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SIMPLE MECHANISMS TO IMPLEMENT THE CORE OF COLLEGE ADMISSIONS PROBLEMS

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Abstract

This paper analyzes simple mechanisms implementing (subselections of) the core correspondence of matching markets. We provide a sequential mechanism which mimics a matching procedure for many-to-one real life matching markets. We show that only core allocations should be attained when agents act strategically faced with this mechanism. We also provide a second mechanism to implement the core correspondence in Subgame Perfect Equilibrium.

Keywords:Matching Markets, College Admissions Problems, Mechanism Design

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1 Introduction

In this paper we introduce two mechanisms that implement the stable correspondence in a college admissions problem. These are stylized versions of contractual processes in bilateral markets where monetary transfers are either irrelevant or can be embodied on agents' preferences.

Matching markets have been extensively analyzed from a game-theoretical point of view (see Roth and Sotomayor [10] for a detailed state of the art until 1990). In this framework, Roth [8] and Alcalde and Barberà [2] have shown the existence of incentives for agents to misreport their true preferences. These incentives appear when agents are faced with some mechanisms selecting allocations that satisfy certain "desired" properties. These results give us a reason to study the implementability of stable solutions for the college admissions problem.

Core correspondence implementability in college admissions problems was first approached by Kara and Sönmez [6]. They showed that the core correspondence can be implemented in Nash equilibrium. However, they do not provide a simple mechanism that can be used in real life situations. They also show that no subselection of the core is Nash implementable. There are two problems that are still open in the college admissions problems framework. The first one is the design of natural mechanisms to solve the implementability of the core, and the second one is the analysis of implementability of core subselections. This paper provides positive answers to both of them.

Alcalde [1] studied both questions for a particular case of matching problems, the marriage market problem. He provides some positive answers to the implementation problem using simple mechanisms. In particular, he implements the core correspondence. He also provides a mechanism to implement the extreme points in the core correspondence.

A way to deal with the problem is to analyze simple real life mechanisms. Romero–Medina [7] studies the mechanism employed by the Spanish University system to allocate new students to colleges. He shows that this mechanism can select unstable outcomes. While, core allocations would be reached when students behave strategically. This results cannot be applied to the more general framework in which we are interested on, since the matching procedure studied in [7] does not allow universities to behave strategically.

Let us now introduce the mechanism that implements the core correspondence in bilateral matching markets. It is a procedure in which agents take decisions sequentially. In the first stage, students simultaneously send an application form to no more than one college. In the second stage, each college selects its best set of students, among applicants. Colleges' decisions determine the matching which results from agents interaction. Thus, the mechanism to be analyzed captures essential aspects which hold in real life college admissions problems. Firstly, we model sequential interactions among agents on both sides of the market, reflecting an adjustment process to reach stable allocations. Secondly, agents on one side of the market (students) adopt an "active" role, making offers, whereas the aptitude shown by agents on the other side (colleges) can be considered as "passive": they only accept or reject the offers they receive. In fact, the mechanisms analyzed below reflect the idea of the classical algorithm developed by Gale and Shapley [4].

Since our mechanism reflects a sequential decision process, the first equilibrium concept we study is Subgame Perfect Nash Equilibrium (SPE). Therefore, our results can not be seen as a consequence of those obtained by Kara and Sönmez [6]. We are also interested in analyzing the effects of the agents' commitments when selecting the strategies to be played. Thus, we also analyze the outcomes that can be expected when agents select strategies that constitute a Strong Subgame Perfect Nash Equilibrium (SSPE) for this game. That is, strategies that are not only a SPE but also Strong Nash Equilibrium for this mechanism. Under such an equilibrium concept, this mechanism selects the students' optimal stable matching. This matching is the (unique) matching weakly preferred by every student to any other stable matching.

Notice that the main feature of the mechanisms that we present in this paper is that they constitute reasonable proposals for effective design. Following Jackson [5], the mechanisms used to implement social choice correspondences should have "natural" features. Applicability to real life situations can be one of these features. The mechanism we present in this paper satisfies this requirement, and furthermore it is traditionally used in many real life situations. There are two main reasons for that: first, it is straightforward to obtain the message space of each agent from its own preferences. Second, any individual is able to evaluate the consequences of her strategy without using a sophisticated analysis of the mechanism.

The rest of the paper is organized as follows. Section 2 introduces the basic model. Section 3 presents and analyzes the mechanism, to be called "the students-propose-and-colleges-choose" mechanism. Section 4 studies the effects of an interchange in agents' roles in the mechanism introduced in Section 3. Conclusions are in Section 5.

2 The model

Consider a college admission problem with n colleges and m students. Let $C = \{c_1, ..., c_n\}$ and $S = \{s_1, ..., s_m\}$ be the set of colleges and students, respectively. Each college has preferences, $P(c)$, over sets students. $P(c)$ is assumed to be a linear order on 2^S. Each student's preferences, $P(s)$, is described by a linear order on $C \cup \{s\}$. A college admissions problem is fully described by a triplet $\left\{C, S; \underset{\sim}{P}\right\}$, where $\underset{\sim}{P} = \{P(c_1), ..., P(c_n), P(s_1), ..., P(s_m)\}$ is a list containing a full description of the agents' preferences and is called a profile.

An allocation for such a problem, or *matching*, is a mapping μ from $C \cup S$ into $2^S \cup C$ satisfying

(i) for all $c \in C$, $\mu(c) \in 2^S$,

(ii) for all $s \in S$, $\mu(s) \in C \cup \{s\}$, and

(iii) for each pair $(c, s) \in C \times S$, $[\mu(s) = c \Longleftrightarrow s \in \mu(c)]$.

>From now on we will consider C and S to be fixed sets. Thus we can identify a college admissions problem $\left\{C, S; \underset{\sim}{P}\right\}$ with the preference profile $\underset{\sim}{P}$.[1] Let $\mathcal{M}$ be the set of all possible matchings μ. Finally, $\underset{\sim}{\mathbb{P}}$ denotes the set of (potential) matching markets.

[1] For the sake of simplicity, we use the same notation for a preference profile and for the related college admissions problem. The context will precise if $\underset{\sim}{P}$ denotes a matching problem or simply a preference profile.

Let $\underset{\sim}{P}$ be a matching market. Given a set of students $A \subseteq S$, we denote by $Ch_c(A)$ the maximal element on 2^A under the linear order $P(c)$.

Definition 1 *A matching μ is said to be individually rational for $\underset{\sim}{P}$ iff*

(i) $Ch_c(\mu(c)) = \mu(c)$ for all $c \in C$, and

(ii) for all $s \in S$, $c \in C$ $\quad [sP(s)c \Longrightarrow s \notin \mu(c)]$.

Definition 2 *Let μ be a matching for $\underset{\sim}{P}$. We say that μ is blocked by a pair $(c, s) \in C \times S$ iff*

(i) $c\ P(s)\ \mu(s)$, and

(ii) $s \in Ch_c\left(\mu(c) \cup \{s\}\right)$.

A pair (c, s) which satisfies the above two conditions is called a blocking pair for μ.

Definition 3 *Let μ be a matching for $\underset{\sim}{P}$. We say that μ is (pair-wise) stable if when is individually rational and there is no pair blocking it. Let $\mathcal{N}\left(\underset{\sim}{P}\right)$ denote the set of stable allocations for the problem $\underset{\sim}{P}$.*

Finally, we assume that colleges' preferences, regarding to groups of students, are substitutive. That is, for any two students $s \neq s'$ if s belongs to $Ch_c(A)$, then she will also belong to $Ch_c(A \setminus \{s'\})$. This is an usual assumption in the related literature and it guarantees non-emptiness of the set of stable allocations (see Theorem 6.5 in [10]). Notice that when colleges' preferences are substitutive, the set of (pair-wise) stable allocations coincides with the core of that college admissions problem.[2] That is, given a stable allocation, no group of

[2] Proposition 6.4 in Roth and Sotomayor [10] establishes that stability and pair-wise stability are equivalent concepts in college admissions problems with substitutive preferences.

agents can find a matching to improve the utility of all its members without being matched with agents outside this group. Furthermore, if colleges' preferences satisfy substitutability, the set of stable allocations has a latticial structure. This property guarantees (i) the existence of an unique stable allocation which is Pareto optimal from the point of view of students and, (ii) the existence of an unique allocation which is Pareto optimal from the point of view of colleges (when restricted to the set of stable matchings).

The concept of implementation used throughout the paper is well-known in the literature. We next formalize this concept for both the Subgame Perfect Nash Equilibrium (SPE) and the Strong Subgame Perfect Nash Equilibrium (SSPE) cases. Let $\mathcal{E}_k$ be the set of strategies for agent k and let $\mathcal{E} = \underset{x \in C \cup S}{\times} \mathcal{E}_x$ be the set of strategy profiles. Associated to each strategy profile $\tilde{e} \in \mathcal{E}$ we can define a message profile $m(\tilde{e})$, or simply $\tilde{m}$, which describes the action taken by each individual given the strategy they choose. A matching mechanism is described by the set of strategies available to each agent, and an outcome function γ that assigns a matching to each profile of messages. We say that a matching mechanism *implements* a solution concept, say χ, in the (Strong) Subgame Perfect Nash Equilibrium if (i) for any $\tilde{e}$, (Strong) Subgame Perfect Equilibrium of the game $\Gamma := \left\{C, S; \underset{\backsim}{P}; \gamma\right\}$, $\gamma(m(\tilde{e}))$ belongs to $\chi\left(\underset{\backsim}{P}\right)$ and (ii) for each μ in $\chi\left(\underset{\backsim}{P}\right)$ there exists a (Strong) SPE for Γ, say $\tilde{e}'$, such that $\gamma(m(\tilde{e}')) = \mu$.

3 The "students-propose-and-colleges-choose" mechanism

This section is devoted to analyze a matching mechanism that mimics real-life matching procedures. In the mechanism we propose, each student selects the college in which she wants study. Then, once each college had received all its application forms, it accepts its most preferred set of students.

Let us introduce the mechanism, that we are going to call the students-propose-and-colleges-choose mechanism. This is a two-stage game. In the first stage, students have to decide. Each student's message space coincides with the set of colleges and she being unmatched, $C \cup \{s\}$. In the second stage colleges, knowing students' messages, select the set of students that they want to admit. Thus, each college's message space coincides with 2^S. Let $m(k)$ denote the message of agent $k \in C \cup S$, and $\tilde{m}$ be an ordered vector containing the messages of all the agents.

The outcome function, denoted by Φ^{SC}, selects a matching which is defined as follows: $\Phi^{SC}(\tilde{m}) = \mu_{\tilde{m}}$, where for any s in S,

$$\mu_{\tilde{m}}(s) = \begin{cases} m(s) & \text{if } s \in m(m(s)) \\ s & \text{otherwise} \end{cases}$$

and, for each c in C,

$$\mu_{\tilde{m}}(c) = \{s \in m(c) \mid c = m(s)\}$$

Theorem 4 *The "students-propose-and-colleges-choose" mechanism implements in SPE the core correspondence of college admissions problems.*

Proof. First, we show that every SPE outcome is a stable matching relative to agents' preferences. Let $\tilde{m}'$ be a SPE for $\Gamma^{SC} := \left\{C, S; \underset{\sim}{P}; \Phi^{SC}\right\}$. One can check that, at the second stage, each college has a dominant strategy, namely $m'(c) = Ch_c\left(\{s \in S \mid c = m'(s)\}\right)$.[3] Thus, $\Phi^{SC}(\tilde{m}')$ should be an individually rational matching for $\underset{\sim}{P}$.

Let us suppose that $\Phi^{SC}(\tilde{m}')$ is not in $\mathcal{N}\left(\underset{\sim}{P}\right)$, then there should be a blocking pair, say (c, s), in $C \times S$. Since all the students play simultaneously, this can not be the case, because student s can reach higher utility by playing $m''(s) = c$. Notice that, in the second stage, c's message has to contain such a student. A contradiction.

On the other hand, let μ be a stable matching for $\underset{\sim}{P}$. Let us consider the following strategies for the agents. Each student message (and strategy) is $m(s) = \mu(s)$. In the second stage any college's strategy is its dominant strategy, its message being $m(c) = \mu(c)$. This constitutes a SPE for the related game whose outcome coincides with μ, which yields the desired result. ∎

Since the Social Choice Correspondence that we study is the core, we analyze the influence of agents' behavior on the expected outcome when commitment is allowed. In such a case Strong Subgame Nash Equilibrium seems to be a minimal requirement to be fulfilled by our predictions. The analysis of such an equilibrium concept is the aim of Theorem 5.

Theorem 5 *The "students-propose-and-colleges-choose" mechanism implements in SSPE the students optimal stable allocation.*

[3] Notice that such a strategy is not its unique best response. In fact, a necessary and sufficient condition for $m(c)$ to be a college c's best response to students' strategies, $m(s)$, is to satisfy that $m(c) \cap \{s \in S \mid c = m(s)\} = Ch_c(\{s \in S \mid c = m'(s)\})$. Nevertheless, all these messages are strategically equivalent. Since we are interested in equilibrium payoffs rather than equilibrium strategies, we do not pay attention to these strategies.

Proof. First, we are going to show that the students' optimal stable matching can be supported by a SSPE. Let $\underset{\sim}{P}$ be a matching market, and μ^S be its students' optimal stable allocation. Consider the following strategies: for any s in S, $m(s) = \mu^S(s)$ and, for each c in C, $m(c) = \arg\max P(c)$ on $\{s \in S \text{ s.t. } c = m(s)\}$. It is straightforward to see that these strategies constitute a SSPE whose outcome is μ^S.

On the other hand, let $\tilde{m}'$ be a SSPE yielding $\mu \neq \mu^S$ as the outcome. We will show that it is not possible. Notice that every SSPE is a SPE. Thus, by Theorem 4, μ has to be stable. Let denote $S' := \left\{s \in S : \mu^S(s)\, P(s)\, \mu(s)\right\}$ the set of students preferring their allocation under μ^S rather than under μ. Since $\mu \neq \mu^S$, S' is non-empty. Now consider the following strategies: every s in S' plays $m''(s) = \mu^S(s)$, and any s in $S \backslash S'$ plays $m'(s)$. Following the latticial structure of the core, it holds that $m'(s) = \mu^S(s)$ for all s not in S'. Given that colleges play their dominant strategies (see the proof of Theorem 4 above), the outcome when agents in S' shift their strategy and play $m''(s)$ is μ^S. A contradiction. ∎

4 The "colleges-propose-and-students-choose" mechanism

This section introduces a mechanism implementing the core correspondence of college admissions problems. The idea underlying this mechanism is very similar to the one analyzed in Section 3. In this case offers are made by colleges and each student selects her "best college", from the proposals she receives. That is, the main formal difference between this mechanism and the one studied in Section 3 is that we shift the order in which agents on both sides of the market make their decisions.

A formal description of the mechanism, named "the colleges-propose-and-students-choose" mechanism, follows. It is a two stage game. In the first stage, colleges have to decide simultaneously. Each college message space coincides with the set of potential teams of students, 2^S. In the second stage, students, knowing colleges' messages, select simultaneously the college in which they want to study. Thus, each student message space coincides with $C \cup \{s\}$. Let $m(k)$ denote the message by agent $k \in C \cup S$, and $\tilde{m}$ be an ordered vector containing agents' messages.

The outcome function, denoted by Φ^{CS}, selects a matching which is defined as follows: $\Phi^{CS}(\tilde{m}) = \mu_{\tilde{m}}$, where for any s in S,

$$\mu_{\tilde{m}}(s) = \begin{cases} m(s) & \text{if } s \in m(m(s)) \\ s & \text{otherwise} \end{cases}$$

and, for each c in C,

$$\mu_{\tilde{m}}(c) = \{s \in m(c) \mid c = m(s)\}$$

The next result analyzes the equilibria outcomes of this mechanism when no commitment by agents is allowed. In some sense, the result can be interpreted as an equivalence between this mechanism and the "students-propose-and-colleges-choose" mechanism. The proof for Theorem 6, is omitted but it can be built in a similar way as the one for Theorem 4.

Theorem 6 *The mechanism described above implements in SPE the core correspondence.*

The relationship found between the mechanisms studied in Theorems 4 and 6 does not hold when agents are allowed to commit on deciding which strategies have to be played. As

Example 7 shows, cooperation among agents does not necessarily reduce the set of possible outcomes.

Example 7 *Let us consider the following five students and three colleges market.*

$$\begin{array}{ll} P(s_1) = c_1 & P(c_1) = s_1 s_2 s_3 s_4 s_5 \\ P(s_2) = c_3 c_1 c_2 & P(c_2) = (s_2 s_3)(s_4 s_5) s_1 s_2 s_3 s_4 s_5 \\ P(s_3) = c_3 c_1 c_2 & P(c_3) = (s_4 s_5)(s_2 s_3) s_1 s_2 s_3 s_4 s_5 \\ P(s_4) = c_2 c_1 c_3 & \\ P(s_5) = c_2 c_1 c_3 & \end{array}$$

It is straightforward to see that there is a Strong Subgame Perfect Nash Equilibrium yielding each stable matching. For instance, the matching μ^S in which $\mu^S(c_1) = s_1$, $\mu^S(c_2) = (s_4 s_5)$ and $\mu^S(c_3) = (s_2 s_3)$ can be supported in SSPE by strategies $m(c_1) = (s_1 s_2 s_3 s_4 s_5)$, $m(c_2) = (s_4 s_5)$, $m(c_3) = (s_2 s_3)$ and, for each student s, $m(s) = \arg\max P(s)$ on $\{c \in C \text{ s.t. } s \in m(c)\} \cup \{s\}$. In a similar way, we can support the colleges' optimal stable matching μ^C in which $\mu^C(c_1) = s_1$, $\mu^C(c_2) = (s_2 s_3)$ and $\mu^S(c_3) = (s_4 s_5)$ by a SSPE described by strategies $m(c_1) = s_1$, $m(c_2) = (s_2 s_3)$, $m(c_3) = (s_4 s_5)$ and, for each student s, $m(s) = \arg\max P(s)$ on $\{c \in C \text{ s.t. } s \in m(c)\} \cup \{s\}$.

5 Final Remarks

This paper introduces two mechanisms implementing the core correspondence of matching markets. The results solve two essential questions. First, the core of such games can be implemented in Subgame Perfect Equilibrium. And, second, it provides simple mechanisms to implement such a solution concept.

The first mechanism that we introduce implements a particular selection of the core, namely the students' optimal stable matching. Thus, this paper also provides a positive answer to the implementability of a selection of the core in matching markets. Notice that Kara and Sönmez [6] prove that no selection of the core can be implemented in Nash Equilibrium.

Unfortunately a symmetric result cannot be provided for the set of colleges. This result points out (as Roth [9] did) the asymmetry holding among both sides of the market. Moreover, we can also state, in the words of Roth, that "*the college admissions problem is not equivalent to the marriage problem.*" Note that, in the particular case of marriage markets (colleges have only one position each), a symmetrical result for Theorem 6 can be stated by exchanging the role of students and colleges.

To conclude, we want to mention that Alcalde, Pérez-Castrillo and Romero-Medina [3] analyzed two mechanisms for job matching markets that were inspired in the mechanisms in this paper. They shown that, under some conditions, the results presented in this paper can be extended to the case in which monetary transfers play an essential role. Nevertheless, as the reader can see, the mathematical tools employed in bot papers are very different. So, even if the results in both papers have interpretative similarities, none of them can be considered a particular case of the other.

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