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Nonlinear axisymmetric vibrations of a hyperelastic orthotropic cylinder

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Abstract

In this paper we investigate the large-amplitude axisymmetric free vibrations of an incompressible nonlinear elastic cylindrical structure. The material behavior is described as orthotropic and hyperelastic using the physically-based invariants proposed by Rubin and Jabareen (J. Elast. 90(1):1-18, 2007; J. Elast. 103(2):289-294, 2010). The cylinder is modeled using the theory of a generalized Cosserat membrane, which allows for finite deformations that include displacements along the longitudinal axis of the structure. The bidimensional approach represents a significant contribution with respect to most works published in this field, which approach the problem at hand assuming plane strain conditions along the axis of the cylinder. We have carried out a systematic analysis of the parameters that govern the dynamic behavior of the structure, paying specific attention to those describing the orthotropy of the material and the dimensions of the cylinder. Using Poincaré maps, we have shown that the motion of the structure can turn from periodic to quasi-periodic and chaotic as a function of the initial conditions, the elastic and kinetic energy supplied to the specimen, the dimensions of the cylinder and the degree of mechanical orthotropy of the material.

Keywords: Nonlinear elasticity, Anisotropy, Large-amplitude vibrations, Cosserat membrane, Chaotic motion

1. Introduction

The analysis of the nonlinear dynamics of hyperelastic shells started with the pioneering works of Knowles [1, 2], and the subsequent developments of Zhong-Heng and Solecki [3], Wang [4], Balakrishnan and Shahinpoor [5] and Shahinpoor and Balakrishnan [6]. These authors focused their attention on the large-amplitude radial vibrations of incompressible, thin-walled and thick-walled, cylindrical and spherical shells. Free and forced oscillations, the later with a Heaviside radial pressure, were explored. Closed-form analytical solutions for the phase diagrams and the period of oscillation could be obtained due to the incompressibility condition imposed on the constitutive model. Moreover, these results revealed the existence of a critical pressure that leads to the loss of the oscillatory behavior of the structure. More recently, the contributions of Beatty [7, 8], Verron et al. [9] and Aranda-Iglesias et al. [10, 11] have pointed out the strong dependency of this critical pressure on the strain-energy function used to model the material behavior. Further information about the

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literature published until 2013 on the oscillatory behavior of incompressible hyperelastic shells can be found in the review of Alijani and Amabili [12].

In addition, several relevant works have been published on this topic since 2013. For instance, Breslavsky et al. [13, 14] studied, analytically and numerically, free and forced nonlinear bending vibrations of thin square plates made of hyperelastic materials. They considered in their analyses Neo-Hookean, Mooney Rivlin and Ogden strain energy functions. Breslavsky and co-workers [13, 14] concluded that the best approximation to the real response of the structure was provided by the Ogden model since it correctly reproduces the behavior of the plate at large strains, including the increase in stiffness of the material after a given strain threshold is reached. This research was continued, shortly after, by Balasubramanian et al. [15] who revisited the nonlinear bending vibrations of a thin square silicone rubber plate analytically, numerically and, also, experimentally. The authors reported that the lack of experimental studies, on square inflated plates in particular, inspired their experimental work and provided a strong innovative character to their research. They conducted an experimental modal analysis on a plate that was fixed to a metallic frame and pre-loaded in its plane. For the analytical model, the equations of motion were obtained by a unified energy approach in which the material and geometrical nonlinearities were considered using the Mooney-Rivlin model and the Novozhilov nonlinear shell theory, respectively. On the other hand, the numerical model was developed using a finite element code. Balasubramanian et al. [15] showed that with a correct parameter identification their analytical and numerical models accurately predict the experimental response of the plate up to large deformations. Moreover, Breslavsky et al. [16] studied the dynamic response of a circular cylindrical shell made of an hyperelastic bio-material described as a combination of Neo-Hookean and Fung models. Free and forced vibrations around preloaded configurations were analyzed. In both cases, the nonlinearity of the single-mode (driven mode) response of the preloaded shell was quite weak, but a resonant regime with both driven and companion modes active was found with more complicated nonlinear dynamics.

The extension of the works of Knowles [1, 2], Zhong-Heng and Solecki [3], Wang [4], Balakrishnan and Shahinpoor [5] and Shahinpoor and Balakrishnan [6] to compressible hyperelasticity was conducted by Haddow and co-workers [17, 18, 19, 20]. The material compressibility precludes obtaining closed-form solutions of the shell motion so that the problem has to be solved numerically. The authors showed that the response of compressible and incompressible spherical shells subjected to radially symmetric dynamic inflation is significantly different. For incompressible specimens, the effect of the applied pressure is felt instantaneously throughout the shell. While, for the case of a compressible material, the applied pressure leads to the formation of a shock wave, which propagates back and forth through the thickness of the specimen. Moreover, the recent paper of Aranda-Iglesias et al. [21], which studied the vibrations of spherical shells subjected to constant inner pressure, revealed that the material compressibility decreases the critical pressure value that leads to the loss of the oscillatory behavior of the structure.

On the other hand, the influence of material anisotropy on the large-amplitude vibrations of hyperelastic shells has received much less attention. The first paper on this topic was published by Huilgol [22], who studied the problem of axisymmetric oscillations of an infinitely long cylindrical thick-walled tube, which was curvilinearly transverse-isotropic, i.e., the anisotropy was assumed to exist in the radial direction. The author derived the conditions that strain-energy functions must satisfy for the existence of periodic solutions. A few years later, Shahinpoor [23] approached the problem of large-amplitude oscillations of a longitudinally anisotropic thin-walled cylindrical structure subjected to radial pressure and plane strain with no axial strain (plane strain was also considered by Huilgol [22]). Exact expressions for the dynamic deformations were obtained by explicitly solving the non-autonomous time-dependent differential equation that arises when the applied pressure was considered to decay exponentially in time. From this point on, most likely due to the extended use of composite materials such as carbon and glass fiber reinforced polymers in the industry, the works dealing with large-amplitude vibrations of anisotropic elastic materials have mostly used linear elasticity. For example, see the recent works of Toorani [24], Jansen [25, 26], Amabili and Reddy [27], and other references included in the review of Alijani and Amabili [12]. The number of recent contributions to the field of anisotropic nonlinear elasticity is small. Within this field, we should highlight the work of Mason and Maluleke [28], who investigated the radial oscillations of transversely isotropic incompressible thin-walled cylindrical tubes with a generalized Mooney-Rivlin strain-energy function and subjected to a time dependent applied pressure. Transversely isotropic cylindrical tubes in the radial, tangential and longitudinal directions were considered. As in the previous works of Huilgol [22] and Shahinpoor [23], the cylinder was subjected to plane strain conditions. The attention of Mason and Maluleke [28] was focused on calculating the conditions on the strain-energy function and the applied pressure such that the differential equation of motion has a Lie point symmetry generator. Under these conditions, the problem can be reduced to an autonomous system for which analytical solutions exist.

In the present paper we extend the works of Huilgol [22], Shahinpoor [23] and Mason and Maluleke [28] to a 2D framework. We study the axisymmetric free vibrations of a nonlinear orthotropic hyperelastic thick-walled cylindrical structure. Unlike in Huilgol [22], Shahinpoor [23] and Mason and Maluleke [28] no plane strain hypothesis is assumed, i.e., the cylinder deforms homogeneously in the axial direction. The orthotropic mechanical response of the material is modeled following the constitutive theory developed by Rubin and Jabareen [29, 30]. Using Poincaré maps and Lyapunov exponents, see for instance [31, 32], we have carried out a systematic analysis to evaluate, for a wide range of initial conditions, the influence of the following factors on the oscillatory response of the structure: (1) the initial elastic and kinetic energies supplied to the specimen, (2) the dimensions of the shell and (3) the anisotropy of the material. Our results reveal that, depending on the energy of the system, the material constants and the geometrical parameters, the response of the structure can be periodic, quasi-periodic or chaotic. However, it is not known if the additional surface vibrations that occur when the surfaces of the cylinder are traction-free pointwise will prevent the large amplitude oscillations analyzed here from being observable.

2. Problem formulation

Consider a cylindrical structure with inner radius A, outer radius B and uniform axial height H, in its stress-free reference configuration. For nonlinear axisymmetric deformations, the cylinder has inner radius a(t), outer radius b(t) and variable axial height h(t) in its present configuration at time t. The cylinder is modeled using the theory of a generalized Cosserat membrane, which allows for finite deformations including displacements along the longitudinal axis of the structure. Further information and details on the use and application of this theory can be found in Chapter 4 of Rubin's book [33]. The balance laws of Cosserat theory can be developed by the writing Bubnov-Galerkin type weak forms of the three-dimensional balance of linear momentum and a weighted average through the thickness of the balance of linear momentum but the constitutive equations are always developed by the direct approach to be consistent with hyperelastic constitutive equations based on a strain energy function. Moreover, the kinematics of the Cosserat theory are similar to those in the three-dimensional theory so the nonlinear orthotropic constitutive equations developed in [29] and [30] can easily be proposed with the context of a generalized Cosserat membrane used here.

Within the context of the Cosserat theory, **X** is the position vector of a material point in the axial middle surface of the tube and \mathbf{D}_3 is the director vector of a material line element along the longitudinal axis of the cylinder in its reference configuration, which are functions of two convected coordinates θ^{α} with ($\alpha = 1, 2$)

$$\mathbf{X} = \mathbf{X}(\theta^{\alpha}). \tag{1}$$

In the present configuration at time t, the material point \mathbf{X} is located by \mathbf{x} and the director \mathbf{D}_3 is deformed to \mathbf{d}_3 , such that

$$\mathbf{x} = \mathbf{x}(\theta^{\alpha}, \mathbf{t}), \quad \mathbf{d}_3 = \mathbf{d}_3(\theta^{\alpha}, \mathbf{t}).$$
 (2)

Also, the velocity \mathbf{v} and director velocities \mathbf{w}_i are defined by

$$\mathbf{v} = \dot{\mathbf{x}}, \quad \mathbf{w}_{\alpha} = \mathbf{v}_{,\alpha}, \quad \mathbf{w}_3 = \dot{\mathbf{d}}_3, \tag{3}$$

where a superposed dot denotes material time differentiation, holding θ^{α} fixed, and a comma denotes partial differentiation with respect to θ^{α} .

The reference directors $\mathbf{A}_{\alpha} = \mathbf{D}_{\alpha}$, the unit normal \mathbf{A}_{3} to the middle surface, the present directors $\mathbf{a}_{\alpha} = \mathbf{d}_{\alpha}$ and the unit normal to the middle surface \mathbf{a}_{3} are defined by

$$\mathbf{A}_{\alpha} = \mathbf{D}_{\alpha} = \mathbf{X}_{,\alpha}, \quad \mathbf{A}_{3} = \mathbf{A}^{-1/2}(\mathbf{A}_{1} \times \mathbf{A}_{2}), \quad \mathbf{A}^{1/2} = \|\mathbf{A}_{1} \times \mathbf{A}_{2}\|, \tag{4}$$

$$\mathbf{a}_{\alpha} = \mathbf{d}_{\alpha} = \mathbf{x}_{,\alpha}, \quad \mathbf{a}_{3} = a^{-1/2} (\mathbf{a}_{1} \times \mathbf{a}_{2}), \quad a^{1/2} = \|\mathbf{a}_{1} \times \mathbf{a}_{2}\|, \tag{5}$$

where the directors $\{D_3, d_3\}$ are restricted so that D_i and d_i form linearly independent sets of vectors

$$\mathbf{D}^{1/2} = \mathbf{D}_1 \times \mathbf{D}_2 \cdot \mathbf{D}_3 > 0, \quad \mathbf{d}^{1/2} = \mathbf{d}_1 \times \mathbf{d}_2 \cdot \mathbf{d}_3 > 0.$$
(6)

Throughout the text, $a^{1/2}$ is considered to be related to the deformed middle surface area of the tube and should not be confused with the square root of the inner radius a. In addition, the reciprocal vectors $\{\mathbf{A}^{i},\mathbf{a}^{i},\mathbf{D}^{i},\mathbf{d}^{i}\}$ are defined by

$$\mathbf{A}^{1} = \mathbf{A}^{-1/2}(\mathbf{A}_{2} \times \mathbf{A}_{3}), \quad \mathbf{A}^{2} = \mathbf{A}^{-1/2}(\mathbf{A}_{3} \times \mathbf{A}_{1}), \quad \mathbf{A}^{3} = \mathbf{A}_{3},$$
 (7)

$$\mathbf{D}^{1} = \mathbf{D}^{-1/2}(\mathbf{D}_{2} \times \mathbf{D}_{3}), \quad \mathbf{D}^{2} = \mathbf{D}^{-1/2}(\mathbf{D}_{3} \times \mathbf{D}_{1}), \quad \mathbf{D}^{3} = \mathbf{D}^{-1/2}(\mathbf{D}_{1} \times \mathbf{D}_{2}),$$
 (8)

with $\{\mathbf{a}^{i}, \mathbf{d}^{i}\}$ defined by similar equations obtained by replacing $\{A^{1/2}, \mathbf{A}_{i}\}$ by $\{a^{1/2}, \mathbf{a}_{i}\}$ and $\{D^{1/2}, \mathbf{D}_{i}\}$ by $\{d^{1/2}, \mathbf{d}_{i}\}$, respectively. Using these expressions, the deformation gradient **F**, dilatation J, velocity gradient tensor **L** and rate of deformation tensor **D** are defined by

$$\mathbf{F} = \mathbf{d}_{i} \otimes \mathbf{D}^{i}, \quad \mathbf{J} = \det \mathbf{F} = \frac{d^{1/2}}{D^{1/2}}, \quad \dot{\mathbf{J}} = \mathbf{J} \, \mathbf{D} \cdot \mathbf{I}, \tag{9}$$

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{w}_{i} \otimes \mathbf{d}^{i}, \quad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^{T}), \tag{10}$$

where $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of two vectors $\{\mathbf{a}, \mathbf{b}\}$, $\mathbf{A} \cdot \mathbf{B} = tr(\mathbf{AB}^{T})$ denotes the inner product of two second order tensors $\{\mathbf{A}, \mathbf{B}\}$ and \mathbf{I} is the unit second order tensor. Note that the usual summation convention is used for repeated indices, with Greek indices taking the values $\{\alpha = 1, 2\}$ and Latin indices taking the values $\{i = 1, 2, 3\}$.

The balance of linear momentum and the balance of director momentum are given, respectively, by

$$\mathbf{m}\dot{\mathbf{v}} = \mathbf{m}\mathbf{b} + \mathbf{t}^{\alpha}_{,\alpha},\tag{11}$$

$$my^{33}\dot{\mathbf{w}}_3 = m\mathbf{b}^3 - \mathbf{t}^3. \tag{12}$$

In these equations, m is a measure of the mass of the cylinder, y^{33} is the constant director inertia, which controls inertia to displacements along the axial direction of the cylinder, the assigned fields {mb, mb³} are due to body force and tractions on the top and bottom surfaces of the cylinder, and the kinetic quantities t^{i} are defined in terms of a symmetric tensor **T** by

$$\mathbf{t}^{i} = \mathbf{a}^{1/2} \mathbf{T} \mathbf{d}^{i}, \quad \mathbf{T}^{T} = \mathbf{T},$$
(13)

which requires a constitutive equation. It can be shown that $a^{1/2}\mathbf{T}$ is related to an average of the threedimensional Cauchy stress through the height of the tube. Specifically, for a hyperelastic material with three-dimensional strain energy function Σ^* per unit mass, the strain energy function Σ of the tube is specified by

$$\Sigma = \Sigma^*(\mathbf{C}), \quad \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \tag{14}$$

and **T** is determined by the derivatives of Σ

$$a^{1/2}\mathbf{T} = a^{1/2}\widehat{\mathbf{T}} = 2\mathbf{m}\mathbf{F}\frac{\partial\Sigma^*}{\partial\mathbf{C}}\mathbf{F}^{\mathrm{T}}.$$
(15)

In addition, the boundary conditions required for the balance of linear momentum (11) require specification of the force **t** per unit arclength ds of the curve ∂P defining the boundary of the tube with (see Eq. (4.4.10) in [33])

$$\mathbf{t} = \mathbf{N}\mathbf{n}, \quad \mathbf{N} = \mathbf{a}^{-1/2}\mathbf{t}^{\alpha} \otimes \mathbf{d}_{\alpha}, \tag{16}$$

where **n** is the unit outward normal vector to the curve ∂P , which is tangent to the middle surface P of the membrane.

For a cylinder made of an incompressible material, the deformation is constrained to be isochoric

$$\mathbf{J} = 1, \quad \mathbf{D} \cdot \mathbf{I} = 0, \tag{17}$$

and \mathbf{T} is expressed in the form (see [33, Sec. 4.9]).

$$\mathbf{a}^{1/2}\mathbf{T} = \mathbf{a}^{1/2}\widehat{\mathbf{T}} - \gamma \mathbf{I} \tag{18}$$

where $\widehat{\mathbf{T}}$ is determined by the constitutive equation (15) and γ is constraint response, which is an arbitrary function of $(\theta^{\alpha}, \mathbf{t})$.

The balance of mechanical power can be derived by taking the dot product of the balance of linear momentum (11) with the velocity \mathbf{v} , the dot product of the balance of director momentum (12) with the director velocity \mathbf{w}_3 and adding the results of both operations to obtain

$$\mathbf{m}\dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{m}\mathbf{y}^{33}\dot{\mathbf{w}}_3\cdot\mathbf{w}_3 = \mathbf{m}\mathbf{b}\cdot\mathbf{v} + \mathbf{m}\mathbf{b}^3\cdot\mathbf{w}_3 + (\mathbf{t}^{\alpha}\cdot\mathbf{v})_{,\alpha} - \mathbf{t}^i\cdot\mathbf{w}_i,$$
(19)

Then, using equations (9), (10), (13) and (16), and due to the facts that $\{m, y^{33}\}$ are constants and $\mathbf{d}_{\alpha} = \mathbf{a}_{\alpha}$, this equation can be rewritten in the form

$$\mathbf{a}^{1/2}\mathbf{T}\cdot\mathbf{D} + \frac{\mathrm{d}}{\mathrm{dt}}\left[\frac{1}{2}\mathrm{m}\left(\mathbf{v}\cdot\mathbf{v} + \mathrm{y}^{33}\mathbf{w}_{3}\cdot\mathbf{w}_{3}\right)\right] = \mathrm{m}\mathbf{b}\cdot\mathbf{v} + \mathrm{m}\mathbf{b}^{3}\cdot\mathbf{w}_{3} + \left(\mathrm{a}^{1/2}\mathbf{v}\cdot\mathbf{N}\mathbf{a}^{\alpha}\right)_{,\alpha}.$$
 (20)

Also, for a hyperelastic material

$$a^{1/2}\mathbf{T} \cdot \mathbf{D} = 2m\mathbf{F} \frac{\partial \Sigma^*}{\partial \mathbf{C}} \mathbf{F}^{\mathrm{T}} \cdot \mathbf{D} = m\dot{\Sigma}^*.$$
 (21)

Then, using the divergence theorem for a tensor \mathbf{B} (see [33, Sec. 4.4])

$$\int_{\partial \mathbf{P}} \mathbf{B} \mathbf{n} \, \mathrm{ds} = \int_{\mathbf{P}} \mathbf{a}^{1/2} \mathrm{div}_{\mathbf{n}} \mathbf{B} \, \mathrm{d}\theta^{1} \mathrm{d}\theta^{2}, \quad \mathbf{a}^{1/2} \mathrm{div}_{\mathbf{n}} \mathbf{B} = \left(\mathbf{a}^{1/2} \mathbf{B} \mathbf{a}^{\alpha}\right)_{,\alpha},\tag{22}$$

Equation (20) can be multiplied by $d\theta^1 d\theta^2$ and integrated over the region P to deduce the global form of the balance of energy

$$\dot{\mathcal{U}} + \dot{\mathcal{K}} = \mathcal{W},\tag{23}$$

where \mathcal{U} is the total strain energy, \mathcal{K} is the total kinetic energy in the cylinder and \mathcal{W} is the rate of work done by body forces and surface tractions on the cylinder.

$$\mathcal{U} = \int_{\mathcal{P}} \mathbf{m} \Sigma^* \, \mathrm{d}\theta^1 \mathrm{d}\theta^2, \quad \mathcal{K} = \int_{\mathcal{P}} \frac{1}{2} \mathbf{m} (\mathbf{v} \cdot \mathbf{v} + \mathbf{y}^{33} \mathbf{w}_3 \cdot \mathbf{w}_3) \, \mathrm{d}\theta^1 \mathrm{d}\theta^2, \tag{24}$$

$$\mathcal{W} = \int_{\mathcal{P}} \mathbf{m} (\mathbf{b} \cdot \mathbf{v} + \mathbf{b}^3 \cdot \mathbf{w}_3) \, \mathrm{d}\theta^1 \mathrm{d}\theta^2 + \int_{\mathcal{P}} \mathbf{t} \cdot \mathbf{v} \mathrm{ds}.$$
(25)

2.1. Orthotropic constitutive model

The cylindrical structure is modeled as a nonlinear orthotropic hyperelastic material using the physicallybased invariants developed in [29, 30]. Using the work of Flory [34], the unimodular part \mathbf{C}' of \mathbf{C} , which is a pure measure of distortional deformation, can be defined as

$$\mathbf{C}' = \mathbf{J}^{-2/3}\mathbf{C}, \quad \det\left(\mathbf{C}'\right) = \mathbf{1}.$$
(26)

In a three-dimensional context, the Cauchy stress tensor \mathbf{T}^* can be expressed in the form

$$\mathbf{T}^* = \mathbf{T}^{*\prime\prime} - \mathbf{p}^* \mathbf{I}, \quad \mathbf{T}^{*\prime\prime} \cdot \mathbf{I} = 0, \tag{27}$$

where p^* is the pressure and $T^{*''}$ is the deviatoric part of T^* . General hyperelastic materials experience no distortion when subjected to pure hydrostatic pressure

$$\mathbf{C}' = \mathbf{I} \text{ for } \mathbf{T}^{*''} = \mathbf{0} \text{ and } \mathbf{J} \neq \mathbf{1}.$$
(28)

It is further assumed that the hyperelastic material is a solid that has non-zero stiffness to all distortional modes of deformation. In contrast, when an orthotropic hyperelastic material is subjected to hydrostatic pressure it experiences distortion

$$\mathbf{C}' \neq \mathbf{I} \text{ for } \mathbf{T}^{*''} = \mathbf{0} \text{ and } \mathbf{J} \neq \mathbf{1}.$$
 (29)

The main idea in [29, 30], is to develop invariants of deformation, which are based on the additional distortions required to cause deviatoric stress in an orthotropic solid. Let \mathbf{p}_i (i = 1, 2, 3) be an orthonormal triad of vectors that characterize the principal directions of orthotropy of the material in its reference configuration and let \mathbf{N}_i be the associated structural tensors defined by

$$\mathbf{N}_{i} = \mathbf{p}_{i} \otimes \mathbf{p}_{i} \text{ (no sum on } i = 1, 2, 3).$$
(30)

Specifically, an orthotropic solid will be in a hydrostatic state of stress if an only if \mathbf{C}' has the form

$$\mathbf{C}' = \eta_1^2 \mathbf{N}_1 + \eta_2^2 \mathbf{N}_2 + \eta_3^2 \mathbf{N}_3, \tag{31}$$

where η_i are positive functions of the dilatation J satisfying the restrictions

$$\eta_{\rm i} = \eta_{\rm i}({\rm J}), \quad \eta_1 \eta_2 \eta_3 = 1, \quad \eta_{\rm i}(1) = 1.$$
 (32)

It was shown by [30] that the strain energy Σ^* per unit mass of a general orthotropic material can be expressed as a function of seven invariants

$$\Sigma^* = \Sigma^* (\mathbf{J}, \beta_{\mathbf{i}}), \tag{33}$$

where the invariants $\beta_i (i = 1, 2, ..., 6)$ are defined by

$$\beta_{i} = \left(\frac{1}{\eta_{i}^{2}}\mathbf{C}' + \eta_{i}^{2}\mathbf{C'}^{-1}\right) \cdot \mathbf{N}_{i}, \quad \beta_{i} \ge 2 \quad (\text{no sum on } i = 1, 2, 3), \tag{34}$$

$$\beta_4 = \frac{(\mathbf{N}_1 \mathbf{C} \mathbf{N}_2 + \mathbf{N}_2 \mathbf{C} \mathbf{N}_1) \cdot \mathbf{C}}{2(\mathbf{C} \cdot \mathbf{N}_1)(\mathbf{C} \cdot \mathbf{N}_2)}, \quad \beta_5 = \frac{(\mathbf{N}_1 \mathbf{C} \mathbf{N}_3 + \mathbf{N}_3 \mathbf{C} \mathbf{N}_1) \cdot \mathbf{C}}{2(\mathbf{C} \cdot \mathbf{N}_1)(\mathbf{C} \cdot \mathbf{N}_3)}, \tag{35}$$

$$\beta_6 = \frac{(\mathbf{N}_2 \mathbf{C} \mathbf{N}_3 + \mathbf{N}_3 \mathbf{C} \mathbf{N}_2) \cdot \mathbf{C}}{2(\mathbf{C} \cdot \mathbf{N}_2)(\mathbf{C} \cdot \mathbf{N}_3)}.$$
(36)

Moreover, recalling the definition of the auxiliary functions $n_i(J)$ it follows that

$$\frac{\partial \mathbf{J}}{\partial \mathbf{C}} = \frac{1}{2} \mathbf{J} \mathbf{C}^{-1}, \quad \mathbf{n}_{i} = \frac{3\mathbf{J}}{\eta_{i}} \frac{d\eta_{i}}{d\mathbf{J}}, \quad \mathbf{n}_{1} + \mathbf{n}_{2} + \mathbf{n}_{3} = 0,
\frac{\partial \eta_{i}}{\partial \mathbf{C}} = \frac{1}{6} \mathbf{n}_{i} \eta_{i} \mathbf{J}^{-2/3} \mathbf{C}'^{-1} \quad (\text{no sum on } i = 1, 2, 3).$$
(37)

Also, the derivatives of β_i are given by

$$\frac{\partial \beta_{i}}{\partial \mathbf{C}} = \mathbf{B}_{i} \quad (i = 1, 2, ..., 6), \tag{38}$$

where

$$\mathbf{B}_{i} = J^{-2/3} \left[\left(\frac{1}{\eta_{i}^{2}} \mathbf{N}_{i} - \eta_{i}^{2} \mathbf{C'}^{-1} \mathbf{N}_{i} \mathbf{C'}^{-1} \right) - \frac{1}{3} (1 + n_{i}) \left(\frac{1}{\eta_{i}^{2}} (\mathbf{C'} \cdot \mathbf{N}_{i}) - \eta_{i}^{2} (\mathbf{C'}^{-1} \cdot \mathbf{N}_{i}) \right) \mathbf{C'}^{-1} \right],$$
(39)
(no sum on i = 1, 2, 3).

$$\mathbf{B}_{4} = \frac{(\mathbf{N}_{1}\mathbf{C}\mathbf{N}_{2} + \mathbf{N}_{2}\mathbf{C}\mathbf{N}_{1}) - \beta_{4}(\mathbf{C}\cdot\mathbf{N}_{2})\mathbf{N}_{1} - \beta_{4}(\mathbf{C}\cdot\mathbf{N}_{1})\mathbf{N}_{2}}{(\mathbf{C}\cdot\mathbf{N}_{1})(\mathbf{C}\cdot\mathbf{N}_{2})},\tag{40}$$

$$\mathbf{B}_{5} = \frac{(\mathbf{N}_{1}\mathbf{C}\mathbf{N}_{3} + \mathbf{N}_{3}\mathbf{C}\mathbf{N}_{1}) - \beta_{5}(\mathbf{C}\cdot\mathbf{N}_{3})\mathbf{N}_{1} - \beta_{5}(\mathbf{C}\cdot\mathbf{N}_{1})\mathbf{N}_{3}}{(\mathbf{C}\cdot\mathbf{N}_{1})(\mathbf{C}\cdot\mathbf{N}_{3})},\tag{41}$$

$$\mathbf{B}_{6} = \frac{(\mathbf{N}_{2}\mathbf{C}\mathbf{N}_{3} + \mathbf{N}_{3}\mathbf{C}\mathbf{N}_{2}) - \beta_{6}(\mathbf{C}\cdot\mathbf{N}_{3})\mathbf{N}_{2} - \beta_{6}(\mathbf{C}\cdot\mathbf{N}_{2})\mathbf{N}_{3}}{(\mathbf{C}\cdot\mathbf{N}_{2})(\mathbf{C}\cdot\mathbf{N}_{3})}.$$
(42)

Specifically, it is noted that the invariants β_i are constants and \mathbf{B}_i vanish when \mathbf{C}' takes the form (31)

$$\beta_{i} = 2, \quad \beta_{i+3} = 0 \quad \text{for } i = 1, 2, 3, \quad \mathbf{B}_{i} = 0 \quad \text{for } i = 1, 2, ..., 6.$$
 (43)

2.2. Specific constitutive equations

As a special case, consider a compressible orthotropic material and take Σ^* in the form

$$2\rho_0^* \Sigma^* = \sum_{i=1}^3 K_i(\beta_i - 2) + \sum_{i=4}^6 K_i \beta_i + K_7 (J - 1)^2, \quad K_i \ge 0 \quad (i = 1, 2, ..., 7),$$
(44)

where K_i are non-negative material constants and ρ_0^* is the three-dimensional reference mass density. In [33, Sec. 4.16], it is shown that for a cylinder, the mass quantity m and the director inertia y^{33} in the balance laws (11) and (12) are defined by

$$m = \rho_0^* D^{1/2} H, \quad y^{33} = Y H^2, \quad Y = \frac{1}{12},$$
(45)

where it is noted that for this case $A^{1/2} = D^{1/2}$. It then follows that **T** in (15) is given by

$$a^{1/2}\mathbf{T} = D^{1/2}H\left(\sum_{i=1}^{6} K_i \mathbf{F} \mathbf{B}_i \mathbf{F}^{T} + J K_7 (J-1)\mathbf{I}\right)$$
(46)

In addition, consider the special case when n_i are constants so that integration of (37) yields

$$\eta_{\rm i} = {\rm J}^{{\rm n}_{\rm i}/3}, \quad {\rm n}_1 + {\rm n}_2 + {\rm n}_3 = 0.$$
 (47)

This model has nine material constants

$$\{K_i (i = 1, 2, ..., 7), n_1, n_2\},$$
(48)

which can be related to the components of the small deformation stiffness

$$\rho_0^* \Sigma^* = \frac{1}{2} \mathbf{K}_{ijkl} \mathbf{E}_{ij} \mathbf{E}_{kl}, \quad \mathbf{E}_{ij} = \mathbf{E} \cdot \mathbf{p}_i \otimes \mathbf{p}_j, \tag{49}$$

where E_{ij} are the components of the strain tensor **E** relative to \mathbf{p}_i (see [29, 30]).

2.3. Radially symmetric deformation

For the problem under consideration here, body force is neglected and tractions are applied on the top and bottom surfaces of the cylinder, which are consistent with axisymmetric vibrations. Specifically, axial stress $\sigma(\mathbf{R},t)\mathbf{e}_{\mathbf{z}}$ is applied to the top of the tube and axial stress $-\sigma(\mathbf{R},t)\mathbf{e}_{\mathbf{z}}$ is applied to the bottom of the tube to maintain a uniform deformed axial height h(t) of the cylinder. In these expressions R is the reference radius of a material point. This property can be dependent on time, so the assigned fields are given by (see [33, Sec. 4.3])

$$\mathbf{m}\mathbf{b} = 0, \quad \mathbf{m}\mathbf{b} = \mathbf{H}\frac{\partial \mathbf{r}}{\partial \mathbf{R}}\mathbf{r}\sigma\mathbf{e}_{\mathbf{z}}.$$
 (50)

Furthermore, for axisymmetric deformations, θ^{α} and the vectors $\{\mathbf{X}, \mathbf{D}_3, \mathbf{x}, \mathbf{d}_3\}$ are expressed in the forms

$$\theta^1 = \mathbf{R}, \quad \theta^2 = \theta, \quad \mathbf{X} = \mathbf{R}\mathbf{e}_{\mathbf{r}}, \quad \mathbf{D}_3 = \mathbf{e}_{\mathbf{z}},$$
(51)

$$\mathbf{x} = r(\mathbf{R}, t) \mathbf{e}_{r}, \quad \mathbf{d}_{3} = \lambda_{z}(t) \mathbf{e}_{z}, \quad \lambda_{z}(t) = \frac{h(t)}{H},$$
(52)

$$\mathbf{v} = \dot{\mathbf{r}} \mathbf{e}_{\mathbf{r}}, \quad \mathbf{w}_3 = \dot{\lambda}_z \mathbf{e}_z, \tag{53}$$

where the base vectors $\{\mathbf{e}_{\mathbf{r}}, \mathbf{e}_{\theta}, \mathbf{e}_{z}\}$ of a cylindrical polar coordinate system satisfy the equations

$$\mathbf{e}_{\mathrm{r}} \times \mathbf{e}_{\theta} \cdot \mathbf{e}_{\mathrm{z}} = 1, \quad \frac{\mathrm{d}\mathbf{e}_{\mathrm{r}}}{\mathrm{d}\theta} = \mathbf{e}_{\theta}, \quad \frac{\mathrm{d}\mathbf{e}_{\theta}}{\mathrm{d}\theta} = \mathbf{e}_{\mathrm{r}}.$$
 (54)

Then, using (4)-(10) it follows that

$$\mathbf{D}_1 = \mathbf{e}_r, \quad \mathbf{D}_2 = \mathrm{R}\mathbf{e}_\theta, \quad \mathrm{D}^{1/2} = \mathrm{R}, \quad \mathbf{D}^1 = \mathbf{e}_r, \quad \mathbf{D}^2 = \frac{1}{\mathrm{R}}\mathbf{e}_\theta, \quad \mathbf{D}^3 = \mathbf{e}_z,$$
 (55)

$$\mathbf{d}_1 = \frac{\partial \mathbf{r}}{\partial \mathbf{R}} \mathbf{e}_{\mathbf{r}}, \quad \mathbf{d}_2 = \mathbf{r} \mathbf{e}_{\theta}, \quad \mathbf{d}_3 = \lambda_z \mathbf{e}_z, \quad \mathbf{a}^{1/2} = \mathbf{r} \frac{\partial \mathbf{r}}{\partial \mathbf{R}}, \tag{56}$$

$$d^{1/2} = \frac{\partial \mathbf{r}}{\partial \mathbf{R}} \mathbf{r} \lambda_{\mathbf{z}}, \quad \mathbf{d}^{1} = \frac{1}{\partial \mathbf{r} / \partial \mathbf{R}} \mathbf{e}_{\mathbf{r}}, \quad \mathbf{d}^{2} = \frac{1}{\mathbf{r}} \mathbf{e}_{\theta}, \quad \mathbf{d}^{3} = \frac{1}{\lambda_{\mathbf{z}}} \mathbf{e}_{\mathbf{z}}, \tag{57}$$

where r is the current radius of a material point. Therefore, the deformation gradient tensor assumes the following spectral decomposition

$$\mathbf{F} = \frac{\partial \mathbf{r}}{\partial \mathbf{R}} \mathbf{e}_{\mathbf{r}} \otimes \mathbf{e}_{\mathbf{r}} + \frac{\mathbf{r}}{\mathbf{R}} \mathbf{e}_{\theta} \otimes \mathbf{e}_{\theta} + \lambda_{\mathbf{z}} \mathbf{e}_{\mathbf{z}} \otimes \mathbf{e}_{\mathbf{z}}.$$
(58)

Now, for an incompressible material, the constraint (17) can be integrated to deduce that

$$J = \frac{\partial r}{\partial R} \frac{r}{R} \lambda_z = 1, \quad r^2 = a^2(t) + \frac{1}{\lambda_z(t)} (R^2 - A^2).$$
(59)

Therefore, the deformed outer radius b(t) of a cylinder with reference outer radius B is given by

$$b^{2}(t) = a^{2}(t) + \frac{1}{\lambda_{z}(t)}(B^{2} - A^{2})$$
(60)

Next, the principal directions of orthotropy are specified by

$$\mathbf{p}_1 = \mathbf{e}_r, \quad \mathbf{p}_2 = \mathbf{e}_\theta, \quad \mathbf{p}_3 = \mathbf{e}_z. \tag{61}$$

Hence, with the help of (18), (46) and (55) the tensor **T** is given by

$$a^{1/2}\mathbf{T} = RH(T_{rr}\mathbf{e}_{r}\otimes\mathbf{e}_{r} + T_{\theta}\mathbf{e}_{\theta}\otimes\mathbf{e}_{\theta} + T_{zz}\mathbf{e}_{z}\otimes\mathbf{e}_{z}),$$
(62)

where

$$T_{\rm rr} = -\frac{\gamma}{R \, \rm H} - K_1 \left(\frac{n_1 - 2}{3}\right) \left(\frac{1}{\lambda_z^2} \frac{R^2}{r^2} - \lambda_z^2 \frac{r^2}{R^2}\right) - K_2 \left(\frac{1 + n_2}{3}\right) \left(\frac{r^2}{R^2} - \frac{R^2}{r^2}\right) - K_3 \left(\frac{1 - n_1 - n_2}{3}\right) \left(\lambda_z^2 - \frac{1}{\lambda_z^2}\right), \tag{63}$$

$$T_{\theta\theta} = -\frac{\gamma}{R H} - K_1 \left(\frac{1+n_1}{3}\right) \left(\frac{1}{\lambda_z^2} \frac{R^2}{r^2} - \lambda_z^2 \frac{r^2}{R^2}\right) - K_2 \left(\frac{n_2-2}{3}\right) \left(\frac{r^2}{R^2} - \frac{R^2}{r^2}\right) - K_3 \left(\frac{1-n_1-n_2}{3}\right) \left(\lambda_z^2 - \frac{1}{\lambda_z^2}\right), \quad (64)$$

$$T_{zz} = -\frac{\gamma}{R H} - K_1 \left(\frac{1+n_1}{3}\right) \left(\frac{1}{\lambda_z^2} \frac{R^2}{r^2} - \lambda_z^2 \frac{r^2}{R^2}\right) - K_2 \left(\frac{1+n_2}{3}\right) \left(\frac{r^2}{R^2} - \frac{R^2}{r^2}\right) + K_3 \left(\frac{2+n_1+n_2}{3}\right) \left(\lambda_z^2 - \frac{1}{\lambda_z^2}\right).$$
(65)

Furthermore, using (13), (55), (59) and (62) it can be shown that

$$\mathbf{d}^{1} = \lambda_{\mathbf{z}} \frac{\mathbf{r}}{\mathbf{R}} \mathbf{e}_{\mathbf{r}}, \quad \mathbf{d}^{2} = \frac{1}{\mathbf{r}} \mathbf{e}_{\theta}, \quad \mathbf{d}^{3} = \frac{1}{\lambda_{\mathbf{z}}} \mathbf{e}_{\mathbf{z}}, \tag{66}$$

$$\mathbf{t}^{1} = \mathrm{H}\lambda_{\mathrm{z}}\,\mathrm{r}\,\mathrm{T}_{\mathrm{rr}}\mathbf{e}_{\mathrm{r}}, \quad \mathbf{t}^{2} = \mathrm{H}\frac{\mathrm{R}}{\mathrm{r}}\mathrm{T}_{\theta\theta}\mathbf{e}_{\theta}, \quad \mathbf{t}^{3} = \mathrm{H}\frac{\mathrm{R}}{\lambda_{\mathrm{z}}}\mathrm{T}_{\mathrm{zz}}\mathbf{e}_{\mathrm{z}}. \tag{67}$$

Then, with help of (45), (50), (66), (55) and (59), the equations of motion (11) and (12) are reduced to the following two scalar equations

$$\rho_0^* \ddot{\mathbf{r}} = \frac{\partial \mathbf{T}_{\mathrm{rr}}}{\partial \mathbf{r}} + \frac{\mathbf{T}_{\mathrm{rr}} - \mathbf{T}_{\theta\theta}}{\mathbf{r}},\tag{68}$$

$$\rho_0^* H^2 Y R \ddot{\lambda}_z = \frac{\partial r}{\partial R} r \sigma - \frac{R}{\lambda_z} T_{zz}.$$
(69)

Moreover, the equation of director momentum (69), which determines the uniform stretch λ_z , is solved only in an integral sense by requiring

$$\rho_0^* \mathrm{H}^2 \mathrm{Y}\pi \left(\mathrm{B}^2 - \mathrm{A}^2\right) \ddot{\lambda}_z = \mathrm{F} - \frac{2\pi}{\lambda_z} \int_{\mathrm{A}}^{\mathrm{B}} \mathrm{T}_{zz} \mathrm{R} \mathrm{d} \mathrm{R}, \quad \mathrm{F} = 2\pi \int_{\mathrm{a}}^{\mathrm{b}} \sigma \mathrm{r} \mathrm{d} \mathrm{r}, \tag{70}$$

where F is the resultant force applied in \mathbf{e}_z direction on the cylinder's top surface and in the $-\mathbf{e}_z$ direction on the cylinder's bottom surface. Also, it is noted that the balance of linear momentum (68) is satisfied pointwise in the radial direction. In contrast, the balance of director momentum (69) is an ordinary differential equation which assumes that the stretch λ_z is uniform through the height of the membrane. This eliminates wave propagation through the height. Furthermore, using the definition (16) for \mathbf{t} and the expressions (51), (55) and (66), the balance of linear momentum (68) is solved subject to the boundary conditions

$$\mathbf{t} = hP_{a}(t)\mathbf{e}_{r} \quad \text{on} \quad r = a \text{ with } \mathbf{n} = \mathbf{e}_{r} \to T_{rr}(a, t) = -P_{a}(t), \tag{71}$$

$$\mathbf{t} = hP_{b}(t)\mathbf{e}_{r} \quad \text{on} \quad r = b \text{ with } \mathbf{n} = -\mathbf{e}_{r} \to T_{rr}(b,t) = -P_{b}(t), \tag{72}$$

where $\{P_a, P_b\}$ are the pressures applied to the cylinder's inner and outer boundaries, respectively. Subsequently, notice from (62) that

$$T_{\theta\theta} = T_{rr} - K_1 \left(\frac{1}{\lambda_z^2} \frac{R^2}{r^2} - \lambda_z^2 \frac{r^2}{R^2} \right) + K_2 \left(\frac{r^2}{R^2} - \frac{R^2}{r^2} \right),$$
(73)

$$T_{zz} = T_{rr} - K_1 \left(\frac{1}{\lambda_z^2} \frac{R^2}{r^2} - \lambda_z^2 \frac{r^2}{R^2} \right) + K_3 \left(\lambda_z^2 - \frac{1}{\lambda_z^2} \right).$$
(74)

It then follows that the solution of (68) and (69) only depend on the three material constants $\{K_1, K_2, K_3\}$, which control the orthotropic response of the incompressible material. In particular, the equations (68) and (69) can be integrated to obtain

$$\frac{P_{a} - P_{b}}{K_{1}} = \frac{1}{8} \left(3\frac{\lambda_{a}^{2}}{\lambda_{z}^{2}}\dot{\lambda}_{z}^{2} - 2\frac{\lambda_{a}^{2}}{\lambda_{z}}\ddot{\lambda}_{z} \right) \frac{\Lambda_{B}^{2} - 1}{\lambda_{a}^{2}\lambda_{z}}
+ \frac{1}{2} \left[\lambda_{a}^{-2}\lambda_{z}^{-2} - \lambda_{z}^{-1} + \frac{K_{2}}{K_{1}} \left(\lambda_{a}^{-2} - \lambda_{z} \right) + \dot{\lambda}_{a}^{2} + \frac{\lambda_{a}}{\lambda_{z}}\dot{\lambda}_{a}\dot{\lambda}_{z} + \frac{1}{4}\frac{\lambda_{a}^{2}}{\lambda_{z}^{2}}\dot{\lambda}_{z}^{2} \right] \left(\frac{1 - \Lambda_{B}^{2}}{\lambda_{a}^{2}\lambda_{z} + \Lambda_{B}^{2} - 1} \right)
+ \frac{1}{2} \left[-\lambda_{z}^{-1} - \frac{K_{2}}{K_{1}}\lambda_{z} + \dot{\lambda}_{a}^{2} + \lambda_{a}\ddot{\lambda}_{a} + \frac{\lambda_{a}}{\lambda_{z}}\dot{\lambda}_{a}\dot{\lambda}_{z} - \frac{1}{2}\frac{\lambda_{a}^{2}}{\lambda_{z}^{2}}\dot{\lambda}_{z}^{2} + \frac{1}{2}\frac{\lambda_{a}^{2}}{\lambda_{z}}\ddot{\lambda}_{z} \right] \ln \left(1 + \frac{\Lambda_{B}^{2} - 1}{\lambda_{a}^{2}\lambda_{z}} \right)
+ \frac{1}{2} \left(\lambda_{z} + \frac{K_{2}}{K_{1}}\lambda_{z}^{-1} \right) \ln \left(\Lambda_{B}^{2} \right),$$
(75)

$$\begin{split} &\frac{1}{24}\Lambda_{\rm H}^{2}(\Lambda_{\rm B}^{2}-1)\ddot{\lambda}_{z} = \frac{\rm F}{2\pi{\rm K}_{1}{\rm A}^{2}} + \left[\frac{1}{2}\frac{{\rm P}_{a}}{{\rm K}_{1}}\lambda_{a}^{2} + \frac{1}{4}\lambda_{z}^{-2} - \frac{1}{4}\lambda_{a}^{2}\lambda_{z} + \frac{{\rm K}_{2}}{4{\rm K}_{1}}\left[1 + \lambda_{a}^{2}\left(\lambda_{z}^{-1}-2\lambda_{z}\right)\right] - \frac{{\rm K}_{3}}{2{\rm K}_{1}}\lambda_{a}^{2}\left(\lambda_{z}^{2}-\lambda_{z}^{-2}\right) \\ &+ \frac{1}{4}\lambda_{a}^{2}\left(2\dot{\lambda}_{a}^{2} + \lambda_{a}\ddot{\lambda}_{a} + 2\frac{\lambda_{a}}{\lambda_{z}}\dot{\lambda}_{a}\dot{\lambda}_{z} + \frac{1}{2}\frac{\lambda_{a}^{2}}{\lambda_{z}^{2}}\dot{\lambda}_{z}^{2}\right)\right]\frac{\Lambda_{\rm B}^{2}-1}{\lambda_{a}^{2}\lambda_{z}} + \frac{1}{32}\lambda_{a}^{4}\left(3\frac{\dot{\lambda}_{z}^{2}}{\lambda_{z}^{2}} - 2\frac{\ddot{\lambda}_{z}}{\lambda_{z}}\right)\left[1 - \left(1 + \frac{\Lambda_{\rm B}^{2}-1}{\lambda_{a}^{2}\lambda_{z}}\right)^{2}\right] \\ &+ \frac{1}{4}\left[\lambda_{z}^{-2} - \frac{\lambda_{a}^{2}}{\lambda_{z}} - \frac{{\rm K}_{2}}{{\rm K}_{1}}\left(1 - \lambda_{a}^{2}\lambda_{z}\right) - \lambda_{a}^{2}\left(\dot{\lambda}_{a}^{2} + \frac{\lambda_{a}}{\lambda_{z}}\dot{\lambda}_{a}\dot{\lambda}_{z} + \frac{1}{4}\frac{\lambda_{a}^{2}}{\lambda_{z}^{2}}\dot{\lambda}_{z}^{2}\right)\right]\ln\left(1 + \frac{\Lambda_{\rm B}^{2}-1}{\lambda_{a}^{2}\lambda_{z}}\right) \\ &+ \frac{1}{4}\left[\lambda_{z}^{-2} - \frac{\lambda_{a}^{2}}{\lambda_{z}} - \frac{{\rm K}_{2}}{{\rm K}_{1}}\left(1 - \lambda_{a}^{2}\lambda_{z}\right) - \lambda_{a}^{2}\left(\dot{\lambda}_{a}^{2} + \frac{\lambda_{a}}{\lambda_{z}}\dot{\lambda}_{a}\dot{\lambda}_{z} + \frac{1}{4}\frac{\lambda_{a}^{2}}{\lambda_{z}^{2}}\dot{\lambda}_{z}^{2}\right)\right]\ln\left(1 + \frac{\Lambda_{\rm B}^{2}-1}{\lambda_{a}^{2}\lambda_{z}}\right) \\ &+ \frac{1}{4}\left[\lambda_{z}^{-2} - \frac{\lambda_{a}^{2}}{\lambda_{z}} - \frac{{\rm K}_{2}}{{\rm K}_{1}}\left(\lambda_{a}^{2} + \lambda_{a}\dot{\lambda}_{a} + \frac{\lambda_{a}}{\lambda_{z}}\dot{\lambda}_{a}\dot{\lambda}_{z} - \frac{1}{2}\frac{\lambda_{a}^{2}}{\lambda_{z}^{2}}\dot{\lambda}_{z}^{2}\right]\left(1 + \frac{\Lambda_{B}^{2}-1}{\lambda_{a}^{2}\lambda_{z}}\right) \\ &- \left[\frac{1}{4}\lambda_{a}^{2}\left(\lambda_{z} + \frac{{\rm K}_{2}}{{\rm K}_{1}}\lambda_{z}^{-1}\right)\frac{\Lambda_{B}^{2}-1}{\lambda_{a}^{2}\lambda_{z}} - \frac{1}{2}\left(\frac{1}{2} - \lambda_{a}^{2}\lambda_{z} - \frac{{\rm K}_{2}}{2{\rm K}_{1}}\lambda_{z}^{-2}\right)\right]\ln\left(\Lambda_{\rm B}^{2}\right), \end{split}$$

where $\lambda_a = a/A$ stands for the circumferential stretch in the inner face of the cylinder, and the following non-dimensional parameters have been introduced

$$\Lambda_{\rm B} = \frac{\rm B}{\rm A}, \quad \Lambda_{\rm H} = \frac{\rm H}{\rm A}, \quad \tau = t \sqrt{\frac{\rm K_1}{\rho_0^* {\rm A}^2}}.$$
(77)

Moreover, the superposed dots in Eqns. (75) and (76) now denote derivation with respect to the nondimensional time τ . Hence, the equations of motion (75) and (76) form a system of two second order ordinary differential equations for the functions $\lambda_{\rm a}(\tau)$ and $\lambda_{\rm z}(\tau)$, which can be solved by a plethora of well-established implicit or explicit methods.

The Eqns. (75-76) in the Cosserat theory include limiting cases that have been considered in the literature. For instance, taking the limits $\Lambda_{\rm H} \to \infty$ and $\Lambda_{\rm B} \to 1$ in the system (75-76), the formulation for an anisotropic, thin-walled and infinitely long cylinder presented in [22, 28, 35] is recovered. Moreover, the limit of $\Lambda_{\rm H} \to \infty$ with $K_1 = K_2 = K_3$ corresponds to the infinitely long, isotropic and thick-walled cylinder first introduced in [1, 2] and used after by many others [36, 7, 10]. Therefore, the formulation presented in this paper includes all the cases cited above and generalizes them to bi-dimensional vibrations. As will be seen in further sections, the introduction of the axial degree of freedom completely modifies the nonlinear dynamics of the problem allowing quasi-periodic and chaotic behaviors.

Recall that the axial stress distribution ensures that each horizontal plane in the cylinder remains flat. In the reminder of this work, this total applied force F in (76) is set equal to zero. Furthermore, it can be deduced from (75) and (76) that the parametric dependence of the problem has been reduced to the following six non-dimensional parameters: two material constants K_2/K_1 and K_3/K_1 , two geometrical parameters Λ_B and Λ_H , and the normalized pressures P_a/K_1 and P_b/K_1 . In particular notice that even when the stresses depend on n_1 , n_2 and n_3 (see Eqns. (63) – (65)), these parameters do not appear in the equations of motion due to the symmetry of the problem.

For the development of the results presented in Section 4, it is convenient to introduce expressions for the energies involved in the deformation process. Therefore, we complete the problem formulation with the balance of energy (23) that, in dimensionless form, can be written as

$$\overline{\mathcal{W}} = \frac{\mathrm{d}\left(\overline{\mathcal{U}} + \overline{\mathcal{K}}\right)}{\mathrm{d}\tau},\tag{78}$$

where the work exerted by the external pressures $\overline{\mathcal{W}}$, the total strain energy $\overline{\mathcal{U}}$ and the kinetic energy $\overline{\mathcal{K}}$ in dimensionless form are given by

$$\overline{\mathcal{W}} = \pi \Lambda_{\rm H} \lambda_{\rm z} \left[\frac{\mathrm{P}_{\rm a}}{\mathrm{K}_1} \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\lambda_{\rm a}^2 \right) - 2 \frac{\mathrm{P}_{\rm b}}{\mathrm{K}_1} \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\lambda_{\rm a}^2 + \frac{\Lambda_{\rm B}^2 - 1}{\lambda_{\rm z}} \right) \right],\tag{79}$$

$$\overline{\mathcal{U}} = \frac{1}{2} \pi \Lambda_{\rm H} \lambda_{\rm a}^2 \lambda_{\rm z} \left[\left(1 + \frac{\mathrm{K}_2}{\mathrm{K}_1} \right) \left(\lambda_{\rm z} + \lambda_{\rm z}^{-1} - 2 \right) + \frac{\mathrm{K}_3}{\mathrm{K}_1} \left(\lambda_{\rm z}^2 + \lambda_{\rm z}^{-2} - 2 \right) \right] \left(\frac{\Lambda_{\rm B}^2 - 1}{\lambda_{\rm a}^2 \lambda_{\rm z}} \right)
+ \frac{1}{2} \pi \Lambda_{\rm H} \lambda_{\rm z} \left(\lambda_{\rm z}^{-2} + \frac{\mathrm{K}_2}{\mathrm{K}_1} \right) \left(1 - \lambda_{\rm a}^2 \lambda_{\rm z} \right) \ln \left(1 + \frac{\Lambda_{\rm B}^2 - 1}{\lambda_{\rm a}^2 \lambda_{\rm z}} \right)
+ \frac{1}{2} \pi \Lambda_{\rm H} \left(\lambda_{\rm z}^2 + \frac{\mathrm{K}_2}{\mathrm{K}_1} \right) \left(\lambda_{\rm a}^2 - \lambda_{\rm z}^{-1} \right) \ln \left(\Lambda_{\rm B}^2 \right),$$
(80)

$$\overline{\mathcal{K}} = \frac{1}{2} \pi \Lambda_{\rm H} \lambda_{\rm a}^2 \lambda_{\rm z} \left[\frac{1}{24} \Lambda_{\rm H}^2 \dot{\lambda}_{\rm z}^2 - \frac{1}{2} \lambda_{\rm a} \lambda_{\rm z}^{-1} \dot{\lambda}_{\rm a} \dot{\lambda}_{\rm z} - \frac{1}{4} \lambda_{\rm a}^2 \lambda_{\rm z}^{-2} \dot{\lambda}_{\rm z}^2 \right] \left(\frac{\Lambda_{\rm B}^2 - 1}{\lambda_{\rm a}^2 \lambda_{\rm z}} \right)
+ \frac{\pi}{16} \Lambda_{\rm H} \lambda_{\rm a}^4 \lambda_{\rm z}^{-1} \dot{\lambda}_{\rm z}^2 \left[\left(1 + \frac{\Lambda_{\rm B}^2 - 1}{\lambda_{\rm a}^2 \lambda_{\rm z}} \right)^2 - 1 \right]
+ \frac{1}{2} \pi \Lambda_{\rm H} \lambda_{\rm a}^2 \lambda_{\rm z} \left[\dot{\lambda}_{\rm a}^2 + \lambda_{\rm a} \lambda_{\rm z}^{-1} + \frac{1}{4} \lambda_{\rm a}^2 \lambda_{\rm z}^{-2} \dot{\lambda}_{\rm z}^2 \right] \ln \left(1 + \frac{\Lambda_{\rm B}^2 - 1}{\lambda_{\rm a}^2 \lambda_{\rm z}} \right).$$
(81)

Henceforth, we focus on the case of zero internal and external pressures, $P_a = P_b = 0$. Under these conditions, the work of the external forces \overline{W} vanishes and the total normalized energy remains constant

$$\overline{\mathcal{E}}_{\mathrm{T}} = \overline{\mathcal{U}} + \overline{\mathcal{K}} = \text{constant.}$$
 (82)

Note that the parameters associated with the orthotropic constitutive model K_3/K_1 and K_2/K_1 only appear in the total strain energy $\overline{\mathcal{U}}$.

3. Numerical solution

In the previous section we showed that the motion of the cylindrical structure, which is defined by (75) and (76), has two degrees of freedom $\lambda_{\rm a}(\tau)$ and $\lambda_{\rm z}(\tau)$. The corresponding four-dimensional phase space is formed by the variables $\{\lambda_{a}, \dot{\lambda}_{a}, \lambda_{z}, \dot{\lambda}_{z}\}$. Since we limit our attention to the case of free vibrations with no external applied loads on the cylinder, the total energy is constant (82) and one of the variables can be expressed as a function of the others. For the particular case developed in this article, let us take $\lambda_z = \lambda_z(\lambda_a, \lambda_z, \lambda_a)$. Hence, for a given energy level, the phase space can be reduced from four to three dimensions without any loss of information. In this new three dimensional phase space $\{\lambda_a, \lambda_z, \lambda_a\}$, when the motion of the cylinder is periodic or quasi-periodic, the trajectory of the structure lies on the surface of a torus [37, 38, 39]. As an example, let us consider $\Lambda_{\rm H} = \Lambda_{\rm B} = 2$, $K_2/K_1 = K_3/K_1 = 1$, $\mathcal{E}_{\rm T} = 8$, $\lambda_{\rm a}(0) = 1$, $\lambda_{\rm z}(0) = 1$ and $\dot{\lambda}_{\rm a}(0) = 1$. We substitute these geometrical parameters, material constants, total energy and initial conditions in (82)to obtain $\lambda_z(0) = 1.4140$. Then, we solve the system (75)-(76) to obtain the solution represented in Fig. 1(a), which shows the torus (depicted in blue) and the trajectory during the first oscillation (represented by the red curve). To visualize the torus, we have plotted the trajectory that flows on its surface for a time long enough to cover almost all the surface. This structure of the solution naturally leads to the concept of Poincaré Surface of Section (SOS). Poincaré SOS correspond to 2D representations of the motion of the structure that are constructed from the intersection between the trajectory of the cylinder in the 3D phase space and a given plane. Throughout this document we will choose the $\lambda_z = 1$ plane. In this way, the Poincaré SOS reduces a 3D trajectory in the phase space $\{\lambda_a, \lambda_z, \dot{\lambda}_a\}$ to a discrete 2D mapping in the phase space $\{\lambda_a, \lambda_a\}$ that contains all the dynamical information of the system. As an example, Fig. 1(b) shows the Poincaré SOS corresponding to the plane $\lambda_z = 1$, for $\lambda_z > 0$ and the initial conditions listed above.

As mentioned, Poincaré SOS shown in this paper correspond to the intersection of the trajectories of the cylinder with the $\lambda_z = 1$ plane. This ensures that, irrespective of the energy of the system, the phase space includes all the possible trajectories described by the structure. We have checked that, in most cases, considering a plane different from $\lambda_z = 1$ does not modify the main results and trends that will be shown in Section 4. However, it may be the case that, for small values of the energy supplied to the system, some (or all) trajectories of the cylinder would not intersect the selected plane, which would lead to the partial (or total) loss of information on the motion of the structure. Similarly, all the Poincaré SOS of this paper consider that $\dot{\lambda}_z > 0$, this condition being necessary to avoid overlapping between the Poincaré SOS arising from trajectories with the same energy level but different initial conditions.

The representation of the Poincaré SOS is the main technique used in the following section of the paper to analyze the influence of the initial conditions, the total energy supplied to the system, the dimensions of the cylinder and the degree of mechanical orthotropy of the material on the dynamical response of the structure.



Figure 1: Geometrical parameters $\Lambda_H = \Lambda_B = 2$. Material parameters $\frac{K_2}{K_1} = \frac{K_3}{K_1} = 1$. Total energy $\mathcal{E}_T = 8$. Initial conditions $\lambda_a(0) = 1$, $\lambda_z(0) = 1$, $\dot{\lambda}_a(0) = 1$ and $\dot{\lambda}_z(0) = 1.4140$. (a) Three-dimensional torus in the $(\lambda_a, \lambda_z, \dot{\lambda}_a)$ phase space corresponding to a quasi-periodic motion. The trajectory of the first oscillation is indicated in red. (b) Poincaré SOS corresponding to the plane $\lambda_z = 1$ with the condition $\dot{\lambda}_z > 0$.

For a better understanding of the Poincaré SOS that will be presented in Section 4, let us introduce the following concepts:

- A single point in the Poincaré SOS corresponds to a periodic motion of the structure, which is characterized by the period T.
- μ isolated points in the Poincaré SOS correspond to a periodic motion of the structure, which is characterized by the period μ T.
- A closed curve in the Poincaré SOS (as the one presented in Fig. 1(b)) corresponds to a quasi-periodic motion of the structure.
- Finally, a filled region in the Poincaré SOS corresponds to a chaotic motion of the structure.

To obtain the Poincaré SOS, the system (75)-(76) has been solved with the fourth order Runge-Kutta method pre-implemented in Wolfram Mathematica[®]. To guarantee the quality of the numerical integration, we have checked that all the resultant trajectories remain in their respective energy hypersurfaces up to an error of 10^{-6} . The computations presented in this paper have been carried out using the system formed by Eqns. (75-76) directly. Furthermore, the same computations have been conducted using the formulation of

the \mathbb{R}^4 state-space (i.e., after decoupling $\ddot{\lambda}_k$, k = a, z). It has been proved that both solutions lead to the same results preserving the energy up to an error of 10^{-6} .

4. Sample results

In the examples presented in this section, we take as a reference the geometrical parameters, material constants and energy level used in Fig. 1, i.e., $\Lambda_{\rm H} = \Lambda_{\rm B} = 2$, $K_2/K_1 = K_3/K_1 = 1$ (isotropic material) and $\mathcal{E}_{\rm T} = 8$. The influence of these parameters on the response of the structure is explored in the following examples, where we change one parameter while keeping the other reference parameters fixed.

4.1. Reference case

Figure 2(a) depicts the Poincaré SOS corresponding to the reference geometrical parameters, material constants and energy level. The active 2D phase space includes all the possible combinations of $\dot{\lambda}_{a}$ and λ_{a} that fulfill the imposed energy level $\mathcal{E}_{T} = 8$ and the fact that λ_{z} must be greater than 0 (as discussed in the previous section). The blue curve enclosing the active phase space has been obtained from (82) as the combinations of λ_a and $\dot{\lambda}_a$ that minimize $\dot{\lambda}_z$. Following the indications provided in the previous section for the interpretation of the Poincaré SOS, we observe that the phase space is divided into a chaotic domain (plotted in red), and several regions corresponding to periodic and quasi-periodic trajectories (plotted in brown, blue, purple...). It becomes apparent that the nature of the radial motion of the structure depends on the initial conditions considered. This is clearly illustrated in Fig. 2(b), which shows the time evolution of λ_a for those two trajectories whose initial conditions are explicitly labeled in Fig. 2(a). The solid blue curve corresponds to a periodic solution (single point in the phase space). The oscillatory motion of the structure shows a clear repetitive pattern in which the amplitude and period of the oscillations do not depend on τ . The dashed red curve, on the contrary, does not show any clear pattern; the amplitude and period of the oscillations are strongly dependent on the loading time τ . These qualitative observations can be quantified using the maximal Lyapunov characteristic exponent (LCE) that provide a measurement of the exponential divergence in time of two trajectories starting from arbitrary-close initial conditions. While a rigorous derivation of the LCE can be found elsewhere [38, 39, 37], the basic formulation and the renormalization scheme used to compute the LCE are presented in Appendix A for completeness. A positive LCE indicates that the response of the system is chaotic, while a zero LCE implies that the response of the system is periodic or quasi-periodic. This is exactly what we observe in Fig. 2(c), which shows, for the two trajectories investigated in Fig. 2(b), the evolution of their corresponding maximal Lyapunov exponents, denoted by λ_k , where k is a coefficient that varies linearly with time (see Appendix A). The LCE approaches zero for the periodic orbit as $k \to \infty$, and it tends to 0.033 ± 0.0001 for the chaotic orbit, where the error has been computed as the typical deviation of the last 200 values of k considered.

In the next section we investigate the influence of the total energy supplied to the cylinder in the dynamic response of the system.



Figure 2: (a) Poincaré SOS corresponding to the reference geometrical parameters, material constants and energy level: $\Lambda_H = \Lambda_B = 2$, $K_2/K_1 = K_3/K_1 = 1$ and $\mathfrak{E}_T = 8$. (b) Time evolution of the stretch in the inner face of the cylinder λ_a for the periodic and chaotic trajectories indicated in subfigure (a). (c) Maximal Lyapunov exponent λ_k as a function of k for the periodic and chaotic trajectories indicated in subfigure (a). (For interpretation of the references to color in the text, the reader is referred to the web version of this article).

4.2. Influence of the energy supplied to the system

Figure 3 presents the Poincaré SOS corresponding to the reference geometrical parameters $\Lambda_{\rm H} = \Lambda_{\rm B} = 2$, material constants $K_2/K_1 = K_3/K_1 = 1$, and two different values of the total energy supplied to the system. $\mathcal{E}_{T} = 6$ (lower than the reference value) is considered in Fig. 3(a), and $\mathcal{E}_{T} = 10$ (greater than the reference value) is considered in Fig. 3(b). The black dashed line in Figs. 3(a) and 3(b) corresponds to the curve enclosing the active phase space of the reference configuration presented in Fig. 2(a). Note that, from this point on, this curve will be plotted in all the Poincaré SOS presented in the paper with the aim to compare the size of the active phase spaces. A decrease in the energy supplied to the system reduces the active area of the phase space $\{\lambda_{a}, \lambda_{a}\}$ and promotes the development of periodic and quasi-periodic motions rather than chaotic motion. For the case of $\mathcal{E}_{T} = 6$ shown in Fig. 3(a), the cylinder does not exhibit chaotic response for any set of initial conditions. The whole chaotic region observed in Fig. 2(a) for $\mathcal{E}_T = 8$ has been transformed for the case of $\mathcal{E}_{\rm T} = 6$ to a set of closed curves and single points, which reveal the periodic and quasiperiodic nature of the radial motion of the structure. Note that the closed curves and single points, which are generated by the same trajectory, are plotted with the same color. Similarly, an increase in the energy supplied to the system enlarges the active area of the phase space $\{\lambda_a, \lambda_a\}$ and allows for the development of chaotic trajectories. For the case of $\mathcal{E}_T = 10$ shown in Fig. 3(b), the response of the structure is chaotic for most of the initial conditions and the periodic and quasi-periodic trajectories are confined within a few (relatively) small islands. We have checked that for values of \mathcal{E}_{T} greater than 10 the chaotic region continues growing until it virtually covers the whole phase space.

The process followed by the system from the structured Poincaré SOS presented in Fig. 3(a) to the extended chaotic behavior of Fig. 3(b) is a well-known route to chaos usually referred to as break-down of quasi-periodic tori [38, 39, 37]. For a system with two degrees of freedom, the Kolmogorov-Arnold-Moser (KAM) theorem [40] states that some phase space tori, those associated with quasi-periodic motion, survive when a previously integrable system is made slightly nonintegrable. However, as the perturbation increases (in this case the applied energy), the original KAM-tori decomposes into smaller and smaller tori in a manner consistent with the Poincare-Birkhoff theorem [41]. Specifically, each KAM-torus is replaced by an even number of new fixed-points, one half of which are elliptic and the other hyperbolic. Around each elliptic point will be a series of elliptic orbits, while associated with the hyperbolic points will be a series of heteroclinic orbits. It is the tangle of the stable and unstable manifolds of the hyperbolic points that is mainly responsible for the emergence of chaos (see Melnikov Theory for further details [42]). This structure of the solution has been identified in Fig. 2(a) where the elements of one chain of alternating elliptic and hyperbolic points have been labeled. It can be seen that, for $\mathcal{E}_{\rm T} = 8$, chaotic behavior has already started in the small regions close to the labeled hyperbolic points.

In the next section we investigate the influence of the dimensions of the cylinder on the dynamic response of the system.



Figure 3: Poincaré SOS corresponding to the reference geometrical parameters $\Lambda_H = \Lambda_B = 2$ and material constants $K_2/K_1 = K_3/K_1 = 1$, and two different values of the total energy supplied to the system. (a) $\mathcal{E}_T = 6$ and (b) $\mathcal{E}_T = 10$. (For interpretation of the references to color in the text, the reader is referred to the web version of this article).

4.3. Influence of the specimen dimensions

Figure 4 depicts the Poincaré SOS corresponding to the reference non-dimensional thickness $\Lambda_{\rm B}$ = 2, material constants $K_2/K_1 = K_3/K_1 = 1$, supplied energy $\mathcal{E}_T = 8$, and two different values of the nondimensional height of the cylinder $\Lambda_{\rm H}$. Namely, $\Lambda_{\rm H} = 1.7$ (lower than the reference value) is considered in Fig. 4(a) and $\Lambda_{\rm H} = 15$ (greater than the reference value) is considered in Fig. 4(b). A decrease in the height of the cylinder enlarges the active surface of the phase space since a smaller structure develops radial oscillations which are faster and have greater amplitude for the same supplied energy than for a larger structure. Moreover, note that a decrease in $\Lambda_{\rm H}$ allows for the development of chaotic motions. For the case of $\Lambda_{\rm H} = 1.7$ shown in Fig. 4(a), the response of the structure is chaotic for most of the initial conditions. Furthermore, we have checked that, for values of $\Lambda_{\rm H}$ smaller than 1.7, the chaotic region extends virtually to the entire phase space. Similarly, an increase in the height of the cylinder reduces the active surface of the phase space since, as expected, a bigger structure develops radial oscillations which are slower and have smaller amplitude. An increase in $\Lambda_{\rm H}$ promotes the development of periodic and quasi-periodic trajectories rather than chaotic. In fact, for the case of $\Lambda_{\rm H} = 15$ shown in Fig. 4(b), the response of the system is never chaotic. In addition, we have checked that for a value of $\Lambda_{\rm H}$ equal to or greater than 15 (the tube becomes very long) the three dimensional tori, which describes the motion of the structure in the phase space $\{\lambda_{a}, \lambda_{z}, \lambda_{a}\}$, collapse to different curves contained in the plane $\lambda_{z} = 1$. In other words, the response of the cylinder approaches plane strain conditions, i.e., $\lambda_z \sim 1$ and $\lambda_a \gg \lambda_z$. Therefore, the curves depicted in Fig. 4(b) are no longer Poincaré SOS but *complete* phase portraits in the { $\dot{\lambda}_a$, λ_a } phase space. Each closed curve in Fig. 4(b) corresponds to a periodic solution, i.e., we recover the 1D solution developed by Knowles [1]. It is only in the vicinity of the central point (1,0) where the plane strain conditions are not satisfied since both stretches are again of the same order of magnitude $O(\lambda_a) \sim O(\lambda_z) \sim 1$.



Figure 4: Poincaré SOS corresponding to the reference non-dimensional thickness $\Lambda_B = 2$, material constants $K_2/K_1 = K_3/K_1 = 1$ and supplied energy $\mathfrak{E}_T = 8$, and two different values of the non-dimensional height of the cylinder. (a) $\Lambda_H = 1.7$ and (b) $\Lambda_H = 15$. Note: figure (b) is not really a Poincaré SOS but a complete phase portrait in the { $\dot{\lambda}_a, \lambda_a$ } space.

In Fig. 5 we show the Poincaré SOS corresponding to the reference non-dimensional height $\Lambda_{\rm H} = 2$, material constants $K_2/K_1 = K_3/K_1 = 1$, supplied energy $\mathcal{E}_{\rm T} = 8$, and two different values of the nondimensional thickness of the cylinder $\Lambda_{\rm B}$. Specifically, Fig. 5(a) depicts the solutions for the case of $\Lambda_{\rm B} = 1.7$ (lower than the reference value), while Fig. 5(b) depicts $\Lambda_{\rm B} = 15$ (greater than the reference value). As expected, a decrease in the thickness of the hollow cylinder increases the active surface of the phase space and promotes chaotic response of the structure. For the case of $\Lambda_{\rm B} = 1.7$, shown in Fig. 5(a), the response of the structure is chaotic for most of the initial conditions. On the other hand, an increase of $\Lambda_{\rm B}$ reduces the surface of the active phase space and promotes the development of quasi-periodic and periodic responses. In fact, for the case of $\Lambda_{\rm B} = 15$, depicted in Fig. 5(b), the structure does not present chaotic motion for any initial conditions. Furthermore, we have checked that for a value of $\Lambda_{\rm B}$ equal to or greater than 15 (the tube becomes very short; like a large plate) the three dimensional tori, which describe the motion of the structure in the phase space $\{\lambda_{a}, \lambda_{z}, \dot{\lambda}_{a}\}$, collapse to different curves contained in the plane $\lambda_{z} = 1$ (as for the case of $\Lambda_{H} = 15$ discussed in previous paragraph). The curves depicted in Fig. 5(b) are *complete* phase portraits in the $\{\dot{\lambda}_{a}, \lambda_{a}\}$ space and define periodic solutions. In other words, the problem becomes 1D. As in the previous case, in the vicinity of the central point (1,0) both stretches are again of the same order of magnitude $O(\lambda_{a}) \sim O(\lambda_{z}) \sim 1$.

It is apparent from this analysis that when the volume of the structure is large enough, either because the tube is infinitely long in the axial direction or infinitely thick in the radial one, the system tends to show periodic behaviors.



Figure 5: Poincaré SOS corresponding to the reference non-dimensional height $\Lambda_H = 2$, material constants $K_2/K_1 = K_3/K_1 = 1$ and supplied energy $\mathfrak{E}_T = 8$, and two different values of the non-dimensional thickness of the cylinder. (a) $\Lambda_B = 1.7$ and (b) $\Lambda_B = 15$. Note: figure (b) is not really a Poincaré SOS but a complete phase portrait in the { $\dot{\lambda}_a, \lambda_a$ } space.

In the next section we investigate the influence of the mechanical anisotropy of the material on the dynamic response of the system.

4.4. Influence of the material anisotropy

Figure 6 presents the Poincaré SOS corresponding to the reference geometrical parameters $\Lambda_{\rm H} = \Lambda_{\rm B} = 2$, circumferential stiffness $K_2/K_1 = 1$, supplied energy $\mathcal{E}_{\rm T} = 8$, and two different values of the axial stiffness K_3/K_1 . Fig. 6(a) depicts $K_3/K_1 = 0.5$ (smaller than the reference value) and Fig. 6(b) depicts $K_3/K_1 = 100$ (greater than the reference value). Notice that the area of the active phase space in the $\{\dot{\lambda}_a, \lambda_a\}$ plane remains unchanged under variations in the axial stiffness. This area may be understood as combined measure of the kinetic and strain energies associated with the radial motion, which is revealed to be independent of the axial stiffness. A decrease in the axial stiffness allows for the development of chaotic motion. In fact, for the case of $K_3/K_1 = 0.5$ illustrated in Fig. 6(a), the structure presents chaotic response for most of the initial conditions. On the other hand, an increase in the axial stiffness promotes periodic and quasi-periodic trajectories. In particular, for the case of $K_3/K_1 = 100$ illustrated in Fig. 6(b), the tori describing the 3D motion of the structure collapse to several curves contained in the plane $\lambda_z = 1$ (as for the cases of $\Lambda_H = 15$ and $\Lambda_B = 15$ presented in Section 4.3) and the motion of the cylinder is periodic. The curves depicted in Fig. 6(b) are *complete* phase portraits in the { $\dot{\lambda}_a, \lambda_a$ } space and define periodic solutions. In the limit of $K_3/K_1 >> K_2/K_1$ the problem becomes 1D as the vibration in the axial direction is negligible compared with the radial one. Note that due to the mechanical anisotropy of the material, the central point of the orbits is (1,0.85) instead of (1,0) in the examples presented in Section 4.3, where the material behavior was considered isotropic.



Figure 6: Poincaré SOS corresponding to the reference geometrical parameters $\Lambda_H = \Lambda_B = 2$, circumferential stiffness $K_2/K_1 = 1$ and supplied energy $\mathcal{E}_T = 8$, and two different values of the axial stiffness. (a) $K_3/K_1 = 0.5$ and (b) $K_3/K_1 = 100$. Note: figure (b) is not really a Poincaré SOS but a complete phase portrait in the $\{\dot{\lambda}_a, \lambda_a\}$ space.

Finally, in Fig. 7 we present the Poincaré SOS corresponding to the reference geometrical parameters $\Lambda_{\rm H} = \Lambda_{\rm B} = 2$, axial stiffness $K_3/K_1 = 1$, supplied energy $\mathcal{E}_{\rm T} = 8$, and two different values of the circumferential stiffness. Namely, Fig. 7(a) depicts $K_2/K_1 = 0.4$ (smaller than the reference value), while Fig. 7(b) presents

the solutions for $K_2/K_1 = 100$ (greater than the reference value). A decrease in the circumferential stiffness increases the active surface of the phase space due to the increase of the amplitude of the radial oscillations, i.e., the phase space enlarges along the λ_a axis. A decrease in K_2/K_1 also promotes the development of chaotic motions. In particular, for the case of $K_2/K_1 = 0.4$, depicted in Fig. 7(a), the chaotic region extends practically to the entire phase space. On the other hand, an increase in the circumferential stiffness reduces the amplitude of the radial oscillations and, therefore, the active area of the phase space. An increase of K_2/K_1 also promotes periodic and quasi-periodic motions. Specifically, for the case of $K_2/K_1 = 100$ depicted in Fig. 7(b), the structure does not show chaotic motion for any initial conditions. We have checked that for values of K_2/K_1 equal to or greater than 100 the tori, which describe the 3D motion of the structure, collapse to different curves contained in the plane $\lambda_z = 1$ (as for the cases of $\Lambda_H = 15$, $\Lambda_B = 15$ and $K_3/K_1 = 100$ discussed in previous paragraphs) and the motion of the cylinder becomes periodic. The curves depicted in Fig. 7(b) are again *complete* phase portraits in the { $\dot{\lambda}_a, \lambda_a$ } space, i.e., the problem becomes 1D. Note that, due to the mechanical anisotropy of the material, the central point of the orbits is (1,0.35) instead of (1,0) in the examples presented in Section 4.3.



Figure 7: Poincaré SOS corresponding to the reference geometrical parameters $\Lambda_H = \Lambda_B = 2$, axial stiffness $K_3/K_1 = 1$ and supplied energy $\mathfrak{E}_T = 8$, and two different values of the circumferential stiffness. (a) $K_2/K_1 = 0.4$ and (b) $K_2/K_1 = 100$. Note: figure (b) is not really a Poincaré SOS but a complete phase portrait in the $\{\dot{\lambda}_a, \lambda_a\}$ space.

5. Summary and conclusions

In this paper we have investigated the large-amplitude axisymmetric free vibrations of an incompressible nonlinear elastic cylindrical structure. The material has been described as orthotropic and hyperelastic using the constitutive model developed by Rubin and Jabareen [29, 30]. The cylinder has been modeled using the theory of a generalized Cosserat membrane which allows for finite deformations that include uniform stretching along the longitudinal axis of the structure. We have conducted a parametric analysis to identify the influence that the initial conditions, the energy supplied to the system, the dimensions of the specimen and the material anisotropy have on the dynamic response of the structure. We have used Poincaré maps and Lyapunov exponents to assess the nature of the motion of the cylinder and the following conclusions have been obtained:

- Initial conditions: the dynamic response of the system has been proved to be very sensitive to the initial conditions. For a given set of energy supplied to the system, geometrical parameters and material constants, the response of the cylinder may turn from periodic to quasi-periodic and chaotic with slight variations in the stretch and stretch rate initially imposed to the structure.
- Energy supplied to the system: as the energy supplied to the system increases, the motion of the system turns from periodic and quasi-periodic to chaotic. The tori which contain the periodic and quasi-periodic trajectories of the structure are gradually destroyed with the increase of the supplied energy, which gives rise to stochastic behavior in the dynamic response of the system. We have shown that for sufficiently small energies the response of the structure is always periodic or quasi-periodic, while for sufficiently large energies it is always chaotic.
- Specimen dimensions: as the volume of the cylinder decreases, the system is more prone to develop chaotic motion. The smaller the size of the cylinder, the smaller the amount of energy required to destroy the tori which contain the periodic and quasi-periodic trajectories of the structure. It is noted that the general 2D formulation developed in this paper includes the specific cases of an infinitely long cylinder and an infinitely large plate for which the problem can be modeled within a 1D framework and the response of the structure becomes periodic.
- Material anisotropy: the structure is more prone to develop chaotic motion as the stiffness along the axial and circumferential direction decreases with respect to the stiffness along the radial direction. On the other hand, if the axial stiffness or the circumferential stiffness of the material are sufficiently high, the response of the cylinder becomes periodic. This is a key result of this research that demonstrates the influence of anisotropic material properties on the nature of the dynamic response of the system.

This research has generalized the 1D approaches developed by Huilgol [22], Shahinpoor [23] and Mason and Maluleke [28] –who studied infinitely long samples assuming plane strain conditions along their axial direction– to a 2D framework which considers structures of finite axial height. Inclusion of uniform stretch in the axial direction of the structure enables the system to show quasi-periodic and chaotic responses, which do not appear in the 1D approximation. The extension of the present analysis to include external forcing through the applied pressures and material dissipation would be interesting to consider in future work. However, it is not known if the additional surface vibrations that occur when the surfaces of the cylinder are traction-free pointwise will limit the utilization of our results in practical applications.

Appendix A. Lyapunov exponents

Consider a fiducial trajectory \mathbf{x}_{f} and a test trajectory \mathbf{x}_{t} that start very close to each other. Both trajectories are solutions of the equations of motion (75)-(76)

$$\frac{\mathrm{d}\mathbf{x}_{\mathrm{f}}}{\mathrm{d}\tau} = \mathbf{G}(\mathbf{x}_{\mathrm{f}}), \qquad \frac{\mathrm{d}\mathbf{x}_{\mathrm{t}}}{\mathrm{d}\tau} = \mathbf{G}(\mathbf{x}_{\mathrm{t}}), \tag{A.1}$$

For the problem addressed in this paper we have that $\mathbf{x}(\tau) = (\lambda_{a}(\tau), \lambda_{z}(\tau), \dot{\lambda}_{a}(\tau), \dot{\lambda}_{z}(\tau))$. The distance between these two trajectories at any time τ is $\mathbf{d}(\tau) = \mathbf{x}_{t}(\tau) - \mathbf{x}_{t}(\tau)$, which evolves in time according to

$$\dot{\mathbf{d}}(\tau) = \dot{\mathbf{x}}_{f}(\tau) - \dot{\mathbf{x}}_{t}(\tau) = \mathbf{G}(\mathbf{x}_{f}) - \mathbf{G}(\mathbf{x}_{t}).$$
(A.2)

Considering that both trajectories remain close enough to each other, we can use the following linear approximation

$$\mathbf{G}(\mathbf{x}_{t}) \approx \mathbf{G}(\mathbf{x}_{f}) + \left. \frac{\partial \mathbf{G}}{\partial \mathbf{x}} \right|_{\mathbf{x}_{f}} \mathbf{d}(\tau).$$
 (A.3)

After substituting Eq. (A.3) in Eq. (A.2), we obtain the following relation

$$\dot{\mathbf{d}}(\tau) = \left. \frac{\partial \mathbf{G}}{\partial \mathbf{x}} \right|_{\mathbf{x}_{\mathrm{f}}} \mathbf{d}(\tau). \tag{A.4}$$

The previous ordinary differential equation admits the following solution

$$\mathbf{d}(\tau) = \mathbf{d}_0 \mathbf{e}^{\boldsymbol{\lambda}\tau},\tag{A.5}$$

where $d_0 = \|\mathbf{d}(0)\|$ is the initial distance between the trajectories and $\boldsymbol{\lambda} = \frac{\partial \mathbf{G}}{\partial \mathbf{x}}\Big|_{\mathbf{x}_f}$ is the so-called locally exponential divergence rate, which can also be written as

$$\boldsymbol{\lambda} = \frac{1}{\tau} \ln \left(\frac{\mathbf{d}(\tau)}{\mathbf{d}_0} \right). \tag{A.6}$$

Note that periodic and quasi-periodic trajectories may present a transient behavior for short loading times that looks like chaotic, but it is not. Furthermore, the initial distance between the fiducial and the test trajectory must be infinitesimal to ensure the validity, at any time, of the linear approximation (A.3). In order to take into account these two specific issues in the definition of the Lyapunov exponents λ_L , we take the following limits

$$\boldsymbol{\lambda}_{L} = \lim_{\substack{\mathbf{d}_{0} \to 0\\ \tau \to \infty}} \left[\frac{1}{\tau} \ln \left(\frac{\mathbf{d}(\tau)}{\mathbf{d}_{0}} \right) \right].$$
(A.7)

In a n-dimensional phase space there is a *spectrum* of n Lyapunov exponents –one per dimension– which indicate whether or not there is an exponential separation in each dimension between trajectories upon time. The maximum of this set is the so-called maximal Lyapunov exponent or the Lyapunov characteristic exponent (LCE). When one of the Lyapunov exponents tends to a positive value, the system is referred to as chaotic. On the other hand, when one of the Lyapunov exponents tends to zero, the others do also (see Liouville theorem at, e.g., [38, 39, 37]), and the system is referred to as periodic or quasi-periodic.

In order to compute numerically the Lyapunov exponents, we have followed the renormalization scheme introduced by Benettin and collaborators [43, 44, 45, 46]. While this method can be used to obtain the whole set of Lyapunov exponents, in this work we have implemented a simplified version to compute only the Lyapunov characteristic exponent. The main steps to obtain the LCE for an n-dimensional phase space are described below, while in Fig. A.8 we present a scheme of these steps for the case of a simplified bi-dimensional phase space $\{\dot{\lambda}_a, \lambda_a\}$:

- We define the initial conditions of the fiducial and test trajectories as $\mathbf{x}_{f0} = (\lambda_a(0), \dot{\lambda}_a(0), \lambda_z(0), \dot{\lambda}_z(0))$ and $\mathbf{x}_{t0} = \mathbf{x}_{f0} + \mathbf{d}_0$.
- At time τ_r the positions of both trajectories are given by $\mathbf{x}_{f1} = (\lambda_a(\tau_r), \dot{\lambda}_a(\tau_r), \lambda_z(\tau_r), \dot{\lambda}_z(\tau_r))$ and $\mathbf{x}_{t1} = \mathbf{x}_{f1} + \mathbf{d}_1$, where $\mathbf{d}_1 = \mathbf{d}(\tau_r)$ is the distance vector at time τ_r . Then, the position of the test trajectory is *renormalized* along the vector \mathbf{d}_1 until the distance between both trajectories becomes \mathbf{d}_0 again (see Fig. A.8).
- The process is systematically repeated at integer multiples of τ_r . For each time $\tau = k\tau_r$ with k = 1, 2, 3...the normalized position vector of the test trajectory is computed as

$$\mathbf{x}_{tk}' = \mathbf{x}_{fk} + \frac{d_0}{d_k} \mathbf{d}_k, \tag{A.8}$$

where $d_k = \|\mathbf{d}_k\|$.

• Then, the local exponential divergence rate at each time $\tau = k\tau_r$ is given by

$$\lambda_{k} = \frac{1}{k\tau_{r}} \sum_{j=1}^{k} \ln\left(\frac{d_{j}}{d_{0}}\right). \tag{A.9}$$

• Finally, the Lyapunov characteristic exponent (λ_{LCE}) is computed as the limit of λ_k when $k \to \infty$.



Figure A.8: Schematic representation of the renormalization scheme developed by Benettin and co-workers [43, 44, 45, 46].

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