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THE SHAPLEY GROUP VALUE

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Key words: Game Theory, TU games, Shapley value, group values**.**

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Abstract

Following the original interpretation of the Shapley value (Shapley, 1953a) as a priori evaluation of the prospects of a player in a multi-person interaction situation, we propose a *group value*, which we call *the Shapley group value*, as a priori evaluation of the prospects of a group of players in a coalitional game when acting as a unit. We study its properties and we give an axiomatic characterization. We motivate our proposal by means of some relevant applications of the Shapley group value, when it is used as an objective function by a decision maker who is trying to identify an optimal group of agents in a framework in which agents interact and the attained benefit can be modeled by means of a transferable utility game. As an illustrative example we analyze the problem of identifying the set of key agents in a terrorist network.

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1 Introduction

One of the most important one-point solution concepts in the framework of coalitional games with side payments is the Shapley value (Shapley, 1953a), which proposes an allocation of resources in multiperson interactions taking into account the power of players in their various cooperation opportunities. Since the pioneering work of Lloyd S. Shapley, many different values have been proposed, as for example the nucleolus (Schmeidler, 1969); the τ-value (Tijs, 1981) or the least square values (Ruiz, Valenciano and Zarzuelo, 1998). There is also a vast literature focused on extensions, modifications and generalizations of the Shapley value: weighted Shapley values (Shapley, 1953b); semi-values (Dubey, Neyman and Weber, 1981); the value for large games (Aumann and Shapley, 1974); values for NTU games which generalize the Nash bargaining solution (Nash, 1950) and the Shapley value (Harsanyi, 1963; Shapley, 1969; Maschler and Owen,

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1992); the Myerson value for games with graph restricted communication (Myerson, 1977), values for games with coalition structures (Aumann and Drèze, 1974; Owen, 1977; Hart and Kurz, 1983), and others.

In this paper, we propose an extension of the Shapley value inspired in the question originally addressed by Shapley in his seminal paper (see Hart 1987): *How would one evaluate the prospects of a player in a multiperson interaction, that is, in a game?* Following this interpretation, we propose a *group value* as a priori evaluation of the prospects of a group of players in a multi-person game when acting as a group, which takes into account the power of groups in their various cooperation opportunities without imposing on the other agents any concrete coalition structure. Mathematically, our proposal is related with the Shapley value of certain quotient games (Owen, 1977), also called merging games by Derks and Tijs (2000), which capture the situation when all the members of a group commit themselves to bargain with the others as a unit.

A key observation in our proposal is that we do not need to suppose necessarily that the players know each other nor agree to act jointly; instead, we assume the existence of an *e*xternal agent, the decision maker, that is able to coordinate the actions of the members of the group. This is the case for instance of terrorist organizations like Al Qaeda, or secret societies; in which there exists a leader (or a set of leaders) who sends a common signal that all the agents in the group are willing to follow. In this work we describe alternative situations in which this type of external coordination occurs.

Following Roth (1977), the ultimate aim of our proposal is to serve as an utility function for a decision maker who is trying to find an optimal group under certain conditions - for instance, of a given size- in those situations in which agents are immersed in a cooperative game. To be specific, we illustrate its applicability in three different settings which share two relevant features: (i) the objective is the selection of an optimal group, rather than the best individual; and (ii) the performance of a group depends on its interaction with the rest of agents, which can also coordinate themselves to form other coalitions. In this context, to maximize the characteristic function entails a too restrictive assumption over the rest of agent's behavior, i.e., only one scenario (mainly, the worst one) is evaluated. On the contrary, we show that maximizing the *value of a group* allows us to consider a more general setting, in which more than one scenario concerning other agents' actions is taken into account.

Section 2 is devoted to a general presentation of the problem we deal with. We first introduce some standard concepts and notation on Game Theory that will be used throughout this paper, and then we describe three different cases in which the need for a group valuation arises. In Section 3 we define the notion of *group value*, we introduce our proposal, which we call the *Shapley group value*, we give an axiomatic characterization for it, and we analyze its properties both, as a value, and as a set function. In Section 4 we explore one of its potential applications previously considered in Section 2. Section 5 concludes the paper.

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2 Motivation and notation

An n-person cooperative game in coalitional game with side payments, or with transferable utility (TU game), is an ordered pair (N, v), where N is a finite set of n players and $v : \mathcal{P}(N) \to \mathbb{R}$ is a map assigning a real number $v(S)$, called the value of S, to each coalition $S \subseteq N$, and where $v(\emptyset) = 0$. The real number $v(S)$ represents the reward that coalition S can achieve by itself if all its members act together.

Taking into account our main purpose, what is really measuring $v(S)$ merits a detailed comment. According to von Neumann-Morgenstern approach, "$v(S)$ describes what a given collection of players (specifically, the set S) can obtain from their *opponents* (the set $-S$)". Then, they propose to consider the zero-sum two person game which results when all players in S cooperate with each other on the one hand, and all players in $N \setminus S$ $(-S)$ cooperate on the other hand; in these conditions, the value of that game will be $v(S)$. Although assuming a zero-sum two-person game would suggest a very pessimistic view, in which the value of coalition S is evaluated considering the rest of agents to be opponents, in general a maximin criteria is used for determining the value of coalition S. In that case, a more neutral scenario in which players in $N \setminus S$ do not interact with players in S, which act as a unit, is taken into account in order to evaluate $v(S)$. In any case, using the characteristic function of the game as the utility function to evaluate the prospects of a player in a game implies the valuation of only one scenario, which entails a too restrictive assumption over the rest of agent's behavior. In order to overcome this limitation, L. Shapley introduces in 1953 the notion of *value* of the game, which can be used as the utility function for games, and is -in words of Roth (1977)- "compatible with the existing utility function that defines the games".

Let $\mathcal{G}_n$ be the vector space of all TU games with fixed player set N, and identify $(N, v) \in \mathcal{G}_n$ with its characteristic function v when no ambiguity appears. A *value* φ for TU games is an assignation which associates to each n-person game $(N, v) \in \mathcal{G}_n$ a vector $\varphi(N, v) \in \mathbb{R}^n$, where $\varphi(N, v)_i \in \mathbb{R}$ represents the *value* of player i, $i \in N$. Shapley (1953a) defines his value as follows:

$$\phi_i(N, v) = \sum_{\substack{S \subseteq N \\ i \notin S}} \frac{s!(n-s-1)!}{n!} \big(v(S \cup \{i\}) - v(S)\big), \quad i = 1, \ldots, n, \tag{1}$$

where $s = |S|$ denotes the cardinality of coalition $S \subseteq N$, and being $(N, v) \in \mathcal{G}_n$. The *value* $\phi_i(N, v)$ of each player admits different interpretations, such as the *payoff* that player i receives when the Shapley value is used to predict the allocation of resources in multiperson interactions, or his *power* when averages are used to aggregate the power of players in their various cooperation opportunities. As we announced in the introduction, our proposal is based on the question originally addressed by Shapley in his seminal paper: we interpret the value as the *expectations* of a player in a game (N, v), so we will refer to $\phi(N, v) \in \mathbb{R}^n$ as a *valuation vector*. Then, $\phi(N, v)_i \in \mathbb{R}$ will measure the (a priori) value that playing the game (N, v) has for player i, and can be used as an objective function for selecting *key players*.[1] The approach just discussed is undertaken in the next three cases, already considered in the literature:

[1]Note that in general $v(i) \neq \phi_i(N, v)$.

(i) In Lindelauf, Hamers and Husslage (2013), the authors introduce a game-theoretic approach to identify the key players in a terrorist network. They considered four different weighted extensions of the *connectivity game* (Amer and Gimenez, 2004) to capture the structure of the terrorist organization as well as additional individual information about the terrorists, and then they proposed to calculate the Shapley value of each game in order to identify the key players. In Section 4, where we analyze in detail this application, we recover the formal definitions of those games.

(ii) In Narayanam and Narahari (2011), the authors also introduce a game-theoretic approach to address the *target set selection problem* in the framework of diffusion of information. They consider the *linear threshold model* (Schelling (1971), Granovetter (1978), and Kempe, Kleinberg and Tardos (2005)) to model the role of the social structure in the sharing of information and the formation of opinions dynamics. The propagation process in this case is characterized by a set of agents N, which are connected through a social network. The flow of available information is captured by a weight matrix $\boldsymbol{W}$ whose entries are understood as *influence weights*; in particular, w_{ij} quantifies the weight that agent i assigns to agent j. It is assumed that these weights are normalized in such a way that $\sum_{j \in N_i} w_{ij} \leq 1$, where N_i represents the set of neighbors of agent i, for all $i \in N$. In this model, it is also assumed that each agent has two possible states: *active*, if he has adopted the information that is being propagated, and *inactive* otherwise. From a dynamic point of view, it is assumed that the status of the agents may change as time goes by. At each date, agents communicate with their neighbors in the social network and update their state. The updating process is simple: all agents that were active in step $(t-1)$ remain active at step t; and every inactive agent at step $(t-1)$ becomes active if the sum of the weights of his active neighbors' from the previous period is at least θ_i, a threshold which represents the weighted fraction of the neighbors of i that must become active in order to activate agent i.

The authors define a TU game (N, v), where the value of coalition S is defined to be the expected number of active nodes at the end of the diffusion process when initially all agents in S are active, whereas all agents in $N \setminus S$ are inactive, and assuming all thresholds θ_i are chosen uniformly at random from the interval $[0, 1]$ initially. Then, they propose to calculate the Shapley value of the game in order to rank the agents. Taking into account that the k agents with highest Shapley value are not in general the optimal set of k agents, they propose an heuristic procedure, based on the Shapley value of each agent and the social network structure, to select the *key set* of k agents.

(iii) In Conklin, Powaga and Lipovetski (2004) the authors introduce a game-theoretic approach for the identification of sets of key drivers in customer satisfaction analysis, on the basis of Kano's theory (Kano *et al.*, 1984) of the relationship between product quality and customer satisfaction, and the information given by a random sample of customers. The attributes take the role of the players, and the characteristic function of the game measures the ability of every group of attributes to predict the dissatisfaction level of customers. To be specific,

$$v(S) = \underbrace{P\{\sum_{i \in S} M_i > 0 \;/\; D = 1\}}_{reach} - \underbrace{P\{\sum_{i \in S} M_i > 0 \;/\; D = 0\}}_{noise},$$

where $M_i \in \{0,1\}$ takes the value 1 if attribute i has failed, and 0 otherwise; and $D \in \{0,1\}$ reflects the overall dissatisfaction ($D = 1$) with the product. The above probabilities are estimated through the opinions of a random sample of customers as proportions of the failed within those who are dissatisfied (*reach*) and non-dissatisfied (*noise*). Then, the authors propose to use the Shapley value for ordering the attributes. Then, they add attributes to the list of key dissatisfiers following that order until a point where the added noise overwhelms the added reach. This problem is in fact a particular instance of a multicriteria decision problem, in which the aim is to rank or score alternatives according to several (often conflicting) points of view, called criteria. This is done on the basis of the information given by the decision maker, who essentially gives his preference over a given set of typical alternatives. For a more general application of the Shapley value to multicriteria problems and subjective evaluation the reader is referred to Grabisch *et al.* (2002) and Grabisch *et al.* (2008).

Note that in the examples considered above, there exists in fact an *external decision maker* who is interested in finding an optimal *group* of agents, rather that an optimal agent:

(i) In that case, the police wants to identify a small group of agents to neutralize in order to break up the criminal organization. Or, it could be the case, that they were interested in selecting a small group of agents to mislead in order to optimally diffuse their own information through the network (by using them as seed). In any case, the objective is to find a key *group* of terrorists.

(ii) In the *target set selection problem*, the goal is to find a set of k key agents that would maximize the spreading of information through the network.

(iii) In Conklin, Powaga and Lipovetski (2004) the authors' goal is the identification of *sets* of key drivers in customer satisfaction analysis taking into account Kano's theory about the relationship between product quality and customer satisfaction.

Remark that a direct use of the characteristic function $v(\cdot)$ to measure the value of a group is not in general the best approach to solve this problem. Let us think for instance in the second example. In that case, to maximize $v(\cdot)$ implies a pessimistic scenario in which none of the agents out of coalition S whose diffusion power is being evaluated adopts the product spontaneously. The same argument remains valid for the other two situations considered above. Thus, since measuring the value of a group is a relevant question, and taking into account that the k more valuable agents (from the individual point of view) do not form in general the most valuable group of k agents, the need for a specific group valuation is clear. We propose to extend the Shapley value to deal with groups in order to account for all possible scenarios. The group value we propose serves the decision maker as a valuation of the average performance of each group.

3 The Shapley group value: definition and axiomatic characterization

In this framework, we give the following definition of a *group value*.

Definition 1. Let $\mathcal{G}_n$ be the set of TU games with fixed player set N. A *group value* is a mapping that assigns for every $N \subset \mathbb{N}$ and every $v \in \mathcal{G}_n$ a *valuation vector* $\xi^g(v) \in \mathbb{R}^{2^n}$, where the real number $\xi^g_C(v)$ represents the *a priori value* of group $C \subseteq N$ in v, and such that $\xi^g_\emptyset(v) = 0$

In order to define our proposal, we must consider what does group integration mean for the application we have in mind. In this framework, group integration does not necessarily implies that agents in C make an agreement to act jointly. For instance, going back to the diffusion of information case, there exists a external agent who can activate the nodes to be used as seeds to diffuse the innovation through the network, and the activated nodes are not in general aware about the other selected seeds' identities. The same occurs when the police selects a group of terrorists to turn back into double agents, or to misinform in order to spread their misinformation through the criminal organization network. Therefore, when evaluating group C's expectations we will evaluate them like a *unit* anyway, and we will adopt the *merging of players* approach of Derks and Tijs (2000). In that case all the agents of C are replaced by a single "*super–player*" c, who can act as a proxy of any agent in C. Formally,

Definition 2. (Derks and Tijs, 2000) Let $(N, v) \in \mathcal{G}_n$, and let $C \subseteq N$ be any non-trivial coalition, $1 < |C| < n$. Then, in the *merging game* with respect to C, (N_C, v_C), the agent set N will become $N_C = (N \setminus C) \cup \{c\}$, and the characteristic function describing the new situation will be given by

$$v_C(S) = \begin{cases} v(S), & \text{if } c \notin S, \\ v(S \cup C) & \text{if } c \in S, \end{cases} \quad \forall\, S \subseteq N_C. \tag{2}$$

The previous merging game is also called the *quotient game* (Owen, 1977) of (N, v) with respect to the *coalition structure*, or in other terms to the *system of a priori unions*, on $N \subset \mathbb{N}$ given by the partition $\mathcal{P}_C = \{C, \{j\}, j \notin C\}$ of N. We trivially extend the previous definition to comprise the extreme cases $C = \{i\}$, $i \in N$, and $C = N$.

Definition 3. We define the *Shapley group value* to be the group value that assigns for every $N \subset \mathbb{N}$ and every $v \in \mathcal{G}_n$ the valuation vector $(\phi^g_C(N, v))_{C \subseteq N}$ given by:

$$\phi^g_C(N, v) = \phi_c(N_C, v_C), \text{ for all coalition } \emptyset \neq C \subseteq N \text{ and } \phi^g_\emptyset(N, v) = 0,$$

where (N_C, v_C) is the merging game with respect to C.

Our goal will be to find an axiomatic characterization of the defined Shapley group value. We first propose and analyze some interesting properties of a group value, which try to extend to this setting the main properties involved in the many axiomatic approaches which have been provided for the Shapley value.

Properties

Let ξ^g be a group value defined over $\bigcup\limits_{n \geq 1} \mathcal{G}_n$, and let (N, v) be any game in $\mathcal{G}_n$, with $n \geq 1$. Then, ξ^g verifies:

(i) *G-structural equivalence*, if $\sum_{i \in N} \xi_i^g(N,v) = \xi_N^g(N,v)$;

(ii) *G-dummy player*, if $\xi_{C \cup i}^g = \xi_C^g + v(i)$, for any group $C \subseteq N$ with $i \notin C$, whenever i is a dummy player for N (i.e., $v(S \cup i) = v(S) + v(i)$, for all $S \subseteq N$);

(iii) *G-null player*, if $\xi_{C \cup i}^g = \xi_C^g$, for any group $C \subseteq N$ with $i \notin C$, whenever i is a null player for N (i.e., $v(S \cup i) = v(S)$, for all $S \subseteq N$);

(iv) *G-anonymity*, if for all $C \subseteq N$, and for all permutations $\pi \in \Pi_n$ of N, $\xi_{\pi(C)}^g(\pi(N), \pi v) = \xi_C^g(N,v)$, where $\pi v(S) := v(\pi(S))$, and being $\pi(S) = \{\pi(i) \ / \ i \in S\}$;

(v) *G-additivity*, if $\xi_C^g(N, v+w) = \xi_C^g(N,v) + \xi_C^g(N,w)$, for every different games v and w defined over a set of players N and for every $C \subseteq N$, where $v+w$ is given by $(v+w)(S) = v(S) + w(S)$, for all $S \subseteq N$;

(vi) *G-coalitional balanced contributions* (or G-CBC for short), if for every two different players i and j in N and any group C such that $i, j \notin C$, we have

$$(\xi_{C \cup i}^g(N,v) - \xi_C^g(N,v)) - (\xi_{C \cup i}^g(N \backslash j, v_{-j}) - \xi_C^g(N \backslash j, v_{-j})) =$$
$$(\xi_{C \cup j}^g(N,v) - \xi_C^g(N,v)) - (\xi_{C \cup j}^g(N \backslash i, v_{-i}) - \xi_C^g(N \backslash i, v_{-i})), \quad (3)$$

where v_{-i} stands for the restriction of the characteristic function v to the set of players $N \backslash i$;

(vii) *G-symmetry over pure bargaining games* (or G-SPB for short), if $\xi_C^g(N, u_N) = \frac{1}{n-c+1}$, for every group $C \subset N$ such that $1 \leq |C| = c \leq n = |N|$, and for all $n \geq 1$, where (N, u_N) is the unanimity game with respect to the grand coalition;

$(viii)$ *G-coalitional monotonicity*, if $\xi_C^g(N,v) \leq \xi_C^g(N,w)$, for every group $C \subseteq T$, and for all games $(N,v), (N,w) \in G_n$ such that $v(S) = w(S)$, for all $S \neq T$, and $v(T) < w(T)$;

(ix) *G-strong monotonicity*, if $\xi_C^g(N,v) \leq \xi_C^g(N,w)$, for every group $C \subseteq N$, and for all games $(N,v), (N,w) \in G_n$ for which $v(S \cup C) - v(S) \leq w(S \cup C) - w(S)$, for all $S \subseteq N \setminus C$.

(x) *G-positivity*, if $\xi_C^g(N,v) \geq 0$, for all $C \subseteq N$, whenever the game (N,v) is *monotonic* (i.e., $v(T) \geq v(S)$, for each T and S such that $T \supseteq S$).

(xi) *G-relative invariance with respect to strategic equivalence*, if $\xi_C^g(N,w) = a\xi_C^g(N,v) + \sum_{i \in S} b_i$, for every $(N,v) \in G_n$, $a > 0$ and $b \in \mathbb{R}^n$, where w is given by $w(S) = av(S) + \sum_{i \in S} b_i$, for all $S \subseteq N$;

(xii) *G-coalitional strategic equivalence*, if $\xi_C^g(N,v) = \xi_C^g(N, v + \lambda u_T)$, for every $(N,v) \in G_n$, $\lambda \in \mathbb{R}$, $C \subseteq N$ and $\emptyset \neq T \subseteq N \setminus C$;

$(xiii)$ *G-fair ranking*, if for every pair of games (N,v), (N,w) such that $v(S) = w(S)$, for all $S \neq T$, $\xi_{C_1}^g(N,v) > \xi_{C_2}^g(N,v)$ implies $\xi_{C_1}^g(N,w) > \xi_{C_2}^g(N,w)$, for all $C_1, C_2 \subseteq T$ with $|C_1| = |C_2|$;

G-structural equivalence adapts *coalitional structure equivalence* (Albizuri, 2008; see also Hart and Kurz, 1983) for this setting. Coalitional structure equivalence is a property of values for games with coalition structures that requires the solution to give the same results both when all the players are joined and when nobody is joined.

G-dummy, G-null, G-anonymity, G-additivity, and G-relative invariance with respect to strategic equivalence generalize their counterparts for individual values for general TU games. Note that additivity of the group value may be replaced by the following property, which is consistent with the interpretation of the group value as the expected utility of playing a game: "the group value of a probabilistic mixture of two games (i.e., with probability p the game v is played, and with probability $p' = 1 - p$, the game w is played) is the same mixture of the group values of the two games" (Hart and Kurz, 1983).

G-coalitional strategic equivalence, which generalizes G-relative invariance with respect to strategic equivalence to coalitions $T \subseteq N$ with $|T| \geq 2$, adapts the same property of individual values, introduced by Chun (1989) to characterize the Shapley value by adding triviality (the value of a null game is the null vector), efficiency and fair ranking -also defined by Chun (1989), and that can be generalized to G-fair ranking-.

G-coalitional monotonicity and G-strong monotonicity generalize the corresponding monotonicity properties considered by Young (1985), who characterizes the Shapley value by means of efficiency, symmetry and strong monotonicity. Moreover, G-positivity extends positivity, introduced by Kalai and Samet (1987).

G-coalitional balanced contributions, which together with G-symmetry over pure bargaining games, plays a crucial role in the axiomatic characterization of the Shapley group value, generalizes the balanced contribution property which adding efficiency characterizes the Shapley value (Myerson, 1977). G-CBC states that the impact of player j's presence over the interest of group $C \subseteq N \setminus \{i, j\}$ to rely on player i equals the impact of player i's presence over the interest of group the same group C to rely on player j, when those interests are measured by the corresponding marginal increments.

G-SPB can be seen as a restriction[2] of the *neutrality to strategic risk* property considered by Roth (1977) to characterize the Shapley value as a utility function. Formally, let $\mathcal{R}$ be a preference relation of a decision maker over positions and games. That is, a total preorder on $N \times \mathcal{G}_n$, where an element (i, v) represents a position (the position i) in the game (N, v), that the decision maker is evaluating (and comparing with other positions and other games). In this setting, Roth distinguishes between two kinds of risks: *ordinary risk*, involving the uncertainty which arises from the chance mechanism involved in lotteries; and *strategic risk*, that involves the uncertainty which arises from the interaction in a game of the strategic players.[3] Roth (1977) states that a preference relation $\succeq$ is *neutral to strategic risk* if $(i, u_S) \sim (i, \frac{1}{s}u_{\{i\}})$, for all unanimity games u_S (seen as pure bargaining games). That is, the *certainty equivalent* of playing u_S in position i should be to receive $\frac{1}{s}$ for sure.

Proposition 1. *The previous properties, (i) to $(xiii)$, hold for ϕ^g.*

Proof. Observe that the individual value defined by ϕ^g over the one-person coalitions is the Shapley value, so property (i), G-structural equivalence, follows from the Shapley value efficiency,

[2] In the sense that it is only imposed for pure bargaining games over the grand coalition N.

[3] Roth (1977) only considers as strategic players those who are not null.

taking into account that $\phi^g_N(N,v) = v(N)$.

We check conditions (ii) to (v), and $(viii)$ to $(xiii)$, using the following common argument: taking into account that ϕ^g_C is the Shapley value of the superplayer $c \in N_C$ in (N_C, v_C), to analyze the implications of the considered property over the merging game and, then, to rely on the individual Shapley value properties. For instance, the G-null player property is clear because $\phi^g_{C\cup i}$ is the Shapley value of the superplayer (c,i) in $(N_{C\cup i}, v_{C\cup i})$, and this is equal to the Shapley value of the superplayer c in (N_C, v_C) as the marginal contributions of i are zero.

With respect to property (vi), G-CBC, first let us remark that condition (3) is equivalent to:

$$\begin{aligned}\xi^g_{C\cup i}(N,v) - \xi^g_{C\cup j}(N,v) = \\ (\xi^g_{C\cup i}(N\backslash j, v_{-j}) - \xi^g_C(N\backslash j, v_{-j})) - (\xi^g_{C\cup j}(N\backslash i, v_{-i}) - \xi^g_C(N\backslash i, v_{-i})).\end{aligned} \quad (4)$$

Also note that, by definition of the quotient game and the Shapley value, we obtain in this case the following equalities:

$$\begin{aligned}\phi^g_{C\cup i}(N,v) = \sum_{\substack{S\subseteq N\backslash C \\ i,j\notin S}} \Big(\frac{s!(n-c-s-1)!}{(n-c)!} (v(S\cup C\cup i) - v(S)) + \\ \frac{(s+1)!(n-c-s-2)!}{(n-c)!} (v(S\cup C\cup i\cup j) - v(S\cup j)) \Big),\end{aligned}$$

$$\begin{aligned}\phi^g_C(N\backslash j, v_{-j}) = \sum_{\substack{S\subseteq N\backslash C \\ i,j\notin S}} \Big(\frac{s!(n-c-s-1)!}{(n-c)!} (v(S\cup C) - v(S)) + \\ \frac{(s+1)!(n-c-s-2)!}{(n-c)!} (v(S\cup C\cup i) - v(S\cup i)) \Big),\end{aligned}$$

$$\phi^g_{C\cup j}(N\backslash i, v_{-i}) = \sum_{\substack{S\subseteq N\backslash C \\ i,j\notin S}} \frac{s!(n-c-s-2)!}{(n-c-1)!} (v(S\cup C\cup j) - v(S)),$$

where the cardinals of N, C and S are respectively designed by n, c and s. Analogous expressions are obtained for $\phi^g_{C\cup j}(N,v)$, $\phi^g_C(N\setminus i, v_{-i})$ and $\phi^g_{C\cup i}(N\setminus j, v_{-j})$. Now it is enough to check that for every $S \in N\backslash$ with $i,j \notin S$ the coefficients of $v(S\cup C\cup i\cup j)$, $v(S\cup C\cup i)$, $v(S\cup C\cup j)$, $v(S\cup C)$, $v(S\cup i)$, $v(S\cup j)$ and $v(S)$ are the same in both sides of the equation in (4), and this is easily deduced from the previous expressions. We leave the details to the reader.

It remains to check (vii), G-SPB. Consider the unanimity game with respect to the grand coalition (N, u_N), and a group C such that $1 \leq c \leq n$. It is straightforward to see that the quotient game $(N_C, (u_N)_C)$ is the unanimity game (N_C, u_{N_C}) with respect to the grand coalition N_C, so $\phi^g_C(N, u_N) = \phi_c(N_C, (u_N)_C) = \frac{1}{n-c+1}$, as desired. □

Next, we give our characterization of the Shapley group value. It must be remarked that it is not a trivial extension of a characterization of the Shapley value, since we do not impose any condition about the value of the individual agents out of group S when that group forms. We have been forced to use in the same characterization group additivity and group coalitional balanced contributions axioms. Thus, we should carefully check that all the considered axioms are necessary to guarantee the uniqueness of the group value ϕ^g (see the Appendix).

Theorem 1. *The unique group value over* $\bigcup_{n\geq 1} \mathcal{G}_n$ *verifying G-null player, G-additivity, G-CBC, and G-SPB, is the Shapley group value* ϕ^g.

Proof. We have proved that the axioms hold for the Shapley group value, so we are left with the question of uniqueness.

First, we will prove uniqueness for all the elements of the basis[4] $\{(N, u_S)\}_{\substack{S\subseteq N\\ S\neq\emptyset}}$, for all $n \geq 2$. The proof will consist in a double induction over the cardinality of the player set N and the cardinality of the unanimous coalition $S \subseteq N$.

Let ξ^g be a group value verifying $(ii), (iii), (iv)$ and (v). Then, we will check that

$$\xi^g_C(N, u_S) = \phi^g_C(N, u_S) := \phi_c(N_C, (u_S)_C) = \begin{cases} \dfrac{1}{s - |S \cap C| + 1}, & \text{if } S \cap C \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases},$$

for all $C \subseteq N$, for all $S \subseteq N$, and for all $n = |N| \geq 2$.

Let us prove that ξ^g coincides with ϕ^g for two-person games. For the unanimity game $(\{i\}, u_i)$ with just one player, G-symmetry condition (v) implies that $\xi^g_i(N, u_i) = 1$. For the trivial game $(\{i\}, u^0)$, whit $u^0(i) = 0$, G-null player implies $\xi^g_i(N, u_i) = 0$. Thus, let (N, v) be a two-person game with $N = \{i, j\}$. For the unanimity game with respect to N, G-symmetry condition (v) implies that $\xi^g_{ij}(N, u_N) = \phi^g_{ij}(N, u_N)$. For the unanimity game (N, u_i), G-null player implies

$$\xi^g_j(N, u_i) = \xi^g_\emptyset(N, u_i) := 0 \text{ and } \xi^g_i(N, u_i) = \xi^g_{ij}(N, u_i) \tag{5}$$

Now, G–CBC condition implies

$$\xi^g_i(N, u_i) - \xi^g_j(N, u_i) = (\xi^g_i(\{i\}, u_i) - \xi^g_\emptyset(\{i\}, u_i) - (\xi^g_j(\{j\}, u^0) - \xi^g_\emptyset(\{j\}, u^0)) = 1 - 0 - 0 + 0,$$

and, therefore, taking into account (5), $\xi^g(\{i, j\}, u_i) = \phi^g(\{i, j\}, u_i)$ holds.

So we may assume by induction that for every unanimity game (N, u_S) with $|N| < r$ we have $\xi^g_C(N, u_S) = \phi^g_C(N, u_S)$ for any group C in N.

Consider then a unanimity game (N, u_S) with $|N| = r > 2$. If $S = N$, then G-SPB implies $\xi^g_C(N, u_N) = \phi^g_C(N, u_N)$ for any group C in N. Consider now the case $S \subsetneq N$, in which there

[4] Recall that a game (N, v) is an *unanimity game* if there exists a coalition S such that for every $T \subseteq N$, $v(T) = 1$ if $S \subseteq T$, and $v(T) = 0$ otherwise; in this case, the game is usually denoted by (N, u_S). Unanimity games are a basis of the vector space of the TU-games.

is at least a player $j \in N\backslash S$ which by definition of unanimity game must be null. Let i be a player in S. Again by G-CBC, taking $C = \emptyset$, since $\xi^g_j(N, u_S) = \xi^g_\emptyset(N, u_S) = 0$, we obtain that $\xi^g_i(N, u_S) = \xi^g_i(N\backslash j, u_S|_{N\backslash j})$, which in turn is equal to $\Phi_i(N\backslash j, u_S|_{N\backslash j}) = \frac{1}{s}$ by induction, and then to $\Phi_i(N, u_S)$. So we are done with the individual case $C = \{i\}$ for $|N| = r$, for any coalition $S \subseteq N$.

Now, let $S \subseteq N$ with $1 \leq s < n$. Then, in order to prove

$$\xi^g_C(N, u_S) = \phi^g_C(N, u_S), \text{for all group } C \text{ with } c = |C| > 1.$$

we proceed by induction on the cardinality of C. So we take now $1 < r' \leq r$, and we may assume that $\xi^g_C(N, u_S) = \phi^g_C(N, u_S)$ if $|N| = r$ and $|C| < r'$. We will check that $\xi^g_D(N, u_S) = \phi^g_D(N, u_S)$ for all D with $|D| = r'$. Since $S \subsetneq N$, again there is a null player j in N. So, if $j \in D$, then $D \setminus j$ is a coalition of cardinal $r' - 1$ and, thus, G-null player and the second induction hypothesis imply

$$\xi^g_D(N, u_S) = \xi^g_{D\backslash j}(N, u_S) = \phi^g_{D\backslash j}(N, u_S) = \phi^g_D(N, u_S).$$

Otherwise, if D does not contain any null player, then let i be a player in $D \cap S$. By the second induction hypothesis and G-null player property it holds

$$\xi^g_{(D\backslash i)\cup j}(N, u_S) = \xi^g_{(D\backslash i)}(N, u_S) = \phi^g_{(D\backslash i)}(N, u_S) = \phi^g_{(D\backslash i)\cup j}(N, u_S).$$

Hence, by G-CBC, taking $C = D\backslash i$, and the first induction,

$$\begin{aligned}\xi^g_D(N, u_S) - \phi^g_{(D\backslash i)\cup j}(N, u_S) &= (\phi^g_D(N\backslash j, u_S|_{N\backslash j}) - \phi^g_{D\backslash i}(N\backslash j, u_S|_{N\backslash j}))- \\ &(\phi^g_{(D\backslash i)\cup j}(N\backslash i, u_S|_{N\backslash i}) - \phi^g_{D\backslash i}(N\backslash i, u_S|_{N\backslash i})) = \phi^g_D(N, u_S) - \phi^g_{(D\backslash i)\cup j}(N, u_S),\end{aligned}$$

and we are done.

So we have proved the uniqueness for the unanimity games. As these games are a base of $\mathcal{G}_n$, the additivity axiom guarantees such uniqueness for all the games in $\mathcal{G}_n$. So we are done. □

Now, we will analyze the behavior of the Shapley group value as a set function. Since one of its main purposes is to serve as an objective function in order to find an optimal group, the *individual marginal contributions to a given group* deserve a careful study.

Proposition 2. *Let ϕ^g be the Shapley group value, and let (N, v) be any game in $\mathcal{G}_n$, with $n \geq 1$. Then,*

(i) Monotonicity*: $\phi^g_C(N, v) \leq \phi^g_D(N, v)$, for every pair of coalitions $C \subseteq D$, if the game (N, v) is monotonic, and*

(ii) Group Rationality*: $\phi^g_C(N, v) \geq v(C)$, for every $C \subseteq N$, if the game (N, v) is superadditive.*

Proof. Monotonicity follows from being

$$\phi^g_{C\cup i}(N,v) - \phi^g_C(N,v) = \sum_{\substack{S\subseteq N\setminus C\\ i\notin S}} \frac{s!(n-c-s)!}{(n-c+1)!}\Big(v(S\cup i\cup C) - v(S\cup C)\Big) + \frac{(s+1)!(n-c-s-1)!}{(n-c+1)!}\Big(v(S\cup i) - v(S)\Big) \geq 0, \quad (6)$$

for all coalitions $C \subseteq N$, and all players $i \notin C$, whenever the game (N,v) is monotonic.

Group rationality follows from the individual rationality of the Shapley value. Note that every quotient game (N_C, v_C), $C \subseteq N$, is superadditive if it is (N,v). □

Superadditivity, i.e., $\phi^g_C(N,v) + \phi^g_D(N,v) \leq \phi^g_{D\cup C}(N,v)$, for every pair of disjoint coalitions $C \cap D = \emptyset$, can not be assured in general. In fact, when restricted to one person coalitions, superadditivity is called *mergeability* by Derks and Tijs (2000). In that case, taking into account Theorem 2 (Derks and Tijs, 2000), it follows that group $C \subseteq N$ is *mergeable* in a game (N,v), i.e.,

$$\sum_{i\in C} \phi^g_i(N,v) \leq \phi^g_C(N,v),$$

whenever all coalitions with positive Harsanyi dividend are either contained in C or have at most one player in common with C. Derks and Tijs propose some interesting types of games for which every coalition is mergeable, or mergeability can be guaranteed for certain kinds of coalitions.

As we have announced before, the k more valuable players are not in general the most valuable group of k players, since individual mesures do not take into account the presence of *complementarities* and *substitutabilities* among the players. Proposition 4 shows how the Shapley group value incorporates those relations among the players when evaluating the value of a group. Before proving that result, we recall some concepts which will be used in the proposition.

Grabisch and Roubens (1999) introduce the notion of *interaction indexes*. Formally:

Definition 4 (Grabisch and Roubens, 1999)**.** An *interaction index* I^v of the game $(N,v) \in \mathcal{G}_n$ is a real valued function on $\mathcal{P}(N)$.

Here, $I^v(S)$ should be interpreted as a measure of the extent of the profitability of the cooperation among the members of $S \subseteq N$. In particular, they obtain the following result, which relays on the *second-order difference operator* for a pair of players $i, j \in N$ considered by Segal (2003), which is defined as a composition of marginal contribution operators (i.e., first-order difference operators) as follows:

$$\Delta^2_{ij}(S,N,v) = v(S\cup\{i,j\}) - v(S\cup j) - v(S\cup i) + v(S) = \Delta^2_{ji}(S,N,v), \quad \forall\, S \subseteq N\setminus\{i,j\},$$

and it is interpreted by Segal as a measure of complementarity of players i and j with respect to the players in S.

Proposition 3 (Grabisch and Roubens, 1999)**.** *Let I^v be an interaction index. If it verifies:*

(LI) Linearity*: I^v is linear on G_n.*

(D) Dummy*: If $i \in N$ is a dummy player (i.e., $v(S \cup i) = v(S) + v(i)$), then for every $S \subseteq N \setminus \{i\}$, $S \neq \emptyset$, $I^v(S \cup \{i\}) = 0$, and $I^v(i) = v(i)$.*

(S) Symmetry*: If $I^v(S) = I^{\pi v}(\pi S)$, for all $S \subseteq N$, and for all order $\pi \in \Pi(N)$; where the game πv is defined by $\pi v(\pi S) = v(S)$, and being $\pi S = \{\pi(i),\, i \in S\}$.*

Then, there exist real constants $p_s(n)$, $s = 0, 1, \ldots, n-2$, such that

$$I^v(\{i,j\}) = \sum_{S \subseteq N \setminus \{i,j\}} p_s(n) \Delta^2_{ij}(S, N, v), \text{ for all } i, j \in N.$$

Specifically, we use a linear, dummy and symmetric interaction index defined as

$$\psi_{ij}(N, v) := \sum_{S \subseteq N \setminus \{i,j\}} \frac{s!(n-s-1)!}{n!} \Delta^2_{ij}(S, N, v), \quad \text{for all } i \neq j \in N, \tag{7}$$

applied to the pair of players $i \neq c \in N_C$ of the game (N_C, v_C). Here, $\psi_{ci}(N_C, v_C)$ measures the extent of the profitability of the cooperation among the members of $\{i, c\} \subseteq N_C$. The concept of interaction index for two players was implicitly first considered by Owen (1972) who defined the *co-value* $q_{ij}(v)$ of i and j. The interaction index $\psi_{ij}(N, v)$ differs from Owen's co-value in the number of orders in which $\Delta^2_{ij}(S)$ is considered. Owen's co-value takes into account only those orders in which i arrives in the first place, then S players arrive, and then j arrives.

Now we are ready to present the desired result:

Proposition 4. *Let ϕ^g be the Shapley group value, and let (N, v) be any game in $\mathcal{G}_n$, with $n \geq 1$. Let $C \subseteq N$ be any group in N, and let $i \notin C$. Then, the* marginal contribution *of player $i \in N \setminus C$ to the group value of C equals:*

$$MC^g_i(C, N, v) := \phi^g_{C \cup i}(N, v) - \phi^g_C(N, v) = \phi_i(N \setminus C, v|_{N \setminus C}) + \psi_{ci}(N_C, v_C). \tag{8}$$

Proof. Adding and subtracting the amount

$$\sum_{\substack{S \subseteq N \setminus C \\ i \notin S}} \frac{s!(n-c-s-1)!}{(n-c)!} \Big(v(S \cup i) - v(S) \Big)$$

to the above expression (6) of the marginal contribution $MC_i^g(C,N,v)$ follows that:

$$MC_i^g(C,N,v) = \sum_{\substack{S\subseteq N\setminus C\\ i\notin S}} \frac{s!(n-c-s-1)!}{(n-c)!}\Big(v(S\cup i)-v(S)\Big)+$$
$$\sum_{\substack{S\subseteq N\setminus C\\ i\notin S}} \frac{s!(n-c-s)!}{(n-c+1)!}\Big(v(S\cup i\cup C)-v(S\cup C)-v(S\cup i)+v(S)\Big).$$

The first term is precisely the Shapley value of player i in the restricted game $(N\setminus C, v|_{N\setminus C})$, where players in C do not play a role. The second term can be expressed by means of the *second-order difference operators* for the pair of players $i,c\in N_C$, as follows:

$$\sum_{\substack{S\subseteq N\setminus C\\ i\notin S}} \frac{s!(n-c-s)!}{(n-c+1)!}\Big(v(S\cup i\cup C)-v(S\cup C)-v(S\cup i)+v(S)\Big) =$$
$$\sum_{\substack{S\subseteq N\setminus C\\ i\notin S}} \frac{s!(n-c-s)!}{(n-c+1)!}\Delta_{ic}^2(S,N_C,v_C) =: \psi_{ci}(N_C,v_C).$$

□

Let us illustrate the previous result by means of two simple examples in which the two more valuable players do not form the most valuable group of two players.

Example 1. Let us consider the following 4-person game (N,v), with $v(1)=100$, $v(2)=v(3)=v(4)=0$, $v(N)=150$, and intermediate coalitions' values:

S	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$
$v(S)$	100	110	120	0	0	0	140	150	120	0

The above example is a slight modification of a horse market game in which Player 1 (the seller) has a horse which has a value of 100 for him. Players 3 and 4 (the buyers) value the horse at 150 and 140, respectively, in case of a swift transaction. Otherwise, if the transaction is delayed, the horse's value decrease to 120 and 110 for each player, respectively. Player 2 acts as an intermediary who is able to accelerate the transaction procedure.

In this case, the two more valuable players, according to their individual Shapley values are the seller (player 1) and the intermediary (player 2): $\phi(v)=(124\frac{1}{6},12\frac{1}{2},4\frac{1}{6},9\frac{1}{6})$. However, the most valuable group of two agents is the one composed by the seller and the buyer who values more the horse. In fact,

$$\phi_{\{1,2\}}^g(v) = 131\frac{2}{3} < 124\frac{1}{6}+12\frac{1}{2} = \phi_1(v)+\phi_2(v)$$

$$\phi_{\{1,4\}}^g(v) = 135 > 124\frac{1}{6}+9\frac{1}{6} = \phi_1(v)+\phi_4(v),$$

Players 1 and 4 are strong complementary players. Players 1 and 2 are however less complementary:

$$MC_2^g(\{1\}, N, v) = \phi_2(N \setminus \{1\}, v|_{N\setminus 1}) + \psi_{12}(N, v) = 0 + 7\frac{1}{2},$$

$$MC_4^g(\{1\}, N, v) = \phi_4(N \setminus \{1\}, v|_{N\setminus 1}) + \psi_{14}(N, v) = 0 + 10\frac{5}{6}.$$

Now, let us consider the following social network represented in Figure 1 as an *undirected graph* (N, Γ), and the *connectivity game* (Amer and Gimenez, 2004), which is defined as

$$v(S) = \begin{cases} 1, & \text{if } S \text{ is connected in } \Gamma \text{ and } |S| > 1, \\ 0, & \text{otherwise,} \end{cases}, \quad \text{for all } S \subseteq N.$$

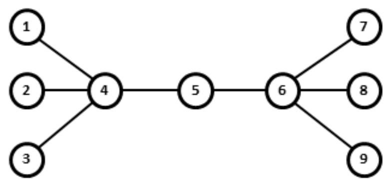

Figure 1: Social Network (N, Γ)

Here, $S \subseteq N$ is a *connected* coalition in (N, Γ) if for every two players $i \neq j$ in S, $\{i, j\} \in \Gamma$, or there exists a *path* between them which consists of nodes in S. That is, there exists a a sequence of nodes and edges $\pi(i, j) = \{i = i_1, i_2, \ldots, i_{k-1}, i_k = j\}$, with $k \geq 2$ satisfying the property that for all $1 \leq r \leq k - 1$, $\{i_r, i_{r+1}\} \in \Gamma$, and $i_r \in S$, for all $2 \leq r \leq k - 1$.

In that case, the two more valuable players, according to their individual Shapley values are the two centers of the satellite stars, players 4 and 6. $\phi_i(v) = -\frac{8}{360}$, for all the leafs $i = 1, 2, 3, 7, 8, 9$, $\phi_4(v) = \phi_6(v) = \frac{139}{360}$, for the two centers, and $\phi_5(v) = \frac{130}{360}$ for the hub which intermediates between players 4 and 6. However, the most valuable group of two agents is the one composed by the hub and one out of the two centers. In fact,

$$\phi^g_{\{4,6\}}(v) = \frac{1}{2} < \frac{1}{2} + \frac{1}{21} = \phi^g_{\{4,5\}}(v).$$

4 Application: Detecting a target group in terrorist networks

We will illustrate now the application of the Shapley group value to the two terrorist networks which have been considered by Lindelauf *at al.* (2013): the operational network of Jemaah Islamiyah's Bali bombing and the network of hijackers of Al Qaeda's 9/11 attack.

For the first case, Jemaah Islamiyah's Bali bombing attack, the authors use the game (N, v^{wconn}). Let (N, Γ) be the *undirected graph* which represents the terrorist network. The nodes in the finite set $N = \{1, \ldots, n\}$ are the terrorists, whereas the *edges* -i.e., unordered pairs of distinct nodesrepresent the known relationships between the terrorists. In Figure 2 is represented the terrorist network they work with.

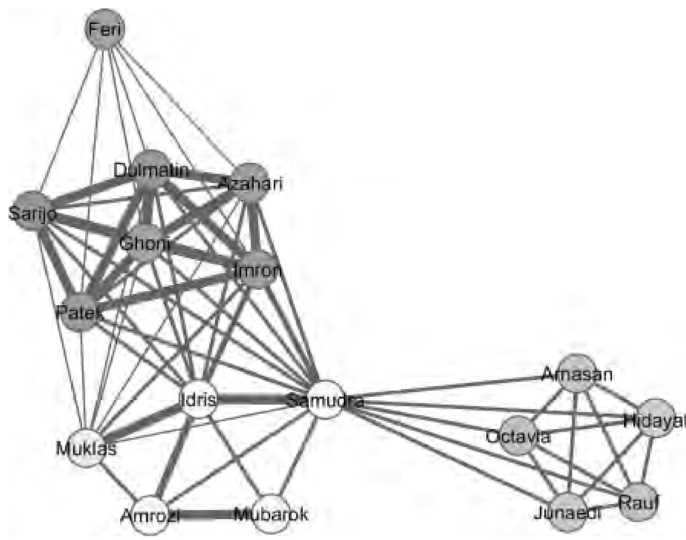

Figure 2: Operational network of JI's Bali attack. Image taken from Lindelauf *et al.* (2013)

Then, Lindelauf *et al.* (2013) define the game (N, v^{wconn}), which extends the connectivity game of Amer and Gimenez (2004) using information about relationships. In that game, a coalition must be connected in order to achieve a non-zero value. That is, players in coalition S must rely only upon their own connections in order to communicate among themselves. Then, since a terrorist cell tries to prevent discovery during the planning and execution phase of an attack, and taking into account the available data about the existing relationships[5], the authors define the power of a coalition as the total number of relationships that exist within that coalition divided by the sum of the weights (representing frequency and duration of interaction) on those relationships;

$$v^{wconn}(S) = \begin{cases} \dfrac{\sum_{\substack{i,j \in S \\ i \neq j}} I_{ij}}{\sum_{\substack{i,j \in S \\ i \neq j}} f_{ij}}, & \text{if } S \text{ is connected in } \Gamma \text{ and } |S| > 1, \\ 0, & \text{otherwise,} \end{cases}, \quad \text{for all } S \subseteq N, \tag{9}$$

where f_{ij} is the weight assigned to relation $\{i, j\} \in \Gamma$ in the terrorist network, $I_{ij} = 1$, for all edge $\{i, j\}$ in Γ, and 0 otherwise.

We obtain the following results concerning groups from one to four individuals. Following Castro, Gomez and Tejada (2009), and taking into account that the marginal contributions in the extended connectivity games are computable in polynomial time, we have estimated with Monte Carlo simulation the Shapley group value of the examples, also in polynomial time. The results obtained are represented in Table 1, which includes the records for the best groups arranged in decreasing order of importance.

[5]The authors collected the strength of existing relationships from Koschade (2006).

Individuals	Two agents	Three agents	Four agents
Samudra, 0,358	{Samudra, Muklas}, 0,453	{Samudra, Muklas,Azahari}, 0,442	{Samudra, Muklas,Feri,Azahari}, 0,466
Muklas, 0,048	{Samudra, Azahari}, 0,392	{Samudra, Muklas,Sarijo}, 0,435	{Samudra, Muklas,Feri,Sarijo}, 0,460
Feri, 0,032	{Samudra, Sarijo}, 0,386	{Samudra, Muklas,Patek}, 0,435	{Samudra, Muklas,Feri,Patek}, 0,460
Azahari, 0,012	{Samudra, Patek}, 0,386	{Samudra, Feri,Azahari}, 0,430	{Samudra, Muklas,Feri,Ghoni}, 0,453
Sarijo, 0,005	{Samudra, Rauf}, 0,384	{Samudra, Muklas,Ghoni}, 0,429	{Samudra, Muklas,Azahari,Sarijo}, 0,429

Table 1: Operational network of JI's Bali attack rankings

According to the individual rankings for the JI network based on the Shapley value, the five most valuable terrorists were, in decreasing order of importance: Samudra, Muklas, Feri, Azahari and Sarijo.

With respect of groups of two terrorists, the most valuable group is that composed by the two more important agents, {Samudra, Muklas}. However, the second group of size two in importance is {Samudra, Azahari}, improving the group value of {Samudra, Feri}, which equals 0, 350, and takes the 15th place. In fact, Samudra has all direct contacts has Feri, and therefore Feri's presence in a group is somehow redundant if Samudra is already in it. According to what it is known about the attack, "*Samudra, an engineering graduate, played a key role in the bombings*", whereas Azahari is the bomb expert who was considered the "brain" behind the entire operation.

Again, the most valuable group of three terrorist is {Samudra, Muklas,Azahari}. However, when considering a bigger group of four terrorist, then {Samudra, Muklas, Feri, Azahari} has the highest Shapley group value.

In the analysis of the terrorist network of the 11S, Lindelauf *et al.*'s starting point was the version of the network in Figure 3, whose links come from terrorists that lived or learned together (black edges) as well as some temporary links that were only activated just before the attack in order to coordinate the cells. See Krebs (2002) for further information. The authors use the game (N, v^{wconn2}), which uses information about the individuals:

$$v^{wconn2}(S) = \begin{cases} \sum\limits_{i \in S} w_i, & \text{if } S \text{ is connected in } \Gamma \text{ and } |S| > 1, \\ 0, & \text{otherwise}, \end{cases} \quad , \quad \text{for all } S \subseteq N, \tag{10}$$

where w_i is the weight assigned to terrorist $i \in N$. The authors also determine the terrorist weights in their analysis (see Table 5 in Lindelauf *et al.*, 2013).

The results obtained by means of Monte carlo simulation (see Castro, Gomez and Tejada, 2009) are depicted in Table 2, which includes the records for the best groups arranged in decreasing order of importance.

According to the individual rankings for the Al Qaeda's 9/11 network based on the Shapley value, the most valuable terrorists were, in decreasing order of importance: A. Aziz Al-Omari (WTC North cell), H. Al-Ghamdi (WTC South cell), Wd. Al-Shehri (WTC North cell), H. Hanjour (Pentagon cell), M. Al-Shehhi (WTC South cell) and M. Atta (WTC North cell).

With respect of groups of two terrorists, the most valuable group is that composed by the two more important agents, {A. Aziz Al-Omari, Al-Ghamdi}. However, the second group of

Figure 3: 11S Social Network. Image taken from V.E. Krebs (Copyright ©2002, First Monday)

Individuals	Two agents	Three agents
A. Aziz Al-Omari (WTC-N), 6,096	{Al-Omari, Al-Ghamdi}, 7,405	{Al-Omari, H. Hanjour, M. Al-Sehhi}, 9,236
H. Al-Ghamdi (WTC-S), 5,578	{Al-Omari, M. Al-Sehhi}, 7,392	{Al-Omari, H. Al-Ghamdi, Wd. Al-Shehri}, 9,153
Wd. Al-Shehri (WTC-N), 5,563	{Al-Omari, H. Hanjour}, 7,368	{M. Al-Shehhi, H. Al-Ghamdi, Wd. Al-Shehri}, 9,140
H. Hanjour (Pent), 5,402	{ Aziz Al-Omari, Wd. Al-Shehri}, 7,324	{ H. Al-Ghamdi, Wd. Al-Shehri, H. Hanjour}, 9,074
M. Al-Shehhi (WTC-S), 2,202	{Al-Omari, M. Atta}, 7,156	{ H. Al-Ghamdi, H. Hanjour, N. Al-Hazmi}, 8,986
M. Atta (WTC-North), 1,600	{ H. Al-Ghamdi, H. Hanjour}, 7,044	{ Al-Omari, H. Al-Ghamdi, H. Hanjour}, 8,963

Table 2: 11S-hijackers network rankings

size two in importance is {Aziz Al-Omari, M. Al-Sehhi}, improving the group value of {Aziz Al-Omari, Wd. Al-Shehri}, which takes the 4th place. In fact, Wd. Al-Shehri is one out of the three hijackers that crashed the plane into WTC South which forms a cycle in the terrorist network that is connected to the rest of terrorist only via A. Aziz Al-Omari, who also belongs to the WTC North cell. Thus, Wd. Al-Shehri presence in a group is not so necessary if A. Aziz Al-Omari is already in it.

The most valuable group of three terrorist is {A. Aziz Al-Omari, H. Hanjour, M. Al-Sehhi}. In that case, Wd. Al-Shehri, from WTC North cell, and H. Al-Ghamdi, from WTC South cell, are displaced. Now, H. Hanjour, who is known to be the leader of WTC South cell, has displaced H. Al-Ghamdi.

When considering a bigger group of four terrorist, then {A. Aziz Al-Omari, H. Hanjour, M. Al-Sehhi, Wd. Al-Shehri} has the highest Shapley group value. The first group with one representative per each cell occupies the 13th place, with a Shapley group value of $\phi^g_D(N, w^{conn2}) = 10,6311$, being {A. Aziz Al-Omari, H. Hanjour, M. Al-Sehhi, Z. Jarrah}. Note that Z. Jarrah, who is known to be the leader of Pennsylvania cell is individually in the 8th position. The group of the four cell's leaders L = {M. Atta, H. Hanjour, M. Al-Sehhi, Z. Jarrah} is in the 48th position with a group value of $\phi^g_L(N, w^{conn2}) = 9,1313$.

Recall that Lindelauf *et al.* (2013) carried out the analysis on the terrorist network of the nineteen hijackers which prepared and executed the attack (distributed in four cells). We extend their analysis to a more dense network (see Figure 4) which included some people which did not take direct part in the attack, but support the terrorists. In this event, the relative positions of the hijackers change: the two poor connected terrorists from the WTC North cell, A. Aziz Al-Omari and Wd. Al-Shehri, are not so relevant in the new network, since they are now better connected through non-hijackers terrorists. According to the rankings for the Al Qaeda's 9/11 hijackers based on the individual Shapley value and the extended network[6], the most valuable hijackers were, in decreasing order of importance, N. Al-Hazmi (Pentagon), H. Hanjour (Pentagon), M. Atta (WTC North), H. Al-Ghamdi (WTC South), Wd. Al-Shehri (WTC North) and Z. Jarrah (Pennsylvania).

Figure 4: 11S extended SN. Image taken from V.E. Krebs (Copyright ©2002, First Monday)

The results obtained by means of Monte carlo simulation (see Castro, Gomez and Tejada, 2009) are depicted in Table 3, which includes the records for the best groups arranged in decreasing order of importance.

With respect to groups of two terrorists, the most valuable group is {N. Al-Hazmi, M. Atta}, which does not coincide with the two more important terrorists' group. In that case, however, the three and four people most valuable 3-group and 4-group are composed of the three and four, respectively, most important hijackers from an individual point of view. Those groups, C_3 and C_4, have group values of $\phi^g_{C_3}(N \cup M, w^{conn2}) = 10,675$ and $\phi^g_{C_4}(N \cup M, w^{conn2}) = 12,529$. Now,

[6] In which we have again considered the w^{conn2} game, with a zero weight for all the terrorist who do not take direct part in the attacks.

Individuals	Two agents
N. Al-Hazmi (Pent), $6,132$	$\{$ N. Al-Hazmi, M. Atta$\}$, $8,424$
H. Hanjour (Pent), $6,089$	$\{$ N. Al-Hazmi, H. Hanjour$\}$, $8,342$
M. Atta (WTC-N), $5,926$	$\{$ H. Hanjour, M. Atta$\}$, $8,152$
H. Al-Ghamdi (WTC-S), $1,844$	$\{$ N. Al-Hazmi, H. Al-Ghamdi$\}$, $7,201$
Wd. Al-Shehri (WTC-N), $1,701$	$\{$ H. Hanjour, Z. Jarrah$\}$, $7,048$
Z. Jarrah (Penn), $1,688$	$\{$ M. Atta, Z. Jarrah$\}$, $7,013$

Table 3: 11S extended network rankings

the group L is in the twentieth position with a value of $\phi_L^g(N \cup M, w^{conn2}) = 10,066$.

5 Conclusions

In this paper we have introduced an extension of the classical game-theoretic concept of value to the framework of groups. We define the group value as a vector whose components measure the expectations of the coalition inside a TU-game, i.e. an evaluation of the prospects of the group if they act together as an individual; in this sense, the group value must be understood as a *v*aluation vector. A key observation in our proposal is that we do not need to suppose necessarily that the agents know each other nor agree to act jointly; instead, we assume the existence of an *e*xternal decision maker that is able and willing to evaluate the value of the different groups, and in his case, find the optimal group who undertakes the appropriate collective action he is trying to promote.

The main motivation of our work and in particular of the previous definitions was to obtain a (marginalistic) extension of the Shapley value to the context of groups. Following the original formulation of Shapley, who intended to apply his value to measure the expectations of players in a game, and also keeping in mind the mentioned idea of the external decision maker, we have performed the generalization of the Shapley value by means of the merging game defined by Derks-Tijs in 2000. In this work, the authors develop a concept of super-player, who acts as a proxy in a quotient game of all the players of the coalition whose value we do want to compute.

In order to show that our extension of Shapley is valid and interesting, we have proved that the natural generalizations to the group framework of the usual properties of the (individual) Shapley value hold for the Shapley group value. Moreover, we offer an axiomatic characterization of the Shapley group value which, although includes the group version of null-player and additivity properties, cannot be directly deduced from the usual characterizations in the individual context. It is likely other axiomatic descriptions of the Shapley group value may exist.

We finish our paper by testing the validity of our methods in the identification of influent groups inside a real terrorist network. The flexibility of the proposed approach allows to suppose that our measure will be effective and usual in a variety of contexts and making use of different interpretations of the Shapley group value. Let us think for instance in two of them.

In a global economy context, in which many firms present a complex interlocked shareholding structure, it may be difficult to asses a firm's controllers. However, "*a common intuition among scholars and in media sees the global economy as being dominated by a handful of powerful transnational corporation*" (Vitali, Glattfelder and Battiston, 2011). In that case, following the approach of Crama and Leruth (2007), and Levy (2011), we can make use of the Shapley group value to detect a small group of firms which in fact have a dominating power.

Another relevant application arises in the context of transportation network's operation, where the identification of sets of stations to defense (or maintain) in order to maximally preserve the network's operation is a relevant question to network protection against natural and human-caused hazards, which has become a topical research topic in engineering and social sciences, as Liu, Fan and Ordonẽz (2009) point out. In that case, following a game theoretical approach as in Perea and Puerto (2013), the Shapley group value can serve security agencies for selecting a group of stations to defense (or maintain).

However, it should be remarked that the problem of finding the optimal group, according to some prearranged criteria, is a combinatorial problem that merits a more careful study. We are aware of the need of heuristics in order to apply the Shapley group value to the group selection problem.

Appendix

In the following we should check that all the axioms considered in Theorem 1 are necessary to guarantee the uniqueness of the group value ϕ^g.

G**-null player**. Let $\alpha \in (0,1)$. Define a value ξ^g in the following manner. If (N,v) is a null game, $\xi^g_C(N,v)$=0 for every $C \in N$. Given a non-null unanimity game u_S, with $S \subsetneq N$, define $\xi^g_C(u_S)$ as

$$\xi^g_C(N,u_S) = \begin{cases} \displaystyle\sum_{k=0}^{c-1} \alpha^{n-k}, & \text{if } C \cap S = \emptyset, \\ \displaystyle\frac{1}{s-|S\cap C|+1} + \sum_{k=0}^{n-s-1} \alpha^{n-k}, & \text{if } C \cap S \neq \emptyset, \end{cases} \tag{11}$$

$\xi^g_C(N,u_N) = \frac{1}{n-c+1}$, for all $C \subseteq N$, and then extend the value by additivity. It is clear that $\xi^g_i(N,u_S) = \alpha^n > 0$ for all $i \notin S$. So, G-null player does not hold. Let us check that ξ^g verifies G-CBC over the class of unanimity games. If (N,u_N), then G-CBC condition (4) trivially holds. Let (N,u_S) be a unanimity game with $S \subsetneq N$. If $|S| = n-1$, then two cases are possible:

a) If $i,j \in S$, then $\xi^g_{C\cup i}(N,u_S) = \frac{1}{s-|S\cap C|} + \sum_{k=0}^{n-s-1} \alpha^{n-k} = \xi^g_{C\cup j}(N,u_S)$ and (4) holds, since $u_S|_{N\setminus i}$ and $u_S|_{N\setminus j}$ are null games.

b) If $i \in S$ and $j \notin S$, then $S = N\setminus j$, $u_S|_{N\setminus i} \equiv 0$ and $u_S|_{N\setminus j}$ is a unanimity game w.r.t. the grand coalition $N \setminus j$. Thus (4) holds:

$$\xi^g_{C\cup i}(N \setminus j, u_{N\setminus j}|_{N\setminus j}) - \xi^g_C(N \setminus j, u_{N\setminus j}|_{N\setminus j}) =$$

$$\frac{1}{(n-1)-(c+1)+1} - \frac{1}{(n-1)-c+1} = \xi^g_{C\cup i}(N, u_{N\setminus j}) - \xi^g_{C\cup j}(N u_{N\setminus j}).$$

If $|S| < n-1$, then three cases are possible:

a) If $i, j \in S$, then $\xi^g_{C\cup i}(N, u_S) = \frac{1}{s-|S\cap C|} + \sum_{k=0}^{n-s-1} \alpha^{n-k} = \xi^g_{C\cup j}(N, u_S)$ and (4) holds.

b) If $i \in S$ and $j \notin S$, then

$$\xi^g_{C\cup i}(N \setminus j, u_S|_{N\setminus j}) = \frac{1}{s - |S\cap C|} + \sum_{k=1}^{n-s-1} \alpha^{n-k}$$

and

$$\xi^g_C(N \setminus j, u_S|_{N\setminus j}) = \begin{cases} \sum_{k=1}^{c} \alpha^{n-k}, & \text{if } C \subseteq N \setminus S, \\ \frac{1}{s - |S\cap C| + 1} + \sum_{k=1}^{n-s-1} \alpha^{n-k}, & \text{otherwise.} \end{cases}$$

Thus, if $C \subseteq N \setminus S$:

$$\xi^g_{C\cup i}(N \setminus j, u_S|_{N\setminus j}) - \xi^g_C(N \setminus j, u_S|_{N\setminus j}) =$$

$$\frac{1}{s} + \sum_{k=1}^{n-s-1} \alpha^{n-k} - \sum_{k=1}^{c} \alpha^{n-k} = \xi^g_{C\cup i}(N, u_S) - \xi^g_{C\cup j}(N, u_S)$$

If $C \cap S \neq \emptyset$, then

$$\xi^g_{C\cup i}(N \setminus j, u_S|_{N\setminus j}) - \xi^g_C(N \setminus j, u_S|_{N\setminus j}) =$$

$$\frac{1}{s - |S\cap C|} + \sum_{k=1}^{n-s-1} \alpha^{n-k} - \Big(\frac{1}{s - |S\cap C| + 1} + \sum_{k=1}^{n-s-1} \alpha^{n-k} = \xi^g_{C\cup i}(N, u_S) - \xi^g_{C\cup j}(N, u_S)\Big)$$

c) If $i, j \in N \setminus S$, then $C \cup i \subseteq N \setminus S$ if, and only if, $C \cup j \subseteq N \setminus S$ and, therefore condition (4) can be easily checked.

Now, since G-CBC condition is additive ξ^g satisfies it over $\bigcup_{n\geq 1} \mathcal{G}_n$

G**-additivity**. Let ξ^g be another group value over (N, v) which is defined in the following way:

(A_1) If there is at least a null player in N, or (N, v) is the unanimity game with respect to the grand coalition N, then $\xi^g_C(N,v) = \phi^g_C(N,v)$ for every group C in N.

(A_2) Otherwise, $\xi^g_C(N,v) = \phi^g_C(N,v) + k$, being $k \neq 0$ a fixed constant.

It is easily checked that G-null-player and G-SPB hold for ξ^g. We will check that the property of coalitional balanced contributions G-BMC also holds for this value. Note that all differences in (4) match $\xi^g_D(L,w)$ with $\xi^g_{D'}(L,w)$, for some coalition $L \in \{N, N \setminus i, N \setminus j\}$ and some game $w \in \{v, v_{-i}, v_{-j}\}$.

Taking into account that $\phi^g = \xi^g$ in case (N,v) is some of the games in the first case (A_1), and in the second one the $k's$ cancel, it holds:

$$\xi^g_{C\cup i}(N,v) - \xi^g_{C\cup j}(N,v) = \phi^g_{C\cup i}(N,v) - \phi^g_{C\cup j}(N,v),$$

$$\xi^g_{C\cup i}(N \setminus j, v_{-j}) - \xi^g_{C}(N \setminus j, v_{-j}) = \phi^g_{C\cup i}(N \setminus j, v_{-j}) - \phi^g_{C}(N \setminus j, v_{-j}),$$

$$\xi^g_{C\cup j}(N \setminus i, v_{-i}) - \xi^g_{C}(N \setminus i, v_{-i}) = \phi^g_{C\cup j}(N \setminus i, v_{-i}) - \phi^g_{C}(N \setminus i, v_{-i}),$$

G-CBC property holds for ϕ^g, and so the property does so for ξ^g, and we are done.

Note that we can modify ξ^g defining $\xi^g_i(N,v) = \Phi_i(N,v)$, for all $i \in N$, and $\xi^g_N(N,v) = v(N)$, for all $N \subseteq \mathbb{N}$, and the same result holds.

G-**CBC**. Define a value ξ^g in the following manner. If (N,v) is a null game, $\xi^g_C(N,v)$=0 for every $C \in N$. Given a non-null unanimity game u_S with $S \subsetneq N$, define $\xi^g_C(u_S)$ as

$$\xi^g_C(u_S) = \begin{cases} 0 & \text{if } C \cap S = \emptyset, \\ \xi^g_C(u_S) = 1, & \text{if } S \subseteq C, \\ |S \cap C| \times |S \backslash C| & \text{if } C \cap S \neq \emptyset \text{ and } C \cap S \neq S, \end{cases} \tag{12}$$

$\xi^g_C(N,u_N) = \frac{1}{n-c+1}$, for all $C \subseteq N$, and then extend the value by additivity.

Observe that all axioms but G-CBC hold. The unique one that needs a bit of discussion is the G-null player axiom, which holds when considering the base of unanimity games because in u_S the null players are precisely the players outside S, and therefore:

$$\begin{aligned} &\xi^g_{C\cup i}(N,u_S) = 0 = \xi^g_C(N,u_S), \text{ if } S \cap C = \emptyset, \\ &\xi^g_{C\cup i}(N,u_S) = 1 = \xi^g_C(N,u_S), \text{ if } S \subseteq C, \text{ and} \\ &\xi^g_{C\cup i}(N,u_S) = |S \cap (C \cup i)| \times |S \backslash (C \cup i)| = |S \cap C| \times |S \backslash C| = \xi^g_C(N,u_S), \text{ otherwise,} \end{aligned}$$

for all player $i \notin S$. Then, taking into account that the Harsanyi dividend $c_S(N,v)$ of any coalition S containing null players in the game (N,v) equals zero, G-null player property holds for any n-person game $(N,v) \in \mathcal{G}_n$, for all $N \subseteq \mathbb{N}$.

Let us check by means of a concrete example that the G-CBC axiom fails in this case. Consider (N,u_S) with $|N| = 3$, and $S = \{1,2\}$. In the notation of the axiom, take $C = \{1\}$, $i = 2$, and $j = 3$.

Then:

$$\xi^g_{\{1,2\}}(N, u_S) = 1,\ \xi^g_{\{1,3\}}(N, u_S) = 1 \times 1 = 1,\ \xi^g_{\{1,2\}}(N\backslash 3, u_S|_{N\backslash 3}) = 1,\ \xi^g_1(N\backslash 3, u_S|_{N\backslash 3}) = 1/2,$$

and $\xi^g_{\{1,3\}}(N\backslash 2, u_S|_{N\backslash 2}) = 0 = \xi^g_1(N\backslash 2, u_S|_{N\backslash 2})$, because the game $(N\backslash 2, u_S|_{N\backslash 2})$ is null. It is clear now that the two sides of the equalities that define the axiom do not coincide in this case.

Note that we can modify ξ^g defining $\xi^g_i(N, u_S) = \Phi_i(N, v)$, for all $i \in N$, and $\emptyset \neq S \subsetneq N$, for all $N \subseteq \mathbb{N}$, and the same result holds. Probably it is easy to find more examples by defining the value over $C \cap S$ (when $C \cap S \neq \emptyset$) as another appropriate function of $(|C \cap S|, |S \backslash C|)$.

*G***-symmetry over pure bargaining games**. Define a value ξ^g in the following manner. If (N, v) is a null game, $\xi^g_C(N, v)$=0 for every $C \in N$. Given a non-null unanimity game u_S, define $\xi^g(N, u_S)$ as $\xi^g_C(N, u_S) = \frac{|C\cap S|}{|S|}$, for all group $C \subseteq N$, and then extend the value by additivity.

Observe that all axioms but G-SPB hold. The G-null player axiom trivially holds when considering the base of unanimity games. Then, since $c_S(N, v) = 0$ for all S containing null players in the game (N, v), G-null player property holds in general. Moreover, G-additivity follows from the definition.

Let us check that the G-CBC axiom holds. Let (N, v) be any n-person game with $n \geq 2$. Let C be any group in N of cardinality $c \leq n - 2$, and let $i, j \in N \setminus C$. Then:

- If $i, j \in N \setminus S$, then (4) holds since all the involved differences are zero because i and j are null players in the three games.
- If $i, j \in S$, then $|(C \cup i) \cap S| = |C \cap S| + 1 = |(C \cup j) \cap S|$. Thus, $\xi^g_{C\cup i}(N, u_S) - \xi^g_{C\cup j}(N, u_S) = 0$, and (4) holds because the games $(N \setminus i, u_S|_{N\setminus i})$ and $(N \setminus j, u_S|_{N\setminus j})$ are null.
- If $i \in S$ and $j \notin S$, then $\xi^g_{C\cup i}(N, u_S) - \xi^g_{C\cup j}(N, u_S) = \frac{1}{s} = \xi^g_{C\cup i}(N \setminus j, u_S|_{N\setminus j}) - \xi^g_C(N \setminus j, u_S|_{N\setminus j})$, and (4) holds because $u_S|_{N\setminus i} \equiv 0$.

Clearly, G-SPB fails, so we are done.

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