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BOOTSTRAP PREDICTIVE INFERENCE FOR ARIMA PROCESSES.*

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Abstract

We introduce a new bootstrap strategy to obtain prediction intervals in ARIMA (p,d,1) processes. Its main advantages over previous resampling proposals for ARI (p,d) models are that it incorporates variability due to parameter estimation and it makes unnecessary the process backward representation to resample the series. Consequently, the method is very flexible and can be extended to general models not having a backward representation. Moreover, our bootstrap technique allows to obtain the prediction density of processes with moving average components. Its implementation is computationally very simple. The asymptotic properties of the bootstrap prediction distributions are proved. Extensive finite sample Monte Carlo experiments are carried out to compare the performance of this method versus alternative techniques for ARI (p,d) processes. Our method either behaves similarly or outperforms in most cases previous proposals.

Keywords:
Forecasting; Non Gaussian distributions; Resampling methods; simulation.

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1. INTRODUCTION

Forecasting is one of the main goals in univariate time series analysis. The problem consists in providing information about the distribution of the variable Y_{T+k} conditional on a realization of the past variables $\mathbf{Y}_T = \{Y_1, \ldots, Y_T\}$. In particular, the objective is to construct prediction intervals $I(\mathbf{Y}_T) = [L(\mathbf{Y}_T), U(\mathbf{Y}_T)]$ designed to capture the future value of Y_{T+k} with a fixed probability (the nominal coverage). We will focus on prediction of future values of time series generated by autoregressive integrated moving average processes (ARIMA) with possibly non-Gaussian innovations.

The standard prediction approach for ARIMA processes (Box and Jenkins 1976) assumes Gaussian innovations and known parameters. Consequently, the resulting prediction intervals are centered around the conditional expectation which is a linear function of past observations and do not incorporate the uncertainty due to parameter estimation. Thus, their accuracy may be very low if the variables are not normally distributed.

Alternatively, bootstrap based methods provide prediction intervals without any distributional assumption on the innovations. There are several bootstrap alternatives in the literature to construct prediction intervals for autoregressive models of order p (AR(p)). Findley (1986), Stine (1987), Masarotto (1990) and Grigoletto (1998) use bootstrap methods to estimate the density of the prediction errors considering the uncertainty due to parameter estimation. As in the standard method, they center the forecast intervals at a linear combination of past observations. However, when the errors are nonnormal, the conditional expectation may not be linear and, consequently, these intervals may not be correctly centered. Alternatively, Thombs and Schucany (1990) and Breidt *et al.* (1995) estimate directly the distribution of Y_{T+k} conditional on $\mathbf{Y}_T$. In an AR(p) process, conditioning on $\mathbf{Y}_T$ is equivalent to conditioning on the last p observations. Consequently, Thombs and Schucany (1990) and Breidt *et* *al.* (1995) use the backward representation of AR(p) models (Box and Jenkins 1976) to generate bootstrap series that mimic the structure of the original data with fixed last p observations. McCullogh (1994) applies the results in Thombs and Schucany and Breidt *et al.* (1995) to real data implementing also the bias-correction bootstrap of Efron (1982). García-Jurado *et al.* (1995) extend the bootstrap approach of Thombs and Schucany (1990) to autoregressive integrated (ARI) processes. They use the backward representation of the autoregressive model to construct bootstrap replicates of the differenced variable k periods ahead and then obtain bootstrap samples of the original variable, Y_{T+k}, by solving a $(k+d) \times (k+d)$ linear system, where d is the number of unit roots. When forecasting far ahead, the huge dimension makes the system difficult to handle. The need to use the backward representation to generate bootstrap series makes all these methods computationally expensive and, what is more important, restricts their applicability to models having a backward representation, excluding, for example, generalized autoregressive conditionally heteroscedastic processes (GARCH). Furthermore, the prediction of moving average (MA) processes can not be handled by these methods because the infinite order of their autoregressive representation requires that, at least theoretically, the whole sample should be fixed to generate bootstrap replicates. An additional difficulty with this backward representation approach is that, although in AR(p) processes the distribution of Y_{T+k} conditional on $\mathbf{Y}_T$ coincides with the distribution conditional on the last p observations under known parameters, if the parameters are estimated these distributions are different in finite sample sizes. Kabaila (1993) questions whether predictive inference should be carried out conditioning on the last p observed values. Finally, Cao *et al.* (1997) present an alternative bootstrap method for constructing prediction intervals for stationary AR(p) models which does not requires the backward representation. However, their intervals do not incorporate the variability due to parameter estimation.

In this paper, we propose a simple resampling procedure for ARIMA processes to estimate the conditional distribution of Y_{T+k} incorporating the variability due to parameter estimation. Our strategy makes unnecessary the backward representation and, as a consequence, this bootstrap procedure can be easily extended to forecasting with more general models.

The paper is organized as follows. Section 2 presents the resampling procedure to estimate prediction distributions and establishes its asymptotic validity for AR(p) processes. In section 3, we prove the asymptotic validity of this method when it is extended to ARI processes. Section 4 contains an extensive Monte Carlo simulation study which compares the performance of all available bootstrap prediction techniques for different ARIMA models and error distributions. Finally, the conclusions and some ideas for further research can be found in Section 5.

2. BOOTSTRAP PREDICTION INTERVALS FOR STATIONARY AR(P) PROCESSES

Let $\mathbf{y}_T = \{y_1, \ldots, y_T\}$ be a sequence of T observations generated by an AR(p) process given by

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + a_t, \; t = ..., -2, -1, 0, 1, 2, \ldots, \tag{1}$$

where $\{a_t\}$ is a sequence of zero-mean independent random variables with common distribution function F_a such that $E[a_t^2] = \sigma_a^2 < \infty$ and $(\phi_0, \phi_1, \ldots, \phi_p)$ are unknown parameters. In this section, we assume stationarity, i.e., all the roots of the polynomial $\Phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$ lie outside the unit circle.

Given $\mathbf{Y}_T = \{Y_1, ..., Y_T\}$, the predictor of Y_{T+k} with minimum mean square error (MSE) is given by the conditional mean of Y_{T+k},

$$\widetilde{Y}_{T+k} = E(Y_{T+k} \mid \mathbf{Y}_T). \tag{2}$$

Under normality of a_t, the conditional mean is a linear function of past observations and the minimum MSE predictor of Y_{T+k} is given by

$$\widetilde{Y}_{T+k} = \phi_0 + \phi_1 \widetilde{Y}_{T+k-1} + \cdots + \phi_p \widetilde{Y}_{T+k-p}, \tag{3}$$

where $\widetilde{Y}_{T+j} = Y_{T+j}$ for $j \leq 0$. The prediction error is a combination of future innovations a_{T+j}, $j = 1, ..., k$, given by

$$\widetilde{e}_{T+k} = Y_{T+k} - \widetilde{Y}_{T+k} = \sum_{i=0}^{k-1} \Psi_i a_{T+k-i}, \tag{4}$$

where Ψ_i are the coefficients of the moving average representation of the AR(p) model obtained from $\Psi(B) = \Phi(B)^{-1}$, where B is the backshift operator. The prediction MSE is given by

$$MSE(\widetilde{e}_{T+k}) = \sigma_a^2 \sum_{i=0}^{k-1} \Psi_i^2. \tag{5}$$

In practice, predictions are made with estimated parameters. The actual predictor is then given by

$$\widehat{Y}_{T+k} = \widehat{\phi}_0 + \widehat{\phi}_1 \widehat{Y}_{T+k-1} + \cdots + \widehat{\phi}_p \widehat{Y}_{T+k-p}, \tag{6}$$

where $(\widehat{\phi}_0, \widehat{\phi}_1, \ldots, \widehat{\phi}_p)$ are the parameter estimators and $\widehat{Y}_{T+j} = Y_{T+j}$ for $j \leq 0$. The corresponding prediction error can be decomposed into two parts by writing

$$\widehat{e}_{T+k} = Y_{T+k} - \widehat{Y}_{T+k} = (Y_{T+k} - \widetilde{Y}_{T+k}) + (\widetilde{Y}_{T+k} - \widehat{Y}_{T+k}). \tag{7}$$

The first term in (7) is the prediction error in (4). The second term appears because parameter estimates are used instead of true values. So, in practice, this uncertainty due to parameter estimation has to be included in the expression of the prediction MSE.

The prediction intervals for Y_{T+k} constructed using the Box-Jenkins (1976) procedure are given by

$$\left[\widehat{Y}_{T+k} - z_{\alpha/2}\left(\widehat{\sigma}_a^2 \sum_{j=0}^{k-1} \widehat{\Psi}_j^2\right)^{1/2}, \ \widehat{Y}_{T+k} + z_{\alpha/2}\left(\widehat{\sigma}_a^2 \sum_{j=0}^{k-1} \widehat{\Psi}_j^2\right)^{1/2}\right], \tag{8}$$

where $z_{\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution, $\widehat{\sigma}_a^2$ is the usual estimate of the innovations variance and $\widehat{\Psi}_j$ are the estimated coefficients of the moving average representation. The prediction intervals in (8) just consider the MSE in (5) and replace the unknown parameters by appropriate estimates. However, they do not incorporate the variability due to parameter estimation. Moreover, these intervals have two additional problems when the distribution of a_t is not normal. First, the value of the standard normal quantile may not be appropriate. To handle this question, Findley (1986), Stine (1987), Masarotto (1990) and Grigoletto (1998) proposed different ways of bootstrapping from the residuals of the estimated model,

$$\widehat{a}_t = y_t - \widehat{\phi}_0 - \widehat{\phi}_1 y_{t-1} - \cdots - \widehat{\phi}_p y_{t-p}, \ t = p+1, \ldots, T, \tag{9}$$

to estimate the distribution function of the prediction error. The second difficulty is that these bootstrap prediction intervals are still centered in (6) and when the innovation distribution is not normal, the conditional mean in (2) may not be a linear combination of past observations.

To solve this problem, Thombs and Schucany (1990) introduced a bootstrap method based on directly estimating the distribution of Y_{T+k} conditional on the available variables $\mathbf{Y}_T$. To incorporate the uncertainty due to parameter estimation in the prediction intervals, they generated bootstrap replicates $\mathbf{y}_T^* = \{y_1^*, ..., y_T^*\}$ that mimic the structure of the original series. Since the prediction is conditional on the last p values of the series, all the bootstrap replicates are generated fixing the last p values.

Consequently, they needed to use the backward representation of stationary AR(p) models, where Y_t is expressed as a linear combination of future values plus an error term. Using the backward representation makes the procedure computationally demanding and it constitutes an obstacle to extend the resampling procedure to models without backward representation. To overcome this problem, Cao *et al.* (1997) proposed a fast procedure (called conditional bootstrap) to generate prediction intervals based on resampling residuals but their method does not incorporate variability due to parameter estimation.

In this section, we introduce a new resampling strategy to build prediction intervals in AR(p) models. Our method is based on fixing the last p observations to obtain bootstrap replicates of future values Y_{T+k} but the estimated parameters are bootstraped without fixing any observation in the sample. As a consequence, we do not need the backward representation of the model and, therefore, the method can be easily extended to more general models.

Our proposal to obtain bootstrap replicates of the series is as follows. Given a set of estimates of the AR(p) model, obtain the residuals by (9) and centre and rescale them, as suggested by Stine (1987), by the factor $[(T-p)/(T-2p)]^{1/2}$. From a set of p initial values, say $\mathbf{y}_0^* = \{y_{-p+1}^*, \ldots, y_0^*\}$, construct a bootstrap series $\{y_1^*, \ldots, y_T^*\}$ from

$$Y_t^* = \widehat{\phi}_0 + \widehat{\phi}_1 Y_{t-1}^* + \cdots + \widehat{\phi}_p Y_{t-p}^* + \widehat{a}_t^*, \ t = 1, \ldots, T, \tag{10}$$

where $\widehat{a}_t^*$ are independent observations obtained by resampling from $\widehat{F}_a$, the empirical distribution function of the centered and rescaled residuals. Once the parameters of this bootstrap series are estimated, say $(\widehat{\phi}_0^*, \widehat{\phi}_1^*, \ldots, \widehat{\phi}_p^*)$, we forecast through the recursion of the autoregressive model with the bootstrap parameters and fixing the last p observations of the original series,

$$Y_{T+k}^* = \widehat{\phi}_0^* + \sum_{j=1}^{p} \widehat{\phi}_j^* Y_{T+k-j}^* + \widehat{a}_{T+k}^*, \tag{11}$$

with $\hat{a}^*_{T+k}$ being a random draw from $\widehat{F}_a$ and $Y^*_{T+h} = y_{T+h}$, $h \leq 0$. Once we obtain a set of B bootstrap replicates $\left\{y^{*(1)}_{T+k}, \ldots, y^{*(B)}_{T+k}\right\}$, we proceed as in Thombs and Schucany (1990). The prediction limits are defined as the quantiles of the bootstrap distribution function of Y^*_{T+k}. More specifically, if $G^*(h) = \Pr(Y^*_{T+k} \leq h)$ is the distribution function of Y^*_{T+k} and its Monte Carlo estimate is $G^*_B(h) = \#(y^{*(b)}_{T+k} \leq h)/B$, a $100\beta\%$ prediction interval for Y^*_{T+k} is given by

$$[L^*_B(\mathrm{y}), U^*_B(\mathrm{y})] = \left[Q^*_B\left(\frac{1-\beta}{2}\right), Q^*_B\left(\frac{1+\beta}{2}\right)\right], \tag{12}$$

where $Q^*_B = G^{*-1}_B$. The main difference between our bootstrap strategy and Thombs and Schucany's (1990) is that our bootstrap parameter estimates are not conditional on the last p observations and this allows us to overcome the computational burden associated with resampling through the backward representation; moreover, this procedure can be extended to forecasting with more general and complex models.

Summarizing, the steps for obtaining bootstrap prediction intervals are:

Step 1. Compute the residuals $\hat{a}_t$ as in (9). Let $\widehat{F}_a$ be the empirical distribution function of the centered and rescaled residuals.

Step 2. Generate a bootstrap series using the forward recursion in (10) and compute the estimates $(\hat{\phi}^*_0, \hat{\phi}^*_1, \ldots, \hat{\phi}^*_p)$.

Step 3. Compute a bootstrap future value by expression (11). (Note that the last p values of the series are fixed in this step but not in the previous one.)

Step 4. Repeat the last two steps B times and then go to Step 5.

Step 5. The endpoints of the prediction interval are given by quantiles of G^*_B, the bootstrap distribution function of Y^*_{T+k}.

Next, we study the asymptotic properties of this bootstrap resampling strategy.

Theorem 1 . *Let $\{y_{T-n+1}, \ldots, y_T\}$ be a realization from a stationary p-th order autoregressive process $\{Y_t\}$ with $E[a_t] = 0$ and $E[|a^\alpha_t|] < \infty$, for some $\alpha > 2$.*

(a) Let $(\widehat{\phi}_0, \widehat{\phi}_1, \ldots, \widehat{\phi}_p)$ be the OLS estimate of $(\phi_0, \phi_1, \ldots, \phi_p)$ and let Y^*_{T+k} be obtained following steps 1 to 5. Then, for almost all sample sequences, $Y^*_{T+k} \to Y_{T+k}$ in distribution as $n \to \infty$.

(b) Let $(\widehat{\phi}_0, \widehat{\phi}_1, \ldots, \widehat{\phi}_p)$ be any M-estimate of $(\phi_0, \phi_1, \ldots, \phi_p)$ and let Y^*_{T+k} be obtained following steps 1 to 5. Then, for any distance d metrizing weak convergence, $d(Y^*_{T+k}, Y_{T+k}) \to 0$ in probability as $n \to \infty$.

Proof. **(a)** Consistency of the OLS bootstrap estimate in conditional probability for almost all sample sequences follows from Freedman (1985). We will express Y^*_{T+k} as a sum involving the available and fixed values $y_{T-n+1,\ldots,}\ y_T$, continuous functions of the bootstrap parameter estimates $(\widehat{\phi}^*_0, \widehat{\phi}^*_1, \ldots, \widehat{\phi}^*_p)$ and the independent random draws $\widehat{a}^*_{T+j}$. Before proving the result for a general k, we present the idea for $k = 1$ and $k = 2$. If $k = 1$, we have

$$Y^*_{T+1} = \widehat{\phi}^*_0 + \widehat{\phi}^*_1 y_T + \cdots + \widehat{\phi}^*_p y_{T-p+1} + \widehat{a}^*_{T+1}.$$

The first term $\widehat{\phi}^*_0$, converges in conditional probability to ϕ_0 for almost all samples. Also, $\widehat{\phi}^*_j y_{T-j+1} \to \phi_j y_{T-j+1}, j = 1, \ldots, p$ in conditional probability, for almost all samples. Thus, Y^*_{T+1} is a sum whose first $p + 1$ terms converge in probability to $\phi_0 + \sum \phi_j y_{T-j+1}$almost surely and $\widehat{a}^*_{T+1} \to a_{T+1}$ in distribution almost surely. Therefore, from the bootstrap version of Slutsky's Theorem, $Y^*_{T+1} \to Y_{T+1}$ in distribution as $n \to \infty$ almost surely. For $k = 2$,

$$Y^*_{T+2} = \widehat{\phi}^*_0 \left(1 + \widehat{\phi}^*_1\right) + g_1 \left(\widehat{\phi}^*_1, \ldots, \widehat{\phi}^*_p\right) y_T + \cdots + g_p \left(\widehat{\phi}^*_1, \ldots, \widehat{\phi}^*_p\right) y_{T-p+1} + \widehat{\phi}^*_1 \widehat{a}^*_{T+1} + \widehat{a}^*_{T+2}.$$

The sum involving the first $p + 1$ terms converges in conditional probability to $\phi_0 (1 + \phi_1) + \sum g_j \left(\phi_1, \ldots, \phi_p\right) y_{T-j+1}$ almost surely. Moreover, $\widehat{\phi}^*_1 \widehat{a}^*_{T+1} \to \phi_1 a_{T+1}$ and $\widehat{a}^*_{T+2} \to a_{T+2}$, both in distribution almost surely. Furthermore, these last two terms are independent, so $\widehat{\phi}^*_1 \widehat{a}^*_{T+1} + \widehat{a}^*_{T+2} \to \phi_1 a_{T+1} + a_{T+2}$ in distribution almost surely. Hence, $Y^*_{T+2} \to Y_{T+2}$ in distribution as $n \to \infty$ almost surely. Similarly, for a general k, we have that

$$\begin{aligned} Y^*_{T+k} &= g_0\left(\widehat{\phi}^*_1, \ldots, \widehat{\phi}^*_p\right) + g_1\left(\widehat{\phi}^*_1, \ldots, \widehat{\phi}^*_p\right) y_T + \cdots + g_p\left(\widehat{\phi}^*_1, \ldots, \widehat{\phi}^*_p\right) y_{T-p+1} + \\ &\quad + h_1\left(\widehat{\phi}^*_1, \ldots, \widehat{\phi}^*_p\right) \widehat{a}^*_{T+1} + \cdots + h_{k-1}\left(\widehat{\phi}^*_1, \ldots, \widehat{\phi}^*_p\right) \widehat{a}^*_{T+k-1} + \widehat{a}^*_{T+k}. \end{aligned}$$

The functions h_j and g_j are different for each prediction horizon, but for simplicity we use the same notation. First, $g_0\left(\widehat{\phi}^*_1, \ldots, \hat{\phi}^*_p\right)$ converges in probability almost surely to $g_0\left(\phi_1, \ldots, \phi_p\right)$. The products of fixed values y_{T-j} and the bootstrap parameter estimates $g_j\left(\widehat{\phi}^*_1, \ldots, \widehat{\phi}^*_p\right) y_{T-j+1}$ converge in probability almost surely to $g_j\left(\phi_1, \ldots, \phi_p\right) y_{T-j+1}$. The terms $\widehat{a}^*_{T+j}$ converge in distribution almost surely to a_{T+j} and the products $h_j\left(\widehat{\phi}^*_1, \ldots, \widehat{\phi}^*_p\right) \widehat{a}^*_{T+j}$ converge in distribution to $h_j\left(\phi_1, \ldots, \phi_p\right) a_{T+j}$, by Slutsky's Theorem. Since these last terms are independent, part **(a)** follows.

(b) Kreiss and Franke (1992) proved weak consistency in probability for the bootstrap version of any M-estimate of $(\phi_0, \phi_1, \ldots, \phi_p)$. Part **(b)** follows by using the distance d metrizing weak convergence and arguing as in the proof of part **(a)**. ∎

3. EXTENSION TO INTEGRATED AUTOREGRESSIVE PROCESSES

In this section we generalize the resampling scheme introduced above to ARI(p, d) processes given by

$$\nabla^d Y_t = \phi_0 + \phi_1 \nabla^d Y_{t-1} + \cdots + \phi_p \nabla^d Y_{t-p} + a_t, \; t = ..., -2, -1, 0, 1, 2, \ldots \qquad (13)$$

where $\nabla = (1-B)$. To obtain the conditional density of Y_{T+k}, we replace expressions (10) and (11) of the bootstrap procedure described in Section 2 by the appropriate recursions. For example, for the ARI$(1,1)$ model, (10) becomes

$$Y^*_t = \widehat{\phi}_0 + (1 + \widehat{\phi}_1)\, Y^*_{t-1} - \widehat{\phi}_1\, Y^*_{t-2} + \widehat{a}^*_t, \qquad t = 1, \ldots, T, \qquad (14)$$

and (11) is replaced by the following recursions,

$$
\begin{aligned}
Y^*_{T+1} &= \widehat{\phi}^*_0 + (1+\widehat{\phi}^*_1)y_T - \widehat{\phi}^*_1 y_{T-1} + \widehat{a}^*_{T+1}, \\
Y^*_{T+2} &= \widehat{\phi}^*_0 + (1+\widehat{\phi}^*_1)Y^*_{T+1} - \widehat{\phi}^*_1 y_T + \widehat{a}^*_{T+2},
\end{aligned}
\tag{15}
$$

and so on. The validity of the proposed method is established in the following theorem.

Theorem 2 *Let $\{y_{T-n+1}, ..., y_T\}$ be a realization of an ARI(p, d) process with $E[a_t] = 0$ and $E[|a_t|^\alpha] < \infty$, for some $\alpha > 2$.*

(a) Let $(\widehat{\phi}_0, \widehat{\phi}_1, \ldots, \widehat{\phi}_p)$ be the OLS estimate of $(\phi_0, \phi_1, \ldots, \phi_p)$ and let Y^*_{T+k} be obtained following steps 1 to 5. Then, for almost all sample sequences, $Y^*_{T+k} \to Y_{T+k}$ in distribution as $n \to \infty$.

(b) Let $(\widehat{\phi}_0, \widehat{\phi}_1, \ldots, \widehat{\phi}_p)$ be any M-estimate of $(\phi_0, \phi_1, \ldots, \phi_p)$ and let Y^*_{T+k} be obtained following steps 1 to 5. Then, for any distance d metrizing weak convergence, $d(Y^*_{T+k}, Y_{T+k}) \to 0$ in probability as $n \to \infty$.

Proof. (a) We express Y^*_{T+k} as a sum involving the available and fixed last $p + d$ values y_{T-j+1} for $j = 1, ..., p + d$, continuous functions of the bootstrap parameter estimates $\left(\widehat{\phi}^*_0, \widehat{\phi}^*_1, \ldots, \widehat{\phi}^*_p\right)$ and the independent random draws $\widehat{a}^*_{T+j}$. To illustrate the proof, consider first an ARI$(1, 1)$ process without constant. For the prediction horizon $k = 1$, we have

$$Y^*_{T+1} = (1 + \widehat{\phi}^*_1)y_T - \widehat{\phi}^*_1 y_{T-1} + \widehat{a}^*_{T+1}.$$

The values (y_{T-1}, y_T) can be seen as constants in the first two terms. Thus, $(1 + \widehat{\phi}^*_1)y_T \to (1 + \phi_1)y_T$ in conditional probability as $n \to \infty$ for almost all sample paths; the same holds for the second term and, finally, $\widehat{a}^*_{T+1} \to a_{T+1}$ in distribution as $n \to \infty$ almost surely. Therefore, from Slutsky's Theorem, $Y^*_{T+1} \to Y_{T+1}$ in distribution as $n \to \infty$ almost surely. For $k = 2$, we have

$$Y^*_{T+2} = \left(1 + \widehat{\phi}^*_1 + \widehat{\phi}^{*2}_1\right) y_T - \widehat{\phi}^*_1(1 + \widehat{\phi}^*_1)y_{T-1} + \left(1 + \widehat{\phi}^*_1\right) \widehat{a}^*_{T+1} + \widehat{a}^*_{T+2}.$$

The sum involving the first two terms converges in conditional probability almost surely to $(1+\phi_1+\phi_1^2)y_T - \phi_1(1+\phi_1)y_{T-1}$. Moreover, $\left(1+\widehat{\phi}_1^*\right)\widehat{a}_{T+1}^* \to (1+\phi_1)\, a_{T+1}$ and $\widehat{a}_{T+2}^* \to a_{T+2}$, both in distribution for almost all sample paths and, since these two terms are independent, $\left(1+\widehat{\phi}_1^*\right)\widehat{a}_{T+1}^* + \widehat{a}_{T+2}^* \to (1+\phi_1)\, a_{T+1} + a_{T+2}$ in distribution almost surely; it follows that $Y_{T+2}^* \to Y_{T+2}$ in distribution as $n \to \infty$ almost surely. For a general lead k the proof is similar. In this case, we have that

$$\begin{aligned} Y_{T+k}^* &= g_1\left(\widehat{\phi}_1^*,\ldots,\widehat{\phi}_p^*\right) y_T + g_2\left(\widehat{\phi}_1^*,\ldots,\widehat{\phi}_p^*\right) y_{T-1} + \\ &\quad + h_1\left(\widehat{\phi}_1^*,\ldots,\widehat{\phi}_p^*\right)\widehat{a}_{T+1}^* + \cdots + h_{k-1}\left(\widehat{\phi}_1^*,\ldots,\widehat{\phi}_p^*\right)\widehat{a}_{T+k-1}^* + \widehat{a}_{T+k}^*. \end{aligned}$$

As before, the products of fixed values (y_{T-1}, y_T) and continuous functions of the bootstrap parameter estimates, denoted by $g_1\left(\widehat{\phi}_1^*,\ldots,\widehat{\phi}_p^*\right) y_T$ and $g_2\left(\widehat{\phi}_1^*,\ldots,\widehat{\phi}_p^*\right) y_{T-1}$, converge in probability almost surely to $g_1\left(\phi_1,\ldots,\phi_p\right) y_T$ and $g_2\left(\phi_1,\ldots,\phi_p\right) y_{T-1}$, respectively. For the remaining terms, $\widehat{a}_{T+k}^* \to a_{T+k}$ in distribution almost surely, and the products $h_j\left(\widehat{\phi}_1^*,\ldots,\widehat{\phi}_p^*\right)\widehat{a}_{T+j}^*$ converge in distribution almost surely to $h_j\left(\phi_1,\ldots,\phi_p\right) a_{T+j}$. Finally, independence gives, as above, that $Y_{T+k}^* \to Y_{T+k}$ in distribution as $n \longrightarrow \infty$ almost surely. The proof for a general ARI(p,d) process is straightforward, following the previous steps.

(b) The proof follows along the lines of the proof of part **(a)**, as in Theorem 1. ■

4. SIMULATION RESULTS

The coverage of prediction intervals for finite samples is usually different from the asymptotic nominal coverage and depends on the model, the distribution of the innovations and the parameter estimation method. In this section, we present several Monte Carlo experiments carried out to analyze the finite sample behavior of the proposed bootstrap estimates of prediction densities of ARIMA(p,d,q) processes. First, we will focus on stationary AR(p) processes and we will compare our proposal (PRR) with Box-Jenkins intervals (BJ) and with alternative bootstrap intervals introduced by Thombs and Schucany (1990) (TS), Breidt, Davis and Dunsmuir (1995) (BDD) and Kabaila (1993)(KAB) and Stine (1987) (STI). Then, we will consider integrated autoregressive models and compare PRR intervals with BJ intervals and with intervals constructed following García-Jurado *et al.* (1995) (GGP). Finally, the behavior of our technique will be analyzed in forecasting future values of MA(q) models. As far as we know, the prediction density of MA(q) models has not been previously estimated by bootstrap methods; therefore, we only present PRR and BJ prediction intervals.

To compare the different prediction intervals, we use their coverage and length, and the proportion of observations lying out to the left and to the right of them. We compare these measures with those corresponding to the empirical prediction distribution obtained for a particular series generated by a specified process, sample size and error distribution F_a, generating R=1000 future values y_{T+k} from that series. Then, for that particular series and for each of the methods considered, we obtain a $100\beta\%$ prediction interval denoted by (L^*, U^*) (based on B=1000 replicates in the case of bootstrap intervals) and estimate the conditional coverage for each procedure by

$$\widehat{\beta}^* = \#\left\{L^* \leq y^r_{T+k} \leq U^*\right\}/R,$$

where y^r_{T+k} ($r = 1,\ldots,R$) are the series values generated previously. We have carried out 1000 Monte Carlo experiments and report average coverage, average length and average proportion of observations on the left and on the right for each method and for the empirical distribution.

4.1 Autoregressive models

First, we consider the AR(2) process in Thombs and Schucany (1990),

$$y_t = 1.75 y_{t-1} - 0.76 y_{t-2} + a_t.$$

The innovations distributions F_a are normal, exponential and a contaminated distribution $.9F_1+.1F_2$, with $F_1 \sim N(-1,1)$ and $F_2 \sim N(9,1)$. Each distribution has been centered to have zero mean. The sample sizes considered are 25, 50 and 100, the prediction horizons are k=1 and 3, and we construct intervals with nominal coverage β equal to 0.80 and 0.95. The results of these experiments appear in tables 1 to 12 where can be observed that the behavior of all bootstrap prediction intervals, except for Kabaila (1993), is rather similar for all horizons, nominal coverages and distributions considered. The intervals constructed by Kabaila's method, although asymptotically correct, are, in general, too wide for moderate sample sizes. When looking at the results for Gaussian innovations, we may see that even though the standard intervals are built assuming the correct error distribution, the bootstrap intervals have better properties for a 80% nominal coverage. This may be due to the fact that standard intervals do not incorporate the variability due to parameter estimation and bootstrap intervals do; also, to the well known good bootstrap behavior for small samples. Moreover, we may observe that the standard intervals have worse coverage properties than PRR intervals when forecasting three periods ahead. Notice that when model parameters are estimated, the distribution of the forecasting errors is not normal even if the innovations are Gaussian. This is due to the fact that the predictors are linear combinations of products of asymptotically normal random variables which, in general, are nonnormal. This is the reason why, even for Gaussian innovations distribution, constructing bootstrap forecasting intervals could improve the forecast properties. Looking at tables 5 to 8 which report results for the contaminated distribution, we observe that the standard intervals are too wide and still are not able to cope with the shape of the error distribution. This can be seen more clearly in Figure 1 where we represent the prediction densities of the one-step ahead predictions when the sample size is 100, estimated by our bootstrap procedure and by the standard methodology together with the empirical density. Finally, comparing the PRR, TS and BDD intervals, we may observe that for all distributions, sample sizes and coverages considered, the behavior of the three methods is very similar. Our procedure does not work worse and, in some cases, seems to be slightly better than the others. The potential gains of PRR over TS and BDD could be due to the fact that the variance of the parameter estimates is reduced when the last p observations are not fixed to obtain bootstrap estimates of the parameters. Since our method is much simpler to implement and less computationally demanding than the other bootstrap methods, it seems to be an interesting alternative even for AR(p) models. Similar results and comments apply to innovations with exponential distribution.

4.2 Integrated models

We analyze the behavior of our bootstrap method for non-stationary series with the ARI$(1, 2)$ process

$$\nabla^2 y_t = 0.5\nabla^2 y_{t-1} + a_t.$$

We use the same error distributions, sample sizes and nominal coverages than before, and compare PRR forecast intervals with standard intervals and intervals built by the method proposed by García-Jurado *et al.* (1995). The results of our simulation study appear in tables 13 to 18. In these tables, we may observe that, even for Gaussian errors, the standard intervals deteriorate very seriously when predicting three-steps ahead. As expected from results in the stationary case, the behavior of standard intervals is even worse when the error distribution is not Gaussian. Comparing PRR and GGP intervals, we observe that both are very similar. However, constructing GGP intervals requires solving a system which could be difficult to handle when forecasting far in the future, complicating the implementation of the method. Furthermore, in the cases considered in this paper, the PRR intervals slightly outperforms the GGP intervals. The simulation results for ARI(p, d) models are illustrated in Figure 2 where we represent the one-step ahead prediction densities estimated using the standard, GGP and PRR proposals together with the empirical density for a sample size of 100 and exponential innovations.

Finally, as in the stationary case, we may observe that as the sample size increases, the average coverage and average length converge to the empirical values supporting the asymptotic properties stated in section 2.

4.3 Moving average models

Finally, we carry out Monte Carlo experiments to check the behavior of our technique when predicting the future of processes with moving average components. In this case, there are no other alternative bootstrap methods proposed in the literature and, consequently, we only compare our strategy with standard intervals.

To predict future values of a moving average process, we need estimates of the within-sample innovations. This is an additional source of uncertainty in forecasting MA processes which makes the construction of forecast intervals a difficult task. However, our bootstrap method is easy to implement even in the presence of moving average components and, as we will see, it works reasonably well. There are several alternatives to estimate the innovations of MA processes and, in this paper, we consider the simplest one which consists in conditioning on the value of all innovations previous to the sample period being equal to their expected value zero. The estimation of model parameters is carried out by conditional quasi-maximum likelohood. We analyze the model

$$y_t = a_t - .3a_{t-1} + .7a_{t-2}$$

under the same design as in previous Monte Carlo experiments. The results of the Monte Carlo simulations, reported in tables 19 to 24, are similar to those previously obtained. Standard intervals are not able to deal with asymmetric distributions. Figure 3 shows the standard and bootstrap densities together with the empirical density for the one-step ahead predictions built with a sample size of 100 and exponential innovations; it is clear that the standard density does not mimic the empirical prediction distribution.

Finally, we check our method for ARMA(p,q) processes by considering the ARMA$(1,1)$ process

$$y_t = .7y_{t-1} + a_t - .3a_{t-1}.$$

The corresponding Monte Carlo results can be found in tables 25 to 30. Conclusions are the same as for the pure moving average model. Again, in the Gaussian case our bootstrap intervals compite well with the standard intervals, but when the error distribution is not Gaussian, the bootstrap intervals are more accurate than the standard ones.

It is important to note that the results presented in this section have been obtained using OLS or conditional quasi-maximum likelihood estimates of the parameters. It seems clear that these results could be improved by using estimates more appropriate for nonnormal innovations. Moreover, in moving average models we estimated the innovations conditioning on pre-sample values being zero. These estimates can be improved resampling from the unconditional residuals, which can be obtained, for example, via the Kalman filter. In this paper, our goal is just to present the resampling plan for ARIMA processes and the technical question of estimation will be the subject for further research.

5. CONCLUSIONS

A new bootstrap approach to estimate the prediction density of ARIMA processes has been presented in this paper. The density is estimated directly from the bootstrap predictions and thus the intervals are not centered around the point linear predictor which could be incorrect under nonnormal error distributions. The proposed bootstrap prediction intervals also incorporate the uncertainty due to parameter estimation. The main advantage of this prediction resampling strategy with respect to previous bootstrap prediction methods with similar properties is that the process backward representation is not required to obtain bootstrap replicates of the series. Consequently, our method is flexible and easy to implement, allowing the generalization to models with moving average components and also to processes without a backward representation. We have established the asymptotic properties of the bootstrap prediction intervals and carried out Monte Carlo experiments to analyze their behavior for finite samples. We have compared them with standard intervals as proposed by Box and Jenkins (1976) and with bootstrap intervals based on Thombs and Schucany (1990). The results of these experiments show that for nonnormal innovations, Box-Jenkins prediction intervals can be heavily distorted. We have seen that all bootstrap intervals have rather similar properties for ARI(p, d) processes and our intervals are slightly better in some cases. Moreover, our proposal is more flexible allowing for the construction of prediction densities for processes with moving average components which previous methods cannot handle. Monte Carlo simulations show that the proposed bootstrap prediction intervals work well in forecasting future values of processes with MA components.

Finally, the flexibility of this method allows to extend the construction of prediction intervals even for models without a backward representation, such as GARCH models. Miguel and Olave (1998) study a bootstrap procedure for GARCH processes based on Cao *et al.* (1995), where the resampling is conditional on the parameter estimates. Presently, we are investigating the application of our bootstrap strategy to prediction densities of GARCH processes with very promising results.

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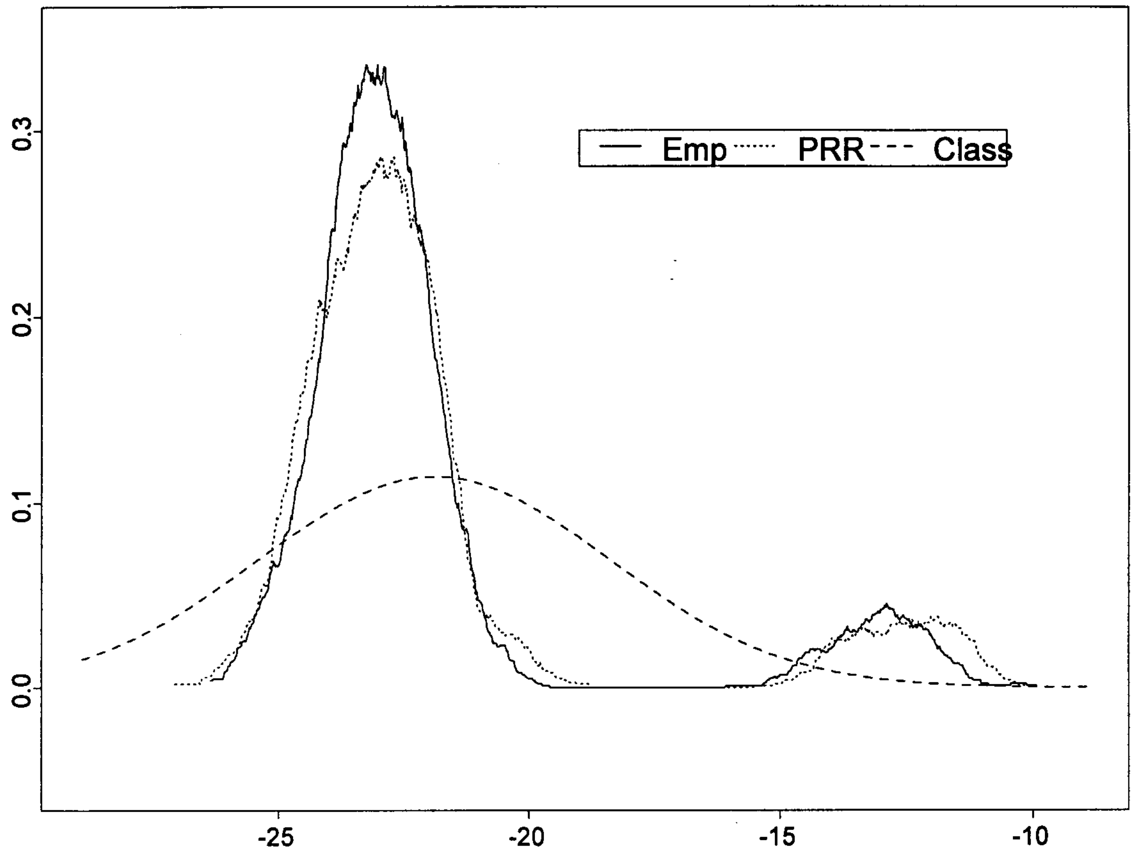

FIG. 1. Empirical, standard and bootstrap densities of one-step ahead predictions of AR(2) process with contaminated normal distribution and sample size 100.

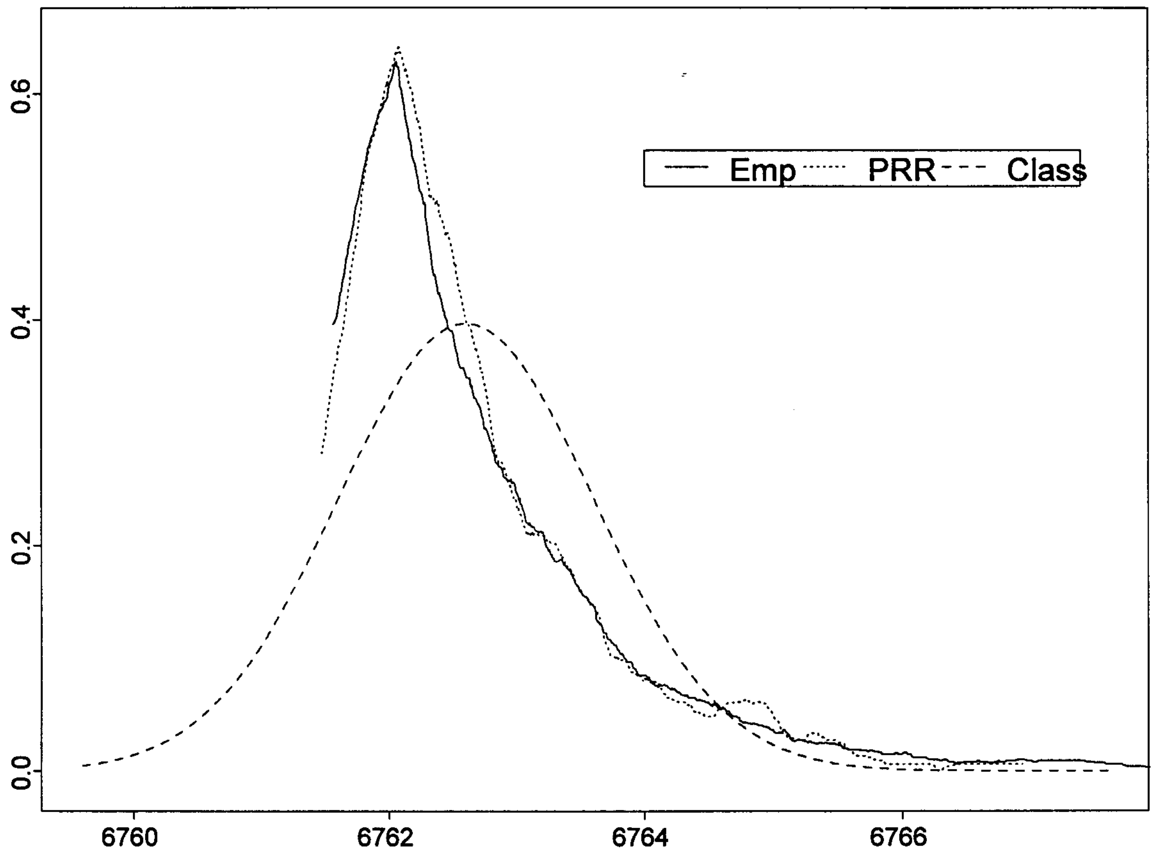

FIG. 2. Empirical, standard and bootstrap densities of one-step ahead predictions of ARI(1,2) process with exponential distribution and sample size 100.

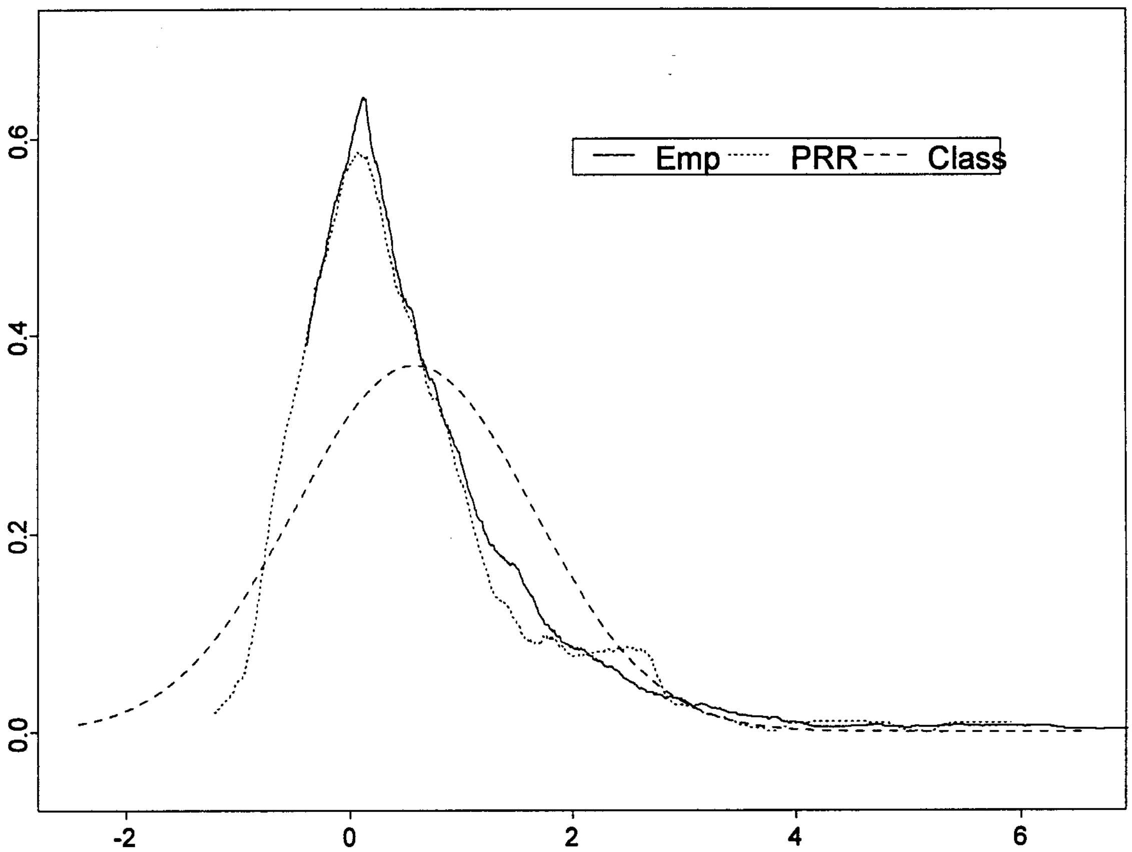

FIG. 3. Empirical, standard and bootstrap densities of one-step ahead predictions of MA(2) process with exponential distribution and sample size 100.

Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
n	**Empirical**	**95%**	**2.5%/2.5%**	**3.92**
25	PRR	92.07(.07)	4.2/3.7	4.16(.94)
	TS	90.67(.07)	5.0/4.3	3.84(.68)
	BDD	91.04(.06)	4.7/4.3	3.81(.67)
	KAB	90.65(.09)	4.9/4.4	7.51(7.43)
	ST	92.16(.06)	4.1/3.7	3.88(.60)
50	PRR	93.58(.03)	3.3/3.1	3.97(.53)
	TS	92.97(.04)	3.6/3.4	3.86(.51)
	BDD	92.83(.04)	3.6/3.6	3.85(.51)
	KAB	92.73(.05)	3.9/3.4	5.11(2.13)
	ST	93.88(.03)	3.1/3.0	3.92(.42)
100	PRR	94.06(.02)	2.9/3.1	3.92(.38)
	TS	93.90(.02)	2.97/3.12	3.90(.38)
	BDD	93.92(.03)	3.0/3.1	3.90(.38)
	KAB	93.36(.03)	3.2/3.4	4.31(.68)
	ST	94.50(.02)	2.7/2.8	3.92(.29)

Table 1. Model $y_t = 1.75y_{t-1} - .76y_{t-2} + a_t$; Gaussian errors; Forecast horizont 1. Nominal coverage 95%.

Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
n	**Empirical**	**95%**	**2.5%/2.5%**	**11.98**
25	PRR	89.27(.11)	5.7/5.1	12.57(4.10)
	TS	87.54(.10)	6.6/5.8	11.12(2.48)
	BDD	89.22(.09)	5.7/5.1	11.31(2.25)
	KAB	83.01(.13)	8.9/8.1	16.84(16.82)
	ST	87.44(.11)	6.7/5.9	11.19(2.35)
50	PRR	92.66(.05)	3.7/3.6	12.03(2.01)
	TS	91.54(.05)	4.3/4.2	11.43(1.69)
	BDD	92.22(.05)	3.9/3.9	11.72(1.73)
	KAB	86.72(.07)	6.9/6.4	12.46(5.8)
	ST	92.02(.05)	4.1/3.9	11.63(1.60)
100	PRR	93.73(.03)	2.9/3.3	11.92(1.23)
	TS	93.35(.03)	3.1/3.6	11.76(1.19)
	BDD	93.54(.03)	3.05/3.4	11.82(1.21)
	KAB	88.23(.04)	5.7/6.1	10.75(1.87)
	ST	93.72(.03)	3.0/3.1	11.85(1.11)

Table 2. Model $y_t = 1.75y_{t-1} - .76y_{t-2} + a_t$; Gaussian errors; Forecast horizont 3. Nominal coverage 95%.

Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
n	**Empirical**	**80%**	**10%/10%**	**2.56**
25	PRR	77.84(.09)	11.5/10.6	2.71(.54)
	TS	75.63(.09)	12.6/11.8	2.55(.44)
	BDD	76.11(.09)	12.3/11.6	2.53(.45)
	KAB	75.38(.15)	12.7/11.9	4.38(3.68)
	ST	76.01(.08)	12.4/11.6	2.53(.39)
50	PRR	78.89(.06)	10.7/10.4	2.61(.34)
	TS	78.06(.06)	11.1/10.8	2.55(.32)
	BDD	77.89(.06)	11.1/11.0	2.54(.33)
	KAB	77.04(.10)	12.0/10.1	3.20(1.06)
	ST	78.37(.05)	10.9/10.7	2.55(.27)
100	PRR	79.37(.04)	10.1/10.5	2.58(.23)
	TS	79.10(.04)	10.2/10.7	2.56(.23)
	BDD	79.07(.04)	10.3/10.6	2.56(.23)
	KAB	77.58(.06)	11.1/11.3	2.81(.39)
	ST	79.26(.04)	10.1/10.6	2.56(.19)

Table 3. Model $y_t = 1.75y_{t-1} - .76y_{t-2} + a_t$; Gaussian errors; Forecast horizont 1. Nominal coverage 80%.

Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
n	**Empirical**	**80%**	**10%/10%**	**7.83**
25	PRR	73.31(.14)	13.9/12.8	8.07(2.43)
	TS	70.14(.13)	15.8/14.1	7.28(1.67)
	BDD	72.73(.11)	14.4/12.9	7.44(1.52)
	KAB	63.68(.19)	18.8/17.5	10.30(9.70)
	ST	70.01(.13)	15.8/14.2	7.31(1.54)
50	PRR	76.92(.08)	11.7/11.3	7.83(1.27)
	TS	75.22(.07)	12.6/12.2	7.49(1.12)
	BDD	76.26(.07)	12.0/11.7	7.68(1.15)
	KAB	66.38(.12)	17.5/16.1	7.70(3.16)
	ST	75.67(.08)	12.3/12.0	7.60(1.04)
100	PRR	78.29(.05)	10.6/11.1	7.80(.79)
	TS	77.64(.05)	10.8/11.6	7.70(.78)
	BDD	77.98(.05)	10.7/11.3	7.74(.79)
	KAB	67.67(.07)	15.9/16.4	6.86(1.10)
	ST	78.03(.05)	10.7/11.3	7.74(.73)

Table 4. Model $y_t = 1.75y_{t-1} - .76y_{t-2} + a_t$; Gaussian errors; Forecast horizont 3. Nominal coverage 80%.

Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
n	**Empirical**	**95%**	**2.5%/2.5%**	**12.57**
25	PRR	91.19(.08)	3.9/4.9	13.28(3.88)
	TS	90.32(.10)	4.7/5.0	12.38(2.87)
	BDD	90.92(.08)	4.1/5.0	12.27(2.81)
	KAB	87.98(.14)	7.7/4.3	22.00
	ST	89.98(.05)	.69/9.33	11.89
50	PRR	92.38(.07)	3.4/4.2	12.85(1.97)
	TS	91.55(.08)	4.3/4.2	12.42(1.88)
	BDD	91.76(.08)	4.0/4.3	12.37(1.89)
	KAB	90.05(.09)	6.3/3.6	16.24(6.09)
	ST	90.10(.04)	.35/9.54	12.20(2.38)
100	PRR	93.77(.04)	2.8/3.4	12.77(.80)
	TS	93.53(.05)	3.1/3.4	12.67(.76)
	BDD	93.56(.05)	3.05/3.4	12.64(.74)
	KAB	91.55(.06)	5.25/3.2	14.18(2.39)
	ST	90.17(.01)	.01/9.82	12.29(1.61)

Table 5. Model $y_t = 1.75y_{t-1} - .76y_{t-2} + a_t$;Contaminated errors; Forecast horizont 1. Nominal coverage 95%.

Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
n	**Empirical**	**95%**	**2.5%/2.5%**	**34.05**
25	PRR	87.73(.13)	6.25/6.02	37.75(15.3)
	TS	86.02(.14)	7.1/6.8	33.99(10.6)
	BDD	88.53(.13)	5.7/5.8	35.33(10.5)
	KAB	78.65(.20)	12.9/8.4	48.20(49.2)
	ST	87.46(.11)	2.3/10.2	34.27(10.8)
50	PRR	91.13(.09)	4.9/4.0	36.57(8.3)
	TS	89.46(.11)	6.3/4.2	34.73(7.07)
	BDD	90.67(.10)	5.5/3.8	35.45(6.92)
	KAB	82.73(.14)	10.5/6.7	37.60(16.9)
	ST	91.03(.08)	.83/8.14	36.34(7.52)
100	PRR	93.03(.06)	3.8/3.2	35.54(5.16)
	TS	92.57(.07)	4.2/3.2	34.93(4.85)
	BDD	92.86(.07)	4.0/3.1	35.06(4.71)
	KAB	85.49(.10)	8.5/6.0	32.69(7.24)
	ST	92.74(.04)	.09/7.16	37.14(5.35)

Table 6. Model $y_t = 1.75y_{t-1} - .76y_{t-2} + a_t$;Contaminated errors; Forecast horizont 3. Nominal coverage 95%.

Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
n	**Empirical**	**80%**	**10%/10%**	**6.50**
25	PRR	77.57(.16)	12.1/10.3	7.34(3.64)
	TS	76.32(.17)	13.2/10.5	6.77(3.43)
	BDD	77.49(.15)	12.3/10.2	6.74(3.39)
	KAB	70.40(.26)	19.1/10.5	12.22(11.5)
	ST	85.77(.10)	3.5/10.7	7.78(2.32)
50	PRR	78.95(.13)	11.1/9.9	7.07(3.34)
	TS	77.17(.15)	12.7/10.1	6.81(3.39)
	BDD	77.52(.14)	12.3/10.2	6.80(3.39)
	KAB	71.77(.22)	18.2/10.02	9.32(4.17)
	ST	88.31(.07)	1.6/10.1	7.69(1.55)
100	PRR	79.47(.09)	10.5/10.02	6.75(3.31)
	TS	78.97(.10)	10.9/10.1	6.67(3.34)
	BDD	79.05(.10)	10.8/10.1	6.66(3.35)
	KAB	73.12(.17)	16.9/10.0	7.81(3.05)
	ST	89.41(.02)	.55/10.03	8.03(1.05)

Table 7. Model $y_t = 1.75y_{t-1} - .76y_{t-2} + a_t$; Contaminated errors; Forecast horizont 1. Nominal coverage 80%.

Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
n	**Empirical**	**80%**	**10%/10%**	**24.20**
25	PRR	69.46(.20)	15.9/14.6	23.81(9.23)
	TS	67.35(.20)	17.2/15.4	21.64(7.28)
	BDD	69.85(.19)	15.6/14.5	22.42(7.49)
	KAB	58.67(.27)	24.4/16.9	29.41(28.6)
	ST	72.57(.17)	10.4/17.1	22.38(7.07)
50	PRR	73.26(.16)	13.8/12.9	23.61(5.46)
	TS	70.72(.17)	15.9/13.3	22.42(5.41)
	BDD	71.98(.17)	14.9/13.1	22.84(5.49)
	KAB	60.65(.21)	24.02/15.3	23.36(14.2)
	ST	79.19(.11)	5.3/15.5	23.73(4.91)
100	PRR	75.97(.12)	12.1/11.9	23.60(3.72)
	TS	75.15(.12)	12.9/11.9	23.22(3.73)
	BDD	75.64(.12)	12.4/11.9	23.31(3.68)
	KAB	62.52(.16)	22.5/14.9	20.94(4.13)
	ST	82.17(.06)	2.7/15.2	24.25(3.49)

Table 8. Model $y_t = 1.75y_{t-1} - .76y_{t-2} + a_t$; Contaminated errors; Forecast horizont 3. Nominal coverage 80%.

Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
n	Empirical	95%	2.5%/2.5%	3.65
25	PRR	91.61(.10)	4.3/4.1	4.18(1.52)
	TS	90.34(.11)	5.2/4.5	3.85(1.31)
	BDD	90.67(.10)	4.9/4.4	3.80(1.29)
	KAB	88.46(.14)	7.9/3.6	7.37(6.96)
	ST	92.28(.07)	.57/7.15	3.77(1.04)
50	PRR	93.27(.07)	3.3/3.4	3.90(.94)
	TS	92.38(.08)	4.05/3.6	3.77(.90)
	BDD	92.39(.08)	4.0/3.6	3.75(.90)
	KAB	90.29(.11)	6.6/3.1	5.08(2.26)
	ST	93.90(.04)	.05/6.04	3.85(.72)
100	PRR	93.88(.06)	3.09/3.03	3.79(.68)
	TS	93.54(.06)	3.4/3.1	3.75(.66)
	BDD	93.46(.06)	3.4/3.1	3.74(.67)
	KAB	91.20(.09)	5.8/3.02	4.25(.94)
	ST	94.37(.02)	.00/5.63	3.87(.53)

Table 9. Model $y_t = 1.75y_{t-1} - .76y_{t-2} + a_t$; Exponential errors; Forecast horizont 1. Nominal coverage 95%.

Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
n	Empirical	95%	2.5%/2.5%	11.62
25	PRR	88.14(.12)	5.9/5.9	12.13(5.01)
	TS	85.99(.12)	7.3/6.7	10.65(3.64)
	BDD	87.96(.11)	6.2/5.8	10.91(3.50)
	KAB	81.38(.16)	10.7/7.9	16.23(16.2)
	ST	87.08(.12)	3.6/9.3	10.79(3.49)
50	PRR	91.56(.07)	4.4/4.1	11.81(2.78)
	TS	90.17(.08)	5.4/4.4	11.23(2.75)
	BDD	91.07(.07)	4.8/4.1	11.48(2.47)
	KAB	85.24(.10)	8.6/6.1	12.22(6.01)
	ST	92.33(.06)	1.08/6.6	11.43(2.31)
100	PRR	93.07(.05)	3.6/3.4	11.65(1.91)
	TS	92.57(.05)	4.0/3.45	11.45(1.85)
	BDD	92.77(.05)	3.87/3.36	11.55(1.88)
	KAB	86.93(.08)	7.57/5.5	10.56(2.30)
	ST	94.17(.03)	.25/5.57	11.70(1.71)

Table 10. Model $y_t = 1.75y_{t-1} - .76y_{t-2} + a_t$; Exponential errors; Forecast horizont 3. Nominal coverage 95%.

Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
n	**Empirical**	**80%**	**10%/10%**	**2.19**
25	PRR	77.08(.16)	11.7/11.2	2.44(.67)
	TS	74.71(.16)	13.8/12.5	2.27(.59)
	BDD	75.25(.16)	13.3/11.4	2.25(.58)
	KAB	73.03(.21)	15.6/11.4	4.11(3.6)
	ST	82.02(.13)	5.2/12.8	2.46(.68)
50	PRR	77.90(.12)	11.4/10.7	2.28(.41)
	TS	76.84(.13)	12.3/10.9	2.22(.41)
	BDD	76.94(.13)	12.1/11.0	2.21(.41)
	KAB	73.74(.16)	15.5/10.7	2.96(1.15)
	ST	86.33(.07)	2.36/11.31	2.51(.47)
100	PRR	78.76(.11)	10.8/10.4	2.23(.30)
	TS	78.22(.10)	11.3/10.4	2.21(.30)
	BDD	78.15(.10)	11.4/10.4	2.21(.31)
	KAB	74.04(.13)	15.2/10.8	2.52(.49)
	ST	88.43(.04)	.81/10.76	2.53(.34)

Table 11. Model $y_t = 1.75y_{t-1} - .76y_{t-2} + a_t$; Exponential errors; Forecast horizont 1. Nominal coverage 80%.

Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
n	**Empirical**	**80%**	**10%/10%**	**7.33**
25	PRR	72.56(.17)	13.8/13.6	7.66(2.80)
	TS	69.41(.16)	16.24/14.34	6.88(2.20)
	BDD	72.27(.15)	14.6/13.1	7.08(2.14)
	KAB	62(.95(.22)	19.4/17.6	9.85(9.30)
	ST	71.11(.16)	12.5/16.4	7.05(2.28)
50	PRR	76.08(.11)	12.2/11.7	7.41(1.53)
	TS	74.02(.12)	13.8/12.1	7.09(1.45)
	BDD	75.33(.11)	12.8/11.8	7.23(1.45)
	KAB	65.34(.15)	19.0/15.7	7.46(3.31)
	ST	78.40(.11)	8.5/13.1	7.46(1.51)
100	PRR	77.67(.08)	11.4/10.9	7.32(1.01)
	TS	76.76(.08)	12.1/11.1	7.20(1.02)
	BDD	77.07(.08)	11.9/11.1	7.26(1.03)
	KAB	66.17(.10)	18.8/15.0	6.56(1.29)
	ST	81.82(.07)	6.4/11.8	7.64(1.12)

Table 12. Model $y_t = 1.75y_{t-1} - .76y_{t-2} + a_t$; Exponential errors; Forecast horizont 3. Nominal coverage 80%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**80%**	**10%/10%**	**2.56**
	25	PRR	76.83(.08)	11.6/11.6	2.52(.45)
		GGP	76.53(.09)	11.8/11.7	2.51(.45)
		ST	77.80(.07)	11.22/10.97	2.54(.38)
	50	PRR	78.61(.06)	10.7/10.7	2.56(.34)
		GGP	78.54(.06)	10.7/10.7	2.56(.34)
		ST	79.01(.05)	10.56/10.44	2.56(.27)
	100	PRR	79.34(.04)	10.25/10.41	2.56(.23)
		GGP	79.33(.04)	10.26/10.40	2.56(.24)
		ST	79.52(.03)	10.15/10.33	2.56(.19)
3	n	**Empirical**	**80%**	**10%/10%**	**12.89**
	25	PRR	75.68(.09)	12.3/12.05	12.34(2.42)
		GGP	75.24(.10)	12.5/12.3	12.35(2.49)
		ST	89.42(.08)	5.3/5.3	17.70(3.60)
	50	PRR	77.82(.06)	11.11/11.07	12.63(1.71)
		GGP	77.74(.07)	11.16/11.10	12.64(1.73)
		ST	91.23(.05)	4.4/4.3	17.94(2.51)
	100	PRR	78.91(.04)	10.4/10.6	12.78(1.22)
		GGP	78.89(.05)	10.5/10.6	12.77(1.22)
		ST	92.09(.03)	3.9/4.01	18.06(1.77)

Table 13. Model $(1-B)^2(1-0.5B)y_t = a_t$; Gaussian errors; Nominal coverage 80%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**95%**	**2.5%/2.5%**	**3.92**
	25	PRR	91.63(.05)	4.3/4.06	3.82(.68)
		GGP	91.39(.06)	4.43/4.2	3.81(.68)
		ST	93.25(.04)	3.44/3.31	3.89(.59)
	50	PRR	93.08(.04)	3.5/3.4	3.87(.54)
		GGP	93.06(.04)	3.5/3.5	3.87(.54)
		ST	94.22(.03)	2.9/2.9	3.92(.41)
	100	PRR	94.04(.03)	2.9/3.04	3.90(.39)
		GGP	94.01(.03)	2.9/3.05	3.90(.39)
		ST	94.64(.02)	2.6/2.7	3.92(.29)
3	n	**Empirical**	**95%**	**2.5%2.5%**	**19.72**
	25	PRR	91.29(.06)	4.09/4.2	18.77(3.61)
		GGP	90.93(.07)	4.5/4.5	18.76(3.69)
		ST	97.91(.04)	1.05/1.03	27.10(5.52)
	50	PRR	93.21(.04)	3.4/3.4	19.32(2.62)
		GGP	93.16(.04)	3.4/3.4	19.31(2.64)
		ST	98.79(.02)	.62/.58	27.47(3.84)
	100	PRR	94.05(.03)	2.9/3.04	19.50(1.89)
		GGP	94.04(.03)	2.9/3.04	19.50(1.90)
		ST	99.16(.01)	.41/.43	27.66(2.71)

Table 14. Model $(1-B)^2(1-0.5B)y_t = a_t$; Gaussian errors; Nominal coverage 95%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**80%**	**10%/10%**	**6.50**
	25	PRR	78.61(.11)	10.85/10.54	6.46(3.70)
		GGP	76.85(.14)	12.66/10.48	6.46(3.71)
		ST	87.12(.07)	2.3/10.6	7.79(2.35)
	50	PRR	78.55(.11)	11.4/10.07	6.76(3.60)
		GGP	77.43(.12)	12.64/9.93	6.76(3.60)
		ST	88.54(.06)	1.41/10.04	7.96((1.55)
	100	PRR	79.41(.08)	10.5/10.1	6.64(3.44)
		GGP	79.11(.09)	10.9/10.0	6.63(3.44)
		ST	98.51(.02)	.46/10.03	8.02(1.04)
3	n	**Empirical**	**80%**	**10%/10%**	**40.47**
	25	PRR	75.00(.14)	11.46/13.53	35.43(12.8)
		GGP	72.70(.16)	13.9/13.4	35.26(13.1)
		ST	88.94(.06)	.66/10.4	54.41(16.6)
	50	PRR	75.64(.13)	12.31/12.04	37.02(10.1)
		GGP	74.38(.14)	13.7/12.0	36.93(10.2)
		ST	89.98(.05)	.49/9.53	55.98(11.3)
	100	PRR	77.51(.10)	11.16/11.3	38.19(7.73)
		GGP	77.14(.10)	11.6/11.24	38.17(7.75)
		ST	90.45(.02)	.06/9.5	56.63(8.12)

Table 15. Model $(1-B)^2(1-0.5B)y_t = a_t$;Contaminated errors; Nominal coverage 80%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**95%**	**2.5%/2.5%**	**12.57**
	25	PRR	91.46(.07)	3.8/4.7	1.99(2.79)
		GGP	90.68(.09)	4.8/4.5	12.03(2.85)
		ST	90.05(.04)	.45/9.5	11.94(3.59)
	50	PRR	92.40(.06)	3.7/3.9	12.26(1.90)
		GGP	91.76(.09)	4.4/3.8	12.27(1.89)
		ST	90.03(.04)	.32/9.7	12.19(2.37)
	100	PRR	93.80(.03)	2.9/3.3	12.58(.74)
		GGP	93.66(.05)	3.2/3.1	12.59(.75)
		ST	90.12(.01)	.00/9.87	12.28(1.60)
3	n	**Empirical**	**95%**	**2.5%/2.5%**	**58.07**
	25	PRR	91.63(.09)	2.9/4.9	58.45(15.1)
		GGP	89.98(.10)	4.9/5.1	57.82(15.9)
		ST	95.14(.05)	.08/4.8	83.31(25.3)
	50	PRR	92.76(.07)	3.7/3.5	59.15(9.64)
		GGP	91.81(.09)	4.7/3.5	59.01(10.12)
		ST	96.47(.04)	.18/3.35	85.72(17.2)
	100	PRR	93.83(.04)	3.3/2.9	59.07(6.4)
		GGP	93.60(.05)	3.5/2.9	59.04(6.47)
		ST	97.24(.02)	.00/2.75	86.71(12.44)

Table 16. Model $(1-B)^2(1-0.5B)y_t = a_t$;Contaminated errors; Nominal coverage 95%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**80%**	**10%/10%**	**2.19**
	25	PRR	76.06(.14)	12.72/11.21	2.23(.61)
		GGP	75.23(.15)	13.68/11.09	2.23(.62)
		ST	84.60(.10)	3.7/11.7	2.46(.68)
	50	PRR	77.96(.11)	11.25/10.97	2.20(.42)
		GGP	77.53(.12)	11.76/10.70	2.20(.42)
		ST	87.44(.07)	1.6/10.98	2.51(.47)
	100	PRR	78.62(.09)	10.99/10.4	2.19(.31)
		GGP	78.27(.10)	11.40/10.33	2.20(.31)
		ST	88.86(.03)	.52/10.62	2.53(.35)
3	n	**Empirical**	**80%**	**10%/10%**	**11.82**
	25	PRR	75.12(.13)	12.92/11.94	11.47(3.16)
		GGP	74.07(.13)	14.02/11.91	11.46(3.46)
		ST	90.14(.08)	1.97/7.9	17.03(5.10)
	50	PRR	77.37(.090)	11.55/11.07	11.63(2.15)
		GGP	76.89(.10)	12.11/11.00	11.61(2.15)
		ST	92.51(.04)	.69/6.8	17.60(3.51)
	100	PRR	78.48(.07)	10.99/10.5	11.73(1.58)
		GGP	78.15(.07)	11.42/10.42	11.74(1.58)
		ST	93.54(.02)	.17/6.3	17.86(2.63)

Table 17. Model $(1-B)^2(1-0.5B)y_t = a_t$; Exponential errors; Nominal coverage 80%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**95%**	**2.5%/2.5%**	**3.65**
	25	PRR	91.02(.09)	4.7/4.3	3.72(1.26)
		GGP	90.59(.10)	5.1/4.3	3.72(1.28)
		ST	93.33(.04)	.14/6.5	3.77(1.04)
	50	PRR	92.65(.07)	3.8/3.5	3.70(.89)
		GGP	92.38(.08)	4.1/3.5	3.70(.91)
		ST	94.03(.03)	.11/5.9	3.84(.72)
	100	PRR	93.45(.06)	3.5/3.05	3.72(.68)
		GGP	93.27(.06)	3.7/3.0	3.72(.68)
		ST	94.44(.02)	.00/5.56	3.87(.53)
3	n	**Empirical**	**95%**	**2.5%/2.5%**	**19.05**
	25	PRR	90.46(.08)	4.4/5.1	18.01(5.72)
		GGP	89.72(.09)	5.13/5.14	17.96(6.13)
		ST	96.48(.03)	.18/3.34	26.09(7.82)
	50	PRR	92.75(.06)	3.7/3.6	18.85(3.90)
		GGP	92.42(.07)	4.01/3.56	18.82(3.91)
		ST	97.41(.02)	.01/2.6	26.96(5.37)
	100	PRR	93.55(.04)	3.3/3.13	18.89(2.96)
		GGP	93.40(.05)	3.5/3.1	18.87(2.95)
		ST	97.75(.01)	.00/2.25	27.35(4.02)

Table 18. Model $(1-B)^2(1-0.5B)y_t = a_t$; Exponential errors; Nominal coverage 95%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**80%**	**10%/10%**	**2.55**
	25	PRR	76.03(.10)	12.29/11.69	2.57(.45)
		ST	76.55(.10)	12.16/11.28	2.60(.40)
	50	PRR	78.15(.06)	11.1/10.7	2.57(.32)
		ST	78.75(.06)	10.8/10.4	2.59(.27)
	100	PRR	79.10(.04)	10.5/10.4	2.57(.23)
		ST	79.50(.04)	10.3/10.2	2.58(.19)
3	n	**Empirical**	**80%**	**10%/10%**	**3.22**
	25	PRR	78.63(.07)	10.61/10.75	3.19(.52)
		ST	81.11(.08)	9.43/9.46	3.38(.58)
	50	PRR	79.51(.05)	10.27/10.22	3.22(.38)
		ST	80.62(.05)	9.72/9.66	3.30(.39)
	100	PRR	79.83(.04)	10.06/10.10	3.23(.27)
		ST	80.29(.04)	9.89/9.81	3.26(.27)

Table 19. Model $y_t = a_t - .3a_{t-1} + .7a_{t-2}$; Gaussian errors; Nominal coverage 80%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**95%**	**2.5%/2.5%**	**3.92**
	25	PRR	91.63(.07)	4.3/4.1	3.99(.74)
		ST	92.38(.07)	4.1/3.5	3.99(.61)
	50	PRR	93.05(.04)	3.5/3.5	3.93(.54)
		ST	94.06(.03)	3.02/2.91	3.97(.42)
	100	PRR	94.05(.02)	2.9/3.0	3.93(.38)
		ST	94.65(.02)	2.7/2.7	3.95(.30)
3	n	**Empirical**	**95%**	**2.5%/2.5%**	**4.91**
	25	PRR	93.42(.04)	3.25/3.33	4.90(.82)
		ST	94.85(.04)	2.57/2.57	5.18(.89)
	50	PRR	94.34(.03)	2.82/2.84	4.95(.60)
		ST	95.03(.03)	2.48/2.49	5.06(.60)
	100	PRR	94.70(.02)	2.59/2.71	4.94(.43)
		ST	95.02(.02)	2.48/2.50	4.99(.41)

Table 20. Model $y_t = a_t - .3a_{t-1} + .7a_{t-2}$; Gaussian errors; Nominal coverage 95%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**80%**	**10%/10%**	**2.19**
	25	PRR	76.24(.17)	12.7/11.1	2.36(.63)
		ST	82.75(.13)	5.75/11.51	2.55(.66)
	50	PRR	76.60(.14)	12.8/10.6	2.26(.44)
		ST	86.15(.09)	3.1/10.7	2.57(.51)
	100	PRR	78.11(.11)	11.5/10.4	2.22(.31)
		ST	88.56(.05)	.94/10.5	2.55(.36)
3	n	**Empirical**	**80%**	**10%/10%**	**2.93**
	25	PRR	78.55(.11)	10.37/11.08	2.98(.74)
		ST	83.44(.09)	5.81/10.75	3.30(.90)
	50	PRR	79.35(.09)	10.08/10.56	2.98(.58)
		ST	84.48(.07)	4.97/10.54	3.28(.69)
	100	PRR	79.63(.07)	10.02/10.34	2.95(.40)
		ST	85.15(.05)	4.31/10.53	3.23(.47)

Table 21. Model $y_t = a_t - .3a_{t-1} + .7a_{t-2}$; Exponential errors; Nominal coverage 80%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**95%**	**2.5%/2.5%**	**3.64**
	25	PRR	91.60(.11)	4.25/4.16	4.03(1.35)
		ST	93.02(.07)	.71/6.26	3.91(1.01)
	50	PRR	92.25(.09)	4.2/3.6	3.86(1.03)
		ST	94.23(.03)	.19/5.6	3.94(.78)
	100	PRR	93.60(.07)	3.4/3.1	3.77(.67)
		ST	94.57(.55)	.01/5.42	3.92(.55)
3	n	**Empirical**	**95%**	**2.5%/2.5%**	**4.82**
	25	PRR	92.80(.06)	2.66/4.54	4.82(1.43)
		ST	94.03(.04)	.72/5.25	5.06(1.38)
	50	PRR	93.98(.05)	2.60/3.43	4.97(1.18)
		ST	94.49(.03)	.50/5.01	5.02(1.06)
	100	PRR	94.48(.03)	2.47/3.05	4.88(.79)
		ST	94.71(.02)	.34/4.95	4.95(.73)

Table 22. Model $y_t = a_t - .3a_{t-1} + .7a_{t-2}$; Exponential errors; Nominal coverage 95%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**80%**	**10%/10%**	**6.51**
	25	PRR	75.63(.19)	14.2/10.4	7.06(3.32)
		ST	85.62(.12)	4.2/10.2	7.99(2.11)
	50	PRR	78.11(.14)	11.9/9.97	7.03(3.32)
		ST	87.97(.06)	1.96/10.06	8.06(1.49)
	100	PRR	79.03(.10)	10.9/10.1	6.76(3.23)
		ST	88.42(.03)	.59/9.98	8.08(1.04)
3	n	**Empirical**	**80%**	**10%/10%**	**10.24**
	25	PRR	75.38(.14)	11.52/13.10	9.13(3.32)
		ST	81.12(.09)	4.66/14.23	10.29(2.93)
	50	PRR	77.73(.10)	10.37/11.89	9.64(2.57)
		ST	82.17(.06)	3.32/14.51	10.28(2.01)
	100	PRR	78.60(.08)	10.28/11.12	9.91(1.83)
		ST	82.38(.04)	2.87/14.75	10.21(1.36)

Table 23. Model $y_t = a_t - .3a_{t-1} + .7a_{t-2}$; Contaminated errors; Nominal coverage 80%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**95%**	**2.5%/2.5%**	**12.56**
	25	PRR	89.60(.11)	5.45/4.95	12.83(3.10)
		ST	89.84(.07)	.97/9.19	12.23(3.23)
	50	PRR	91.72(.08)	3.9/4.4	12.65(1.93)
		ST	90.31(.03)	.18/9.50	12.35(2.28)
	100	PRR	93.61(.05)	3.02/3.4	12.75(.85)
		ST	90.20(.02)	.01/9.8	12.37(1.59)
3	n	**Empirical**	**95%**	**2.5%/2.5%**	**14.80**
	25	PRR	91.99(.09)	3.67/4.35	15.19(3.92)
		ST	91.34(.07)	.98/7.68	15.75(4.49)
	50	PRR	94.08(.05)	2.78/3.14	15.30(2.41)
		ST	92.30(.05)	.37/7.33	15.75(3.06)
	100	PRR	94.70(.03)	2.60/2.70	15.16(1.56)
		ST	92.53(.03)	.15/7.32	15.64(2.09)

Table 24. Model $y_t = a_t - .3a_{t-1} + .7a_{t-2}$; Contaminated errors; Nominal coverage 95%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**80%**	**10%/10%**	**2.18**
	25	PRR	78.24(.14)	10.5/11.2	2.31(.62)
		ST	83.69(.12)	4.5/11.8	2.50(.67)
	50	PRR	79.04(.11)	11.1/10.86	2.25(.44)
		ST	86.84(.07)	2.16/10.99	2.53(.51)
	100	PRR	79.59(.09)	9.06/3.06	2.20(.30)
		ST	88.71(.04)	.55/10.73	2.54(.36)
3	n	**Empirical**	**80%**	**10%/10%**	**2.57**
	25	PRR	78.25(.11)	10.61/11.13	2.65(.72)
		ST	82.22(.10)	6.2/11.6	2.87(.81)
	50	PRR	77.97(.09)	11.1/10.86	2.58(.49)
		ST	84.20(.08)	4.7/11.1	2.86(.59)
	100	PRR	78.46(.07)	10.9/10.6	2.55(.35)
		ST	86.04(.06)	3.3/10.7	2.86(.42)

Table 25. Model $y_t = .7y_{t-1} + a_t - .3a_{t-1}$; Exponential errors; Nominal coverage 80%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**95%**	**2.5%/2.5%**	**3.64**
	25	PRR	93.28(.08)	2.8/3.9	4.05(1.39)
		ST	92.97(.06)	.58/6.44	3.83(1.02)
	50	PRR	94.27(.06)	2.2/3.5	3.86(1.04)
		ST	94.09(.03)	.01/5.81	3.88(.79)
	100	PRR	94.91(.05)	1.97/3.12	3.74(.66)
		ST	94.44(.02)	0.0/5.56	3.89(.55)
3	n	**Empirical**	**95%**	**2.5%/2.5%**	**4.20**
	25	PRR	93.25(.06)	2.7/4.07	4.39(1.39)
		ST	93.64(.04)	.39/5.96	4.40(1.25)
	50	PRR	93.48(.05)	3.08/3.4	4.30(1.01)
		ST	94.28(.03)	.02/5.53	4.38(.91)
	100	PRR	93.94(.04)	2.98/3.08	4.22(.69)
		ST	94.83(.02)	.003/5.14	4.38(.64)

Table 26. Model $y_t = .7y_{t-1} + a_t - .3a_{t-1}$; Exponential errors; Nominal coverage 95%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**80%**	**10%/10%**	**2.56**
	25	PRR	77.68(.08)	11.4/10.9	2.59(.44)
		ST	76.92(.08)	11.9/11.1	2.55(.37)
	50	PRR	78.65(.06)	11.01/10.34	5.57(.30)
		ST	78.62(.05)	11.3/10.3	2.56(.25)
	100	PRR	79.10(.04)	10.4/10.4	2.56(.23)
		ST	79.29(.04)	10.4/10.3	2.56(.19)
3	n	**Empirical**	**80%**	**10%/10%**	**2.85**
	25	PRR	78.61(.07)	10.8/10.6	2.88(.49)
		ST	78.59(.08)	11.1/10.3	2.93(.51)
	50	PRR	78.89(.05)	10.7/10.3	2.85(.35)
		ST	79.45(.06)	10.6/9.9	2.90(.36)
	100	PRR	79.24(.04)	10.3/10.5	2.84(.26)
		ST	79.92(.042)	9.96/10.1	2.88(.26)

Table 27. Model $y_t = .7y_{t-1} + a_t - .3a_{t-1}$; Gaussian errors; Nominal coverage 80%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**95%**	**2.5%/2.5%**	**3.92**
	25	PRR	92.70(.05)	3.7/3.6	3.99(.70)
		ST	92.74(.05)	3.8/3.4	3.90(.57)
	50	PRR	93.46(.03)	3.3/3.2	3.93(.51)
		ST	94.04(.03)	3.1/2.8	3.92(.39)
	100	PRR	94.04(.02)	2.91/3.05	3.91(.37)
		ST	94.49(.02)	2.7/2.7	3.92(.29)
3	n	**Empirical**	**95%**	**2.5%/2.5%**	**4.35**
	25	PRR	93.14(.04)	3.4/3.4	4.38(.77)
		ST	93.59(.05)	3.4/3.0	4.48(.79)
	50	PRR	93.84(.03)	3.1/3.1	4.36(.55)
		ST	94.39(.03)	2.9/2.8	4.44(.54)
	100	PRR	94.27(.02)	2.8/2.9	4.34(.41)
		ST	94.78(.02)	2.6/2.6	4.42(.39)

Table 28. Model $y_t = .7y_{t-1} + a_t - .3a_{t-1}$; Gaussian errors; Nominal coverage 95%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**80%**	**10%/10%**	**6.51**
	25	PRR	79.90(.12)	9.95/10.1	6.95(3.53)
		ST	86.92(.09)	2.6/10.4	7.74(2.21)
	50	PRR	79.91(.11)	9.99/10.1	6.89(3.36)
		ST	88.47(.06)	1.5/10.1	7.95(1.52)
	100	PRR	80.20(.07)	9.7/10.1	6.66(3.29)
		ST	89.55(.02)	.46/9.98	8.02(1.04)
3	n	**Empirical**	**80%**	**10%/10%**	**8.62**
	25	PRR	77.60(.12)	9.9/12.5	8.08(3.17)
		ST	84.83(.08)	2.7/12.4	8.82(2.49)
	50	PRR	77.58(.11)	10.7/11.6	8.23(2.71)
		ST	86.46(.07)	2.1/11.4	9.00(1.77)
	100	PRR	78.38(.09)	10.6/11.0	8.22(2.30)
		ST	87.86(.03)	1.05/11.1	9.05(1.24)

Table 29. Model $y_t = .7y_{t-1} + a_t - .3a_{t-1}$;Contaminated errors; Nominal coverage 80%.

Lead time	Sample Size	Method	Average Coverage(se)	Coverage below/above	Average Length
1	n	**Empirical**	**95%**	**2.5%/2.5%**	**12.56**
	25	PRR	92.17(.07)	3.03/4.8	12.60(2.93)
		ST	90.01(.05)	.58/9.40	11.86(3.38)
	50	PRR	93.16(.06)	2.8/4.1	12.59(1.88)
		ST	90.11(.04)	.33/9.56	12.17(2.33)
	100	PRR	94.37(.03)	2.3/3.3	12.72(.84)
		ST	90.19(.01)	.01/9.8	12.27(1.59)
3	n	**Empirical**	**95%**	**2.5%/2.5%**	**13.52**
	25	PRR	92.81(.06)	2.52/4.66	13.51(3.15)
		ST	90.81(.05)	.17/9.02	13.51(3.82)
	50	PRR	93.54(.05)	2.8/3.6	13.62(2.05)
		ST	91.21(.04)	.25/8.54	13.79(2.72)
	100	PRR	94.22(.03)	2.7/3.02	13.67(1.22)
		ST	91.30(.02)	.01/8.68	13.86(1.99)

Table 30. Model $y_t = .7y_{t-1} + a_t - .3a_{t-1}$;Contaminated errors; Nominal coverage 95%.