Overconvergence of subsequences of rows of Padé approximants with gaps

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Dedicated to Professor Haakon Waadeland on the occasion of his 70th birthday

Abstract

The block structure of the Padé table associated with a formal power series is well known. We study the analytic properties of the given power series in the case that as we travel along a row of the corresponding table, we encounter blocks of increasing size. Thus, we extend to row sequences of Padé approximants some classical results due to Hadamard and Ostrowski related with the overconvergence of subsequences of Taylor polynomials and the analytic properties of the limit function under the presence of gaps in the power series.

1. Introduction

Let

$$f(z)=\sum_{k=0}^{\infty} f_k z^k, \tag{1}$$

be a power series with radius of convergence $R_0>0$. By f we will denote not only the sum of the series (1) in $D_0=\{z\colon |z|<R_0\}$ but also the analytic function determined by the element (f,D_0). Fix a non-negative integer m and consider the mth row of the Padé table of the series (1): $\pi_n=\pi_{n,m}$, $n=0,1,2,\ldots$. That is, for each n,π_n is defined as the ratio p_n/q_n of any two polynomials

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satisfying

- $\deg p_n \leqslant n$, $\deg q_n \leqslant m$, $q_n \not\equiv 0$,
- $(q_n f - p_n)(z) = A_n z^{n+m+1} + \cdots$.

It is well known that π_n is uniquely determined. Let us denote by $D = \{z: |z| < R\}$ the largest disk centered at $z = 0$ inside of which (1) can be extended as a meromorphic function having at most m poles (counting multiplicities). D is called the disk of m-meromorphy of f, and $R = R_m$ is the radius of m-meromorphy.

For $m = 0$, the sequence $\{\pi_n\}$ is that of the Taylor polynomials $\{T_n\}$ relative to f. It is obvious that $f_n = 0$ if and only if $T_{n-1} = T_n$. Therefore, $f_n = 0$, $n_k < n \leqslant n'_k$ is equivalent to the fact that $T_{n_k} = T_n$, $n_k < n \leqslant n'_k$. We study the analytic properties of f under the assumption that for the mth row, there exist two increasing sequences of natural numbers $\{n_k\}$ and $\{n'_k\}$ such that

$$\pi_{n_k} = \pi_n, \quad n_k < n \leqslant n'_k, \ k = 0, 1, 2, \ldots, \tag{2}$$

and, either,

$$\lim_{k\to\infty} \frac{n'_k}{n_k} = \infty, \tag{3}$$

or,

$$\underline{\lim}_{k\to\infty} \frac{n'_k}{n_k} > 1. \tag{4}$$

In correspondence with the standard classical definition, we say that the mth row of Eq. (1) has Ostrowski type gaps if (2) and (3) take place. If, we have (2) and (4), we say that the gaps are of Hadamard type.

Let e be an arbitrary set contained in $\mathbb{C}$. Set

$$\sigma(e) = \inf\left\{\sum_k |U_k|\right\}, \tag{5}$$

where $|U_k|$ denotes the diameter of the disk U_k and the infimum is taken over all coverings $\{U_k\}$ of e by disks. A sequence of functions $\{f_n\}$ is said to converge to f σ-almost uniformly inside some region G if for each compact set $K \subset G$ and $\varepsilon > 0$ there exists a set $e = e(K, \varepsilon)$ such that $\sigma(e) < \varepsilon$ and $\{f_n\}$ converges uniformly to f on $K \backslash e$.

We prove the following extension of a classical theorem of A. Ostrowski. The original result, given for sequences of Taylor polynomials, may be found as Theorem 3.1.1 in [1].

Theorem 1. *Let* (1) *be such that its mth row has Ostrowski type gaps. Then,* (1) *defines a meromorphic function with a simply connected domain of existence G in which it may have no more than m poles* (*counting multiplicities*). *Moreover,* $\{\pi_{n_k}\}$ *converges to* f σ*-almost uniformly inside G.*

In the case that (2) and (4) take place, the existence of regular points on the boundary of D yields overconvergence σ-almost uniformly of the sequence $\{\pi_{n_k}\}$ on a neighborhood of regular points. For the definition of σ-almost uniform convergence see (5). For sequences of Taylor polynomials the corresponding result is due to Hadamard, it is mentioned in [1] and a proof may be found in [6], p. 314.

Theorem 2. *Let* (1) *be such that its mth row has Hadamard type gaps. If a is a regular point or a pole of* (1) *on the boundary of D, then $\{\pi_{n_k}\}$ converges to f σ-almost uniformly inside a neighborhood of a.*

An immediate consequence of Kronecker's theorem (see [5]) is that $\pi_n = \pi_{n_1}$, $n \geqslant n_1$, implies that (1) defines a rational function and the statements of Theorems 1 and 2 take place. Therefore, we shall assume in the sequel, that the sequence $\{\pi_{n_k}\}$ is made up of distinct Padé approximants. In Section 2, we prepare the way for the proof of the main results stated above to which Section 3 is devoted. In the sequel, we maintain the notation introduced above. Recall that m is fixed.

2. Auxiliary results

From the definition of π_n, it is obvious that q_n and p_n may be taken so that they have no common zero except at $z = 0$. On the other hand, if $z = 0$ is a zero of q_n of multiplicity $l_n (0 \leqslant l_n \leqslant \deg q_n \leqslant m)$, then it is also a zero of p_n of multiplicity $\geqslant l_n$. Therefore, there exist relatively prime polynomials P_n, Q_n, such that $\pi_n = P_n/Q_n$ and

- $\deg P_n \leqslant n - l_n$, $\deg Q_n \leqslant m - l_n$, $Q_n \not\equiv 0$,
- $(Q_n f - P_n)(z) = B'_n z^{n+m-1-l_n} + \cdots$,

where $0 \leqslant l_n \leqslant m$.

In the sequel, we will assume that Q_n is normalized by the condition

$$Q_n(z) = \prod_{|z_{n,j}| \leqslant 1} (z - z_{n,j}) \prod_{|z_{n,j}| > 1} \left(\frac{z}{z_{n,j}} - 1\right)$$

(the points $z_{n,j}$ are the poles of π_n). With this normalization, not only π_n is uniquely determined, but also P_n and Q_n. We have that for any compact set $K \subset \mathbb{C}$

$$\|Q_n\|_K = \max_{z \in K} |Q_n(z)| \leqslant C < \infty, \quad n = 0, 1, 2, \ldots, \tag{6}$$

where $C = C(K)$ does not depend on n. In the following, $C_1, C_2, \ldots,$ denote positive constants independent of n.

If $B'_n \neq 0$, it is easy to verify that $l_n \geqslant l_{n+1}$. Therefore, from the definition of the polynomials P_n and Q_n (applied to two consecutive indexes), we have the identity

$$(P_{n+1} Q_n - P_n Q_{n+1})(z) = B_n z^{n+m+1-l_n}, \quad n = 0, 1, 2, \ldots, \tag{7}$$

where $0 \leqslant l_n \leqslant m$. Thus,

$$(\pi_{n+1} - \pi_n)(z) = \frac{B_n z^{n+m+1-l_n}}{(Q_n Q_{n+1})(z)}, \quad n = 0, 1, 2, \ldots.$$

For $B'_n = 0$, $\pi_{n+1} = \pi_n$, and the same relation takes place with $B_n = 0$. Therefore, the convergence of the sequence $\{\pi_n\}$, for z fixed, is equivalent to that of the series

$$\sum_{n=N}^{\infty} \frac{B_n z^{n+m-1-l_n}}{Q_n Q_{n+1}(z)}, \quad 0 \leqslant l_n \leqslant m, \tag{8}$$

where N is a properly chosen non-negative integer (such that $Q_n(z) \neq 0$, $n \geqslant N$).

It is well known (see (24) in [4]), that the sequence $\{\pi_n\}$ converges σ-almost uniformly to f inside D. We also have (see p. 534 in [4])

$$1/R = \overline{\lim_{n\to\infty}} |B_n|^{1/n}. \tag{9}$$

From Eqs. (6) and (9), we deduce that Eq. (8) diverges pointwise for every z such that $|z| > R$. Therefore, $\{\pi_n\}$ cannot converge at any point beyond the closure of D.

Take an arbitrary $\varepsilon > 0$, and define the open set J_ε as follows. For $n = 0, 1, 2, \dots$, let $J_{n,\varepsilon}$ denote the $\varepsilon/6mn^2$-neighborhood of the set $\{z_{n,1}, z_{n,2}, \dots, z_{n,m_n}\}$ of zeros of Q_n, ($m_n = \deg Q_n$). Set

$$J_\varepsilon = \bigcup_{n=0}^{\infty} J_{n,\varepsilon}.$$

It is easy to check that

$$\sigma(J_\varepsilon) < \varepsilon$$

and

$$\sigma(J_{\varepsilon_1}) \leqslant \sigma(J_{\varepsilon_2}), \quad \varepsilon_1 < \varepsilon_2. \tag{10}$$

For any compact set $K \subset \mathbb{C}$, we put

$$K(\varepsilon) = K \backslash J_\varepsilon.$$

From the definition of $K(\varepsilon)$, it follows that

$$\min_{z\in K(\varepsilon)} |Q_n(z)| > C_1 n^{-2m}, \quad n = 0, 1, 2, \dots, \tag{11}$$

holds for every compact set $K \subset \mathbb{C}$. In [4] (see Eq. (23)), it was actually proved that for each compact set $K \subset D$

$$\overline{\lim_{n\to\infty}} \|f - \pi_n\|_{K(\varepsilon)}^{1/n} \leqslant \frac{\max_{z\in K} |z|}{R} (< 1), \tag{12}$$

from which σ-almost uniform convergence inside D immediately follows.

3. Proofs

This paragraph is divided into three sections. In Section 1, we prove Theorem 1. Section 2 is devoted to the proof of Theorem 2. Section 3 is dedicated to some applications of the main results.

Proof of Theorem 1. There is nothing to be proved unless R, the radius of D, is finite. On the other hand, Padé approximants are invariant under linear transformation, therefore without loss of generality, we may assume that $R = 1$. We begin proving the second statement of Theorem 1.

Let G denote the largest region in which (1) may be extended meromorphically (that is, G is made up of the largest region to which the analytic element (f, D_0) may be continued plus the points which are poles of the corresponding analytic function). Obviously, $D \subset G$. Fix an arbitrary compact set $K \subset G$. Since the distance from K to the complement of G is greater than zero, we can find a contour γ_1 contained in G such that the region B_1 bounded by γ_1 satisfies $K \subset B_1 \subset G$ and

$(D \cap B_1) \neq \emptyset$. Choose a closed disk B_2 such that $B_2 \subset ((D \cap B_1) \backslash K$. By γ_2 we denote the boundary of B_2.

Since poles are isolated singularities, f can have in the compact set $B_1 \cup \gamma_1$ at most a finite number of poles. By Q, we denote the monic polynomial whose zeros are the poles of f in $B_1 \cup \gamma_1$, counting multiplicities. For each $n \in \{0, 1, 2, \dots\}$, the function $\log|QQ_n(f - \pi_n)|$ is subharmonic in a neighborhood of $B_1 \cup \gamma_1$. Set

$$||QQ_n(f - \pi_n)||_{\gamma_1} = M_1(n), \qquad ||QQ_n(f - \pi_n)||_{\gamma_2} = M_2(n).$$

Denote $\omega(z)$ the harmonic function defined on $B_1 \backslash B_2$ and continuous up to the boundary, with boundary values 1 on γ_1 and 0 on γ_2. By the two constants theorem (see [2]), we have

$$\log|[QQ_n(f - \pi_n)](z)| \leqslant (\log M_1(n))\omega(z) + (\log M_2(n))(1 - \omega(z)), \tag{13}$$

for all $z \in B_1 \backslash B_2$. Since

$$0 < \inf_{z \in K} \omega(z) \leqslant \sup_{z \in K} \omega(z) < 1,$$

from (13), it follows that there exists a constant c such that

$$||QQ_n(f - \pi_n)||_K \leqslant (M_1(n))^c (M_2(n))^{1-c}. \tag{14}$$

Our next step is to find estimates for $M_1(n)$ and $M_2(n)$.

Let d be the distance from B_2 to the boundary of D, which by construction of B_2 is greater than zero. Choose $\varepsilon > 0$ such that $\varepsilon < d$. By γ_r, we denote the circle of center $z = 0$ and radius r. For each $r < 1$, from (12), it follows that

$$\overline{\lim_{n \to \infty}} ||f - \pi_n||_{\gamma_r(\varepsilon)}^{1/n} \leqslant r \tag{15}$$

(recall that we have assumed that $R = 1$). Since $\varepsilon < d$, it is easy to find $r < 1$ such that $\gamma_r(\varepsilon) = \gamma_r$ and B_2 is contained in the disk defined by γ_r (consider the circular projection of J_ε on a radius of D). Using (15), (6), the definition of Q, and the maximum principle, we obtain

$$\overline{\lim_{n \to \infty}} (M_2(n))^{1/n} \leqslant \overline{\lim_{n \to \infty}} ||QQ_n(f - \pi_n)||_{B_2}^{1/n} \leqslant r. \tag{16}$$

Take $r < r_1 < 1$. Using Eq. (16), we have that there exists a natural number N such that

$$M_2(n) < r_1^n, \quad n \geqslant N. \tag{17}$$

From (2), (16) and the definition of $M_2(n)$, it follows that

$$M_2(n_k) < r_1^{n_k'}, \quad n_k \geqslant N. \tag{18}$$

In order to estimate $M_1(n)$, we proceed as follows. Take $r_2 > 1$ such that the circle γ_{r_2} surrounds γ_1 and $\gamma_{r_2}(\varepsilon) = \gamma_{r_2}$. Therefore, for each $n = 0, 1, 2, \dots$, and $z \in \gamma_{r_2}$

$$\pi_n(z) = \pi_0(z) + \sum_{j=0}^{n-1} \frac{B_j z^{j+m+1-l_n}}{Q_j Q_{j+1}(z)}.$$

By the maximum principle, (9), and (11)

$$\begin{aligned} \| QQ_n\pi_n \|_{\gamma_1} &\leqslant \| QQ_n\pi_n \|_{\gamma_{r_2}} \\ &\leqslant \| QQ_n\pi_0 \|_{\gamma_{r_2}} + \| QQ_n \|_{\gamma_{r_2}} \sum_{j=0}^{n-1} \frac{B_j r_2^{j+m+1-l_j}}{\min_{z \in \gamma_{r_2}} Q_j Q_{j+1}(z)} \leqslant C_2 n^{4m+1}[(1+\varepsilon) r_2]^n. \end{aligned} \tag{19}$$

Since

$$M_1(n) \leqslant \| QQ_n f \|_{\gamma_1} + \| QQ_n \pi_n \|_{\gamma_1},$$

using (6), (19), and the regularity of f on G, we obtain

$$M_1(n) \leqslant C_3 n^{4m-1}[(1+\varepsilon)r_2]^n. \tag{20}$$

From (14), (18), and (20), we arrive to the estimate

$$\| QQ_{n_k}(f - \pi_{n_k}) \|_K \leqslant (C_3 n_k^{4m+1}[(1+\varepsilon)r_2]^{n_k})^{\varepsilon}(r_1^{n_k'})^{1-\varepsilon}, \quad n_k \geqslant N. \tag{21}$$

From (3) and (21), it follows that

$$\lim_k \| QQ_{n_k}(f - \pi_{n_k}) \|_K = 0. \tag{22}$$

Let ε be an arbitrary positive number. We have

$$\begin{aligned}\{z \in K: |(f - \pi_{n_k})(z)| \geqslant \varepsilon\} &\subset \{z \in K: |[QQ_{n_k}(f - \pi_{n_k})](z)| \geqslant \varepsilon |(QQ_{n_k})(z)|\} \\ &\subset \{z \in \mathbb{C}: |(QQ_{n_k})(z)| \leqslant \varepsilon^{-1} \| QQ_{n_k}(f - \pi_{n_k}) \|_K\}.\end{aligned}$$

It is obvious that

$$\begin{aligned}&\sigma(\{z \in \mathbb{C}: |(QQ_{n_k})(z)| \leqslant \varepsilon^{-1} \| QQ_{n_k}(f - \pi_{n_k}) \|_K\}) \\ &\quad \leqslant (\deg(QQ_{n_k}))(\varepsilon^{-1} \| QQ_{n_k}(f - \pi_{n_k}) \|_K)^{1/\deg(QQ_{n_k})},\end{aligned} \tag{23}$$

because each point of the set must lie in one of the at most $\deg(QQ_{n_k})$ disks centered at the zeros of QQ_{n_k} and of radius

$$(\varepsilon^{-1} \| QQ_{n_k}(f - \pi_{n_k}) \|_K)^{1/\deg(QQ_{n_k})}.$$

Hence, using Eqs. (10), (22), and (23), it follows that

$$\lim_k \sigma(\{z \in K: |(f - \pi_{n_k})(z)| \geqslant \varepsilon\}) = 0. \tag{24}$$

Since ε can be taken arbitrarily small, this means that there is σ-almost uniform convergence to f inside G.

It is known (see Lemma 1 in [3]), that from the σ-almost uniform convergence it follows that f is meromorphic in G and it may have there at most m poles (because for all n_k, π_{n_k} has at most m poles). It rests to show that G is simply connected.

Assume, on the contrary, that the complement of G has a bounded connected component. Take a curve γ which surrounds that bounded connected component. Let Q denote the polynomial whose zeros are the poles of f in G. According to Eq. (22)

$$\lim_k \| QQ_{n_k}(f - \pi_{n_k}) \|_\gamma = 0. \tag{25}$$

Take a sequence of indexes Λ such that

$$\lim_{k \in \Lambda} Q_{n_k} = Q'. \tag{26}$$

Using Eqs. (25) and (26), we have

$$\lim_{k \in \Lambda} QQ_{n_k} \pi_{n_k} = QQ' f,$$

uniformly on γ. But the sequence of functions $\{QQ_{n_k}\pi_{n_k}\}$, $k \in \Lambda$, is holomorphic in the region bounded by γ; therefore, by the maximum principle, $QQ_{n_k}\pi_{n_k}$ converges to a holomorphic function F in the region bounded by γ. Since $F = fQQ'$ on G, it follows that f may be extended as a meromorphic function onto the bounded connected component of the complement of G. This contradicts the assumption that G was the largest region in which f could be extended meromorphically. $\square$

Now, we consider the case of Hadamard type gaps.

Proof of Theorem 2. As pointed out at the beginning of the proof of Theorem 1, we may assume that $R = 1$. In the present case, by rotation, we can suppose additionally, that $a = 1$.

In the proof, we will work with two functions and different indexes m. Therefore, in order to avoid confusion, we will indicate the function and the index we are refering to in the notation of R, D, and π_n. Thus, $D_m(f)$ denotes the mth disk of meromorphy of $f, R_m(f)$ its radius, and $\pi_{n,m}(f)$ the nth approximant of the mth row of f. Analogously for any other function.

Let p be the order of the pole which (1) has at point 1 (if 1 is a regular point take $p = 0$). Because of Eq. (4), there exists a positive integer $\lambda > m + p$ such that

$$1 + \frac{1}{\lambda} < \varliminf_{k \to \infty} \frac{n'_k}{n_k}.$$

Since $\lim_{k \to \infty} n_k = \infty$, there exists an N such that

$$\left(1 + \frac{1}{\lambda}\right) + \frac{p+m}{n_k} < \frac{n'_k}{n_k}, \quad n_k \geqslant N.$$

In other words,

$$(1 + \lambda)n_k + (p + m)\lambda + 1 \leqslant \lambda n'_k, \ n_k \geqslant N, \ m + p < \lambda. \tag{27}$$

Consider the polynomial $l(z) = z^{\lambda}(1 + z)/2$ and the function $F = (f \circ l)$ (by $(f \circ l)$, we denote the composition of l and f). Notice that for all z such that $|z| \leqslant 1$, $z \neq 1$, we have that $|l(z)| < 1$ and $l(1) = 1$. The function f has no more than m poles in $D_m(f)$ and 1 is a pole of f of order p; therefore, F has no more than $(m + p)(1 + \lambda)$ poles in a disk centered at the origin of radius larger than one. That is, $R_{(m+p)(1+\lambda)}(F) > 1$. It follows that the sequence $\{\pi_{n,(m+p)(1+\lambda)}(F)\}$, $n = 0, 1, 2, \ldots$, converges σ-almost uniformly to F in a neighborhood of 1.

According to the definition of $\pi_{n_k,m}(f) = p_{n_k}/q_{n_k}$ and (2), we have

$$(q_{n_k}f - p_{n_k})(z) = A_{n_k}z^{n'_k+m+1} + \cdots,$$

where $\deg p_{n_k} \leqslant n_k, \deg q_{n_k} \leqslant m, q_{n_k} \not\equiv 0$. Therefore,

$$q_{n_k}(l(z))F(z) - p_{n_k}(l(z)) = A'_{n_k}z^{(n'_k+m+1)\lambda} + \cdots,$$

where $\deg(p_{n_k} \circ l) \leqslant n_k(1 + \lambda), \deg(q_{n_k} \circ l) \leqslant m(1 + \lambda) \leqslant (m + p)(1 + \lambda)$, $(q_{n_k} \circ l) \not\equiv 0$. From Eq. (27) it follows that

$$n_k(1 + \lambda) + (m + p)(1 + \lambda) + 1 \leqslant (n'_k + m + 1)\lambda, \quad n_k \geqslant N.$$

Hence,

$$\pi_{n_k(1+\lambda),(m+p)(1+\lambda)}(F) = (p_{n_k} \circ l)/(q_{n_k} \circ l), \quad n_k \geqslant N,$$

and

$$\lim_{k\to\infty} \pi_{n_k(1-\lambda),(m+p)(1+\lambda)}(F) = F,$$

σ-almost uniformly on a neighborhood of 1. It follows that

$$\lim_{k\to\infty} \pi_{n_k,m}(f) = f,$$

σ-almost uniformly on a neighborhood of $1(l(1) = 1)$. □

Let us consider some consequences of the results above.

Corollary 1. *A power series (1) which represents an analytic function with singularities other than poles in its domain of existence, or with more than m poles (counting multiplicities) cannot have Ostrowski type gaps in the mth row of its Padé table.*

Proof. It follows directly from the statement of Theorem 1. □

Corollary 2. *A power series (1) which represents a meromorphic function with more than m poles (counting multiplicities) in the closure of its mth disk of meromorphy cannot have Hadamard type gaps in the mth row of its Padé table.*

Proof. To the contrary, assume that (1) has Hadamard type gaps. In D_m the complete mth row of the Padé table of (1) converges σ-almost uniformly. In particular, on a neighborhood of each pole lying in D_m. From Theorem 2, we know that this is also true for the subsequence $\{\pi_{n_k}\}$ of the mth row on a neighborhood of each pole lying on the boundary of D_m.

Lemma 1 in [3] indicates that from σ-almost uniform convergence on a neighborhood of a pole of order p, it follows that for all sufficiently large k, π_{n_k} must have at least p poles in the neighborhood. This is clearly impossible because for all k, the total number of poles of π_{n_k} does not exceed m. □

Corollary 3. *Assume that (1) is such that there exists an increasing sequence of natural numbers $\{n_k\}$ such that the mth row of its Padé table satisfies*

$$\pi_{n_k} = \pi_n, \quad n_k \leqslant n < n_{k+1},$$

and

$$\varliminf_{k\to\infty} \frac{n_{k+1}}{n_k} > 1.$$

Then, all points on the boundary of the mth disk of meromorphy are singular points.

Proof. If there would exist a regular point, then the sequence $\{\pi_{n_k}\}$ would converge σ-almost uniformly on a neighborhood of the regular point. From the assumptions. this sequence coincides with the complete mth row (without repetitions). But as pointed out above the mth row diverges at each point of the complement of the closure of the mth disk of meromorphy. Thus such a regular point cannot exist. □

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