Nonmanipulable decision mechanisms for economic environments

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Abstract. In the social choice literature studying the problem of designing institutions for collective decision making, it is customary to (implicitly) assume that each dimension of the social outcome is of *public interest* (i.e., that each individual's welfare depends on every dimension of the outcome). Thus, the scope of the conclusions obtained is very limited. Here social decision problems with and without that public character are considered and it is shown that the same negative results arise in most cases; namely, that only dictatorial mechanisms are immune to the participants' manipulations. These results are obtained without requiring that the mechanisms produce Pareto optimal outcomes (they must simply be minimally *responsive* to the participants' preferences), which deepens their pessimistic character.

The purpose of this paper is to investigate whether the results obtained in the social choice literature can be extended to social decision problems other than the *purely public* ones – that is, to problems in which some individuals' welfare might not be affected by some dimensions of the outcome. A broad class of social decision problems is considered, and it is shown that the presence of a single public dimension already produces the results previously obtained for the purely public case.

The results reported here build on the seminal papers of Gibbard (1973) and Satterthwaite (1975) and on a recent paper by Barberà and Peleg (1990). The Gibbard-Satterthwaite Theorem states that when participants can claim all possible preferences over the set of alternatives (an assumption that is appropriate only if the social decision problem is purely public), every voting scheme whose range is finite and contains three or more alternatives must be either manipulable

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or dictatorial. Barberà and Peleg have extended the Gibbard-Satterthwaite Theorem to include the cases in which the set of alternatives may be infinite (any metric space), even when individuals' preferences are restricted to be only the continuous ones. The Barberà and Peleg Theorem suggests that this framework might be suitable for the study of the social decision problems traditionally considered in the literature on allocation mechanisms in economics.

Extending the results obtained for the purely public case to more general collective decision problems requires one to deal with two new issues, both related with the presence of dimensions in the social outcome that affect some individuals but not others:

The first issue is to devise a measure of the extent to which a mechanism deals with the conflict of interests present in a social decision problem. For purely public collective decision problems, the cardinality of a mechanism's range is an appropriate measure. The concept of *degree of conflict* introduced here provides a measure for more general problems.

The second issue arises from the fact that the above mentioned impossibility results may be avoided if a mechanism can select some dimensions of the social outcome on the basis of the preferences of individuals whose welfare is not affected by those dimensions. Consider, for example, a social decision problem in which three individuals must decide on the basis of their preferences how to allocate one unit of a *divisible* and *freely disposable* private good. Assume that the individuals admissible preferences are those representable by a continuous function on $[0, 1]$, and write m_i for the smallest maximizer of Individual i's reported utility function.

The mechanism that assigns to individuals 1, 2 and 3, respectively, $\frac{1}{3}(1+m_2-m_3)$, $\frac{1}{3}(1+m_3-m_1)$, and $\frac{1}{3}(1+m_1-m_2)$ units of good, is nondictatorial (no individual gets always his most preferred outcome), and it is not subject to the participants' manipulations – in fact, no one can influence his own outcome. It is, however, a very unsatisfactory mechanism: the conflict among the participants is circumvented *by ignoring the preferences of those directly affected by the choice to be made.* If, for instance, individuals 1 and 2 report $m_1=m_2$, then Individual 3 gets one third of a unit (independently of the utility function he reports), even though the set of feasible outcomes for him (given the outcome the mechanism selects for individuals 1 and 2) is $[0, \frac{1}{3}]$.

We will rule out mechanisms having this feature by requiring that mechanisms be *responsive* to the participants' preferences: The alternative selected must be such that each individual's reported preference is maximized on the set of outcomes that are feasible for him when combined with the outcome the mechanism has selected for all other individuals. Responsive mechanisms therefore must select minimally efficient outcomes, although *they need not always produce Pareto optimal outcomes.*

In the literature on allocation mechanisms, most of the impossibility results are obtained by requiring that the mechanisms produce Pareto optimal outcomes. Hurwicz and Walker (1990), for example, show that nonmanipulable mechanisms generally produce non-optimal outcomes. It seems likely, however, that their result depends upon their requirement of *exact optimality;* might there be mechanisms that are nonmanipulable and produce outcomes that are *not far* (in some sense) from being Pareto optimal? If so, the significance of the Hurwicz and Walker impossibility result would not be so negative.

The results presented here, however, tend to confirm the more pessimistic interpretations of the existing impossibility results for allocation problems: We show that a substantially larger class of nonmanipulable mechanisms (not only the ones that always produce Pareto optimal outcomes, but all the ones that are merely responsive) are very unsatisfactory.

Satterthwaite and Sonnenschein (1981) were the first to extend the social choice impossibility results to collective decision problems that are not purely public. The approach taken here differs from theirs in that we do not restrict the class of mechanisms to the differentiable ones, and more importantly, we do not rule out *bossy*[1] mechanisms (those for which it is possible for an individual to alter the outcome for some other individual without changing his own outcome). The mechanism in the previous example is bossy: an individual can change someone else's outcome and maintain his own unchanged. Its most negative feature, however, is not its bossiness, but the fact that the mechanism selects each individual's outcome ignoring the preferences he declared, and therefore it produces very inefficient outcomes. Responsiveness introduces an efficiency condition that rules out these mechanisms.

Responsiveness is neither weaker nor stronger than nonbossiness. For example, if the private good in the previous example is *non-disposable* (i.e., if the unit of the good *must* be allocated), then the given mechanism will be (vacuously) responsive – the set of outcomes that are feasible for an individual, given the outcomes assigned to the other participants, is a singleton – and it will continue to be bossy. Thus, the responsiveness property does not rule out all bossy mechanisms. An example of a nonbossy nonresponsive mechanism is given in the discussion of Example 1.

It should be pointed out that the results presented here do not apply to social decision problems like the one described in the example, involving more than two individuals and with only *private components*. In fact, when public components are absent from the collective decision problem, there are nonmanipulable, nondictatorial mechanisms that are responsive (and nonbossy).

The results reported here also build on previous papers by Zhou (1991) and by Moreno and Walker (1991). For purely public social decision problems, Zhou shows that whenever the set of feasible alternatives is a subset of an euclidean space, nonmanipulable mechanisms whose range is at least a two-dimensional set must be dictatorial, even if individual preferences are restricted to be the ones representable by quadratic functions. Moreno and Walker extend Zhou's Theorem to cover all collective decision problems for which the set of feasible alternatives is a Cartesian product of the public dimensions and the other dimensions present in the outcome.

The remainder of the paper is organized as follows: In Sect. 1 the model is described; in Sect. 2 the results are stated in discussed; the proofs are given in Sect. 3.

[1] It is not clear why one should be interested in requiring a mechanism to be nonbossy. In fact, there are interesting mechanism that are bossy (e.g., the *competitive mechanism*).

1. The Model

N denotes the set $\{1,\dots,n\}$ of **individuals.** Throughout we assume $n \geq 2$.

We assume that the **set of possible outcomes for individual i** has the form $X_i = X^p \times Y_i \subset \mathbb{R}^m \times \mathbb{R}^{\ell_i}$. Z denotes the **set of feasible alternatives**, a subset of $X^p \times \prod_1^n Y_i$, members of which are written as $\mathbf{x} = (x^p, \mathbf{y}) = (x^p, y_1, \dots, y_n)$. We refer to the coordinates of members of X^p as **public components**[2]. Representing a social decision problem in this way does not impose any restriction: If there are no public components, X^p is a singleton; similarly, if there are *only* public components, the sets Y_i are singletons, and we say that Z is **purely public.** *We assume that Z is compact.*

A **utility function** for individual i is a real-valued function on X_i. For each $i \in N$, a set U_i of **admissible** utility functions is given. $\mathscr{U}$ denotes the product $\prod_i^n U_i$; the members $\mathbf{u} = (u_1, \dots, u_n)$ of $\mathscr{U}$ are called **profiles.** If $\mathbf{u}$ is a profile and $\tilde{u}_i$ is a member of U_i, then $(\tilde{u}_i, \mathbf{u}_{\sim i})$ denotes the profile in which $\tilde{u}_i$ has replaced the i^{th} component of $\mathbf{u}$.

A **mechanism** (or **voting scheme**) is a function $f: \mathscr{U} \to Z$. We denote the range of f by $\mathscr{R}$, and by $\mathscr{R}_i$ and $\mathscr{R}^P$ its projections into, respectively, X_i and X^p. Given $\mathbf{x} \in Z$, we write $x_i = (x^p, y_i)$ and $\mathbf{x}_{\sim i} = (x^p, \mathbf{y}_{\sim i})$ for its projections into, respectively, X_i and $X^p \times \prod_{j \neq i} Y_j$. Similarly, for each $\mathbf{u} \in \mathscr{U}$, we write $f_i(\mathbf{u})$ and $f_{\sim i}(\mathbf{u})$ for the projections of $f(\mathbf{u})$ into, respectively, X_i and $X^p \times \prod_{j \neq i} Y_j$. Finally, we reduce notation by writing $\tilde{u}_i f_i(\mathbf{u})$ for $\tilde{u}_i(f_i(\mathbf{u}))$, i.e., for i's utility (according to the utility function $\tilde{u}_i$) at the outcome associated with the profile $\mathbf{u}$.

A mechanism f is **manipulable** by individual i at profile $\mathbf{u}$ via utility function $\tilde{u}_i$ if $u_i f_i(\tilde{u}_i, \mathbf{u}_{\sim i}) > u_i f_i(\mathbf{u})$. A mechanism f is **nonmanipulable** (NM) if, for each profile $\mathbf{u} \in \mathscr{U}$, each $i \in N$, and each $\tilde{u}_i \in U_i$, f is not manipulable by i at $\mathbf{u}$ via $\tilde{u}_i$.

An individual $i \in N$ is a **dictator** for the mechanism f if for every profile $\mathbf{u} = (u_1, \dots, u_n) \in \mathscr{U}$, $f_i(\mathbf{u})$ maximizes u_i on $\mathscr{R}_i$. A mechanism f is **dictatorial** if there is a dictator for f, otherwise it is **nondictatorial.**

The Gibbard-Satterthwaite Theorem (G-S) and the Barberà and Peleg Theorem (B & P) consider only problems for which the set of social alternatives is purely public. They give conditions under which any voting scheme will be either manipulable or dictatorial:

(G-S) *For each $i \in N$, U_i includes all functions on X_i, and $\mathscr{R}$ is finite and contains more than two alternatives.*

(B & P) *For each $i \in N$, X_i is a metric space, U_i includes all continuous functions on X_i, and $\mathscr{R}$ contains more than two alternatives.*

There are many social decision problems that are left out of the scope of the G-S and the B & P theorems; examples of such problems are those in which the social decision includes components that only single individuals care about (Examples 1 and 2, below) or that include public components but also other "externalities" that are not fully public goods – i.e., components that affect some individuals but not others. The fact that only purely public decision problems are considered in the G-S and the B & P theorems might give the impression that

[2] The definition does not provide an obvious way of checking the presence of public components in the social decision problem: this may require one to find the appropriate representation of the problem.

it is the public relevance of every component of the social decision that drives these results. We shall see that there are many situations other than purely public ones for which the existence of opposing interests among individuals leads to the same negative conclusions.

Responsive Mechanisms

Given an arbitrary set $T \subset X^p \times \prod_1^n Y_i$, we denote by T_i the projection of T into X_i; also, for each $\bar{\mathbf{x}} \in T$, we write $T(\bar{\mathbf{x}}_{\sim i})$ for the set $\{\mathbf{x} \in T \mid \mathbf{x}_{\sim i} = \bar{\mathbf{x}}_{\sim i}\}$, and $T_i(\bar{\mathbf{x}}_{\sim i})$ for the projection of $T(\bar{\mathbf{x}}_{\sim i})$ into X_i. Thus, given the set of feasible alternatives Z, the set $Z_i(\bar{\mathbf{x}}_{\sim i})$ contains all outcomes for Individual i that are feasible when combined with $\bar{\mathbf{x}}_{\sim i}$.

A mechanism f is **responsive**[3] if for each $i \in N$ and each $\mathbf{u} = (u_1, \ldots, u_n) \in \mathscr{U}$, $f_i(\mathbf{u})$ maximizes u_i on $Z_i(f_{\sim i}(\mathbf{u}))$.

Thus, a mechanism is responsive if at every profile, each individual's utility is conditionally maximized, given the outcome assigned to all other individuals. Responsive mechanisms, however, need not always select Pareto optimal outcomes. (An outcome $\mathbf{x} \in Z$ is **Pareto optimal for** $\mathbf{u} \in \mathscr{U}$ if whenever there are $\mathbf{x}' \in Z$ and $i \in N$ such that $u_i(x_i') > u_i(x_i)$, then there is also $j \in N$ such that $u_j(x_j') < u_j(x_j)$.)

Example 1. There is a single unit of a divisible and freely disposable private good to be allocated among two individuals. Hence we write $X_i = [0, 1]$, $i = 1, 2$ (X^p is a singleton that we suppress) and $Z = \{\mathbf{x} \in [0, 1]^2 \mid x_1 + x_2 \leq 1\}$. Assume that all continuous functions on $[0, 1]$ are admissible utility functions for individuals 1 and 2, and let f be a mechanism that for each $\mathbf{u} \in \mathscr{U}$, selects the alternative for which $f_1(\mathbf{u})$ is the maximizer of u_1 closest to one, and $f_2(\mathbf{u})$ is a maximizer of u_2 on $Z_2(f_1(\mathbf{u}))$ (the set $[0, 1 - f_1(\mathbf{u})]$). Clearly f is nonmanipulable and responsive. However, it does not always produce Pareto optimal outcomes: For example, if u_1 is a constant function, then for each $u_2 \in U_2$, one has $f(u_1, u_2) = (1, 0)$; thus, if u_2 is an increasing function, this outcome is Pareto dominated by $(0, 1)$. Note that this is a dictatorial mechanism, but as Theorem 1 below shows, all nonmanipulable responsive mechanisms that can be designed for this social decision problem are dictatorial.

Note that when Z is purely public, then every mechanism is vacuously responsive (because the sets $Z_i(\mathbf{x}_{\sim i})$ are singletons). By contrast, for collective decision problems for which individuals' interests are never in conflict (i.e., when for each $\mathbf{x} \in Z$ and each $i \in N$ one has $Z_i(\mathbf{x}_{\sim i}) = Z_i$), a nonmanipulable responsive mechanism f must be such that for each $i \in N$, $f_i(\mathbf{u})$ is a maximizer of u_i on Z_i; hence, in this case, a nonmanipulable responsive mechanisms must always select Pareto optimal outcomes.

[3] In a previous version of this paper, the term *responsive* was used for a slightly different definition.

Degree of Conflict

The notion of degree of conflict provides of measure of the conflict of interests present in a set of alternatives. Thus, the degree of conflict of the range of a mechanism is a measure of the extent to which the mechanism deals with the conflicting interests present in a social decision problem.

Given an arbitrary set $T \subset X^p \times \prod_1^n Y_i$, $\mathbf{x} \in T$ is said to be a **conflictive outcome for Individual i** if $T_i(\mathbf{x}_{\sim i})$ is a proper subset of T_i. For each $i \in N$ and each $T \subset X^p \times \prod_1^n Y_i$, let $\mathscr{C}_i(T)$ denote the set of conflictive outcomes for Individual i, and write $\mathscr{C}(T)$ for the set $\bigcap_{i \in N} \mathscr{C}_i(T)$ of **conflicting outcomes.**

Let $T \subset X^p \times \prod_1^n Y_i$. T is said to **contain a conflict of degree one** if $\mathscr{C}(T)$ is empty. T is said to satisfy **Condition DC(l)** if for each $i \in N$ and each $C \subset \mathscr{C}(T)$ containing fewer than l elements, the set $\bigcup_{\mathbf{x} \in C} T_i(\mathbf{x}_{\sim i})$ is a proper subset of T_i. T is said to **contain a conflict of degree k**, if it satisfies Condition $DC(k)$, but it does not satisfy Condition $DC(k+1)$.

Note that the concept of degree of conflict resembles the notion of dimension of a vector space: given a set of conflicting outcomes, the set $\bigcup_{\mathbf{x} \in C} T_i(\mathbf{x}_{\sim i})$ is its spanning in T_i; the degree of conflict of a set $T \subset X^p \times \prod_1^n Y_i$ is the minimum number of conflicting outcomes necessary to span every T_i.

When Z is purely public, then the range of a mechanism f contains a degree of conflict equal to its cardinality: If $\mathscr{R}$ is a singleton (i.e., if f is a constant mechanism), then for each $i \in N$ the set $\mathscr{C}_i(\mathscr{R})$ is empty; hence $\mathscr{R}$ contains a conflict of degree one. If $\mathscr{R}$ is not a singleton, then for each $i \in N$ and each $\mathbf{x} \in \mathscr{R}$, one has $\mathscr{R}_i(\mathbf{x}) = \mathbf{x} \neq \mathscr{R}_i = \mathscr{R}$; hence $\mathscr{C}_i(\mathscr{R}) = \mathscr{R} = \mathscr{C}(\mathscr{R})$, and for each i $\in$ N and each proper subset C of $\mathscr{R}$, the set $\bigcup_{\mathbf{x} \in C} \mathscr{R}_i(\mathbf{x}_{\sim i}) = C$. Therefore $\mathscr{R}$ contains a conflict of degree equal to its cardinality.

For more general collective decision problems, the range of a mechanism contains a conflict of degree greater than or equal to the cardinality of $\mathscr{R}^p$ (the projection of $\mathscr{R}$ into X^p).

Theorem 1 below requires one to check whether the range of a mechanism contains a conflict of degree greater than two. This condition plays the same role in Theorem 1 as the cardinality condition on the range of a mechanism plays in both the G-S and the B & P theorems.

It should be noted that the notions of responsiveness and degree of conflict are not independent; in the social decision problem of Example 1, if the set of admissible utility functions is *sufficiently rich* (e.g., if it contains all the continuous utility functions), it can be shown that any responsive and nonmanipulable mechanism must have Z as its range, which contains a conflict of degree greater than two. (Recall $Z = \{\mathbf{x} \in [0,1]^2 \mid x_1 + x_2 \leq 1\}$; thus for $x \in (0,1]$ and $i \in N$, one has $Z_i(x) \neq Z_i$ - see Fig. 1. Hence $\mathscr{C}_1(Z) = \mathscr{C}_2(Z) = \mathscr{C}(Z) = \{\mathbf{x} \in Z \mid x_{\sim i} > 0\}$, and for each $i \in N$ and each $C \subset \mathscr{C}(Z)$, the set $\bigcup_{\mathbf{x} \in C} Z_i(\mathbf{x}_{\sim i})$ is a proper subset of Z_i. Therefore Z contains a conflict of degree greater than two.)

Theorem 1 below will ensure that every nonmanipulable responsive mechanism for this social decision problem will be dictatorial. It is possible, however, to construct nonmanipulable and nondictatorial mechanisms that are nonbossy; for example, for each $\mathbf{u} \in \mathscr{U}$, let $f(\mathbf{u})$ be $(\frac{1}{2}, \frac{1}{2})$ if $u_i(\frac{1}{2}) \geq u_i(0)$, $i = 1, 2$, and $(0, 0)$ otherwise. Thus, whereas for this problem nonmanipulable responsive mechanisms are dictatorial, one can construct nonbossy (but nonresponsive) mechanisms that are nonmanipulable and nondictatorial.

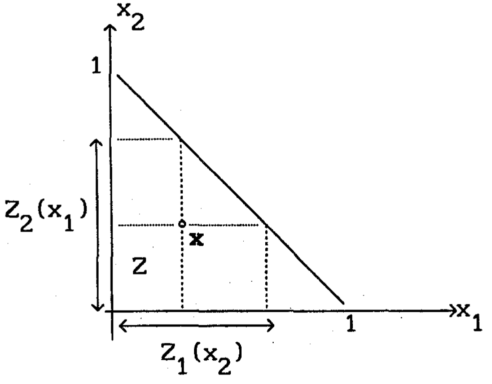

Fig. 1.

Example 2 illustrates further the concept of degree of conflict:

Example 2. There are two individuals whose sets of possible outcomes are $X_i=[0,2]$ (X^p is a singleton, which we suppress). The set of feasible alternatives is $Z=[0,1]^2\cup[1,2]^2$. Individuals' admissible utility functions are all the continuous functions on X_i.

Notice that for each $i\in N$ and $\mathbf{x}\in Z$, either $Z_i(x_{\sim i})=[0,1]$ or $Z_i(x_{\sim i})=[1,2]$, but not both (see Fig. 2); hence $Z_i(x_{\sim i})\neq Z_i=[0,2]$; thus $\mathscr{C}_i(Z)=Z$. Moreover, take $\mathbf{x}\in[0,1]^2$, $\mathbf{x}'\in[1,2]^2$; then $Z_i(x_{\sim i})\cup Z_i(x'_{\sim i})=Z_i$, $i=1,2$. Hence Z contains a conflict of degree 2.

A nonmanipulable, responsive and nondictatorial mechanism can be constructed as follows: for each $i\in N$, $u_i\in U_i$ and $T_i\subset Z_i$, let $m(u_i,T_i)$ denote the smallest maximizer of u_i on T_i and write $m(\mathbf{u})=(m(u_1,Z_1),m(u_2,Z_2))$; for each $\mathbf{u}\in\mathscr{U}$, let the mechanism f be given by

$$f(\mathbf{u})=\begin{cases} m(\mathbf{u}) & \text{if } (m(\mathbf{u})\in Z \\ (m(u_1,Z_1),m(u_2,[1,2])) & \text{if } (m(\mathbf{u})\in[1,2]\times[0,1] \\ (m(u_1,[1,2]),m(u_2,Z_2)) & \text{if } (m(\mathbf{u})\in[0,1]\times[1,2]\ . \end{cases}$$

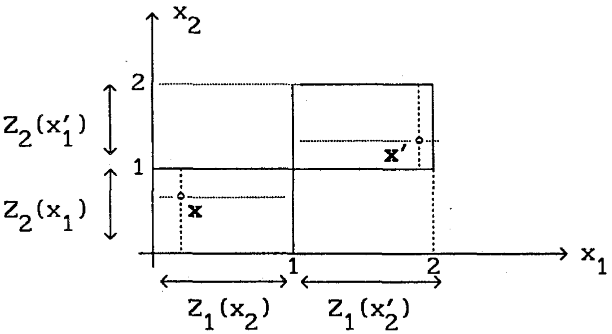

Fig. 2.

The existence of this mechanism shows that the condition on the degree of conflict of the range of a mechanism cannot be removed from Theorem 1 below.

2. The Results

Theorem 1 provides conditions under which a social decision situation involving two individuals cannot be resolved by a mechanism that is simultaneously nonmanipulable and nondictatorial. It applies to social decision problems with and without public components, but as the example in the introduction showed (in the case in which the good is non-disposable), it cannot be extended to problems involving more than two individuals. Theorem 1, however, plays an important role in the proof of Theorem 2, which is applicable to social decision problems with any number of individuals. The proof of Theorem 1 is given in Sect. 3.

Theorem 1. *If $N=\{1,2\}$ and for each $i \in N$, U_i contains all the continuous functions on X_i, then every nonmanipulable responsive mechanism whose range contains a conflict of degree greater than two is dictatorial.*

Remark 1. As shown above, when the set of alternatives is purely public, responsiveness is satisfied vacuously and the range of f contains a conflict of degree equal to the cardinality of $\mathscr{R}$. Thus, Theorem 1 includes the Barberà and Peleg Theorem, *for the case* $n=2$, as a special case.

Remark 2. Mechanisms like the ones described in Example 2 show that there are social decision problems for which it is possible to design nonmanipulable responsive mechanisms whose range contains a conflict of degree less than or equal to two. It is not clear, however, whether responsiveness is a necessary condition in Theorem 1.

Remark 3. Although our notion of dictator is weaker than the one used in most of the related literature, within the assumptions of Theorem 1 the presence of a dictator is unambiguously a very unsatisfactory feature of a mechanism: the requirement that the range of the mechanism contains a conflict of degree greater than two implies that the dictator will sometimes impose upon the other participants an undesired outcome, since whenever a conflict arises the dictator's preferences will prevail.

Example 1 shows a social decision problem that falls within the scope of Theorem 1 and that is not covered by the B & P Theorem. In fact, in this example every nonmanipulable responsive mechanism satisfies $\mathscr{R}=Z$, and therefore its range contains a conflict of degree greater than two, which according to Theorem 1 implies that it is dictatorial.

Theorem 2 extends the conclusion of Theorem 1 to social decision problems with more than two individuals, when the set of alternatives contains at least one public component. (Section 3 contains the proof of the theorem.)

Theorem 2. *If for each $i \in N$, U_1 contains all continuous utility functions on X_i, then every nonmanipulable responsive mechanism for which $\mathscr{R}^P$ contains more than two elements is dictatorial.*

Remark 4. When the set of alternatives is purely public, Theorem 2 becomes just the Barberà and Peleg Theorem. Notice also that the condition on the cardinality of the set $\mathscr{R}^p$ effectively requires than $\mathscr{R}$ contains a conflict of degree greater than two.

Remark 5. The non-disposable version of the example in the introduction shows that the presence of a public dimension cannot be dispensed with in Theorem 2. It is unclear whether responsiveness is required: its only use in the proof by induction of Theorem 2 is to allow us apply Theorem 1 to two-person mechanisms.

The final example describes a social decision situation which falls within the scope of Theorem 2, and it is not covered by any of the existing impossibility results:

Example 3. A community of n farmers controls a water reservoir which is used for recreational pursuits (fishing, swimming, etc.), and also to irrigate all the farmer's crops. Everyone therefore cares about the amount of water in the reservoir, and each farmer cares only about his own allocation of water. A social alternative consists of as specification of the amount of water in the reservoir, x^p, which is a public component, and an allocation of water to farmers $\mathbf{y}=(y_1, y_2, \ldots, y_n)$ for irrigation purposes. The amount of water is limited, so that a social alternative $(x^p, \mathbf{y})$ must satisfy $x^p+\sum_{i=1}^n y_i=b$.

Theorem 2 tells us that every nonmanipulable responsive mechanism whose range contains more than two outcomes that differ in the amount of water in the Reservoir will be dictatorial. The following two remarks comment on the limitations of Theorems 1 and 2.

Remark 6. It seems clear that in many situations other than those covered by Theorems 1 and 2, no satisfactory mechanisms can be designed. Mechanisms such as the one described in the introduction (which is a responsive mechanism in the non-disposable case) show that when public dimensions are absent from the social decision, it is possible to design nonmanipulable and nondictatorial mechanisms that are also responsive. Clearly the mechanism proposed is very unsatisfactory, since an individual's preference is never considered in selecting his outcome. It would be of interest to investigate whether all mechanisms that can be designed for social decision problems involving more than two individuals and without any public component have the same negative features.

Remark 7. An important issue that deserves further study is whether these results arise when the sets of admissible utility functions are restricted in several ways; e.g., to contain only monotonic and/or concave functions, etc.. Extensions of the results in this direction will enable a wider application to economic problems. An extension to concave functions ought to be possible, perhaps by taking an approach similar to the one followed by Zhou for the purely public case with only the quadratic utility functions. It is unclear how (or even whether) similar results can be obtained when the set of utility functions is restricted to contain only monotonic functions.

3. Proofs

Before proving Theorems 1 and 2, two preliminary lemmas are established. Lemma 1 establishes that a NM mechanism must always select each coordinate of the social alternative so as to maximize that individual reported utility function on his *option set* (the set $O_i(\mathbf{u}_{\sim i})$, given for each $i \in N$ and each $\mathbf{u}_{\sim i} \in \prod_{j \neq i} U_j$

by $\{x_i \in X_i \mid \exists\, u_i \in U_i : x_i = f_i(u_i, \mathbf{u}_{\sim i})\}$ – i.e., the set of outcomes directly attainable by Individual i given $\mathbf{u}_{\sim i}$). Lemma 2 establishes that if the U_i's sets are sufficiently rich, then the option sets must be closed.

Lemma 1. *Let f be a* NM *mechanism. Then for each $i \in N$ and each $\mathbf{u} \in \mathscr{U}$: $f_i(\mathbf{u})$ maximizes u_i on $O_i(\mathbf{u}_{\sim i})$.*

Proof. Suppose not; let $\mathbf{u} = (u_i, \mathbf{u}_{\sim i}) \in \mathscr{U}$ and $\tilde{x}_i \in O_i(\mathbf{u}_{\sim i})$ be such that $u_i(\tilde{x}_i) > u_i f_i(\mathbf{u})$. Since $\tilde{x}_i \in O_i(\mathbf{u}_{\sim i})$, let $\tilde{u}_i \in U_i$ be such that $f_i(\tilde{u}_i, \mathbf{u}_{\sim i}) = \tilde{x}_i$. Then we have

$$u_i f(\tilde{u}_i, \mathbf{u}_{\sim i}) = u_i(\tilde{x}_i) > u_i f_i(\mathbf{u}) \ .$$

Hence Individual i can manipulate f at $\mathbf{u}$ via $\tilde{u}_i$, contradicting that f is NM. □

Lemma 2: *Let f be a* NM *mechanism, and assume that for each $i \in N$, U_i contains all the continuous functions on x_i. Then for each $i \in N$ and each $\mathbf{u}_{\sim i} \in U_{\sim i}$: $O_i(\mathbf{u}_{\sim i})$ is closed.*

Proof. Let $i \in N$ and $\mathbf{u}_{\sim i} \in U_{\sim i}$ arbitrary, and let $\bar{x}_i$ be a limit point of $O_i(\mathbf{u}_{\sim i})$. Assume by way of contradiction that $\bar{x}_i \notin O_i(\mathbf{u}_{\sim i})$. Consider the utility function $u_i(x_i) = -\,||\bar{x}_i - x_i||$. Let $f_i(u_i, \mathbf{u}_{\sim i}) = x_i$. Since $\bar{x}_i$ is a limit point of $O_i(\mathbf{u}_{\sim i})$, let $\tilde{x}_i \in O_i(\mathbf{u}_{\sim i})$ be such that $||\bar{x}_i - \tilde{x}_i|| < ||\bar{x}_i - x_i||$, and let $\tilde{u}_i \in U_i^*$ be such that $m(\tilde{u}_i) = \tilde{x}_i$; Lemma 1 yields $f_i(\tilde{u}_i, \mathbf{u}_{\sim i}) = \tilde{x}_i$. Thus we have

$$u_i f_i(\tilde{u}_i, \mathbf{u}_{\sim i}) = u_i(\tilde{x}_i) > u_i(x_i) = u_i f_i(u_i, \mathbf{u}_{\sim i}) \ ,$$

and therefore Individual i can manipulate f at $\mathbf{u}$ via $\tilde{u}_i$. Hence $\bar{x}_i$ must in fact be a member of $O_i(\mathbf{u}_{\sim i})$. □

Proof of Theorem 1. The proof of Theorem 1 follows the lines of the proof of the B & P Theorem. As in their proof, the method of the pivotal-voter (developed in Barberà 1983) plays a fundamental role. Before proving Theorem 1, a number of preliminary results are established. For the remaining of the proof, let f be a mechanism satisfying the assumptions of Theorem 1. Let U_i^* denote the set of continuous utility functions with a unique maximizer (denoted by $m(u_i)$) on $\mathscr{R}_i$, and write $\mathscr{U}^*$ for the set $U_1^* \times U_2^*$.

It is well known that if the social decision problem is purely public, and if the mechanism f is nonmanipulable, then it satisfies the unanimity property that if $\mathbf{u} \in \mathscr{U}^*$ and $x \in \mathscr{R}$ are such that for each $i \in N$, $m(u_i) = x$, then $f(\mathbf{u}) = x$. Lemma 1.1 establishes a generalized version of this property for social decision problems involving two individuals.

Lemma 1.1. *If $\mathbf{u} \in \mathscr{U}$ and $\mathbf{x} \in \mathscr{R}$ are such that for each $i \in N$, $x_i = m(u_i)$, then $f(\mathbf{u}) = \mathbf{x}$.*

Proof. Let $\bar{\mathbf{u}} \in \mathscr{U}$ and $\bar{x} \in \mathscr{R}$ be such that for each $i \in N$, $m_i(u_i) = \bar{x}_i$, and let $\mathbf{u} \in \mathscr{U}$ be such that $f(\mathbf{u}) = \bar{\mathbf{x}}$. Since $\bar{x}_2 = m(\bar{u}_2) \in O_2(u_1)$, Lemma 1 implies that $f_2(u_1, \bar{u}_2) = \bar{x}_2$. If we can show that $\bar{x}_1 = m(\bar{u}_1) \in O_1(\bar{u}_2)$, then another application of Lemma 1 would yield $f_1(\bar{u}_1, \bar{u}_2) = \bar{x}_1$, and since $\bar{x}_2 \in Z_2(\bar{x}_1)$, responsiveness of f would imply $f_2(\bar{u}_1, \bar{u}_2) = \bar{x}_2$, and the proof would be complete.

If $f_1(u_1, \bar{u}_2) = \bar{x}_1$, there is nothing to prove. Suppose that $f_1(u_1, \bar{u}_2) = \hat{x}_1 \neq \bar{x}_1$; for each $O < \varepsilon < ||\bar{x}_1 - \hat{x}_1||$, let $u_1^\varepsilon \in U_1^*$ be such that

(1) $m(u_1^\varepsilon)=\bar{x}_1$,
(2) $\hat{x}_1$ is the unique maximizer of u_1^ε on $\mathscr{R}_1\backslash\bar{B}(\bar{x}_1,\varepsilon)$, and
(3) for $x_1, x_1' \in \bar{B}(\bar{x}_1,\varepsilon)$: $\|x_1-\bar{x}_1\| < \|x_1'-\bar{x}_1\| \Rightarrow u_1^\varepsilon(x_1) > u_1^\varepsilon(x_1')$,

where $\bar{B}(\bar{x}_1,\varepsilon)$ is the set $\{x_1 \in X_1 \mid \|x_1-\bar{x}_1\| \leq \varepsilon\}$.

Since f is nonmanipulable, we must have

$$u_1^\varepsilon f_1(u_1^\varepsilon, u_2) \geq u_1^\varepsilon(\hat{x}_1) \ .$$

Thus, *(2)* implies that either $f_1(u_1^\varepsilon,\bar{u}_2)=\hat{x}_1$ or $f_1(u_1^\varepsilon,\bar{u}_2)\in\bar{B}(\bar{x}_1,\varepsilon)$. Suppose $f_1(u_1^\varepsilon,\bar{u}_2)=\hat{x}_1$; since $\bar{x}_2 \in Z_2(\hat{x}_1)=Z_2(f_1(u_1^\varepsilon,\bar{u}_2))$ (recall $f(u_1,\bar{u}_2)=(\hat{x}_1,\bar{x}_2)$) and f is responsive, we have $f_2(u_1^\varepsilon,\bar{u}_2)=\bar{x}_2$; hence $\bar{x}_1 \in Z_1(f_2(u_1^\varepsilon,\bar{u}_2))$ which contradicts that f is responsive. Therefore $f_1(u_1^\varepsilon,\bar{u}_2)\in\bar{B}(\bar{x}_1,\varepsilon)$.

We now show that in fact $f_1(u_1^\varepsilon,\bar{u}_2)=\bar{x}_1$, and therefore that $\bar{x}_1 \in O_1(\bar{u}_2)$. Suppose that $f_1(u_1^\varepsilon,\bar{u}_2)=\hat{x}_1' \neq \bar{x}_1$. Let $0<\varepsilon' < \|\hat{x}_1'-\bar{x}_1\|$; the previous argument shows that $f_1(u_1^{\varepsilon'},\bar{u}_2)\in\bar{B}(\bar{x}_1,\varepsilon')$; i.e., $\|f_1(u_1^{\varepsilon'},\bar{u}_2)-\bar{x}_1\| < \|\hat{x}_1'-\bar{x}_1\|$. Thus, according to *(3)* we have

$$u_1^\varepsilon f_1(u_1^{\varepsilon'},u_2) > u_1^\varepsilon(\hat{x}_1') = u_1^\varepsilon f_1(u_1^\varepsilon,u_2) \ ,$$

and therefore Individual 1 can manipulate f at $(u_1^\varepsilon,\bar{u}_2)$ via $u_1^{\varepsilon'}$, contradicting that f is NM. Thus $f_1(u_1^\varepsilon,\bar{u}_2)=\bar{x}_1$, and therefore $\bar{x}_1 \in O_1(\bar{u}_2)$. □

Because every continuous function on X_i is included in U_i, we have the following corollary to Lemma 1.1.

Corollary 1.1. $f(\mathscr{U}^*)=\mathscr{R}$.

Lemma 1.2. *For each $u_2 \in U_2^*$, $Z_1(m(u_2)) \subseteq O_1(u_2)$.*

Proof. Suppose not; let $u_2 \in U_2^*$ and $\bar{x}_1 \in Z_1(m(u_2))$, $\bar{x}_1 \notin O_1(u_2)$. Let $\hat{\mathbf{x}} \in \mathscr{R}$ be such that $\hat{x}_2 = m(u_2)$ and $\hat{x}_1 \neq \bar{x}_1$. Lemma 2 allows us to construct a continuous utility function[4] $u_1 \in U_1^*$ satisfying

(1) $m(u_1)=\bar{x}_1$
(2) $\hat{x}_1$ is the unique maximizer of u_1 on $O_1(u_2)$.

Lemma 1 yields $f_1(u_1,u_2)=\hat{x}_1$. Responsiveness of f implies $f_2(u_1,u_2)=\hat{x}_2$. But then $\bar{x}_1 \in Z_1(f_2(u_1,u_2))$ and $u_1(\bar{x}_1) > u_1(\hat{x}_1)$, contradicting that f is responsive. □

Lemma 1.3 establishes that for each $u_2 \in U_2^*$, the set $O_1(u_2)$ is determined solely by $m(u_2)$.

Lemma 1.3. *If $u_2, u_2' \in U_2^*$ are such that $m(u_2)=m(u_2')$, then $O_1(u_2)=O_1(u_2')$.*

Proof. Suppose not; let $u_2, u_2' \in U_2^*$ be such that $m(u_2)=m(u_2')$, and let $\bar{x}_1 \in O_1(u_2)\backslash O_1(u_2')$. Let $\hat{x}_1 \in Z_1(m(u_2'))$; according to Lemma 1.2, we must have $\hat{x}_1 \in O_1(u_2')$. As in Lemma 1.2 we construct a continuous utility function $u_1 \in U_1^*$ satisfying

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[4] For an example see Barberà and Peleg (1990), Lemma 5.6.

(1) $m(u_1)=\tilde{x}_1$
(2) $\hat{x}_1$ is the unique maximizer of u_1 on $O_1(u_2')$.

Lemma 1 implies that $f_1(u_1,u_2)=\tilde{x}_1$. Since $x_1\notin O_1(u_2')$, Lemma 1.2 implies that $\tilde{x}_1\notin Z_1(m(u_2'))=Z_1((m(u_2))$, and therefore $f_2(u_1,u_2)\neq m(u_2)$.

On the other hand, Lemma 1 also implies that $f_1(u_1,u_2')=\hat{x}_1$; thus $m(u_2')\in Z_2(\hat{x}_1)=Z_2(f_1(u_1,u_2'))$, and since f is responsive we must have that $f_2(u_1,u_2')=m(u_2')$. Hence

$$u_2(m(u_2))=u_2f_2(u_1,u_2')>u_2f_2(u_1,u_2)\ ,$$

and therfore Individual 2 can manipulate f at (u_1,u_2) via u_2'. This contradiction establishes the lemma. □

Lemma 1.4. *For each $u_2\in U_2^*$, either $O_1(u_2)=Z_1(m(u_2))$, or $O_1(u_2)=\mathscr{R}_1$.*

Proof. Suppose not; let $u_2\in U_2^*$ and $\tilde{x}_1,\hat{x}_1\in\mathscr{R}_1$ be such that $\hat{x}_1\notin Z_1(m(u_2))$, $\hat{x}_1\in O_1(u_2)$ and $\tilde{x}_1\notin O_1(u_2)$. Let $u_1\in U_2^*$ be such that

(1) $m(u_1)=\tilde{x}_1$
(2) $\hat{x}_1$ is the unique maximizer of u_1 on $O_1(u_2)$.

Lemma 1 implies that $f_1(u_1,u_2)=\hat{x}_1$, and therfore $f_2(u_1,u_2)\in Z_2(\hat{x}_1)$. Let $\tilde{x}_2\in Z_2(\tilde{x}_1)$, and without loss of generality (Lemma 1.3) assume that for each $x_2\in Z_2(\hat{x}_1)$, $u_2(\tilde{x}_2)>u_2(x_2)$. (A continuous utility function with these properties can be constructed since $m(u_2)\notin Z_2(\hat{x}_1)$, and $Z_2(\hat{x}_1)$ is closed - recall that Z is compact.)

If $\tilde{x}_2\in Z_2(\hat{x}_1)=Z_2(f_1(u_1,u_2))$, responsiveness of f implies $f_2(u_1,u_2)=\tilde{x}_2$; hence $\tilde{x}_1\in Z_1(f_2(u_1,u_2))$, which contradicts that f is responsive.

If $\tilde{x}_2\notin Z_2(\hat{x}_1)$, let $\tilde{u}_2\in U_2^*$ be such that $m(\tilde{u}_2)=\tilde{x}_2$; Lemma 1.1 yields $f(u_1,\tilde{u}_2)=(\tilde{x}_1,\tilde{x}_2)$; hence

$$u_2f_2(u_1,\tilde{u}_2)=u_2(\tilde{x}_2)>u_2f_2(u_1,u_2)\ ,$$

and therefore Individual 2 can manipulate f at (u_1,u_2) via $\tilde{u}_2$. This violation of nonmanipulability establishes the lemma. □

Lemma 1.5. *Either for each $u_2\in U_2^*$: $O_1(u_2)=Z_1(m(u_2))$, or for each $u_2\in U_2^*$: $O_1(u_2)=\mathscr{R}_1$.*

Proof. Suppose not; then Lemma 1.4 implies that there are $\bar{u}_2,\hat{u}_2\in U_2^*$ satisfying both $O_1(\bar{u}_2)=Z_1(m(\bar{u}_2))\neq\mathscr{R}_1$ and $O_1(\hat{u}_2)=\mathscr{R}_1\neq Z_1(m(\hat{u}_2))$. Note that for each $u_2\in U_2$, $Z_1(m(u_2))=\mathscr{R}_1(m(u_2))$. (Lemma 1.2 yields $Z_1(m(u_2))\subseteq O_1(u_2)\subseteq\mathscr{R}_1$, and $\mathscr{R}_1\subseteq Z_1$). Thus, one has $\mathscr{R}_1(m(\bar{u}_2))=Z_1(m(\bar{u}_2))\neq\mathscr{R}_1$ and $\mathscr{R}_1(m(\hat{u}_2))$ $=Z_1(m(\hat{u}_2))\neq\mathscr{R}_1$, and therefore if $\mathbf{x}\in\mathscr{R}$ satisfies either $x_2=m(\bar{u}_2)$ or $x_2=m(\hat{u}_2)$, then $\mathbf{x}\in\mathscr{C}_1(\mathscr{R})$.

Since $\mathscr{R}$ contains a conflict of degree greater than two, there is $\tilde{x}_1\in\mathscr{R}_1\backslash[\mathscr{R}_1(m(\bar{u}_2))\cup\mathscr{R}_1(m(\hat{u}_2))]$. Without loss of generality (Lemma 1.3) assume that $u_2(m(\bar{u}_2))>u_2(x_2)$, for each $x_2\in Z_2(\tilde{x}_1)$.

Let $\tilde{u}_1\in U_1^*$ be such that $m(\tilde{u}_1)=\tilde{x}_1$. Since $O_1(\hat{u}_2)=\mathscr{R}_1$, Lemma 1 implies that $f_1(\tilde{u}_1,\hat{u}_2)=\tilde{x}_1$ and therefore $f_2(\tilde{u}_1,\hat{u}_2)\in Z_2(\tilde{x}_1)$. Also since $O_1(\bar{u}_2)=Z_1(m(\bar{u}_2))$ one has $m(\bar{u}_2)\in Z_2(f_1(\tilde{u}_1,\bar{u}_2))$, and responsiveness of f yields $f_2(\tilde{u}_1,\bar{u}_2)=m(\bar{u}_2)$. Hence

$$\hat{u}_2f_2(\tilde{u}_1,\bar{u}_2)=\hat{u}_2(m(\bar{u}_2))>\hat{u}_2f_2(\tilde{u}_1,\hat{u}_2)\ ;$$

thus Individual 2 can manipulate f at $(\tilde{u}_1, \hat{u}_2)$ via $\tilde{u}_2$, which contradicts the nonmanipulability of f and completes the proof of Lemma 1.5. □

We introduce now a concept that will be useful for the remaining of the proof of Theorem 1 as well as for the proof of Theorem 2. Let $f: \mathscr{U} \to Z$ be a mechanism and let $\tilde{\mathscr{U}}$ be an arbitrary proper subset of $\mathscr{U}$. We say that Individual i is a **dictator for f on** $\tilde{\mathscr{U}}$ if for each $\mathbf{u} \in \tilde{\mathscr{U}}$, $f_i(\mathbf{u})$ maximizes u_i on $f(\tilde{\mathscr{U}})$. If there is a dictator for f on $\tilde{\mathscr{U}}$, we say that f is dictatorial on $\tilde{\mathscr{U}}$.

Proof of Theorem 1. The proof that f is dictatorial on $\mathscr{U}^*$ is immediate from Lemma 1.5:

If for each $\mathbf{u} \in \mathscr{U}^*$, $O_1(u_2) = Z_1(m(u_2))$, then one has $m(u_2) \in Z_2(f_1(\mathbf{u}))$, and since f is responsive one has that $f_2(\mathbf{u}) = m(u_2)$. Hence Individual 2 is a dictator for f on $\mathscr{U}^*$.

If for each $\mathbf{u} \in \mathscr{U}^*$, $O_1(u_2) = \mathscr{R}_1$, then Lemma 1 yields $f_1(\mathbf{u}) = m(u_1)$, and therefore Individual 1 is a dictator for f on $\mathscr{U}^*$.

Suppose that Individual 1 is a dictator for f on $\mathscr{U}^*$. We show that he is a dictator for f on $\mathscr{U}$.

It is easy to show that Individual 1 is a dictator for f_1 on $U_1^* \times U_2$. Suppose not; let $(\bar{u}_1, u_2) \in U_1^* \times U_2$ be such that $f_1(\bar{u}_1, u_2) \neq m(\bar{u}_1)$. Clearly $m(\bar{u}_1) \notin Z_1(f_2(\bar{u}_1, u_2))$, for otherwise f would not be responsive. Let $\bar{u}_2 \in U_2^*$ be such that $m(\bar{u}_2) = f_2(\bar{u}_1, u_2)$. Since $m(\bar{u}_2) \in O_2(\bar{u}_1)$, Lemma 1 implies that $f_2(\bar{u}_1, \bar{u}_2) = m(\bar{u}_2)$. Notice that $(\bar{u}_1, \bar{u}_2) \in \mathscr{U}^*$ and recall that $f(\mathscr{U}^*) = \mathscr{R}$ (Corollary 1.1). Furthermore, $m(\bar{u}_1) \notin Z_1(m(\bar{u}_2)) = Z_1(f_2(\bar{u}_1, \bar{u}_2))$; hence $f_1(\bar{u}_1, \bar{u}_2) \neq m(\bar{u}_1)$, which contradicts that Individual 1 is a dictator for f on $\mathscr{U}^*$.

Finally, it is easy to show that Individual 1 is a dictator for f on $\mathscr{U}$, since $\mathscr{R} = f(\mathscr{U}^*) \subseteq f(U_1^* \times U_2)$ and therefore he could otherwise manipulate f by claiming as his utility function a member of U_1^*. □

Proof of Theorem 2. The outline of the proof of Theorem 2 is as follows: first it is shown that the mechanism is dictatorial when restricted to certain domains (Lemmas 2.1 and 2.2); then the theorem is proved by induction on the number of individuals. Henceforth, let f be a mechanism satisfying the assumptions of Theorem 2. We introduce now some additional notation.

For each $\mathbf{u} \in \mathscr{U}$, $f_p(\mathbf{u})$ denotes the public part of the outcome, $f_p(\mathbf{u}) \in X^p$. Let $U_i^p \subseteq U_i$ be the set of all utility functions of the form $u_i(x^p, y_i) = u_i^p(x^p)$, where u_i^p is a continuous real valued function on X^p. We write $\mathscr{U}^p$ for the set $\prod_1^n U_i^p$.

Lemma 2.1. *f is dictatorial on $\mathscr{U}^p$.*

Proof. The proof of Lemma 2.1 is essentially an application of the Barberà and Peleg Theorem to the restriction of f_p to $\mathscr{U}^p$. First, note that for each $i \in N$ and each $\mathbf{u} \in \mathscr{U}^p$, $u_i f(\mathbf{u}) = u_i^p f_p(\mathbf{u})$. Indeed the set $\mathscr{U}^p$ can be put in a one-to-one correspondence with the set of all the continuous utility profiles on X^p, and therefore the restriction of f_p to $\mathscr{U}^p$ (which we denote also by f_p), $f_p: \mathscr{U}^p \to X^p$, is a *mechanism* for a purely public social decision problem. Moreover, f_p is nonmanipulable (since f is nonmanipulable) and $f_p(\mathscr{U}^p) = \mathscr{R}^p$. (Given $\bar{x}^p \in \mathscr{R}^p$, let $\mathbf{u} \in \mathscr{U}$ be such that $f(\mathbf{u}) = \mathbf{x}$, with $x^p = \bar{x}^p$; for each $i \in N$, choose $\bar{u}_i \in U_i^p$ such that $\bar{x}^p$ is the unique maximizer of $\bar{u}_i^p$; nonmanipulability of f_p yields $f_p(\bar{\mathbf{u}}) = \bar{x}^p$.) Since $\mathscr{R}^p$ contains more than two elements, it follows from the

Barberà and Peleg Theorem that f_p must be dictatorial; i.e., there is $j \in N$ such that for each $\mathbf{u} \in \mathscr{U}^p$ and each $(x^p, y_j) \in \mathscr{R}_j$, we have

$$u_j f_j(\mathbf{u}) = u_j^p f_p(\mathbf{u}) \geq u_j^p(x^p) = u_j(x^p, y_j) \ .$$

Thus j is a dictator for f on $\mathscr{U}^p$. □

Lemma 2.2. *Some Individual j is the dictator for f on $U_j^p \times \prod_{i \neq j} U_i$.*

Proof. Without loss of generality, assume that Individual 1 is the dictator for f on $\mathscr{U}^p$. It will shown that he is a dictator for f on $U_1^p \times \prod_2^n U_i$ (i.e., for each $\mathbf{u} \in U_1^p \times \prod_2^n U_i$, $f_p(\mathbf{u})$ maximizes u_1^p). Suppose not; let $\mathbf{u} \in U_1^p \times \prod_2^n U_i$ and $\bar{x}^p \in \mathscr{R}^p$ be such that $u_1^p(\bar{x}^p) > u_1^p f_p(\mathbf{u})$.

Write $f_p(\mathbf{u}) = \tilde{x}^p$, and for $i = 2, \ldots, n$, let $\tilde{u}_i \in U_i^p$ be such that $\tilde{x}^p$ is the unique maximizer of $\tilde{u}_i^p$. Since f is nonmanipulable, for $k = 2, \ldots, n$, one has

$$f_p(\tilde{u}_2, \ldots, \tilde{u}_k, \mathbf{u}_{\sim\{2,\ldots,k\}}) = \tilde{x}^p \ .$$

Thus, $f_p(u_1, \tilde{\mathbf{u}}_{\sim 1}) = \tilde{x}^p \neq \bar{x}^p$. But notice that $(u_1, \tilde{\mathbf{u}}_{\sim 1}) \in \mathscr{U}^p$; therefore Individual 1 is not a the dictator for f on $\mathscr{U}^p$, contrary to our assumption. This contradiction establishes the lemma. □

For the proof of Theorem 2 some additional notation needs to be introduced. Let $i \in N$ and $u_i \in U_i$ arbitrary. Consider the function fu_i defined, for each $\mathbf{u}_{\sim i} \in \prod_{j \neq i} U_j$ by $fu_i(\mathbf{u}_{\sim i}) = f(u_i, \mathbf{u}_{\sim i})$. Although the fu_i functions are not mechanisms *stricto sensu*, the notions of manipulability and dictatorship, as well as the conclusions of the previous two lemmas apply to them. Indeed if f is nonmanipulable and responsive, then for each $i \in N$ and each $u_i \in U_i$, fu_i is nonmanipulable and responsive. Write $\mathscr{R}(fu_i)$ for the range of fu_i, and $\mathscr{R}^p(fu_i)$ and $\mathscr{R}_j(fu_i)$ for its projections into, respectively, X^p and X_j.

Proof of Theorem 2. The theorem is proved by induction. The case $n = 2$ is a simple application of Theorem 1, since the requirement on the cardinality of $\mathscr{R}^p$ implies that the range of the mechanism contains a conflict of degree greater than two. Now we can assume that f is dictatorial for $n-1$, and we must show that it is dictatorial for n.

Without loss of generality, let Individual 1 be the dictator for f on $U_1^p \times \prod_2^n U_i$ (Lemma 2.2). Let $u_2 \in U_2$ be arbitrary. Clearly fu_2 is nonmanipulable and responsive, and it involves only $n-1$ individuals; furthermore, since Individual 1 is a dictator for f on $U_1^p \times \prod_2^n U_i$, we have $\mathscr{R}^p(fu_2) = \mathscr{R}^p$. Thus, the induction assumption implies that fu_2 is dictatorial.

We show that Individual 1 must be the dictator for fu_2. Suppose not; again without loss of generality, let Individual 3 be the dictator for fu_2 and let $\tilde{x}^p \in \mathscr{R}^p$ and $\tilde{u}_3 \in U_3^p$ be such that $\tilde{x}^p$ is the unique maximizer of $\tilde{u}_3^p$. Let $u_1 \in U_1^p$ be such that $u_1^p(\bar{x}^p) > u_1^p(\tilde{x}^p)$, for some $\bar{x}^p \in \mathscr{R}^p \backslash \{\tilde{x}^p\}$. Since Individual 3 is the dictator for fu_2, we have

$$f_p(u_1, u_2, \tilde{u}_3, \mathbf{u}_{\sim\{1,2,3\}}) = \tilde{x}^p \ ,$$

which contradicts the fact that Individual 1 is the dictator for f on $U_1^p \times \prod_2^n U_i$. Thus, for each $u_2 \in U_2$, Individual 1 must be the dictator for fu_2; i.e., for each $\mathbf{u} \in \mathscr{U}$, $f(\mathbf{u})$ maximizes u_1 on $\mathscr{R}_1(fu_2)$.

In order to establish Theorem 2, one has to show that for each $u_2 \in U_2$, $\mathscr{R}_1 = \mathscr{R}_1(fu_2)$. Suppose not; let $\bar{u}_2 \in U_2$ and $\bar{x}_1 = (\bar{x}^p, \bar{y}_1) \in \mathscr{R}_1 \backslash \mathscr{R}_1(f\bar{u}_2)$. Note that since Individual 1 is a dictator for f on $U_1^p \times \prod_2^n U_i$, then $\mathscr{R}^p(f\bar{u}_2) = \mathscr{R}^p$. Let $\hat{x}_1 = (\hat{x}^p, \hat{y}_1) \in \mathscr{R}_1(f\bar{u}_2)$, $\hat{x}^p \neq \bar{x}^p$. Since Individual 1 is also a dictator for $f\bar{u}_2$, then $\mathscr{R}_1(f\bar{u}_2)$ is closed ($\mathscr{R}_1(f\bar{u}_2)$ is Individual 1's option set for $f\bar{u}_2$, which is closed by Lemma 2). Thus, we can construct a utility function $\hat{u}_1 \in U_1$ satisfying

(1.1) $m(\hat{u}_1) = \hat{x}_1$
(1.2) $\hat{x}_1$ is the unique maximizer of $\hat{u}_1$ on $\mathscr{R}_1(f\bar{u}_2)$.

Let $(u_3, \ldots, u_n) = \mathbf{u}_{\sim\{1,2\}} \in \prod_3^n U_j$ be arbitrary. Since Individual 1 is a dictator for $f\bar{u}_2$, one has $f(\hat{u}_1, \bar{u}_2, \mathbf{u}_{\sim\{1,2\}}) = (\hat{x}^p, \hat{y}_1, \ldots, \hat{y}_n)$. Since $\mathscr{R}^p$ contains at least three different outcomes, let $\tilde{x}^p \in \mathscr{R}^p \backslash \{\hat{x}^p, \bar{x}^p\}$ and $\hat{u}_2 \in U_2$ be such that

(2.1) $m(\hat{u}_2) = (\hat{x}^p, \hat{y}_2)$
(2.2) $\hat{u}_2(\bar{x}^p, y_2) > \hat{u}_2(\tilde{x}^p, y_2)$, $\forall y_2 \in Y_2$.

Since f is NM, $f_2(\hat{u}_1, \hat{u}_2, \mathbf{u}_{\sim\{1,2\}}) = (\hat{x}^p, \hat{y}_2)$, and therefore $\bar{x}_1 \notin \mathscr{R}_1(f\hat{u}_2)$. (Otherwise, since Individual 1 is a dictator for $f\hat{u}_2$, one would have $f(\hat{u}_1, \hat{u}_2, \mathbf{u}_{\sim\{1,2\}}) = \bar{x}_1$, and hence $f_p(\hat{u}_1, \hat{u}_2, \mathbf{u}_{\sim\{1,2\}}) = \bar{x}^p \neq \hat{x}^p$.) Let $\tilde{y}_1$ be such that $\tilde{x}_1 = (\tilde{x}^p, \tilde{y}_1) \in \mathscr{R}_1(f\hat{u}_2)$, and let $\bar{u}_1 \in U_1$ be such that

(3.1) $m(\bar{u}_1) = \bar{x}_1$
(3.2) $\tilde{x}_1$ is the unique maximizer of $\bar{u}_1$ on $\mathscr{R}_1(f\hat{u}_2)$.

As Individual 1 is a dictator for $f\hat{u}_2$ and $\bar{x}_1 \notin \mathscr{R}_1(f\hat{u}_2)$, one has $f_1(\bar{u}_1, \hat{u}_2, \mathbf{u}_{\sim\{1,2\}}) = \tilde{x}_1$. Also since $\bar{x}_1 \in \mathscr{R}_1$, let $\tilde{\mathbf{u}} = (\tilde{u}_2, \tilde{\mathbf{u}}_{\sim 2}) \in \mathscr{U}$ be such that $f_1(\tilde{\mathbf{u}}) = \bar{x}_1$. Thus, $\bar{x}_1 \in \mathscr{R}_1(f\tilde{u}_2)$ and since Individual 1 is a dictator for $f\tilde{u}_2$, $f_1(\bar{u}_1, \tilde{u}_2, \mathbf{u}_{\sim\{1,2\}}) = \bar{x}_1$. Hence for some $\bar{y}_2, \tilde{y}_2 \in Y_2$, one has

$$\hat{u}_2 f_2(\bar{u}_1, \tilde{u}_2, \mathbf{u}_{\sim\{1,2\}}) = \hat{u}_2(\bar{x}^p, \bar{y}_2) > \hat{u}_2(\tilde{x}^p, \tilde{y}_2)$$
$$= \hat{u}_2 f_2(\bar{u}_1, \hat{u}_2, \mathbf{u}_{\sim\{1,2\}}) \;,$$

and therefore Individual 2 can manipulate f at $(\bar{u}_1, \hat{u}_2, \mathbf{u}_{\sim\{1,2\}})$ via $\tilde{u}_2$. This contradiction establishes that for each $u_2 \in U_2$, $\mathscr{R}_1(fu_2) = \mathscr{R}_1$, completing the proof of Theorem 2. □

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