Logarithmic asymptotics of contracted Sobolev extremal polynomials on the real line

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Abstract

For a wide class of Sobolev type norms with respect to measures with unbounded support on the real line, the contracted zero distribution and the logarithmic asymptotic of the corresponding re-scaled Sobolev orthogonal polynomials is given.

1. Introduction

Let $\{\mu_k\}_{k=0}^m$ be a family of positive Borel measures supported on the real line. On the linear space $\mathbf{P}$ of polynomials with real coefficients we introduce the Sobolev p-norm

$$\|q\|_S = \left(\sum_{k=0}^{m} \int_{\mathbf{R}} |q^{(k)}(x)|^p \, d\mu_k(x) \right)^{\frac{1}{p}} = \left(\sum_{k=0}^{m} \|q^{(k)}\|_{L^p(\mu_k)}^p \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty, \tag{1}$$

assuming that the integrals of all the polynomials are finite. When $p = 2$ this is said to be a Sobolev norm that is induced by the Sobolev inner product

$$\langle p, q \rangle_S := \sum_{k=0}^{m} \int_{\mathbf{R}} p^{(k)}(x) q^{(k)}(x) \, d\mu_k(x). \tag{2}$$

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In this case it is well known and easy to verify that the monic polynomials that minimize the Sobolev norm are the orthogonal polynomials with respect to the inner product (2).

Asymptotic properties of polynomials orthogonal with respect to a Sobolev inner product have been studied when the supports $\operatorname{supp}(\mu_k)$, $k = 0, 1, \ldots, m$, of the measures are bounded subsets of the real line. In particular

1. If $\{\mu_k\}_{k=1}^m$ are atomic measures, the asymptotic behavior of the polynomials orthogonal with respect to (2) is compared with that of the polynomials orthogonal with respect to μ_0 in [6,12] assuming that $\mu_0' > 0$ a.e. on its supporting interval.
2. The asymptotic zero distribution of the zeros of the Sobolev orthogonal polynomials and their derivatives was analyzed for $m = 1$ in [4] and for general m in [7].
3. The nth root asymptotic behavior of the orthogonal polynomials was analyzed in [7,9] for measures in the class **Reg** of regular measures.
4. For $m = 1$, the strong asymptotic of Sobolev orthogonal polynomials and their first derivative was studied in [13] assuming that the measures belong to the Szegő class. A natural extension when $m > 1$ was given in [14].

When the measures have unbounded support very few results are available and they are restricted to the case $m = 1$. A trivial situation appears when $d\mu_0(x) = d\mu_1(x) = e^{-x^2}\,dx$. Here, the Sobolev orthogonal polynomials are the Hermite polynomials. If $d\mu_0(x) = x^\alpha e^{-x}\,dx$ and $d\mu_1(x) = \lambda x^\alpha e^{-x}\,dx$, in [11] the analytic properties of Laguerre–Sobolev orthogonal polynomials were considered, and in [10] the ratio asymptotics for such polynomials was deduced.

A first example, different from the Laguerre case, was considered in [1] with $d\mu_0(x) = (x^2 + a^2)e^{-x^2}\,dx$, $d\mu_1(x) = \lambda e^{-x^2}\,dx$ in the framework of the so-called symmetric coherent pairs. Relative asymptotics of the corresponding Sobolev orthogonal polynomials with respect to the Hermite polynomials was obtained. An interesting fact is that this asymptotic behavior depends on the parameter λ, which distinguishes it from the bounded case.

Using well-known properties of Freud orthogonal polynomials, relative asymptotics as well as Plancherel–Rotach formulas were deduced in [2] if $d\mu_0(x) = e^{-x^4}\,dx$ and $d\mu_1(x) = \lambda e^{-x^4}\,dx$.

From the general perspective a fundamental breakthrough is given in [5]. There the authors consider general measures of the form $d\mu_0(x) = (\Psi W)^2\,dx$ and $d\mu_1(x) = \lambda W^2\,dx$, where $W = \exp(-Q)$, $Q : I \mapsto [0, +\infty)$ is a convex function, I is an unbounded interval of the real line, and Ψ is a sufficiently smooth function. Let $\{q_n\}$ denote the sequence of orthonormal polynomials with respect to this Sobolev inner product and $\{p_n\}$ the sequence of orthonormal polynomials with respect to $W^2\,dx$, then q_n' behaves like $\lambda^{-\frac{1}{2}}\,p_{n-1}$ for fairly general weights W on I, but some growth restriction on Ψ is necessary. Assuming some extra conditions on Q, the weighted estimate of $q_n' - \lambda^{-\frac{1}{2}} p_{n-1}$ in $L^2(\mathbf{R})$ and $L^\infty(\mathbf{R})$ as well as the strong asymptotics for the re-scaled Sobolev orthogonal polynomials was obtained.

In this paper, we relax the conditions on the weights as compared to [5] aiming for weak asymptotics instead of strong asymptotics. At the same time, we consider more general norms involving derivatives of higher order.

Let $\{w_0, w_1, \ldots, w_m\}$ be a family of positive continuous functions on $\mathbf{R}$ and $p \in [1, +\infty)$. We assume that for each $k = 0, \ldots, m$ the measure $w_k^p\,dx$ has finite moments. For $q \in \mathbf{P}$ we define

$$\|q\|_{\mathrm{S}} = \left(\sum_{k=0}^{m} \int_{\mathbf{R}} |q^{(k)}(x) w_k(x)|^p\,dx\right)^{\frac{1}{p}} = \left(\sum_{k=0}^{m} \|q^{(k)} w_k\|_{L^p(\mathbf{R})}^p\right)^{\frac{1}{p}}. \tag{3}$$

This application induces a norm on $\mathbf{P}$. We say that Q_n is an nth extremal monic polynomial with respect to (3) if $Q_n(x) = x^n + \cdots$ and

$$\|Q_n\|_S = \min\{\|q\|_S : q(x) = x^n + \cdots\}.$$

The existence for each $n \in \mathbf{Z}_+$ of an extremal polynomial is easy to prove. When $1 < p < \infty$ this norm is strictly convex (see [8, Lemma 1]) which yields that for each $n \in \mathbf{Z}_+$ the monic extremal polynomial is uniquely determined. For $p = 1$, when we refer to "the" extremal polynomial of a given degree, we simply mean a representative.

We say that $w \in W(\alpha, \tau)$, $\alpha > 0$, $\tau > 0$, if w is a positive continuous function on $\mathbf{R}$ such that

$$\lim_{|x|\to\infty} \frac{-\log\, w(x)}{\tau|x|^\alpha} = 1.$$

A typical example is $v_{\alpha,\tau}(x) = e^{-\tau|x|^\alpha} \in W(\alpha, \tau)$. In the sequel,

$$\gamma_\alpha := \frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{1+\alpha}{2}\right)},$$

where as usual Γ denotes the Gamma function.

In Section 3, we consider norms of type (3) with $w_k \in W(\alpha_k, \tau_k)$, $0 \leqslant k \leqslant m$. The results depend on the weight which dominates the norm. Let $\overline{k} \in \{0, 1, \ldots, m\}$ be the smallest index such that either

$$\alpha_{\overline{k}} < \min_{\substack{0\leqslant k\leqslant m\\ k\neq\overline{k}}} \alpha_k \quad \text{or} \quad \alpha_{\overline{k}} = \min_{\substack{0\leqslant k\leqslant m\\ k\neq\overline{k}}} \alpha_k \quad \text{and} \quad \tau_{\overline{k}} = \min\{\tau_k : \alpha_k = \alpha_{\overline{k}}\}.$$

Set $\overline{\alpha} = \alpha_{\overline{k}}$ and $\overline{\tau} = \tau_{\overline{k}}$. We prove

Theorem 1.1. *Let Q_n be the nth Sobolev monic extremal polynomial relative to the norm* (3) *where $w_k \in W(\alpha_k, \tau_k)$, $0 \leqslant k \leqslant m$. Then*

$$\lim_{n\to\infty} n^{-1/\overline{\alpha}}\, \|Q_n\|_S^{1/n} = \frac{1}{2}\left(\frac{\gamma_{\overline{\alpha}}}{\overline{\tau}e}\right)^{1/\overline{\alpha}}, \tag{4}$$

and, for all $j \geqslant \overline{k}$

$$\lim_{n\to\infty} n^{-1/\overline{\alpha}}\, \|Q_n^{(j)} v_{\overline{\alpha},\overline{\tau}}\|_{L^\infty(\mathbf{R})}^{1/n} = \frac{1}{2}\left(\frac{\gamma_{\overline{\alpha}}}{\overline{\tau}e}\right)^{1/\overline{\alpha}}. \tag{5}$$

A direct consequence of (5) (see Corollaries 3.1 and 3.2) is the asymptotic contracted limit distribution of the zeros of $Q_n^{(j)}$ and the weak limit of the corresponding contracted extremal polynomials.

In Section 4, we consider a more general class of weights. We say that $w \in W(\alpha)$, $\alpha > 0$, if

$$\lim_{|x|\to\infty} \frac{\log\, \log \frac{1}{w(x)}}{\log\, |x|} = \alpha. \tag{6}$$

Notice that $W(\alpha, \tau) \subset W(\alpha)$ for all $\tau > 0$. In particular, $v_\alpha = e^{-|x|^\alpha} \in W(\alpha)$. Here, we consider norms of type (3) such that $w_k \in W(\alpha_k)$, $0 \leqslant k \leqslant m$. The results also depend on the dominating weight in the norm. Let $\widetilde{k} \in \{0, 1, \ldots, m\}$ be the first index such that

$$\alpha_{\widetilde{k}} = \min\{\alpha_k : 0 \leqslant k \leqslant m\}.$$

Set $\widetilde{\alpha} = \alpha_{\widetilde{k}}$. We prove

Theorem 1.2. *Let Q_n be the nth monic extremal polynomial with respect to the norm* (3) *where $w_k \in W(\alpha_k)$, $0 \leqslant k \leqslant m$. Then*

$$\lim_{n\to\infty} \|Q_n\|_{\mathrm{S}}^{1/(n \log n)} = e^{1/\widetilde{\alpha}}. \tag{7}$$

To make the reading more comprehensive, we include in Section 2 a brief review of known results from potential theory which will be needed. More details may be found in [16]. Sections 3 and 4 are dedicated to the proof of Theorems 1.1 and 1.2, respectively.

2. Auxiliary results

Technically, this paper is a continuation of [7,9] (which in turn were inspired in [4]) to the case of measures with unbounded support. The basic idea is to compare the norm (3) of the extremal polynomials with their weighted uniform norm on $\mathbf{R}$. The reduction is possible due to Markov and Nikolskii type inequalities. For convenience of the reader, we state the corresponding results as lemmas in the form in which they will be used. They are simple reformulations of Theorems VI.5.5 and VI.5.6 in [16].

Lemma 2.1 (*Markov type inequalities*). *Let $v_{\alpha,\tau}(x) = e^{-\tau|x|^\alpha}$, $\alpha > 0$, $\tau > 0$, and let P_n be a polynomial of degree at most $n \in \mathbf{Z}_+$. Then, for every $1 \leqslant p \leqslant \infty$ and $1 \leqslant k \leqslant n$*

$$\|P_n^{(k)}\, v_{\alpha,\tau}\|_{L^p(\mathbf{R})} \leqslant M(n, k, \alpha, \tau, p)\, \|P_n\, v_{\alpha,\tau}\|_{L^p(\mathbf{R})},$$

where $M_n = M(n, k, \alpha, \tau, p)$ is such that

$$\lim_{n\to\infty} M_n^{1/n} = 1.$$

Lemma 2.2 (*Nikolskii type inequalities*). *Let $v_{\alpha,\tau}(x) = e^{-\tau|x|^\alpha}$, $\alpha > 0$, $\tau > 0$, and let P_n be a polynomial of degree at most $n \in \mathbf{Z}_+$. Then, for every $0 < p, q \leqslant \infty$*

$$\|P_n\, v_{\alpha,\tau}\|_{L^p(\mathbf{R})} \leqslant N(n, \alpha, \tau, p, q)\, \|P_n\, v_{\alpha,\tau}\|_{L^q(\mathbf{R})},$$

where $N_n = N(n, \alpha, \tau, p, q)$ satisfies

$$\lim_{n\to\infty} N_n^{1/n} = 1.$$

From formulas (VI.5.7) and (VI.5.16) in [16] the values of M_n and N_n may be specified, but we only need that they have the nth root limit indicated.

Other key ingredients in the proof are the properties of the asymptotically extremal polynomials with respect to weighted L^∞ norms. We say that a positive continuous function w on $\mathbf{R}$ is admissible if

$$\lim_{|x|\to\infty} |x| w(x) = 0.$$

Let $\mathcal{M}(\mathbf{R})$ be the collection of all positive unit Borel measures μ supported on $\mathbf{R}$ and define the weighted energy integral

$$\begin{aligned} I_w(\mu) &:= \int\int \log\left(\frac{1}{|x-t|w(x)w(t)}\right) d\mu(x)\, d\mu(t) \\ &= \int\int \log\left(\frac{1}{|x-t|}\right) d\mu(x) d\mu(t) + 2\int Q(t)\, d\mu(t), \end{aligned}$$

where $w(x) = \exp(-Q(x))$. As usual,

$$U^{\mu}(z) := \int \log\frac{1}{|z-t|}\, d\mu(t)$$

denotes the logarithmic potential of μ.

There exists a unique $\mu_w \in \mathcal{M}(\mathbf{R})$ such that

$$I_w(\mu_w) = \inf\{I_w(\mu) : \mu \in \mathcal{M}(\mathbf{R})\}.$$

Set

$$F_w := I_w(\mu_w) - \int Q(t)\, d\mu_w(t).$$

The measure μ_w is characterized by

$$U^{\mu_w}(x) + Q(x) \begin{cases} \leqslant F_w, & x \in \operatorname{supp}(\mu_w), \\ \geqslant F_w, & \mathbf{R} \setminus E, \end{cases}$$

where E is a set of logarithmic capacity equal to zero (see [16, Theorem I.1.3]). Moreover, the support, $\operatorname{supp}(\mu_w)$, of μ_w is a compact subset of the real line and has positive capacity.

Let $\{P_n\}$, $n \in \mathbf{Z}_+$, $\deg(P_n) = n$, be a sequence of monic polynomials. It is well known (see [16, Theorem I.3.6]) that if w is admissible,

$$\liminf_{n\to\infty} \|P_n w^n\|_{L^\infty(\mathbf{R})}^{1/n} \geqslant e^{-F_w}. \tag{8}$$

The sequence $\{P_n\}$, $n \in \mathbf{Z}_+$, is said to be asymptotically extremal with respect to w if

$$\lim_{n\to\infty} \|P_n\, w^n\|_{L^\infty(\mathbf{R})}^{1/n} = e^{-F_w}. \tag{9}$$

The nth monic weighted Chebyshev polynomial with respect to w is defined as follows:

$$\|T_n w^n\|_{L^\infty(\mathbf{R})} = \inf\{\|P_n w^n\|_{L^\infty(\mathbf{R})} : P_n(x) = x^n + \cdots\}.$$

From [16, Theorem III.3.1], we have

Lemma 2.3. *Let w be admissible, then the sequence $\{T_n\}$, $n \in \mathbf{Z}_+$, is asymptotically extremal with respect to w. In particular,*

$$\lim_{n\to\infty} \|T_n\, w^n\|_{L^\infty(\mathbf{R})}^{1/n} = e^{-F_w}. \tag{10}$$

From [16, Theorem IV.5.1], we have

Lemma 2.4. *Let* $\widehat{v}_\alpha = \exp\left(-\gamma_\alpha |x|^\alpha\right)$, $\alpha > 0$. *Then* $\operatorname{supp}(\mu_{\widehat{v}_\alpha}) = [-1, 1]$,

$$F_{\widehat{v}_\alpha} = \log 2 + 1/\alpha, \tag{11}$$

and

$$d\mu_{\widehat{v}_\alpha}(t) = \left(\frac{\alpha}{\pi}\int_{|t|}^{1} \frac{u^{\alpha-1}}{\sqrt{u^2 - t^2}}\, du\right) dt, \quad t \in [-1, 1]. \tag{12}$$

Furthermore, on $\mathbf{R} \setminus [-1, 1]$,

$$U^{\mu_{\widehat{v}_\alpha}}(x) = -\log|x + \sqrt{x^2 - 1}| - |x|^\alpha \int_0^{1/|x|} \frac{u^{\alpha-1}}{\sqrt{1 - u^2}}\, du + 1/\alpha + \log 2, \tag{13}$$

whereas on $\mathbf{C} \setminus \mathbf{R}$,

$$U^{\mu_{\widehat{v}_\alpha}}(z) = -\log|z + \sqrt{z^2 - 1}| - \operatorname{Re}\left[\int_0^1 \frac{z u^{\alpha-1}}{\sqrt{z^2 - u^2}}\, du\right] + 1/\alpha + \log 2. \tag{14}$$

For $p = 2$ the next lemma is contained in (1.3), Theorem VII.1.3, in [16]. For any $1 \leqslant p < \infty$ the proof is exactly the same using the Nikolskii type inequalities.

Lemma 2.5. *Consider the weight* $v_{\alpha,\tau}$. *Let* L_n *be the nth monic extremal polynomial with respect to the weight* $v_{\alpha,\tau}$ *in the p-norm,* $1 \leqslant p < \infty$, *i.e.*

$$\|L_n v_{\alpha,\tau}\|_{L^p(\mathbf{R})} = \inf\{\|q v_{\alpha,\tau}\|_{L^p(\mathbf{R})} : q(x) = x^n + \cdots\}.$$

Then,

$$\lim_{n\to\infty} n^{-1/\alpha} \|L_n\|_{L^p(\mathbf{R})}^{1/n} = \frac{1}{2}\left(\frac{\gamma_\alpha}{\tau e}\right)^{1/\alpha}.$$

There is a close connection between the asymptotic extremality of a sequence of polynomials, the contracted asymptotic zero distribution of its zeros, and the contracted nth root asymptotics of the polynomials. For details see [16, Theorems III.4.2 and III.4.7(iv)]. Let q be a polynomial of degree n. We define the normalized zero counting measure by

$$\nu_n(q) := \frac{1}{n}\sum_{q(x)=0} \delta_x,$$

where δ_x is the Dirac measure at the point x.

Lemma 2.6. *Let* w *be an admissible weight (on* $\mathbf{R}$*) and* $\{P_n\}$, $n \in \mathbf{Z}_+$, *a sequence of monic asymptotically extremal polynomials with respect to* w. *Set* $\nu_n = \nu_n(P_n)$. *Then*,

$$* \lim_{n\to\infty} \nu_n = \mu_w \tag{15}$$

in the weak topology of measures. Moreover, if z is not a limit point of the zeros of the* P_n*'s, then*

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = e^{-U^{\mu_w}(z)}. \tag{16}$$

3. Sobolev inner product on the real line

Let q be a monic polynomial of degree n. We denote

$$\widehat{q}(\alpha,\tau;x)=\left(\frac{n\,\gamma_\alpha}{\tau}\right)^{n/\alpha} q\left(\left(\frac{\tau}{n\,\gamma_\alpha}\right)^{1/\alpha} x\right).$$

Lemma 3.1. *For every p, $1\leqslant p<\infty$, there exists a constant C such that for all k, $0\leqslant k\leqslant m$,*

$$v_{\alpha_k,\tau_k}(x)\leqslant C v_{\overline{\alpha},\overline{\tau}}(x),\qquad x\in\mathbf{R}, \tag{17}$$
$$\|q\,v_{\alpha_k,\tau_k}\|_{L^p(\mathbf{R})}\leqslant C\|q\,v_{\overline{\alpha},\overline{\tau}}\|_{L^p(\mathbf{R})},\qquad q\in\mathbf{P}, \tag{18}$$
$$\|\widehat{q}(\alpha,\tau;\cdot)\,v_{\alpha,\tau}\|_{L^\infty(\mathbf{R})}=\left(\frac{n\,\gamma_\alpha}{\tau}\right)^{n/\alpha}\|q\,\widehat{v}_\alpha^n\|_{L^\infty(\mathbf{R})},\qquad q(x)=x^n+\cdots. \tag{19}$$

Proof. Suppose that there exists a $k\in\{0,1,\ldots,m\}$, $k\neq\overline{k}$, such that $\alpha_k=\overline{\alpha}$. By the definition of $\overline{k}$, $\overline{\tau}\leqslant\tau_k$; therefore,

$$-\tau_k|x|^{\alpha_k}\leqslant-\overline{\tau}|x|^{\overline{\alpha}},\qquad x\in\mathbf{R},$$

and consequently,

$$v_{\alpha_k,\tau_k}(x)\leqslant v_{\overline{\alpha},\overline{\tau}}(x),\qquad x\in\mathbf{R}. \tag{20}$$

For all k such that $\alpha_k\neq\overline{\alpha}$, by the definition of $\overline{k}$, $\overline{\alpha}<\alpha_k$. Hence,

$$\lim_{|x|\to\infty}\frac{\overline{\tau}|x|^{\overline{\alpha}}}{\tau_k|x|^{\alpha_k}}=0.$$

In particular, there exists a constant $C_1>0$ such that

$$\frac{\overline{\tau}|x|^{\overline{\alpha}}}{\tau_k|x|^{\alpha_k}}\leqslant 1,\qquad |x|\geqslant C_1,$$

or equivalently,

$$\max_{\substack{k\neq\overline{k}\\ \alpha_k>\overline{\alpha}}} v_{\alpha_k,\tau_k}(x)\leqslant v_{\overline{\alpha},\overline{\tau}}(x),\qquad |x|\geqslant C_1. \tag{21}$$

On the interval $|x|\leqslant C_1$, the functions $\dfrac{v_{\alpha_k,\tau_k}}{v_{\overline{\alpha},\overline{\tau}}}$ are continuous and positive. Thus

$$\max_{\substack{k\neq\overline{k}\\ \alpha_k>\overline{\alpha}}}\ \max_{|x|\leqslant C_1}\frac{v_{\alpha_k,\tau_k}}{v_{\overline{\alpha},\overline{\tau}}}\leqslant C_2<+\infty. \tag{22}$$

From (20)–(22), we conclude (17). Relation (18) is a direct consequence of (17).

Finally, (19) follows from the linear substitution $x=(n\,\gamma_\alpha/\tau)^{1/\alpha}\,t$ and the relation

$$v_{\alpha,\tau}(x)=e^{-\tau|x|^\alpha}=(e^{-\gamma_\alpha|t|^\alpha})^n=\widehat{v}_\alpha^n(t).\qquad\square$$

Proof of Theorem 1.1. If $w\in W(\alpha,\tau)$, it is easy to see that for every ε, $\tau-\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that

$$C_\varepsilon^{-1}e^{-(\tau+\varepsilon)|x|^\alpha}\leqslant w(x)\leqslant C_\varepsilon e^{-(\tau-\varepsilon)|x|^\alpha},\qquad x\in\mathbf{R}. \tag{23}$$

By the extremal property of Q_n with respect to the norm (3) and the right-hand side of (23), we have that for each ε $(0 < \varepsilon < \min_{0\leqslant k\leqslant m} \tau_k)$ there exists a constant C_1, such that for any monic polynomial Q, $\deg(Q) = n$,

$$\|Q_n^{(\overline{k})}\, w_{\overline{k}}\|_{L^p(\mathbf{R})}^p \leqslant \|Q_n\|_S^p \leqslant \|Q\|_S^p = \sum_{k=0}^{m} \|Q^{(k)}\, w_k\|_{L^p(\mathbf{R})}^p \leqslant C_1 \sum_{k=0}^{m} \|Q^{(k)}\, v_{\alpha_k,\tau_k-\varepsilon}\|_{L^p(\mathbf{R})}^p .$$

Using Lemma 2.1, (17), and Lemma 2.2, it follows that there exist constants $\widetilde{M}_n$ such that $\lim_{n\to\infty} \widetilde{M}_n^{1/n} = 1$ and

$$\sum_{k=0}^{m} \|Q^{(k)}\, v_{\alpha_k,\tau_k-\varepsilon}\|_{L^p(\mathbf{R})}^p \leqslant \widetilde{M}_n\, \|Q\, v_{\overline{\alpha},\overline{\tau}-\varepsilon}\|_{L^\infty(\mathbf{R})}^p .$$

Let T_n be the nth monic weighted Chebyshev polynomial with respect to $\widehat{v_{\overline{\alpha}}}$. Take $Q = \widehat{T}_n(\overline{\alpha}, \overline{\tau} - \varepsilon; \cdot)$. From (19) and the previous bounds, it follows that

$$\|Q_n^{(\overline{k})}\, w_{\overline{k}}\|_{L^p(\mathbf{R})}^p \leqslant \|Q_n\|_S^p \leqslant C_1 \widetilde{M}_n \left(\frac{n\gamma_{\overline{\alpha}}}{\overline{\tau}-\varepsilon}\right)^{pn/\overline{\alpha}} \|T_n\, \widehat{v}_{\overline{\alpha}}^n\|_{L^\infty(\mathbf{R})}^p .$$

Hence, using (10) and (11), we obtain

$$\limsup_{n\to\infty} \frac{\|Q_n^{(\overline{k})}\, w_{\overline{k}}\|_{L^p(\mathbf{R})}^{1/n}}{n^{1/\overline{\alpha}}} \leqslant \limsup_{n\to\infty} \frac{\|Q_n\|_S^{1/n}}{n^{1/\overline{\alpha}}} \leqslant \frac{1}{2} \left(\frac{\gamma_{\overline{\alpha}}}{e(\overline{\tau}-\varepsilon)}\right)^{1/\overline{\alpha}} .$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\limsup_{n\to\infty} \frac{\|Q_n^{(\overline{k})}\, w_{\overline{k}}\|_{L^p(\mathbf{R})}^{1/n}}{n^{1/\overline{\alpha}}} \leqslant \limsup_{n\to\infty} \frac{\|Q_n\|_S^{1/n}}{n^{1/\overline{\alpha}}} \leqslant \frac{1}{2} \left(\frac{\gamma_{\overline{\alpha}}}{e\,\overline{\tau}}\right)^{1/\overline{\alpha}} . \tag{24}$$

Fix $\varepsilon > 0$. Let $L_{n-\overline{k}}$ be the $(n - \overline{k})$ monic extremal polynomial with respect to the weight $v_{\overline{\alpha},\overline{\tau}+\varepsilon}$ in the norm of $L^p(\mathbf{R})$. By the left-hand side of (23) and the extremal property of $L_{n-\overline{k}}$, there exists a constant C_3 such that

$$\|Q_n\|_S^p \geqslant \|Q_n^{(\overline{k})}\, w_{\overline{k}}\|_{L^p(\mathbf{R})}^p \geqslant C_3\, \|Q_n^{(\overline{k})}\, v_{\overline{\alpha},\overline{\tau}+\varepsilon}\|_{L^p(\mathbf{R})}^p \geqslant C_3 \left(\frac{n!}{(n-\overline{k})!}\right)^p \|L_{n-\overline{k}}\, v_{\overline{\alpha},\overline{\tau}+\varepsilon}\|_{L^p(\mathbf{R})}^p .$$

Lemma 2.5 and these inequalities imply

$$\liminf_{n\to\infty} \frac{\|Q_n\|_S^{1/n}}{n^{1/\overline{\alpha}}} \geqslant \liminf_{n\to\infty} \frac{\|Q_n^{(\overline{k})}\, w_{\overline{k}}\|_{L^p(\mathbf{R})}^{1/n}}{n^{1/\overline{\alpha}}} \geqslant \frac{1}{2} \left(\frac{\gamma_{\overline{\alpha}}}{e(\overline{\tau}+\varepsilon)}\right)^{1/\overline{\alpha}} .$$

Since $\varepsilon > 0$ is arbitrary,

$$\liminf_{n\to\infty} \frac{\|Q_n\|_S^{1/n}}{n^{1/\overline{\alpha}}} \geqslant \liminf_{n\to\infty} \frac{\|Q_n^{(\overline{k})}\, w_{\overline{k}}\|_{L^p(\mathbf{R})}^{1/n}}{n^{1/\overline{\alpha}}} \geqslant \frac{1}{2} \left(\frac{\gamma_{\overline{\alpha}}}{e\,\overline{\tau}}\right)^{1/\overline{\alpha}} . \tag{25}$$

From (24) and (25), we obtain (4) and

$$\lim_{n\to\infty} \frac{\|Q_n^{(\overline{k})}\, w_{\overline{k}}\|_{L^p(\mathbf{R})}^{1/n}}{n^{1/\overline{\alpha}}} = \frac{1}{2} \left(\frac{\gamma_{\overline{\alpha}}}{e\,\overline{\tau}}\right)^{1/\overline{\alpha}} .$$

On the other hand (see the sentence before Theorem VI.6.1 in [16]),

$$\lim_{n\to\infty}\left(\frac{\| Q_n^{(\overline{k})}\, w_{\overline{k}}\|_{L^p(\mathbf{R})}}{\| Q_n^{(\overline{k})}\, v_{\overline{\alpha},\overline{\tau}}\|_{L^p(\mathbf{R})}}\right)^{1/n}=1.$$

Therefore,

$$\lim_{n\to\infty}\frac{\| Q_n^{(\overline{k})}\, v_{\overline{\alpha},\overline{\tau}}\|_{L^p(\mathbf{R})}^{1/n}}{n^{1/\overline{\alpha}}}=\frac{1}{2}\left(\frac{\gamma_{\overline{\alpha}}}{e\,\overline{\tau}}\right)^{1/\overline{\alpha}},$$

and using the Nikolskii type inequalities once more, we obtain (5) for $j=\overline{k}$. For $j\geqslant\overline{k}$

$$\limsup_{n\to\infty}\frac{\| Q_n^{(j)}\, v_{\overline{\alpha},\overline{\tau}}\|_{L^\infty(\mathbf{R})}^{1/n}}{n^{1/\overline{\alpha}}}\leqslant\frac{1}{2}\left(\frac{\gamma_{\overline{\alpha}}}{e\,\overline{\tau}}\right)^{1/\overline{\alpha}} \tag{26}$$

follows on account of the Markov type inequalities. Let S_{n-j} be such that $\widehat{S}_{n-j}(\overline{\alpha},\overline{\tau};\cdot)=\frac{(n-j)!}{n!}Q_n^{(j)}$. From (19),

$$\frac{(n-j)!}{n!}\| Q_n^{(j)}\, v_{\overline{\alpha},\overline{\tau}}\|_{L^\infty(\mathbf{R})}=\left(\frac{(n-j)\gamma_{\overline{\alpha}}}{\overline{\alpha}}\right)^{(n-j)/\overline{\alpha}}\|S_{n-j}\,\widehat{v}_{\overline{\alpha}}^{\,n-j}\|_{L^\infty(\mathbf{R})}. \tag{27}$$

By (8),

$$\liminf_{n\to\infty}\|S_{n-j}\,\widehat{v}_{\overline{\alpha}}^{\,n-j}\|_{L^\infty(\mathbf{R})}^{1/n}\geqslant\frac{1}{2e^{1/\overline{\alpha}}}. \tag{28}$$

Taking nth root in (27) and dividing by $n^{1/\overline{\alpha}}$, we obtain the lower bound which allows to conclude the proof of (5). □

Corollary 3.1. *Let $\{x_{n,1},x_{n,2},\ldots,x_{n,n}\}$ be the set of zeros of $Q_{n+j}^{(j)}$ and denote*

$$Z_{n,\overline{k}}=\{z_{n,i}\}_{i=1}^n,\quad z_{n,i}:=\left(\frac{\overline{\tau}}{n\,\gamma_{\overline{\alpha}}}\right)^{1/\overline{\alpha}}x_{n,i}$$

the set of contracted zeros. Let

$$u_{\overline{k}}(t):=\frac{\overline{\alpha}}{\pi}\int_{|t|}^1\frac{x^{\overline{\alpha}-1}}{\sqrt{x^2-t^2}}\,dx,\qquad t\in[-1,1],$$

be the Ullman distribution associated with the index $\overline{k}$. Let $S_{n,j}$ be such that $\widehat{S}_{n,j}(\overline{\alpha},\overline{\tau},;\cdot)=\frac{n!}{(n+j)!}Q_{n+j}^{(j)}$. For each $j\geqslant\overline{k}$

$$*\lim_{n\to\infty}\overline{v}_n(S_{n,j})=u_{\overline{k}}(t)\,dt. \tag{29}$$

Moreover, if z is not a limit point of the zeros of the $S_{n,j}$'s, $n\in\mathbf{Z}_+$, then

$$\lim_{n\to\infty}|S_{n,j}(z)|^{1/n}=\frac{1}{2e^{1/\overline{\alpha}}}\left|z+\sqrt{z^2-1}\right|e^{\eta_{\overline{\alpha}}(z)}, \tag{30}$$

where $\eta_{\overline{\alpha}}(z)=\mathrm{Re}\left[\int_0^1\frac{zt^{\overline{\alpha}-1}}{\sqrt{z^2-t^2}}\,dt\right]$.

Proof. At the end of the proof of Theorem 1.1 we actually proved that for each $j \geqslant \overline{k}$, the sequence of polynomials $\{S_{n,j}\}$, $n \geqslant j$, is asymptotically extremal with respect to the weight $\widehat{v_{\overline{\alpha}}}$ (see (26)–(28)). From Lemmas 2.4 and 2.6, we conclude that (29) and (30) take place. □

Corollary 3.2. *Let us assume that* $p = 2$ *and the weights* $w_k \in W(\alpha_k, \tau_k)$ *defining the norm* (3) *satisfy* $w_k/w_{k-1} \in L^{\infty}(\mathbf{R})$, $1 \leqslant k \leqslant m$. *Let* $S_{n,j}$ *be such that* $\widehat{S}_{n,j}(\overline{\alpha}, \overline{\tau}, ; \cdot) = \frac{n!}{(n+j)!} Q_{n+j}^{(j)}$. *For each* $j \geqslant 0$

$$* \lim_{n\to\infty} \overline{\nu}_n(S_{n,j}) = u_0(t)\, dt. \tag{31}$$

Moreover, for each $j \geqslant 0$,

$$\lim_{n\to\infty} |S_{n,j}(z)|^{1/n} = \frac{1}{2e^{1/\overline{\alpha}}} |z + \sqrt{z^2 - 1}|\, e^{\eta_{\alpha_0}(z)}. \tag{32}$$

Proof. The condition on the ratio of the weights implies that $\overline{k} = 0$ and (29) is true for all $j \geqslant 0$. On the other hand, it implies that the zeros of Q_n lie in the band $\{z : |\Im(z)| \leqslant C\}$ where C is a constant that does not depend on n. This was proved in [3, Theorem1.1]. Since the zeros of $Q_n^{(j)}$ for all $j \geqslant 0$ are in the convex hull of the set of zeros of Q_n, it follows that for all $j \geqslant 0$ the zeros of $Q_{n+j}^{(j)}$ lie in $\{z : |\Im(z)| \leqslant C\}$. Therefore, the contracted zeros may only have accumulation points on the real line. Whence, using (16), (29) implies the stronger version (32) of (30). □

4. Sobolev inner product with weight in class $W(\alpha)$

In this section we use the norm (3) but now the $m + 1$ functions $\{w_0(x), \dots, w_m(x)\}$ satisfy $w_k \in W(\alpha_k)$, $\alpha_k > 0$, $k = 0, 1, \dots, m$. Let q be a monic polynomial of degree n. We denote

$$\widetilde{q}(\alpha; x) := (n\,\gamma_\alpha)^{n/\alpha}\, q((n\,\gamma_\alpha)^{-1/\alpha}\, x).$$

Set $v_\alpha(x) = e^{-|x|^\alpha}$. Notice that the linear substitution $x = (n\,\gamma_\alpha)^{1/\alpha}\, t$ yields

$$v_\alpha(x) = e^{-|x|^\alpha} = \left(e^{-\gamma_\alpha |t|^\alpha}\right)^n = \widehat{v}_\alpha^n(t).$$

The constants $\widetilde{k}$ and $\widetilde{\alpha}$ were defined in section 1. Using Lemma 3.1 with $\tau_k = 1$, $k = 0, \dots, m$, there exists a constant C such that for all k, $0 \leqslant k \leqslant m$,

$$v_{\alpha_k}(x) \leqslant C v_{\widetilde{\alpha}}(x), \quad x \in \mathbf{R}, \tag{33}$$

$$\|q\, v_{\alpha_k}\|_{L^p(\mathbf{R})} \leqslant C \|q\, v_{\widetilde{\alpha}}\|_{L^p(\mathbf{R})}, \tag{34}$$

$$\|\widetilde{q}(\alpha; \cdot)\, v_\alpha\|^2_{L^\infty(\mathbf{R})} = (n\,\gamma_\alpha)^{n/\alpha} \|q\, \widehat{v}_\alpha^n\|_{L^\infty(\mathbf{R})}, \quad q(x) = x^n + \cdots. \tag{35}$$

Proof of Theorem 1.2. If $w \in W(\alpha)$, it is easy to see that for each ε ($\alpha > \varepsilon > 0$) there is a constant $\widetilde{C}_\varepsilon$ such that

$$\widetilde{C}_\varepsilon^{-1}\, v_{\alpha+\varepsilon}(x) \leqslant w(x) \leqslant \widetilde{C}_\varepsilon\, v_{\alpha-\varepsilon}(x), \quad v_\alpha(x) = e^{-|x|^\alpha}, \quad x \in \mathbf{R}. \tag{36}$$

By the extremal property of Q_n for the given norm and the right-hand side inequality of (36), we have that for each ε, $\widetilde{\alpha} > \varepsilon > 0$, there exists a constant C_1, such that for any monic polynomial Q, $\deg(Q) = n$,

$$\|Q_n\|_{\mathrm{S}}^p \leqslant \|Q\|_{\mathrm{S}}^p = \sum_{k=0}^{m} \|Q^{(k)}\, w_k\|_{L^p(\mathbf{R})}^p \leqslant C_1 \sum_{k=0}^{m} \|Q^{(k)}\, v_{\alpha_k-\varepsilon}\|_{L^p(\mathbf{R})}^p.$$

Using the Markov type inequalities, (33), and the Nikolskii type inequalities, it follows that there exist constants $\widetilde{M}_n$, $\lim_{n\to\infty} \widetilde{M}_n^{1/n} = 1$, such that

$$\sum_{k=0}^{m} \|Q^{(k)}\, v_{\alpha_k-\varepsilon}\|_{L^p(\mathbf{R})}^p \leqslant \widetilde{M}_n \|Q\, v_{\widetilde{\alpha}-\varepsilon}\|_{L^\infty(\mathbf{R})}^p .$$

Let T_n be the Chebychev polynomial of degree n with respect to $\widehat{v}_{\widetilde{\alpha}-\varepsilon}^n$. Take $Q = \widetilde{T}_n(\widetilde{\alpha}-\varepsilon;\cdot)$. By (35) and the previous bounds

$$\|Q_n\|_{\mathrm{S}}^p \leqslant C_1 \widetilde{M}_n\, (n\, \gamma_{\widetilde{\alpha}-\varepsilon})^{pn/(\widetilde{\alpha}-\varepsilon)} \|T_n\, \widehat{v}_{\widetilde{\alpha}-\varepsilon}^n\|_{L^\infty(\mathbf{R})}^p.$$

Therefore,

$$\frac{\log \|Q_n\|_{\mathrm{S}}}{n \log n} \leqslant \frac{\log(C_1 \widetilde{M}_n)^{1/p}}{n\, \log n} + \frac{\|T_n\, \widehat{v}_{\widetilde{\alpha}-\varepsilon}^n\|_{L^\infty(\mathbf{R})}}{n \log\, n} + \frac{1}{\widetilde{\alpha}-\varepsilon}\, \frac{\log(n\, \gamma_{\widetilde{\alpha}-\varepsilon})}{\log n}.$$

Hence, from Lemma 2.3, we obtain

$$\limsup_{n\to\infty}\, \log \|Q_n\|_{\mathrm{S}}^{1/(n \log\, n)} \leqslant \frac{1}{\widetilde{\alpha}-\varepsilon}.$$

Making ε tend to zero, we conclude that

$$\limsup_{n\to\infty}\, \|Q_n\|_{\mathrm{S}}^{1/(n \log\, n)} \leqslant e^{1/\widetilde{\alpha}}. \tag{37}$$

From (3), the left-hand side inequality of (36), and the extremal property of monic extremal polynomials, we have that for every $\varepsilon > 0$ there exists a constant C_2 such that

$$\|Q_n\|_{\mathrm{S}}^p \geqslant \|Q_n^{(\widetilde{k})}\, w_{\widetilde{k}}\|_{L^p(\mathbf{R})}^p \geqslant C_2\, \|Q_n^{(\widetilde{k})}\, v_{\widetilde{\alpha}+\varepsilon}\|_{L^p(\mathbf{R})}^p \geqslant C_2 \left(\frac{n!}{(n-\widetilde{k})!}\right)^p \|L_{n-\widetilde{k}}\, v_{\widetilde{\alpha}+\varepsilon}\|_{L^p(\mathbf{R})}^p,$$

where $L_{n-\widetilde{k}}$ is the monic extremal polynomial of degree $n-\widetilde{k}$ in the $L^p(\mathbf{R})$ norm with respect to $v_{\widetilde{\alpha}+\varepsilon}$. Using Lemma 2.5 and the previous inequalities, it follows that

$$\liminf_{n\to\infty} \frac{\|Q_n\|_{\mathrm{S}}^{1/n}}{n^{1/(\widetilde{\alpha}+\varepsilon)}} \geqslant \frac{1}{2} \left(\frac{\gamma_{\widetilde{\alpha}+\varepsilon}}{e}\right)^{1/(\widetilde{\alpha}+\varepsilon)}. \tag{38}$$

Let δ, $0 < \delta < \frac{1}{2}\left(\frac{\gamma_{\widetilde{\alpha}+\varepsilon}}{e}\right)^{1/(\widetilde{\alpha}+\varepsilon)}$, be arbitrary. From (38), for all $n \geqslant n_0$ we have

$$\frac{\|Q_n\|_{\mathrm{S}}^{1/n}}{n^{1/(\widetilde{\alpha}+\varepsilon)}} \geqslant \frac{1}{2} \left(\frac{\gamma_{\widetilde{\alpha}+\varepsilon}}{e}\right)^{1/(\widetilde{\alpha}+\varepsilon)} - \delta.$$

Taking logarithm on both sides, dividing by $\log n$, and taking limit as n tends to ∞ we get

$$\liminf_{n\to\infty} \frac{\log \|Q_n\|_S}{n \log n} \geqslant \frac{1}{\widetilde{\alpha}+\varepsilon}.$$

Letting ε tend to zero, we conclude that

$$\liminf_{n\to\infty} \|Q_n\|_S^{1/(n \log n)} \geqslant e^{1/(\widetilde{\alpha}+\varepsilon)}$$

which together with (37) implies (7). $\square$

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