})} +$$

$$h_t\left(y^{t-1}\right) \sum_{y_t \in Y} u\left(w_t\left(y^t\right), y_t\right) \pi\left(y_t, a_t\left(y^{t-1}\right)\right)] \prod_{i=\tau}^{t-1} h_i\left(y^{i-1}\right) \pi\left(y_i, a_i\left(y^{i-1}\right)\right).$$

Given that everything is bounded, we will have that for any admissible contract c, the following holds:

$$U_\tau(c, y^{\tau-1}) = (1 - h_\tau(y^{\tau-1})) \underline{U}_{l(y_{\tau-\theta}, \dots, y_{\tau-1})} +$$

$$h_\tau(y^{\tau-1}) \sum_{y_\tau \in Y} [u(w_\tau(y^\tau), y_\tau) + \beta_P U_{\tau+1}(c, y^\tau)] \pi(y_\tau, a_\tau(y^{\tau-1})).$$

Analogously, we can define $V_\tau\left(c, y^{\tau-1}\right)$ as the expected discounted utility of the agent at node $y^{\tau-1}$ and represent it recursively.

Note that the termination decisions play an explicit role in defining expected discounted utilities. For example, if node $y^{\tau-1}$ is a terminal node, i.e., $h_\tau\left(y^{\tau-1}\right) = 0$, we would have that the expected utility of the principal at that node equals his/her reservation value, $U_\tau\left(c, y^{\tau-1}\right) = \underline{U}_{l(y_{t-\theta}, \dots, y_{t-1})}$ since all other right-hand side elements will disappear as multiplied by 0.[4]

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outcomes realized from period 0 to period $t-1$. That is, y^{t-1} is of length $\theta + t$.

[3] As already mentioned before, this is done for convenience, i.e., any supercontract is defined on the same tree of contingencies stemming from some initial history node, since the decision to terminate is endogenous and it does not really matter what the supercontract specifies at contingencies following terminal nodes. Moreover, it is easier to apply the theoretical results of Morfov (2009) to a formulation of this sort.

[4] Note that since the definitions are forward-looking, we will have $U_\tau\left(c, y^{\tau-1}\right)$ well defined even if $y^{\tau-1}$ is not reached according to the supercontract c. This, however, would be of no significance since ultimately we are only interested in what happens on nodes that are actually reached.

Then, the principal's problem at some initial node of contracting l (in the beginning of period 0) is:

[PPa]

$\sup\limits_{c} U_0(c, l)$ s.t.:

$$c \text{ admissible} \tag{1}$$

$$V_\tau\left(h, a, w, y^{\tau-1}\right) \geq V_\tau\left(h, a', w, y^{\tau-1}\right), \\ \forall a' \text{ admissible}, \forall \textit{ non-terminal nai}\,(l) \tag{2}$$

$$V_\tau\left(c, ., \widetilde{l}\right) \geq \underline{V}_{\widetilde{l}}, \forall nai\,(l) \tag{3}$$

$$U_\tau\left(c, ., \widetilde{l}\right) \geq \underline{U}_{\widetilde{l}}, \forall nai\,(l) \tag{4}$$

where "$\forall nai(l)$" should be understood as "for any *node after* and *including* l", that is, $\forall y^{\tau-1} \in l \times Y^\tau$, $\forall \tau = 0, 1, ...$ Here, we follow the convention that a function maximized over an empty set takes an arbitrarily low value, but assume that its multiplication with zero is well defined and is, in fact, 0. For example, if there does not exists a supercontract that satisfies constraints (1)-(4), then the value function of [PPa] equals $\underline{U}_l$ and the supremum is achieved at $h_0(l) = 0$, i.e., the contract is terminated at l. Regarding the constraints, (1) was already discussed above, (3) and (4) are the individual rationality constraints[5] of the agent and, respectively, the principal guaranteeing them an expected discounted utility at every node at least as high as their respective reservation utility. Since, in general not all nodes will be reached, initial incentive compatibility will not be equivalent to incentive compatibility at all nodes, just at the nodes actually reached. Therefore, we impose incentive compatibility at all nodes [constraint (2)] having in mind that what happens on nodes that are not actually reached is immaterial.

Note that in Morfov (2009), the principal is solving for a self-enforcing contract, i.e., he/she considers only contracts that allow for no terminations.[6] Here, in [PPa], the principal is solving for a contract that guarantees that the continuation utilities of both parties are greater or equal to their respective reservation values only at nodes that are actually reached and non-terminal. In a dynamic sense, at every node that is actually reached the parties compare their expected discounted utility values of continuing the relationship with the value of their outside options and are free to walk out of the relationship if optimal.

[5]These constraints are not referred to as participation constraints since they do not guarantee participation. Indeed, each party can receive his/her respective reservation utility by unilaterally terminating the relationship.

[6]Indeed, we needed Assumption 3 in order to guarantee that the set of such contracts is non-empty.

The constraints of [PPa] may look similar to the constraints of [PP] in Morfov (2009), but one should have in mind that here, the expected discounted utilities are specially defined to account for a possible termination, so in fact constraints (2)-(4) are trivially satisfied on terminal nodes. Indeed, on any terminal node $y^{\tau-1} \in Y^{\tau} \times \widetilde{l}$, $V_{\tau}\left(h, a, w, y^{\tau-1}\right) = V_{\tau}\left(h, a', w, y^{\tau-1}\right) = \underline{V}_{\widetilde{l}}$ and $U_{\tau}\left(h, a, w, y^{\tau-1}\right) = \underline{U}_{\widetilde{l}}$. Note that the constraints are defined on all nodes. This is done for convenience since what happens on nodes that are not actually reached is immaterial to the problem.[7]

3 Recursive Form

It is not difficult to prove[8] that for any admissible contract, incentive compatibility at all nodes is equivalent to Green (1987)'s temporary incentive compatibility at all nodes. A plan is temporary incentive compatible on a node if conditional on future compliance, there is no deviation from the recommended action at this node that will make the agent strictly better off. Then, we can replace constraint (2) in [PPa] by:

$$\begin{aligned}
&\forall \textit{ non-terminal nai}\,(l)\,,\; V_{\tau}\left(h, a, w, y^{\tau-1}\right) \geq V_{\tau}\left(h, a', w, y^{\tau-1}\right), \\
&\forall a' : a'_{\tau}\left(y^{\tau-1}\right) \in A \text{ and } \forall\left(nai\left(y^{\tau-1}, y\right) : y \in Y\right),\; a'_{t}\,(.) = a_{t}\,(.)
\end{aligned} \tag{5}$$

Using the arguments of Morfov (2009), we can show that an optimal contract exists and it can be characterized recursively.

Since the reservation utilities depend on the previous θ outcome realizations, we will consider n^{θ} relevant initial history states each being associated to a particular outcome stream of length θ. Naturally, the initial history states are indexed by L. Hereafter, all functions and correspondences with domain Y^{θ} are treated as vectors or Cartesian products of sets indexed by L. Let $l_{+} : L \times Y \rightarrow L$ be a function that maps today's initial histories and current outcomes to future initial histories. For example, if the outcome stream in the θ periods previous to t has been $(y_{t-\theta}, y_{t-\theta+1}, ..., y_{t-1})$ with an index l and y_t is realized at the end of t, the index of $(y_{t-\theta+1}, ..., y_{t-1}, y_t)$ is given by $l_{+}\,(l, y_t)$.

Let V^{APa} be the product of sets of possible expected discounted utilities for the agent signing an admissible incentive-compatible supercontract that is individually rational for the agent and allows for a termination at every node (an APa contract). To be more precise, the supporting supercontracts should

[7] Note that the supercontract c is well defined on all nodes.

[8] Most of the theoretical results mentioned in this section are formally established in Appendix 1.

be admissible on the whole tree of possible histories, should satisfy agent's individual rationality[9] (3) at every node and (temporary) incentive compatibility (5) at all non-terminal nodes. Define V^{2Pa} as the Cartesian product of sets of possible expected discounted utilities for the agent signing a supercontract that triggers termination only at nodes where admissibility, temporary incentive-compatibility, agent's participation, or principal's participation (at least one of these constraints) is violated (a 2Pa contract).[10]

For any $V \in V^{2Pa}$, let $U^{a^*}(V)$ be a vector with a general element $U^{a^*}(V_l, l)$ defined as the maximum utility the principal can get by signing an optimal supercontract of the second type at l offering V_l to the agent. Let $\widehat{U}^{a^*}$ be the extension of U^{a^*} on V^{APa} s.t. $\forall V \in V^{APa}$, $\widehat{U}^{a^*}(V)$ is a vector with a general element $\widehat{U}^{a^*}(V_l, l) = U^{a^*}(V_l, l)$ if $V_l \in V^{2Pa}(l)$ and $\widehat{U}^{a^*}(V_l, l) := -\infty$ otherwise. Now, we define the operators used to characterize the optimal contract.

Let $\widehat{V} := \frac{\nu(\max\{W\}, \min\{A\})}{1-\beta_A}$ and note that $\max_{l \in L}\left\{\max V^{APa}(l)\right\} \leq \widehat{V}$. Let B^a be a set operator such that for any $X \subset \mathbb{R}^{n^\theta}$, $B^a(X) = \{B^a_l(X)\}$ with $B^a_l(X) := \underline{V}_l \cup \{V \in X_l : \exists$ a (static) contract $c_R(V, l) = \{a_-, w_+(y), V_+(y) : y \in Y\}$ s.t.:

$$a_- \in A \tag{6}$$

$$w_+(y) \in W,\ \forall y \in Y \tag{7}$$

$$\sum_{y \in Y}[\nu\left(w_+(y), a'_-\right) + \beta_A V_+(y)]\pi\left(y, a'_-\right) \leq V,\ \forall a'_- \in A \tag{8}$$

$$\sum_{y \in Y}[\nu\left(w_+(y), a_-\right) + \beta_A V_+(y)]\pi\left(y, a_-\right) = V \tag{9}$$

$$V_+(y) \in X_{l_+(l,y)},\ \forall y \in Y \tag{10}$$

hold$\}$.

Note that for each initial history $l \in L$ and utility level V, the static contract mentioned in the definition of the operator above specifies an action a_-, a contingent transfer $w_+(.)$ from the principal to the agent and a contingent continuation utility for the agent $V_+(.)$, where the last two elements are defined

[9] Individual rationality should not be confused with participation. Terminations are practically allowed at every node.

[10] I am keeping some abbreviations from Morfov (2009), but note that here the difference is substantial. The APa contract does not guarantee the participation of the agent at all. It simply allows termination at each contingency. The 2Pa contract does not satisfy the participation constraints of both parties at all nodes. In fact, if any of the constraints (1), (3), (4), or (5) is violated, it triggers termination. That is, unlike the APa contract, the 2Pa contract allows termination only when optimal.

on Y. Then, (6) and (7) are simply admissibility constraints, (8) imposes temporary incentive compatibility, (9) is a promise-keeping constraint. Note that the operator $\widetilde{B}^a$ does not impose individual rationality on part of the agent; it only requires the principal to provide him/her with a current and future utility levels from an initially specified set. Constraint (10) merely guarantees the consistency of the principal's promise. Nevertheless, iterating on $\widetilde{B}^a$ will allow us to recover the equilibrium set of utility promises at a low cost.[11] We just need to choose the proper initial guess, e.g. $\left\{\left[\underline{V}_l, \widehat{V}\right]\right\}$.

Let T^a be an operator such that for any function $U = \{U_l\}$ with $U_l : V^{APa} \to \mathbb{R}$ upper semicontinuous (usc) and bounded with respect to the sup metric, and any $V \in V^{APa}$, $T^a(U)_{(V)} = \left\{T_l^a(U)_{(V_l)}\right\}$ where

$$T_l^a(U)_{(V_l)} := \max_{h(V_l)\in\{0,1\}} \{(1 - h(V_l))\underline{U}_l + h(V_l)[\max_{c_R}\{\sum_{y\in Y}[u(w_+(y), y) +$$

$$\beta_P U_{l_+(l,y)}(V_+(y))]\pi(y, a_-)\}\} \text{ s.t.:}$$

$$(6) - (9) \text{ hold, and}$$

$$V_+(y) \in V^{APa}(l_+(l, y)), \ \forall y \in Y\} \tag{11}$$

Here, constraint (11) is the equivalent of the consistency constraint (10).

Once we have obtained V^{APa}, the operator T^a can easily be shown to be a contraction.[12] Then, if we start with some initial guess for the value function, e.g. $U_0 = \{\underline{U}_l\}$, and successively apply T^a, we will converge to $\widetilde{U}^{a^*}$, the vector of maximum utilities the principal can derive from a 2P contract that allows for inconsistent utility promises. How does the inconsistency enter the picture? Note that for any utility promise to the agent, the principal can (immediately) terminate the relationship if he/she finds proceeding it suboptimal. In general, this may happen for utility promises in V^{APa} such that not all their elements are agent's reservation utilities. Suppose, for example, that for some $V_l \in V^{APa}(l) \setminus \{\underline{V}_l\}$, we have $\widetilde{U}^{a^*}(V_l, l) = \underline{U}_l$ which is uniquely supported by $h(V_l) = 0$. This, however, would not be consistent with the promise to provide the agent with V_l since the principal actually terminates the relationship and the agent only gets $\underline{V}_l < V_l$. Moreover, V_l may be a continuation utility of a unique single-round contract supporting some other initial utility promise in

[11] See Appendix 1 for details.

[12] Note that we cold have alternatively defined the operator as follows:
$T_l^a(U)_{(V_l)} := \max\{\underline{U}_l, \max_{c_R}\{\sum_{y\in Y}[u(w_+(y), y) + \beta_P U_{l_+(l,y)}(V_+(y))]\pi(y|a_-)\}\}$ s.t.:

$$(6) - (9) \text{ and } (11) \text{ hold.}$$

which case this promise would also prove inconsistent. Therefore, we may in fact be facing an "inconsistency cascade".

To deal with this problem, I use a procedure inspired by Rustichini (1998). For any usc function bounded from above, $U : V^{APa} \to \mathbb{R}^{n^{\theta}}$, and any usc function $H : V^{APa} \to \{0,1\}^{n^{\theta}}$ let $\underline{T}^a$ be an operator such that $\underline{T}^a (U,H) = \left\{ \underline{T}^a_l (U,H)_{(V_l)} \right\}$ with

$$\underline{T}^a_l (U,H)_{(V_l)} := \begin{cases} (-\infty, 0) & \text{if } V_l \in V^{APa}(l) \setminus \{\underline{V}_l\} \text{ and } H(V_l, l) = 0 \\ \left(T^a_l (U)_{(V_l)}, h(V_l) \right) & \text{otherwise,} \end{cases}$$

where $h(V_l)$ is such that the max in the definition of $T^a_l (U)_{(V_l)}$ is achieved.

Using this operator, we can recursively clean the inconsistent promises and modify the value function accordingly to obtain $\widehat{U}^{a^*}$. Then, we can recover U^{a^*} and V^{2Pa}.

4 Computation and Results

Since the model cannot be solved analytically, I resort to numerical methods. I focus on CEO compensation (i.e., the principal is a proxy for firm's shareholders, the agent is the company's top executive, the action is interpreted as effort, the monetary transfer as the manager's compensation package, and the outcomes as the company's profit). I parameterize the model as follows:[13] $Y = \left\{ y_{(1)}, y_{(2)}, y_{(3)} \right\} = \{0.55,\ 1.125,\ 1.7\}$, $A = \{\underline{a}, \overline{a}\} = \{0.1253,\ 0.1469\}$, $\pi\left(y_{(1)}, \underline{a}\right) = 0.1891$, $\pi\left(y_{(2)}, \underline{a}\right) = 0.7687$, $\pi\left(y_{(3)}, \underline{a}\right) = 0.0421$, $\pi\left(y_{(1)}, \overline{a}\right) = 0.1555$, $\pi\left(y_{(2)}, \overline{a}\right) = 0.7654$, $\pi\left(y_{(3)}, \overline{a}\right) = 0.0791$, $W = \left[0, \frac{y_{(3)}}{1-\beta_P}\right]$, $\beta_A = \beta_P = 0.96$, $u(w_t, y_t) = y_t - w_t$, $\nu(w_t, a_t) = \log(1 + w_t) - a_t$, $\theta = 1$, $\underline{U}(.) = 0$, $\underline{V}(.) \in$ {L,M,H}, where L := $\frac{\upsilon(\underline{w}) - \max A}{1-\beta_A}$ = -3.6725, M := 0, H := -L and assume a positive correlation between $\underline{V}(.)$ and Y.[14] I also consider three cases: case 1a where the firm cannot offer the manager a compensation that exceeds its highest possible lifetime profit in discounted terms, i.e., $w_t \leq \frac{y_{(3)}}{1-\beta_P}$ (and $\widehat{V} = \frac{\upsilon\left(\frac{y_{(3)}}{1-\beta_P}\right) - \underline{a}}{1-\beta_A}$), case 2 where it can compensate the manager up to the highest possible profit realization, i.e., $w_t \leq y_{(3)}$ (and $\widehat{V} = \frac{\upsilon\left(y_{(3)}\right) - \underline{a}}{1-\beta_A}$), and case 3

[13] The parameterization is based on Aseff and Santos (2005) who take the value of low effort from Margiotta and Miller (2000) and use the results of Hall and Liebman (1998) to derive the stock price distribution conditional on manager's effort.

[14] You may notice that the probabilities conditional on $\underline{a}$ add to 0.9999 instead of 1. The reason is that here I report the numbers at a much lower precision than the one I actually use in the numerical computation. For example, if we round the probabilities to the fifth digit after the decimal point rather than to the fourth, the reported numbers would be 0.18915, 0.76871, and 0.04214 respectively.

where it is essentially prohibited from borrowing, i.e., $w_t\left(y^{t-1}, y\right) \leq y, \forall y \in Y$ (where, as in case 2, we take $\widehat{V} = \frac{v\left(y_{(3)}\right)-\underline{a}}{1-\beta_A}$). Since the results are qualitatively similar, I focus on LMH, case 3, where LMH is the assignment of reservation values to initial histories, namely $\underline{V}_1 =$ L, $\underline{V}_2 =$ M, $\underline{V}_3 =$ H.

Figure 1 in Appendix 2 plots the value functions for the optimal contract allowing for permanent separations (the 2Pa contract) and the optimal self-enforcing contract (the 2P contract in the terminology of Morfov (2009)). For higher initial utilities, the two value functions practically coincide. For smaller utility values, most of which lie outside the state space of the self-enforcing contract, the value function of the contract allowing for permanent separations is still defined but is actually increasing, so these utility promises would be wiped out by re-negotiation if such were possible. Indeed, most of these initial utilities are supported by future separations. Note that the fact that much lower utilities are supported under the 2Pa contract means that the manager's minimum utility is much lower than under the 2P contract, so the insurance against fluctuations in the value of manager's outside options is less pronounced. More precisely, the threat of a separation significantly decreases the smoothing of the manager's lower utility bound across initial histories. Consider the optimal contract that allows for permanent separations. It supports the manager's reservation utility for both medium and high profits. For low profits, the minimum managerial utility is about -0.42, which is above the reservation value of -3.67. But let us compare these results with the situation observed under the optimal self-enforcing contract. There, the manager's reservation utility is only supported for high profits. For both low and medium profits, he/she is guaranteed at least 1.14 given reservation utility values of -3.67 and 0 respectively.

Otherwise, the properties of the optimal contract allowing for separations (the 2Pa contract) are not much different from those of the optimal self-enforcing contract (the 2P contract). Figure 2 shows the optimal level of effort as a function of initial utility (cf. Figure 3 for the self-enforcing contract). The firm finds it optimal to induce high effort for most but the richest (in expected utility terms) managers. While we have a pool of managers who do not participate in the optimal self-enforcing contract and are now motivated to work hard, the cutoff point from where on the managers start to shirk does not differ significantly across the two contracts. As before, executive pay increases with the utility promise (compare Figure 4 with Figure 5; the graph for a future profit $y_{(1)}$ is kinked due to the firm's binding borrowing constraint[15]) and with the end-of-period profit realization (Figure 6 vs. Figure 7). Manager's continuation utility tends to increase in both current utility (cf. Figures 8 and 9) and firm's profit (cf. Figures 10 and 11). The difference with the optimal self-enforcing contract is that now much lower continuation utilities are possible. The reason, as for the initial utility promises, is that they can be supported by future

[15]Note that managerial compensation cannot exceed the current profit realization, i.e., $w_t(.,y) \leq y$, $\forall y \in Y$. So, when the lowest profit, $y_{(1)}$, is realized, the firm's inability to borrow adversely affects the incentive scheme by capping executive pay for all but the poorest (in expected utility terms) managers.

terminations. Indeed, the prospect of permanent separations mostly affects the poorest (in utility terms) managers. For these managers, current compensation does not play any incentive role, they are exclusively motivated by deferred compensation and are fired in case of (a stream of) low profits.

5 Conclusion

In a dynamic model of hidden action with limited commitment and history-dependent reservation utilities, I recursively characterize the optimal contract allowing for permanent separations. I numerically compute the optimal contract for top executives and find little difference with the optimal self-enforcing contract. High effort appears optimal for most but the richest managers. CEO's compensation and future utility increase in both current utility and firm's future profit. The contract provides the manager with a lower level of insurance against fluctuations in the value of his/her outside options than does the respective self-enforcing contract. It offers much lower continuation utility values which are supported by future terminations.

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APPENDIX 1

The first step towards a recursive representation is to establish that an admissible supercontract is incentive compatible on all non-terminal nodes if and only if it is temporary incentive compatible on all non-terminal nodes.

Proposition 1 *For any c admissible, (2)⇔(5).*

Proof. It is trivial to show (2)⇒(5). In the other direction, let (5) hold, but assume that (2) is not satisfied, i.e. there is a non-terminal node $y^{\tau-1}$ s.t. $\exists a'$ admissible on the tree with initial node $y^{\tau-1}$ and $V_\tau\left(h,a',w,y^{\tau-1}\right) > V_\tau\left(h,a,w,y^{\tau-1}\right)$. We have:

$$V_\tau\left(h,a',w,y^{\tau-1}\right) =$$

$$\sum_{t=\tau}^{T}\beta_A^{t-\tau}\sum_{y_\tau\in Y}...\sum_{y_{t-1}\in Y}\left[\left(1-h_t\left(y^{t-1}\right)\right)\underline{V}_{l(y_{t-\theta},\ldots,y_{t-1})}\prod_{i=\tau}^{t-1}h_i\left(y^{i-1}\right)\pi\left(y_i,a_i'\left(y^{i-1}\right)\right)+\right.$$

$$\sum_{t=\tau}^{T}\beta_A^{t-\tau}\sum_{y_\tau\in Y}...\sum_{y_t\in Y}v\left(w_t\left(y^t\right),a_t'\left(y^{t-1}\right)\right)\prod_{i=\tau}^{t}h_i\left(y^{i-1}\right)\pi\left(y_i,a_i'\left(y^{i-1}\right)\right)+$$

$$\beta_A^{T-\tau+1}V_{T+1}\left(h,a',w,y^{\tau-1}\right)\prod_{i=\tau}^{T}h_i\left(y^{i-1}\right)\pi\left(y_i,a_i'\left(y^{i-1}\right)\right)$$

where the last term on the right-hand side can be made arbitrarily small by choosing T big enough. Therefore, $\exists T\in\mathbb{Z}_+$ and an admissible action plan $a'' : a_t''\left(y^{t-1}\right)=a_t'\left(y^{t-1}\right), \forall y^{t-1}\in y^{\tau-1}\times Y^{t-\tau}, \forall t\leq T$, and $a_t''=a_t$ elsewhere, s.t. $V_\tau\left(h,a'',w,y^{\tau-1}\right) > V_\tau\left(h,a,w,y^{\tau-1}\right)$ Then, take $\tau'\in\mathbb{Z}_+ : \tau\leq\tau'\leq T$ s.t. $\exists y^{\tau'-1} : a_{\tau'}''\left(y^{\tau'-1}\right)\neq a_{\tau'}\left(y^{\tau'-1}\right)$ and $\nexists\tau''\in\mathbb{Z}_{++} : \tau'<\tau''\leq T : a_{\tau''}''\left(y^{\tau''-1}\right)\neq a_{\tau''}\left(y^{\tau''-1}\right)$ for some $y^{\tau''-1}\in y^{\tau-1}\times Y^{\tau''-\tau}$. Define an admissible action plan $a''' : a_{\tau'}'''\left(y^{\tau'-1}\right)=a_{\tau'}\left(y^{\tau'-1}\right), \forall y^{\tau'-1}\in y^{\tau-1}\times Y^{\tau'-\tau}$ and $a_t'''=a_t''$ elsewhere. Then, for any $y^{\tau'-1}\in y^{\tau}\times Y^{\tau'-\tau}$ such that $h_{\tau'}\left(y^{\tau'-1}\right)=0$, we have that $V_{\tau'}\left(h,a''',w,y^{\tau'-1}\right)=V_{\tau'}\left(h,a'',w,y^{\tau'-1}\right)=\underline{V}_{l\left(y_{\tau'-\theta},\ldots,y_{\tau'-1}\right)}$, while
if $h_{\tau'}\left(y^{\tau'-1}\right) = 1$, by (5) we obtain that $V_{\tau'}\left(h,a''',w,y^{\tau'-1}\right) \geq$

$V_{\tau'}\left(h, a'', w, y^{\tau'-1}\right)$. Therefore, $V_\tau\left(h, a''', w, y^{\tau-1}\right) \geq V_\tau\left(h, a'', w, y^{\tau-1}\right)$. Proceeding in this way we can eliminate all the deviations (note that $\tau' \in \mathbb{Z}_+ : \tau' \leq T$) to obtain $V_\tau\left(h, a, w, y^{\tau-1}\right) \geq V_\tau\left(h, a'', w, y^{\tau-1}\right)$, i.e. a contradiction. ■

For any $X \in \mathbb{R}^{n^\theta}$ let $B^a(X)$ be a set operator such that $B_l^a(X) := \underline{V}_l \cup \{V \in \left[\underline{V}_l, \widehat{V}\right] : \exists$ a (static) contract $c_R(V, l)$ s.t. (6)-(9) are satisfied and

$$V_+(V, y) \in X_{l_+(l,y)} \cap \left[\underline{V}_{l_+(l,y)}, \infty\right) \tag{12}$$

holds$\}$.

Compare this operator with $\widetilde{B}^a$. Here, $V \in \left[\underline{V}_l, \widehat{V}\right]$ and constraint (12) combines individual rationality on part of the agent with a consistent continuation utility promise by the principal. The operator is more robust than $\widetilde{B}^a$ regarding the initial guess for the equilibrium set. However, given a suitably chosen initial guess, iterating on $\widetilde{B}^a$ leads to the same result at a lower computational cost.

For any $X = \{X_l\} : X_l \in \mathbb{R}, \forall l \in L$ let $B^{a\prime}(X) := \{B_l^{a\prime}(X)\}$ with $B_l^{a\prime}(X) := \{V_l \in \left[\underline{V}_l, \widehat{V}\right] : \exists c_R(V_l, l)$:(6)-(9) and (10) hold$\}$. Note that the only difference between this operator and operator $\widetilde{B}^a$ defined in Section 3 is that $B_l^{a\prime}(X) \subset \left[\underline{V}_l, \widehat{V}\right]$, while $\widetilde{B}_l^a(X) \subset X_l$.

Lemma 1 $V^{APa} \subset B^a\left(V^{APa}\right)$.

Proof. Let $V \in V^{APa}$ and fix an arbitrary $l \in L$. Since $V_l \in V^{APa}(l)$, $\exists c$: (1) holds, (3) $\forall nai(l)$, (5) $\forall$ non-terminal $nai(l)$, and $V_0(c, l) = V_l$. By construction, $V_l \in \left[\underline{V}_l, \widehat{V}\right]$. If $h(l) = 0$, node l is terminal, so $V_l = \underline{V}_l$, and since $\underline{V}_l \in B_l^a\left(V^{APa}\right)$ by definition, the result is trivial. Therefore, let us assume $h(l) = 1$. For any $y \in Y$, let $a_- := a_0(l)$, $w_+(y) := w_0(l, y)$, and $V_+(y) := V_1(c, l, y)$. Given these choices, we immediately have that (9) holds. Moreover, (1) $\Rightarrow$ (6) $\cap$ (7), (5) $\Rightarrow$ (8). Note that for all $y \in Y$, $V^{APa}(l_+(l, y)) \cap \left[\underline{V}_{l_+(l,y)}, +\infty\right)$ $= V^{APa}(l_+(l, y))$. Note that for every $y \in Y$, the truncation of the original supercontract c to the tree with initial node (l, y) satisfies (1) on the tree with initial node $l_+(l, y)$ and is such that (3) $\forall nai(l_+(l, y))$, (5) $\forall$ non-terminal $nai(l_+(l, y))$ and $V_0(c_y, l_+(l, y)) = V_1(c, l, y)$, which means that (12) is satisfied. Therefore, $V_l \in B_l^a\left(V^{APa}\right)$. Since $l \in L$ was chosen randomly, this generalizes to $V \in B^a\left(V^{APa}\right)$. ■

The lemma establishes that V^{APa} is self-generating in the terminology of Abreu, Pearce and Stacchetti (1990).

Lemma 2 *Assume* $X = \{X_l\} : X_l \subset B_l^a(X)$, $\forall l \in L$. *Then,* $B^a(X) \subset V^{APa}$.

Proof. Let the condition of the lemma hold and take $V \in B^a(X)$. Fix an arbitrary $l \in L$. If $V_l = \underline{V}_l$, it is immediate that $V_l \in V^{APa}(l)$ since we can support it by an admissible supercontract with $h(.) = 0$ on all $nai(l)$. If $V_l \neq \underline{V}_l$, then $\exists c_R(V_l, l)$: (6)-(12) hold. By (12) and $X_{l_+(l,y)} \subset B^a_{l_+(l,y)}(X)$, we obtain that $V_{+,l}(V_l, y) \in B^a_{l_+(l,y)}(X)$, $\forall y \in Y$. Consider $y \in Y$. We either have that $V_{+,l}(V_l, y)$ can be supported by a contract $c_{R,}(V_{+,l}(V_l, y), (l, y))$: (6)-(12) hold or $V_{+,l}(V_l, y) = \underline{V}_{l+(l,y)} \in V^{APa}(l_+(l, y))$ since it can supported by an admissible supercontract with $h(.) = 0$ on all $nai(l_+(l, y))$. Proceeding this way, we can consecutively construct a supercontract c after l s.t. (1), (3) $\forall nai$ (l), (5) $\forall$non-terminal nai (l) and $V_0(c, l) = V_l$. Here, it deserves noting that while (12) implies (3) on every node but the initial one, $V_l \in B_l^a(X) \subset \left[\underline{V}_l, \widehat{V}\right]$, from where (3) is also satisfied at l. Therefore, $V_l \in V^{APa}(l)$, which generalizes to $V \in V^{APa}$. ■

The lemma says that the image of every nonempty, self-generating set is a subset of V^{APa}.

Proposition 2 *(a)* $B^a\left(V^{APa}\right) = V^{AP}$; *and (b) if* $\exists X \subset \mathbb{R}^{n^\theta} : B^a(X) = X$, *then* $X \subset V^{APa}$.

Proof of Proposition 2. (a) From Lemmas 1 and 2.
(b) It follows by Lemma 2. ■

This proposition establishes that the set of agent's expected discounted utilities supportable by an APa supercontract is the largest fixed point of B^a.

Lemma 3 *Assume* $X' = \{X'_l\}$ *and* $X'' = \{X''_l\} : X'_l \subset X''_l \subset \mathbb{R}$, $\forall l \in L$. *Then,* $B_l^a(X') \subset B_l^a(X'')$, $\forall l \in L$.

Proof. Trivial. ■

Lemma 4 *Assume* $X = \{X_l\}$: $X_l \subset \mathbb{R}$ *compact,* $\forall l \in L$. *Then,* $B_l^a(X)$ *compact,* $\forall l \in L$.

Proof. Let the condition of the lemma hold and assume $B_l^a(X) \neq \emptyset$ for some $l \in L$. Note that $B_l^a(X) \subset \left[\underline{V}_l, \widehat{V}\right] \subset \mathbb{R}$ is bounded by definition. We should also show that it is closed. Take an arbitrary convergent sequence $\{V_i\}_{i=1}^{\infty}$: $V_i \in B_l^a(X)$, $\forall i \in \mathbb{Z}_{++}$ with $V_i \underset{i\to\infty}{\rightarrow} V_\infty$. We need to prove that $V_\infty \in B_l^a(X)$. If $V_\infty = \underline{V}_l$, then the result is trivial since $\underline{V}_l \in B_l^a(X)$. Therefore, assume that $V_\infty \neq \underline{V}_l$. Then, it should be the case that there exists a subsequence $\left\{V_{i_j}\right\}_{j=1}^{\infty}$ of $\{V_i\}_{i=1}^{\infty}$, such that for every positive integer j, $V_{i_j} \in \left[\underline{V}_l, \widehat{V}\right]$ and $\exists c_{R,i_j}$:(6)-(12) hold at $\left(V_{i_j}, l\right)$. By $V_{i_j} \in \left[\underline{V}_l, \widehat{V}\right]$, $\forall$ positive integer j, we obtain $V_\infty \in \left[\underline{V}_l, \widehat{V}\right]$. By (6), (7), (12), Y finite, and $X_l \subset \mathbb{R}$ compact for any $l \in L$, we have that $\left\{c_{R,i_j}\right\}_{j=1}^{\infty}$ is uniformly bounded, therefore $\exists$ a subsequence $\left\{c_{R,i_{j_k}}\right\}_{k=1}^{\infty}$ of $\left\{c_{R,i_j}\right\}_{j=1}^{\infty}$: $c_{R,i_{j_k}} \underset{k\to\infty}{\rightarrow} c_{R,\infty}$. It is immediate that $c_{R,\infty}$ satisfies (6)-(12) at (V_∞, l). ∎

Proposition 3 *Let* X_0 *compact* : $V^{APa} \subset X_0 \subset \mathbb{R}^{n^\theta}$ *and* $B^a(X_0) \subset X_0$. *Define* $X_{i+1} := B^a(X_i)$, $\forall i \in \mathbb{Z}_+$. *Then,* $X_{i+1} \subset X_i$, $\forall i \in \mathbb{Z}_+$ *and* $X_\infty := \lim\limits_{i\to\infty} X_i = V^{APa}$.

Proof of Proposition 3. For every $l \in L$ and $i \in \mathbb{Z}_+$, denote by $X_{i,l}$ the element of X_i corresponding to initial history l. By the condition of the Proposition, we have $V^{APa}(l) \subset X_{0,l} \subset \mathbb{R}$, $\forall l \in L$. Since by Proposition 2 (a) $B_l^a\left(V^{APa}\right) = V^{APa}(l)$, we can apply Lemma 3 to obtain $V^{APa}(l) \subset X_{1,l} \subset \mathbb{R}$, $\forall l \in L$. Using $X_1 \subset X_0$ and repeating the argument, we reach $V^{AP} \subset X_{i+1} \subset X_i$, $\forall i \in \mathbb{Z}_+$ Then, $\{X_i\}_{i=0}^{\infty}$ is a sequence of non-empty, compact (by Lemma 4 since X_0 compact), monotonically decreasing (nested) sets; therefore it converges to $X_\infty = \overset{\infty}{\underset{i=0}{\cap}} X_i \supset V^{APa}$ with X_∞ compact.

What remains to be shown is that $X_\infty \subset V^{APa}$. By Lemma 2, it is enough to show that $X_\infty \subset B^a(X_\infty)$. Let $V \in X_\infty$. This implies that $V \in X_i$, $\forall i \in \mathbb{Z}_+$. Fix an arbitrary $l \in L$. If $V_l = \underline{V}_l$, we immediately have that $V_l \in B_l^a(X_\infty)$,.therefore assume $V_l \neq \underline{V}_l$.Then, $\exists c_{R,i}$: (6)-(12) hold at (V_l, l). By (6), (7), (12), Y finite, and $X_i \subset X_0 \subset \mathbb{R}^{n^\theta}$ compact, $\forall i \in \mathbb{Z}_+$, we have that $\{c_{R,i}\}_{i\in\mathbb{Z}_+}$ is uniformly bounded; therefore, $\exists$ a subsequence $\left\{c_{R,i_j}\right\}_{j=0}^{\infty}$ of $\{c_{R,i}\}_{i=0}^{\infty}$: $c_{R,i_j} \underset{j\to\infty}{\rightarrow} c_{R,\infty}$. It is immediate that $c_{R,\infty}$ satisfies(6)-(8) at (V_l, l). Moreover, $V_{+,\infty}(y) \geq \underline{V}_{l_+(l,y)}$, $\forall y \in Y$. We also need to show that for every $y \in Y$, $V_{+,\infty}(y) \in X_{\infty,l_+(l,y)}$. Fix an arbitrary $y \in Y$ and assume, on the contrary, that $V_{+,\infty}(y) \notin X_{\infty,l_+(l,y)}$. Since $X_{\infty,l_+(l,y)} = \overset{\infty}{\underset{i=0}{\cap}} X_{i,l_+(l,y)} = \overset{\infty}{\underset{j=0}{\cap}} X_{i_j,l_+(l,y)}$, we have that $\exists j' \in \mathbb{Z}_+ : V_{+,\infty}(y) \notin X_{i_{j'},l_+(l,y)}$. Furthermore, $\left\{X_{i_j}\right\}_{j=0}^{\infty}$ was shown to be a monotonically decreasing (nested) sequence, from where $V_{+,i_j}(y) \in X_{i_j,l_+(l,y)} \subset X_{i_{j'},l_+(l,y)}$, $\forall j \in \mathbb{Z}_+ : j \geq j'$. Since $X_{i_{j'},l_+(l,y)}$ is closed and $V_{+,i_j}(y) \underset{k\to\infty}{\rightarrow} V_{+,\infty}(y)$, we obtain that $V_{+,\infty}(y) \in X_{i_{j'},l_+(l,y)}$, i.e. a contradiction is reached. This proves $V_{+,\infty}(y) \in X_{\infty,l_+(l,y)}$, $\forall y \in Y$. Consequently, (12) holds for $c_{R,\infty}$. Finally, note that $V_l \in \left[\underline{V}_l, \widehat{V}\right]$ follows immediately from $V_l \in X_{1,l}$. Therefore, $V_l \in B^a_l(X_\infty)$, which generalizes to $V \in B^a(X_\infty)$. ■

The proposition says that if we start iterating on B^a taking as an initial guess some compact set X_0 that contains both $B^a(X_0)$ and V^{APa}, we will ultimately converge to the largest fixed point of the operator, V^{APa}. This is sufficient for obtaining V^{APa} since we can always take $X_0 = \{X_{l,0}\} : \left[\underline{V}_l, \widehat{V}\right] \subset X_{l,0} \subset \mathbb{R}$ with $X_{l,0}$ compact, $\forall l \in L$. However, an even more computationally efficient result exists.

For any $X = \{X_l\} : X_l \in \mathbb{R}$, $\forall l \in L$ let $B^{a\prime}(X) := \{B^{a\prime}_l(X)\}$ with $B^{a\prime}_l(X) := \{V_l \in \left[\underline{V}_l, \widehat{V}\right] : \exists c_R$:(6)-(9) and (10) hold at $(V_l, l)\}$. Note that the only difference between this operator and operator $\widetilde{B}^a$ defined in Section 3 is that $B^{a\prime}_l(X) \subset \left[\underline{V}_l, \widehat{V}\right]$, while $\widetilde{B}^a_l(X) \subset X_l$.

Lemma 5 *Take* $X'_0 := \left\{X'_{0,l}\right\}$ *with* $X'_{0,l} := \left[\underline{V}_l, \widehat{V}\right]$, $\forall l \in L$ *and let* $X'_{i+1} := B^{a\prime}(X'_i)$, $\forall i \in \mathbb{Z}_+$. *Then,* $X'_{i+1} \subset X'_i$, $\forall i \in \mathbb{Z}_+$ *and* $X'_\infty := \lim\limits_{i\to\infty} X'_i = V^{APa}$.

Proof. We have that X'_0 is compact and $V^{APa} \subset X'_0 \subset \mathbb{R}^{n^\theta}$. Note that for every $X \subset \mathbb{R}^{n^\theta}$, we have $B^a_l(X) \subset B^{a\prime}_l(X)$. Then, by Lemma 3 and Proposition 2 (a), we obtain $V^{APa} \subset B^a(X'_0) \subset B^{a\prime}(X'_0)$. Using the same arguments plus the monotonicity of $B^{a\prime}$ (trivial), we have $V^{APa} \subset X^{a\prime}_i$, $\forall i \in \mathbb{Z}_+$. Moreover, by construction $B^{a\prime}(X'_0) \subset X'_0$. Then, the condition $B(X'_0) \subset X'_0$ is satisfied. Observe that for every $l \in L$, $X'_{1,l} = \underline{V}_l \cap \{V_l \in \left[\underline{V}_l, \widehat{V}\right] : \exists c_R$ s.t. (6)-(9), (10) hold at $(V_l, l)\} = \underline{V}_l \cap \{V_l \in \left[\underline{V}_l, \widehat{V}\right] : \exists c_R$ s.t. (6)-(9), (12) hold at $(V_l, l)\} = B^a_l(X'_0)$ since, by construction, we have that $X'_0(l_+(l,y)) \cap \left[\underline{V}_{l_+(l,y)}, +\infty\right) = X'_{0,l_+(l,y)}$, $\forall y \in Y$. Furthermore, by $X'_1 \subset X'_0$ and the monotonicity of B', we obtain $X'_{i+1} \subset X'_i$, $\forall i \in \mathbb{Z}_+$. Then, it is trivial that $X'_{i+1} = B^a(X'_i)$, $\forall i \in \mathbb{Z}_+$ Therefore, Proposition 3 applies to $\{X'_i\}_{i=1}^\infty$. ■

Lemma 6 *Let* $\{X'_i\}_{i=1}^\infty$ *be defined as in Lemma 5. Take* $\widetilde{X}_0 := X'_0$ *and let* $\widetilde{X}_{i+1} := \widetilde{B}^a\left(\widetilde{X}_i\right)$, $\forall i \in \mathbb{Z}_+$. *Then,* $\widetilde{X}_i = X'_i$, $\forall i \in \mathbb{Z}_+$.

Proof. Assume $\widetilde{X}_{i-1} = X'_{i-1}$ for some $i \in \mathbb{Z}_{++}$. By Lemma 5, $\emptyset \neq X'_i \subset X'_{i-1}$. Fix $l \in L^{\theta}$ and let $V \in X'_{i,l}$ Then, we have $V \in \widetilde{X}_{i-1,l}$, which together with $V \in B'_l\left(\widetilde{X}_{i-1}\right)$ implies that $V \in \widetilde{B}_l\left(\widetilde{X}_{i-1}\right)$. Since l and V were chosen randomly, this generalizes to $X'_i \subset \widetilde{X}_i$. Note that $\widetilde{X}_{i-1} = X'_{i-1} \subset X'_0$ by Lemma 5. Consequently, $\widetilde{B}_l\left(\widetilde{X}_{i-1}\right) \subset B'_l\left(\widetilde{X}_{i-1}\right)$, i.e. $\widetilde{X}_i \subset X'_i$.

We have that $\widetilde{X}_0 = X'_0$ by definition and have just shown that $\widetilde{X}_{i-1} = X'_{i-1}$ would imply $\widetilde{X}_i = X'_i$; therefore, by induction we obtain that $\widetilde{X}_i = X'_i$, for any $i \in \mathbb{Z}_+$. ■

Proposition 4 *(a) Take* $\widetilde{X}_0 := \left\{\widetilde{X}_{0,l}\right\}$ *with* $\widetilde{X}_{0,l} = \left[\underline{V}_l, \widehat{V}\right]$, $\forall l \in L$ *and let* $\widetilde{X}_{i+1} := \widetilde{B}^a\left(\widetilde{X}_i\right)$, $\forall i \in \mathbb{Z}_+$. *Then,* $\widetilde{X}_{i+1} \subset \widetilde{X}_i$, $\forall i \in \mathbb{Z}_+$ *and* $\widetilde{X}_\infty := \lim_{i\to\infty} \widetilde{X}_i = V^{APa}$. *(b)* $\widetilde{B}^a\left(V^{APa}\right) = V^{APa}$*; and (c) if* $\exists X \subset \widetilde{X}_0 : \widetilde{B}^a(X) = X$*, then* $X \subset V^{APa}$.

Proof of Proposition 4. (a) From Lemmas 5 and 6.

(b) Similarly to the proof of Lemma 1, we can show that $V^{AP} \subset \widetilde{B}\left(V^{AP}\right)$. It is trivial that $\widetilde{B}\left(V^{AP}\right) \subset V^{AP}$.

(c) Since $X \subset \widetilde{X}_0$, we can use the monotonicity of $\widetilde{B}^a$ and $\widetilde{B}^a(X) = X$ to obtain $X \subset \widetilde{X}_i$, $\forall i \in \mathbb{Z}_+$. Then, by (a), we have $X \subset \widetilde{X}_\infty = V^{APa}$. ■

This proposition outlines a practical way of obtaining V^{APa}. Namely, we start with the set $\left\{\left[\underline{V}_l, \widehat{V}\right]\right\}$ and iterate on the set operator $\widetilde{B}^a$ until convergence in a properly defined sense is attained.

APPENDIX 2

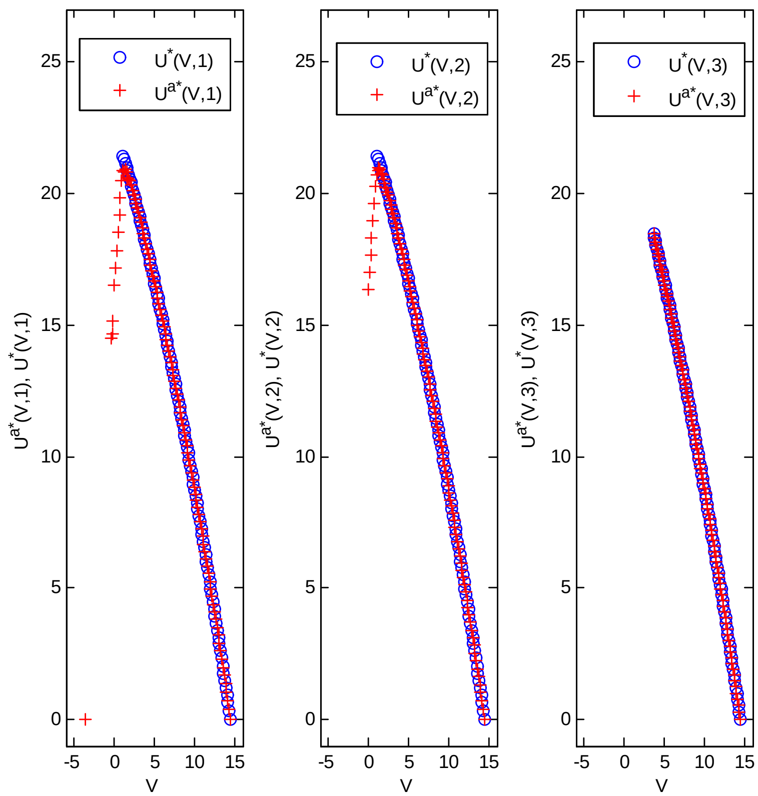

Figure 1: Value functions for the 2Pa (allowing for permanent separations) contract and the 2P (self-enforcing) contract ordered by initial history: $U^{a^*}(.,l)$, and respecively $U^*(.,l)$, $l \in \{1,2,3\}$ (LMH, case 3)

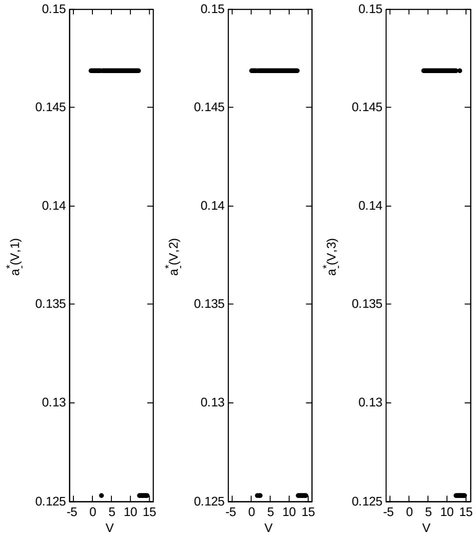

Figure 2: Optimal effort under the 2Pa contract as a function of initial utility promise: $a^*_-(.,l)$, $l \in \{1,2,3\}$ (LMH, case 3)

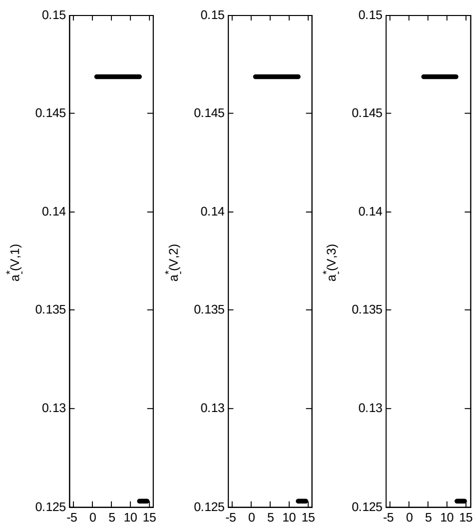

Figure 3: Optimal effort under the 2P contract as a function of initial utility promise: $a^*_-(.,l)$, $l \in \{1,2,3\}$ (LMH, case 3)

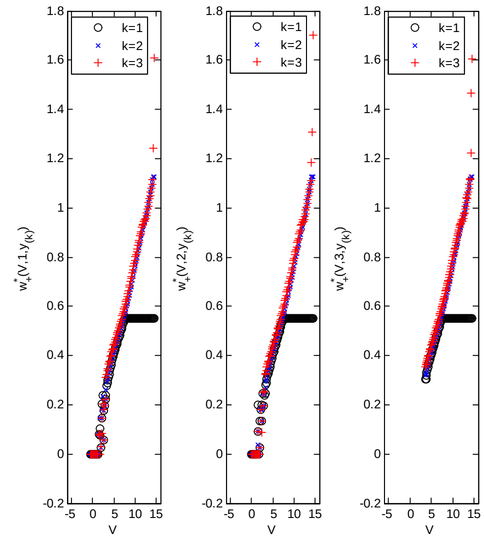

Figure 4: Optimal wage under the 2Pa contract as a function of initial utility promise: $w^*_+\left(.,l,y_{(k)}\right)$, $l,k \in \{1,2,3\}$ (LMH, case 3)

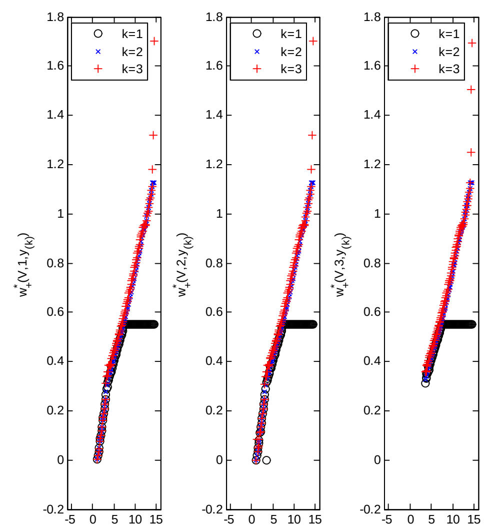

Figure 5: Optimal wage under the 2P contract as a function of initial utility promise: $w^*_+\left(.,l,y_{(k)}\right)$, $l,k \in \{1,2,3\}$ (LMH, case 3)

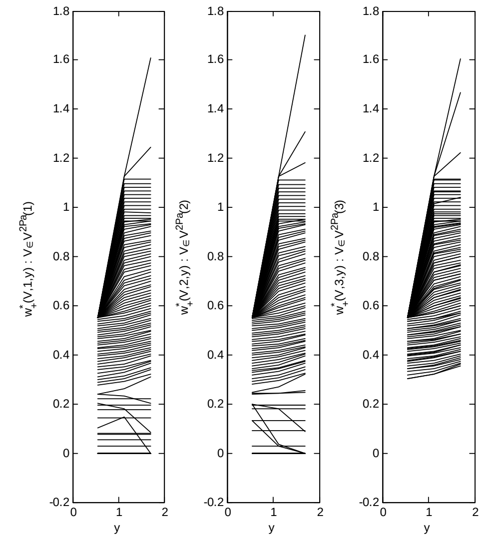

Figure 6: Optimal wage under the 2Pa contract as a function of future profit: $w_+^*(V, l, .)$: $V \in V^{2Pa}(l)$, $l \in \{1, 2, 3\}$ (LMH, case 3)

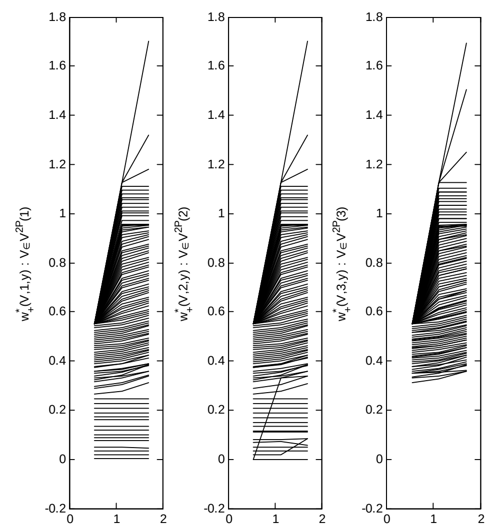

Figure 7: Optimal wage under the 2P contract as a function of future profit: $w_+^*(V, l, .)$: $V \in V^{2P}(l)$, $l \in \{1, 2, 3\}$ (LMH, case 3)

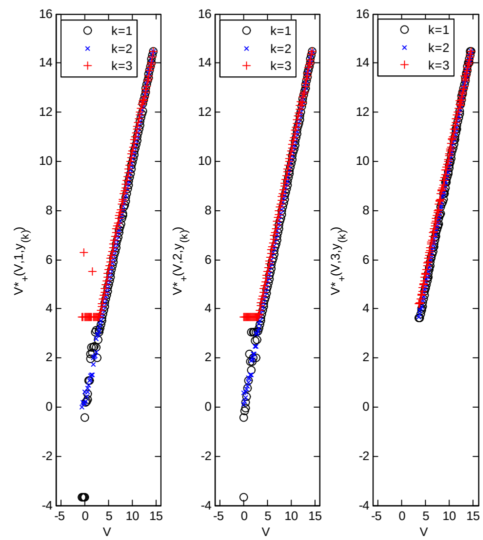

Figure 8: Optimal future utility promise under the 2Pa contract as a function of initial utility promise: $V_+^*\left(.,l,y_{(k)}\right)$, $l,k\in\{1,2,3\}$ (LMH, case 3)

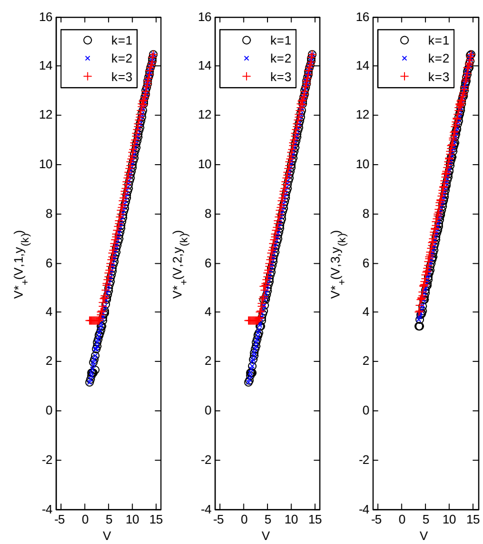

Figure 9: Optimal future utility promise under the 2P contract as a function of initial utility promise: $V_+^*\left(.,l,y_{(k)}\right)$, $l,k\in\{1,2,3\}$ (LMH, case 3)

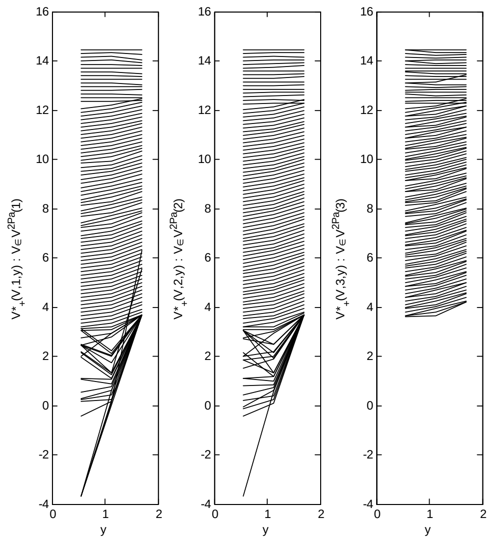

Figure 10: Optimal future utility promise under the 2Pa contract as a function of future profit: $V_{+}^{*}(V,l,.)$: $V \in V^{2Pa}(l)$, $l \in \{1,2,3\}$ (LMH, case 3)

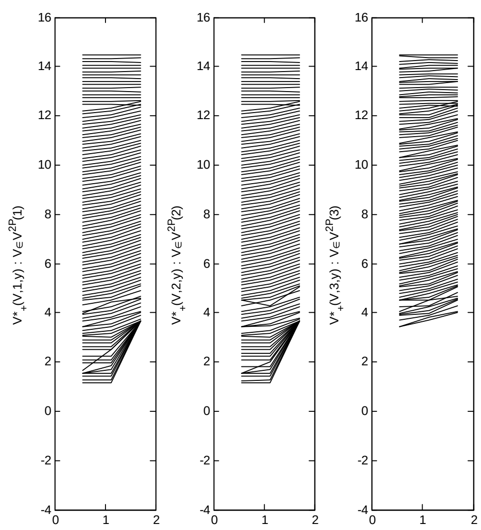

Figure 11: Optimal future utility promise under the 2P contract as a function of future profit: $V_{+}^{*}(V,l,.)$: $V \in V^{2P}(l)$, $l \in \{1,2,3\}$ (LMH, case 3)