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## Stochastic Replicator Dynamics\*

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## **Abstract**

This paper studies the replicator dynamics in the presence of shocks. I motivate the dynamics as the result of a process by which agents change the strategy they use when its performance is not satisfactory. I show that under these dynamics strictly dominated strategies are eliminated even in the presence of nonvanishing perturbations. I also provide sufficient conditions for the existence of a unique ergodic distribution and give examples that show that the stochastic dynamics in this paper have equilibrium selection properties that differ from those of other stochastic dynamics in the literature.

## 1. INTRODUCTION

The current interest in evolutive dynamics was born from a discussion of the foundations of game theory, whose emphasis on equilibria seemed to make excessive demands on the rationality of agents. As an alternative, it was thought that equilibrium may be the result of the repeated interaction of agents who are less strategically sophisticated than the traditional theory supposes. There are precedents here, since Cournot himself gave a dynamic explanation for how the equilibrium for his oligopoly game would come to be.

This paper studies stochastic selection systems, in the context of games. I work with a particular dynamic system, the replicator dynamics, which I will try to show has interpretations beyond the usual evolutionary ones. These dynamics model agents with a very low degree of sophistication. Despite the agents' lack of sophistication, I find that even in the presence of stochastic shocks of several kinds the dynamics give little asymptotic weight to strictly dominated strategies. I also give sufficient conditions for the stochastic dynamics to have an ergodic distribution.

When considering the replicator dynamics it is useful to think of a large population of agents who use pure strategies and are randomly matched to play against each other. The growth rate of the proportion of players using a certain pure strategy is the difference between the expected payoff of that pure strategy, given the proportions of players using every pure strategy, and the average expected payoff in that population. In contrast to other dynamics that have been proposed, like the best-response dynamics of Matsui [16], the fictitious play of Brown [2] and Robinson [23], or the learning papers of

Milgrom and Roberts [17] and Fudenberg and Kreps [12], the replicator dynamics have the characteristic that the strategies whose weight in the population increase need not be best responses to anything, in particular to past outcomes of play. And even in these circumstances, it can be shown, (see Cabrales and Sobel [3]) that if selection operates slowly enough or in continuous time, then all limit points of the dynamics are best responses to time averages of past play, thus giving some support to the notion that agents that are not rational behave as if they were.

The history of stochastic selection processes is not long, in part because the techniques are relative newcomers also. An early article was written by M. J. Farrell [9] at a time when most of the discussion on selection was done in terms of deterministic dynamics. The main concern of the literature on selection dynamics was whether the assumption of profit maximization was a sensible one to use for the theory of the firm. One possible argument in favor of the idea was that non-profit maximizers would tend to grow less or become bankrupt more often, thus making an ever shrinking proportion of the industry. Winter [31] has a more extensive discussion of this argument. Farrell uses branching processes to model a situation where several different "ability" groups are characterized by their probability of success. He then calculates the relative preponderance of the different groups. He also considers the introduction of new entrants to the groups. The fate of strictly dominated actions and the implications of a stream of disoriented new entrants are two of the themes I will address. Farrell's work is concerned with pure decision problems in a small population setting, though. In a more recent paper, Foster and Young [10], develop a model where perturbations, which they describe by Wiener processes, are constantly affecting the replicator dynamics and keeping the

process within the interior of the simplex. Kandori, Mailath and Rob [14] and Young [32] consider explicitly a case with finite populations where the randomness comes from the stochastic replacements of agents by newcomers that start by playing something at random. In the three previous models the processes have ergodic distributions, and the authors arrive at predictions by looking at the limit of these ergodic distributions when the variance of the noise goes to zero. Papers by Samuelson [24], Noldeke, Samuelson and Van Damme [20], and Kandori and Rob [15] apply the techniques in Kandori, Mailath and Rob [14] and Young [32] to a variety of games, including cheap talk, pure coordination games and supermodular games. This approach has proven useful because it has been able to discriminate between strict equilibria, something most refinements and other dynamic systems were unable to distinguish. An exception is Crawford's [4] paper where he shows that in some games strategic uncertainty and adaptive adjustments can give rise to systematic equilibrium selection patterns without having to depend on an ergodic distribution. Crawford [5] shows that in finite populations evolutionary stability is also capable of selecting between strict equilibria.

The model that I will use was first developed by Fudenberg and Harris [11], for a two player, two strategy, symmetric game. I will study games that are not necessarily symmetric with multiple players and strategies. Like Fudenberg and Harris, I define the replicator dynamics in continuous time, and the state variables are points in the simplex. One difference with the Foster and Young paper is that the source of the shocks becomes important, and we distinguish between aggregate shocks to payoffs and mutations. In the fourth section I show that strictly dominated strategies become rare when selection has been operating for a long time. This extends the result found by Fudenberg and Harris for

their class of games. In the sixth section I show that for a two strategy, N-player, symmetric game with two strict equilibria, the equilibrium selected by the dynamics used in my paper is different from the one that the dynamics used by Kandori, Mailath and Rob would select. In the class of games that Fudenberg and Harris study both kinds of dynamics select the same equilibria, if mutations are possible, because in a game with two players the payoffs are linear in the proportion of players using every strategy.

The shocks captured by the model in this paper, as in Fudenberg and Harris, are of two types. There are individual, uncorrelated changes of strategy, produced by the entry of uninformed players. Since I assume there is no correlation in these changes and the population is very large, I model these shocks as deterministic shifts to the replicator dynamics, which is how traditional models of biological selection tend to include mutations. There are also aggregate shocks that affect payoffs in the same way for all users of a strategy. These will not average out; they constitute the part of my model that is explicitly stochastic. For a first approximation they are considered uncorrelated across time. Since the model is formulated in continuous time Wiener processes are an adequate way to model them. The sixth section extends the ergodicity result in Fudenberg and Harris to games with multiple players and strategies.

The second section of the paper will be devoted to a description and motivation of the replicator dynamics. I will argue that the replicator dynamics can be thought of as the reduced form of a process of imitation or of economic survival, and present a class of dynamic systems, first introduced by Smallwood and Conlisk [27], of which the replicator dynamics are a special case. In the third section I will introduce the model with shocks. In the fourth section I will prove that if the variance of the noise is not too large,

when the mutation rates are smaller and smaller the system tends to give less and less weight to strictly dominated strategies. The fifth section deals with the existence and uniqueness of an ergodic measure. The sixth section presents an example that highlights the differences between the model in this paper and other stochastic dynamics. The seventh section shows that for members of the Smallwood-Conlisk family of dynamics other than the replicator dynamics, strictly dominated strategies need not be eliminated. This happens even for dynamics that are close, in a parametric sense, to the replicator dynamics. Then I conclude the paper.

## 2. REPLICATOR DYNAMICS

The game considered here will have finitely many pure strategies and players. There are  $N$  players and the pure strategy set for the  $i$ th player is  $P_i$  which has  $n_i$  strategies. Player  $k$ 's payoff function is  $u_k : \prod_{i=1}^N P_i \rightarrow \mathbb{R}$ . Let  $S^n$  denote the standard  $n-1$  dimensional simplex.  $u_k$  is extended to the space of mixed strategies in the usual way, and  $j \in P_i$  will be identified with the mixed strategy that gives probability one to the pure strategy  $j$ . Suppose there are  $N$  populations of agents, one for each player, and each of them contains a continuum of individuals. The usual interpretation of the replicator dynamics is that they describe the evolution of the proportion of members of each population playing every strategy. Payoffs in that case represent reproductive fitness, or the number of successors for the user of a strategy given the makeup of the population.

Let  $x_j^i(t)$  be the proportion of members of the  $i$ th player population using strategy  $j$  at time  $t$ . The replicator dynamics are defined as follows;

$$\dot{x}_j^i(t) = x_j^i(t) \left[ u_i(j, x(t)^{-i}) - \sum_{k=1}^{n_i} x_k^i(t) u_i(k, x(t)^{-i}) \right].$$

To justify the dynamics imagine that the individuals are randomly matched during period  $t$  to play the game. They learn their payoff. A small portion of them are then taken and randomly paired with members of the same player population. They compare payoffs and the one with lower payoff changes to the strategy of the one with higher payoff with a probability that is proportional to the difference in payoffs. This could happen for example if the agents had an idiosyncratic uniformly distributed cost of changing strategies, and decided to change only when the difference in payoffs were higher than the cost of changing strategies. Nachbar [18] gives a similar interpretation for the replicator dynamics.

For an economist it is difficult to accept an explanation for a model that makes agents behave on the basis of information that is so limited, instead of using more sophisticated information gathering and processing techniques. Not everybody shares this belief, however. Nisbett and Ross [19] report experimental results in which the opinions communicated in person by others have a stronger effect on decision makers than written information that is statistically more relevant. In my opinion, the weakness of the replicator dynamics lies in the fact that the scope of the agents' research is limited, both in the number of people consulted, and the amount of past experience used; and in the uniformity of the learning rule assumed for all the population. The result in section 4 extends one conclusion obtained for the replicator dynamics to a perturbed version of the model, and the example in section 7 shows the necessity of additional assumptions to extend the conclusions even further. This implies that for practical applications the particular way in

which agents adapt needs to be taken into consideration.

I want to consider now a different interpretation for the replicator dynamics, which may help in connecting them to other selection models in the economic literature and illuminate the results in the remaining sections of the paper. The main behavioral hypothesis for this interpretation is that human economic agents are satisficers, and change their actions only when the action they are currently taking does not perform better than a preset standard. Winter [31], for example, proposes a model with this characteristic as an alternative to profit maximization by firms. In a consumers' choice model, proposed by Smallwood and Conlisk [27], the task is to choose between  $N$  different brands, differentiated by their probabilities of performing unsatisfactorily,  $b_i$  for brand  $i$ . A consumer that owns a product that doesn't break down in a certain period purchases the same brand in the next period. If the product breaks down, he chooses next period's brand randomly. One possibility would be to give the same weight to all brands, another would be to purchase the most popular brand. In general the consumer could be somewhat sensitive to market popularity, without necessarily adopting such extreme procedures. Maybe the procedure consists of picking the first brand in the shelf and shelf space is only partially sensitive to market power. Smallwood and Conlisk summarized these possibilities by parametrizing the model in the following way. Let the market share of brand  $i$  be  $m_i$ , then the probability that a consumer chooses  $i$  is

$$\frac{m_i^\alpha}{\sum_{k=1}^N m_k^\alpha}, \text{ where } \alpha \geq 0$$

is the parameter that controls the importance of popularity. When  $\alpha = 0$  popularity is unimportant for the consumers' choice; when  $\alpha$  is infinity only the most popular brand will be chosen; when  $\alpha$  is exactly one the probabilities are exactly the same as the market

shares. Given that the individuals choose independently, if the total number of consumers is large, the law of large numbers guarantees that the error made in identifying the frequencies with which actions are taken in the population with the probabilities that each individual will take them is small. Then the dynamics that regulate the evolution of market shares can be expressed,

$$m_i(t+1) = m_i(t)(1 - b_i) + \sum_j b_j m_j(t) \frac{m_i(t)^\alpha}{\sum_r m_r(t)^\alpha}.$$

Now suppose that instead of a consumer choosing a brand we are looking at player  $i$  in our game who is choosing strategies. Total payoffs for strategy  $j$  are given by  $u_i(j, x^{-i})$  plus an idiosyncratic uniformly distributed random shock with support  $[a, b]$ . This is intended to model the fact that people are taking many decisions at the same time and knowledge about their payoffs in a particular case can be gathered only imperfectly. Agents change their strategies when total payoff is less than a certain acceptable level, call it  $c$ . Let's assume that the constants  $a$  and  $b$  are such that  $\max_{i,j,k} u_i(j, k) \leq c - a$  and  $\min_{i,j,k} u_i(j, k) \geq c - b$ . With these constraints on  $a$  and  $b$  any strategy at any time can either give a payoff above the acceptable level or fail to do so with positive probability. If the performance of a strategy is adequate agents keep using it. If it is not they choose strategy  $j$  in the next period with probability  $\frac{(x_j^i)^\alpha}{\sum_k (x_k^i)^\alpha}$ . The probability that strategy  $j$  fails

for a player is equal to  $\frac{c - u_i(j, x^{-i}) - a}{b - a}$ , and the probability that it doesn't is  $\frac{b - c + u_i(j, x^{-i})}{b - a}$ . The dynamics that result for the population shares are,

$$x_j^i(t+1) = x_j^i(t) \frac{b - c + u_i(j, x^{-i}(t))}{b - a} + \frac{x_j^i(t)^\alpha}{\sum_k x_k^i(t)^\alpha} \sum_l x_l^i(t) \frac{c - u_i(l, x^{-i}(t)) - a}{b - a}.$$

If  $\alpha = 1$  we can rewrite the expression in the following way,

$$x_j^i(t+1) = x_j^i(t) + \frac{x_j^i(t)}{b-a} [u_i(j, x^{-i}(t)) - u_i(x^i(t), x^{-i}(t))].$$

This is the discrete time version of the replicator dynamics. By reducing the period length, and simultaneously reducing the probability of failure at a particular period (say raising  $b$ ), in the appropriate way, we can obtain the continuous time version that I will work with in subsequent sections.

### 3. THE MODEL WITH SHOCKS

I want to consider now the introduction of shocks to the replicator dynamics. The first type of shocks includes those that affect the payoffs of all users of a strategy in the same way. They could be random changes in total demand in an oligopoly game where oligopolists face the same demand curve, or changes in the legal system that make certain strategies more costly, or changes in factor prices that alter the cost of using a technology. The sum of these shocks will be modeled as a Wiener process. These are continuous-time stochastic processes with almost surely continuous sample paths and stationary independent increments with mean zero. Let  $W$  be a  $d$ -dimensional Wiener process,  $\sigma_j^i$  a  $d$ -dimensional vector of positive constants. The instantaneous payoff for the user of strategy  $j$  of population  $i$  at time  $t$  is,

$$d\tilde{u}_i(j, t) = u_i(j, x^{-i}(t))dt + \sum_{l=1}^d \sigma_{jl}^i dW_l(t).$$

At a particular instant in time a player in the  $i$ th population is matched with a player from each of the other  $N-1$  populations. The probability that a randomly chosen member of the  $k$ th population uses strategy  $j$  is  $x_j^k(t)$ . The payoff for a player in the  $i$ th

population using strategy  $j$  will be  $u_i(j, x^{-i}(t))dt$  plus a random quantity  $\sum_{l=1}^d \sigma_{jl}^i dW_l(t)$ .

These random quantities are independent over time, have a zero mean and constant variance.  $W_i(t)$  is independent of  $W_j(t)$ , when  $i$  is different from  $j$ , and  $dW_i(t)$  does not depend on  $x(t)$ .

If payoffs have this structure and  $x(t)$  evolves according to the replicator dynamics,  $x(t)$  will be an Itô diffusion which is the name given to stochastic processes satisfying a stochastic differential equation, in this case;

$$dx_j^i(t) = x_j^i(t) \left[ u_i(j, x^{-i}(t))dt + \sum_{l=1}^d \sigma_{jl}^i dW_l(t) - \sum_{k=1}^{n_i} x_k^i(t) u_i(k, x^{-i}(t))dt - \sum_{k=1}^{n_i} \sum_{l=1}^d x_k^i(t) \sigma_{kl}^i dW_l(t) \right].$$

One of the characteristics of replicator dynamics is that if a strategy disappears or is never in the population it will never reappear again. This happens because you cannot imitate a strategy that nobody is using, or inherit it in a biological context. And it will be true independently of the payoff of that strategy, thus the shocks I modeled above will not change that. A strategy that is represented in the population at the initial time will not disappear, almost surely, in finite time with the replicator dynamics. The reason is that the dynamics are defined in terms of the growth rates, and growth rates are finite, although perhaps negative, so the fastest possible decreasing path for a variable is a negative exponential one, which is zero only in the limit. In similar finite-state models, by contrast, extinction is a definite possibility. For the stochastic dynamics in Farrell's [9] paper, for instance, there is a positive probability that even the users of the best strategies become bankrupt, although the expectation of their share of the total wealth in the market goes to one as time goes to infinity.

I want to incorporate in the model the possibility that strategies that are not used by anybody in a given period start to be used in later periods, while retaining the assumption that the agents are not sophisticated. For this reason I assume that new players replace part of the population at all times and some of them adopt strategies in a random way that is independent of the actions of both old players and other new players. I model the effect of these new players in the dynamics as a deterministic shock that modifies the transition rates for all periods. The aggregate effect of the newcomers that take actions at random is modeled in a deterministic fashion because their actions are assumed to be uncorrelated across individuals and the population is so large that we can invoke the law of large numbers to assume that the average of these actions is not random. By the next time these new players can change their strategies they start behaving like other members of the population. By analogy with the biological literature I call these newcomers mutants, and their actions, mutations. Samuelson and Zhang [25] discuss the issues that come up when mistakes occur at the implementation stage, so that people don't always choose the strategy they intend, but some other. These mistakes don't persist and are not inherited. The distinction appears to be important for the purpose of predicting whether outcomes ruled out by perfectness and other tremble-based refinements will persist in evolutionary contexts. In particular, the presence of mutations does not rule out imperfect equilibria as limit points of the dynamics, while implementation mistakes do.

The new model is then,

$$\begin{aligned}
 dx_j^i(t) = x_j^i(t) & \left[ u_i(j, x^{-i}(t))dt + \sum_{l=1}^d \sigma_{jl}^i dW_l(t) - \sum_{k=1}^{n_i} x_k^i(t) u_i(k, x^{-i}(t))dt \right. \\
 & \left. - \sum_{k=1}^{n_i} \sum_{l=1}^d x_k^i(t) \sigma_{kl}^i dW_l(t) \right] + \sum_{k=1}^{n_i} \lambda_{jk}^i x_k^i(t)dt - \sum_{k=1}^{n_i} \lambda_{kj}^i x_j^i(t)dt.
 \end{aligned} \tag{1}$$

$\lambda_{jk}^i$  is the probability that a member of population  $i$  that is using strategy  $k$  will be replaced in a given period by a player who chooses randomly strategy  $j$ . I will call the  $\lambda_{jk}^i$ 's mutation rates. Notice that  $\lambda_{jj}^i$  can have any value without affecting the dynamics. I will choose to make  $\lambda_{jj}^i = \min_{k \neq j} \lambda_{jk}^i$  to economize notation in the proofs of the propositions.

#### 4. STRICTLY DOMINATED STRATEGIES

One of the first questions to arise when considering selection dynamics that come from less than rational behavior is whether the outcomes generated resemble the ones predicted from a rationality perspective, so that as-if-rational type arguments can be made. I will need some definitions for this discussion.

Strategy  $x' \in S^{n_i}$  is strictly dominated in  $M_i \subset S^{n_i}$  relative to  $M_{-i} \subset \prod_{j \neq i} S^{n_j}$  if there exists  $x \in S^{n_i}$  such that  $u_i(x, y) > u_i(x', y)$  for all  $y \in M_{-i}$ . Let  $\bar{D}_i(M_i, M_{-i})$  be the set of mixed strategies in  $M_i$  that are not strictly dominated in  $M_i$  relative to  $M_{-i}$ . The strategy  $x' \in S^{n_i}$  survives strict iterated admissibility (SIA) if there exist sequences of the form  $S^{n_i} = M_{i0}, M_{i1}, \dots, M_{iT}$  and  $\prod_{k \neq i} M_{k0} = M_{-i0}, M_{-i1}, \dots, M_{-iT}$  where  $M_{i,n+1} = \bar{D}_i(M_{in}, M_{-in})$  and  $M_{-in} = \prod_{k \neq i} M_{kn}$ .

Strategies which do not survive SIA are not justifiable for a rational player, so if a nonnegligible part of the population plays them a nonvanishing proportion of the time the dynamics cannot be thought of as behaving in a way that mimics the traditional economic notion of rationality. The usual justification for strong rationality assumptions is that in the long-run behavior is close to rational due to unspecified selection processes. It is interesting, then, to find whether the replicator dynamics eliminate all but

admissible strategies in the long run. This is true for continuous time replicator dynamics but not for the discrete time case, as shown in Dekel-Scotchmer [6]. Nevertheless, Cabrales and Sobel [3] show that the result can be partially recovered and give sufficient conditions for discrete time dynamics to avoid in the limit strategies which do not survive SIA. The question now is whether a similar result is true for a model like the one proposed in last section.

The payoff function with respect to which I consider the strict domination is the average payoff function,  $u_i(j, x)$ . Total payoff,  $\bar{u}_i(j, x)$ , which includes the aggregate shocks to payoffs, can be different from  $u_i(j, x)$ , although on average they coincide. I will show that the elimination of non-SIA strategies by the replicator dynamics is maintained even when transitory payoff perturbations and mutations are added to the model.

Proposition 1 demonstrates that if mutation rates are small, and selection has been operating for a long time, the probability that nonnegligible proportions of the population are playing a non-SIA strategy is small. I cannot say that the weight of a strictly dominated strategy will be small with probability one because it could happen that a streak of good luck makes the proportion of users of a generally bad strategy grow for a while.

This result does not depend on the existence of an ergodic distribution, and it is not necessary for variances to be infinitely small. This is interesting because many other results in the literature of stochastic dynamics concern the limit of the ergodic distribution as the variance of the aggregate shocks goes to zero.

Let  $r$  be any  $n_i \times 1$  vector and

$$V_r(t) = \prod_{j=1}^{n_i} (x_j^i(t))^{r_j}.$$

Suppose that  $r$  is a strategy vector for player  $i$ . If  $r$  is a non SIA strategy and  $V_r$  is zero at least one of the pure strategies that have positive weight under  $r$  has to be zero.

### Proposition 1

Let strategy  $p \in S^{n_i}$  fail strict iterated admissibility. If  $\text{Max}_j(\lambda_{jk}^i)(\text{Min}_j(\lambda_{jk}^i))^{-1}$  is bounded for all  $k$  having positive weight under  $p$ , as we let  $\lambda \rightarrow 0$ , there is  $\bar{\sigma}_p$  such that if  $\text{Max}_{j,i,l} \{ \sigma_{jl}^i \} < \bar{\sigma}_p$

$$\lim_{\lambda \rightarrow 0} \left[ \limsup_{t \rightarrow \infty} E(V_{2p}(t)) \right] = 0$$

The proposition shows that the probability that the weight of a strictly dominated strategy (or that of at least one member of its carrier in the case of a mixed strategy) is larger than any given positive number  $K$ , which may be as small as we want, will be very close to zero when selection has been operating for a long enough time provided that the variance of the noise is below a certain bound and the mutation rates are both small and not orders of magnitude apart from one another.

I don't need to assume that  $\sigma$  goes to zero, but I need to have a bound on it. The smaller the advantage of the dominating strategy the stricter the bound. The assumption about mutation rates is always satisfied in the games that Fudenberg and Harris study. In games with two strategies the only mutation rate to, say, strategy one is  $\lambda_{12}$ , (remember that  $\lambda_{11}$  is assumed to be equal to the minimum over  $j \neq 1$  of  $\lambda_{1j}$ ), so the maximum and the minimum coincide. They do have an assumption about ratios of mutation rates, but it

refers to ratios of mutation rates to different strategies and it is not used to ensure that strictly dominated strategies have little weight asymptotically. They use it to ensure that the probability of getting out of a certain equilibrium depends mainly on the size of payoffs, and not on an increasing asymmetry of mutation rates.

I need four lemmas and some notation before I can proceed with the proof of the proposition. The first one is a direct application of standard theorems on linear stochastic differential equations, and the proofs of the other ones are given in the appendix.

$$\text{Let } m_j(x, \lambda) = u_i(j, x^{-i}) - u_i(x^i, x^{-i}) - \sum_k \lambda_{kj}.$$

The function  $m_j(x, \lambda)$  gathers all the terms in  $dx_j^i$  that are multiplied by  $x_j^i$ .

$$M = \text{Max}_{x, \lambda, j} \{ |m_j(x, \lambda)| \},$$

$$\sigma = \text{Max}_{j,i,l} \{ \sigma_{jl}^i \}.$$

Let  $\delta_{kj} = 0$  if  $k \neq j$  and  $\delta_{jj} = 1$ .

$$A_{js}^t = \sum_{j=1}^{n_i} \sum_{l=1}^d \int_s^t (\delta_{kj} - x_j^i(u)) \sigma_{jl}^i dW_l(u).$$

$A_{js}^t$  gathers all the stochastic terms in  $dx_j^i$  and integrates them from  $s$  to  $t$ .

Let  $r$  be any  $n_i \times 1$  vector, and,

$$V_r(t) = \prod_{j=1}^{n_i} (x_j^i(t))^{r_j}.$$

By Itô's formula, which is the analog in differential stochastic calculus to the chain rule in ordinary calculus,

$$dV_p(t) = \sum_{j=1}^{n_i} p_j m_j(x^i(t)) V_p(t) dt$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{j=1}^{n_i} \sum_{k=1}^{n_i} \sum_{l=1}^d \left[ \sum_{m=1}^{n_i} (\delta_{jm} - x_m^i(t)) x_j^i(t) \sigma_{ml}^i \right] \left[ \sum_{m=1}^{n_i} (\delta_{km} - x_m^i(t)) x_k^i(t) \sigma_{ml}^i \right] \left[ \frac{p_j}{x_j^i(t)} \frac{p_k}{x_k^i(t)} - \delta_{jk} \frac{p_j}{x_j^i(t)^2} \right] V_p(t) dt \\
& + \sum_{j=1}^{n_i} \frac{p_j}{x_j^i(t)} \left[ \sum_k \lambda_{jk} x_k^i(t) \right] V_p(t) dt + \sum_{j=1}^{n_i} \sum_{l=1}^d \sum_{m=1}^{n_i} (\delta_{jm} - x_m^i(t)) x_j^i(t) \sigma_{ml}^i \frac{p_j}{x_j^i(t)} V_p(t) dW^l(t).
\end{aligned}$$

I am going to collect now some terms and give them a name to save space.

$$\text{Let } A(p)_s^t = \sum_{j=1}^{n_i} \sum_{l=1}^d \sum_{m=1}^{n_i} \int_s^t (\delta_{jm} - x_m^i(u)) x_j^i(u) \sigma_{ml}^i \frac{p_j}{x_j^i(u)} dW^l(u).$$

$A(p)_s^t$  collects the stochastic terms in  $dV_p$  and integrates from  $s$  to  $t$ .

$$\text{Let } \sigma_p(t) = \frac{1}{2} \sum_{j=1}^{n_i} \sum_{k=1}^{n_i} \sum_{l=1}^d \left[ \sum_{m=1}^{n_i} (\delta_{jm} - x_m^i(t)) x_j^i(t) \sigma_{ml}^i \right] \left[ \sum_{m=1}^{n_i} (\delta_{km} - x_m^i(t)) x_k^i(t) \sigma_{ml}^i \right] \left[ \frac{p_j}{x_j^i(t)} \frac{p_k}{x_k^i(t)} - \delta_{jk} \frac{p_j}{x_j^i(t)^2} \right].$$

The function  $\sigma_p$  collects the deterministic terms which are multiplied by the variances of the stochastic shocks. These are the terms that appear in stochastic calculus but would not appear in deterministic calculus when using the chain rule.

In the remainder of the section I will suppress the superindex when it is clear that we are referring to strategies for player  $i$ .

### Lemma 1

$$\text{a) } x_j(t) = \exp \left[ \int_0^t m_j(x(s), \lambda) ds + A_{j0}^t \right] x_j(0) + \int_0^t \exp \left[ \int_s^t m_j(x(u), \lambda) du + A_{js}^t \right] \sum_k \lambda_{jk} x_k(s) ds. \quad (2)$$

$$\begin{aligned}
\text{b) } V_p(t) &= \exp \left[ \int_0^t \left[ \sum_j p_j m_j(x(s), \lambda) + \sigma_p(s) \right] ds + A(p)_0^t \right] V_p(0) \\
&+ \int_0^t \exp \left[ \int_s^t \left[ \sum_j p_j m_j(x(u), \lambda) + \sigma_p(u) \right] du + A(p)_s^t \right] \sum_j \frac{p_j}{x_j(s)} \left[ \sum_k \lambda_{jk} x_k(s) \right] V_p(s) ds.
\end{aligned} \quad (3)$$

$$\text{b')} \quad V_p(t) = \exp \left[ \int_0^t \left[ \sum_j p_j m_j(x(s), \lambda) - f(x(s)) + \sigma_p(s) \right] ds + A(p)_0^t \right] V_p(0) \quad (3')$$

$$\begin{aligned}
& + \int_0^t \exp \left[ \int_s^t \left( \sum_j p_j m_j(x(u), \lambda) - f(x(u)) + \sigma_p(u) \right) du + A(p)_0^s \right] \sum_j \frac{p_j}{x_j(s)} \left[ \sum_k \lambda_{jk} x_k(s) \right] V_p(s) ds \\
& + \int_0^t \exp \left[ \int_s^t \left( \sum_j p_j m_j(x(u), \lambda) - f(x(u)) + \sigma_p(u) \right) du + A(p)_0^s \right] f(x(s)) V_p(s) ds.
\end{aligned}$$

**Proof:** See Gihman and Skorohod [13] p.37

It is easy to see by differentiating that the solution to the ordinary differential equation;

$$\dot{y}(t) = a(t)y(t) + f(t); \quad y(0) = \bar{y}, \quad (4)$$

is given by

$$y(t) = \exp \left[ \int_0^t a(s) ds \right] \bar{y} + \int_0^t \left[ \int_s^t a(u) du \right] f(s) ds. \quad (5)$$

The  $x(t)$  process is the solution of equation (1), which is the stochastic differential version of equation (4). To go from (1) to (2), which is the stochastic analog of (5), since you cannot use differentiation it is necessary to use Itô's rule. Something similar applies to  $V_p(t)$ .

## Lemma 2

- a)  $E(\exp \beta A_{ks}^t) \leq \exp \left[ dn^2 \beta^2 \sigma^2 (t-s) \right].$
- b)  $E(\exp \beta A(p)_s^t) \leq \exp \left[ dn^4 \beta^2 \sigma^2 (t-s) \right].$

**Proof:** See the appendix.

The proof of the proposition is going to proceed by taking expectations in equations (2) and (3). The assumption that strategy  $p$  is strictly dominated will make the terms in those equations that contain the  $m_j$  functions small, but we still need to know what happens with the term that contains the noise. The assumption on the variance plus

Lemma 2 assures that the noise term does not change the conclusions.

### Lemma 3

Let  $c > 0$  and  $C(M, \sigma) = e^{cM} C_1^{1/2} \exp[6c^2 dn^2 \sigma^2]$ , where  $C_1$  is a constant independent of both the time index and the particular stochastic process we consider.

$$E(x_k(t)^{-c}) \leq C(M, \sigma) (\min_j \lambda_{jk})^{-c}.$$

**Proof:** See the appendix.

### Proof of Proposition 1 :

Let  $A_i = \{p \in S^{n_i} : p \text{ fails SIA and for all } \bar{\sigma} > 0 \text{ there is a } \sigma < \bar{\sigma} \text{ with } \lim_{\lambda \rightarrow 0} \limsup_{t \rightarrow \infty} E(V_{2p}(t)) > 0\}$ , and assume that  $\bigcup_{i=1}^N A_i \neq \emptyset$  for a contradiction. Let  $K(p)$  be

such that  $p' \in M_{iK(p)} \cap M_{iK(p)+1}^c$ . Let  $p$  be a minimizer of  $K(p)$  on  $\bigcup_{i=1}^N A_i$ . Let  $r = K(p)$ .

Since  $p \in M_{ir} \cap M_{ir+1}^c$  there is  $p' \in M_{ir}$  such that

$$u_i(p, x) - u_i(p', x) < 0 \text{ for all } x \in M_{ir}. \quad (6)$$

so that  $p'$  strictly dominates  $p$  relative to  $M_{ir}$ . Let  $X$  consist of all those  $x \in \prod_{k \neq i} S^{n_k}$  such that  $x_j^k > 0$  only if  $j \in M_{kr}$ . It follows by equation (6) that

$$u_i(p, x) - u_i(p', x) < 0 \text{ for all } x \in X. \quad (7)$$

Let the set  $C_{jr} = \{x \in P_j \cap M_{jr}^c\}$ .  $C_{jr}$  is the set of pure strategies for player  $j$  that are not in  $M_{jr}$ . Let  $x_c = \sum_{k \neq i} \sum_{j \in C_{kr}} x_j^k$ , that is, the sum of the weights of pure strategies (of players

other than  $i$ ) which are not in the sets  $C_{kr}$ .

$$\text{Let } M_{pp'} = \max_{\substack{x \in \Pi S^{n_j} \\ j \neq i}} |u_i(p, x) - u_i(p', x)| + 1.$$

Then by equation (7)

$$u_i(p, x) - u_i(p', x) - M_{pp'} x_c < 0 \text{ for all } x \in \Pi S^{n_j}_{j \neq i}.$$

For the rest of the proof consider only  $\lambda$  small enough that

$$\sum_j (p_j - p'_j) m_j(x, \lambda) - M_{pp'} x_c < 0.$$

Since

$$\sum_j (p_j - p'_j) m_j(x, \lambda) = u_i(p, x) - u_i(p', x) - \sum_j \sum_k (p_j - p'_j) \lambda_{kj}^i,$$

this is satisfied when

$$\max_{\substack{x \in \Pi S^{n_j} \\ j \neq i}} \{u_i(p', x) - u_i(p, x) - M_{pp'} x_c\} > \sum_j \sum_k (p'_j - p_j) \lambda_{kj}^i.$$

$$\text{Let } m = \max_{x, \lambda} \left\{ \sum_j (p_j - p'_j) m_j(x, \lambda) - M_{pp'} x_c \right\}.$$

By the restriction placed on  $\lambda$ ,  $m < 0$ .

The function  $V_{2(p-p')}$  is something like a ratio of the weights of the dominated to the dominating strategy. Lemma 4 is going to show that the expectation of  $V_{2(p-p')}$  is bounded.

**Lemma 4**

$$\begin{aligned} E(V_{2(p-p')}(t)) &\leq \exp[2(m + 2\sigma^2 n^4 d)t] V_{2(p-p')}(0) \\ &+ \left[ 2M_{pp'} N + 2 \sum_j C(M, \sigma) (\min_k \lambda_{jk})^{-1} \max_k \lambda_{jk} \right] \left[ 2(m + 2\sigma^2 n^4 d) \right]^{-1} \left[ 1 - \exp \left[ 2(m + 2\sigma^2 n^4 d)t \right] \right]. \end{aligned} \quad (8)$$

**Proof:** See the appendix.

Samuelson and Zhang show that strictly dominated strategies disappear by showing that  $V_{p-p'}$  goes to zero when there are no shocks. I can't do that because mutations prevent the weights of strategies from becoming arbitrarily small. But at least I can show that  $E(V_{2(p-p')})$  has a bound that is independent of the mutation rates, if these are not orders of magnitude apart. This happens because far from the boundaries the dynamics tend to make  $V_{p-p'}$  small, and so the first term in equation (8) is small, but near the boundaries the movement depends on mutation rates to a greater extent, and the second term in equation (8) reflects that.

Lemma 4 shows that there will always be enough users of the dominating strategy so that it can be imitated by the users of the dominated strategy. The number of agents that play the dominated strategy could increase for two reasons. Some new players (mutants) choose it by chance and some agents who were doing something else switch to the dominated strategy because they don't know of the dominating strategy. The rest of the proof of Proposition 1 demonstrates that the accumulation of those two types of players is slower than the losses of players who discover that the dominating strategy is better.

Let  $b < t$ . By Lemma 1b,

$$\begin{aligned} V_{2p'}(t) = & \exp \left[ \int_b^t \left( 2 \left( \sum_j p'_j m_j(x(s), \lambda) + \sigma_{2p'}(s) \right) ds + A(2p')_b^t \right) \right] V_{2p'}(b) \\ & + \int_b^t \exp \left[ \int_s^t \left( 2 \left( \sum_j p'_j m_j(x(u), \lambda) + \sigma_{2p'}(u) \right) du + A(2p')_b^s \right) \right] \sum_j \frac{2p'_j}{x_j(s)} \left[ \sum_k \lambda_{jk} x_k(s) \right] V_{2p'}(s) ds \end{aligned}$$

Then by the positivity of  $\lambda$ ,  $p'$ ,  $x$  and the exponential function,

$$1 \geq V_{2p'}(t) \geq \exp \left[ \int_b^t \left[ 2 \left( \sum_j p'_j m_j(x(s), \lambda) + \sigma_{2p'}(s) \right) ds + A(2p') \right]_b^t \right] V_{2p'}(b).$$

By lemma 1b'

$$\begin{aligned} V_{2p}(t) = & \exp \left[ \int_b^t \left[ 2 \left( \sum_j p_j m_j(x(s), \lambda) - M_{pp'} x_c(s) \right) + \sigma_{2p}(s) \right] ds + A(2p) \right]_b^t V_{2p}(b) \\ & + \int_b^t \exp \left[ \int_s^t \left[ 2 \left( \sum_j p_j m_j(x(u), \lambda) - M_{pp'} x_c(u) \right) + \sigma_{2p}(u) \right] du + A(2p) \right]_b^s \sum_j \frac{2p_j}{x_j(s)} \left[ \sum_k \lambda_{jk} x_k(s) \right] V_{2p}(s) ds \\ & + \int_b^t \exp \left[ \int_s^t \left[ 2 \left( \sum_j p_j m_j(x(u), \lambda) - M_{pp'} x_c(u) \right) + \sigma_{2p}(u) \right] du + A(2p) \right]_b^s 2M_{pp'} x_c(s) V_{2p}(s) ds. \end{aligned}$$

Now I divide the first line in the previous equation by,

$$\exp \left[ \int_b^t \left[ 2 \left( \sum_j p'_j m_j(x(s), \lambda) + \sigma_{2p'}(s) \right) ds + A(2p') \right]_b^t \right] V_{2p'}(b)$$

and since we showed that the last expression is less than one,

$$\begin{aligned} V_{2p}(t) \leq & \exp \left[ \int_b^t \left[ 2 \left( \sum_j (p_j - p'_j) m_j(x(s), \lambda) - M_{pp'} x_c(s) \right) + \sigma_{2(p-p')}(s) \right] ds + A(2(p-p')) \right]_b^t V_{2(p-p')}(b) \\ & + \int_b^t \exp \left[ \int_s^t \left[ 2 \left( \sum_j p_j m_j(x(u), \lambda) - M_{pp'} x_c(u) \right) + \sigma_{2p}(u) \right] du + A(2p) \right]_b^s \sum_j \frac{2p_j}{x_j(s)^{1-p_j}} \left[ \sum_k \lambda_{jk} x_k(s) \right] ds \\ & + \int_b^t \exp \left[ \int_s^t \left[ 2 \left( \sum_j p_j m_j(x(u), \lambda) - M_{pp'} x_c(u) \right) + \sigma_{2p}(u) \right] du + A(2p) \right]_b^s 2M_{pp'} x_c(s) ds. \end{aligned}$$

Taking expectations and applying lemmas 2 and 3, by definition of  $M$ , and taking the summation over  $j$  only over those indices for which  $p_j$  is strictly positive,

$$\begin{aligned} E(V_{2p}(t)) \leq & \exp[(2m + 4\sigma^2 n^4 d)(t - b)] E(V_{2(p-p')}(b)) \quad (9) \\ & + \sum_j C(M, \sigma) 2(\text{Min}_k \lambda_{jk})^{-(1-p_j)} \text{Max}_k \lambda_{jk} \left[ 2(M + 2\sigma^2 n^4 d) \right]^{-1} \left[ -1 + \exp \left[ 2(M + 2\sigma^2 n^4 d)(t - b) \right] \right] \\ & + 2M_{pp'} \sum_{j \neq i} \sum_{k \in C_{jr}} (\text{Max}_{s \in (b, t)} E(x_k^j(s))^2)^{1/2} \left[ 2(M + 2\sigma^2 n^4 d) \right]^{-1} \left[ -1 + \exp \left[ 2(M + 2\sigma^2 n^4 d)(t - b) \right] \right] \end{aligned}$$

I have to show that for any positive number  $\alpha$ , for all  $t$  larger than some  $t_\alpha$  and  $\lambda$  smaller than some  $\lambda_\alpha$ ,  $E(V_{2p}(t))$  is smaller than  $\alpha$ . Lemma 4 shows that  $E(V_{2(p-p)}(b))$  is bounded by a constant  $C$  which depends only on  $m, M$  and  $\sigma$  when  $b$  is above some  $b_\alpha$ . Choose  $t'$  such that  $t' - b_\alpha > 0$  and

$$\exp[(2m + 4\sigma^2 n^4 d)(t' - b_\alpha)]C < \alpha.$$

Then for all  $t > t'$  choose  $b$  such that  $t - b = t' - b_\alpha$ . This guarantees that the first line in equation (9) is strictly smaller than  $\alpha$ . Having chosen  $b$ , notice  $(\min_k \lambda_{jk})^{-(1-p_i)} \max_k \lambda_{jk}$

$$= \left[ \frac{\min_k \lambda_{jk}}{\max_k \lambda_{jk}} \right]^{-(1-p_i)} \max_k \lambda_{jk}^{p_j} \text{ so by taking } \lambda' \text{ sufficiently small, if } t > t', \text{ for all } \lambda < \lambda' \text{ the}$$

sum of the first two lines in equation (9) will be smaller than  $\alpha$ . The third line in equation (9) can also be made as small as needed for all  $\lambda$  smaller than some  $\lambda''$  when  $t$  is larger than some  $t'' > t'$  because for all  $j \neq i$  and all  $k$  in  $C_{jr}$ ,  $\lim_{\lambda \rightarrow 0} \limsup_{t \rightarrow \infty} E(x_k^j(t)^2) = 0$ . Let  $t_\alpha$  be larger than  $t''$ , and  $\lambda$  smaller than  $\lambda'$  and  $\lambda''$  and the result follows.

The following interpretation can be given to the proof. The first line of equation (9) says that few of the initial users of strategy  $p$  are still using it or have been replaced by imitators. The second says that the inexperienced new players and their imitators cannot replace them, unless the initial level of  $p$ -strategists was very low. The third line allows us to extend the argument to strategies that are strictly dominated only after other strictly dominated strategies have been eliminated.

## 5. ERGODICITY

Stochastic dynamics sometimes have the property that the time average of the

probability that the process hits a certain set goes to a limit that is independent of the starting point. This is useful because it allows the modeler to make unique limiting predictions. It is also interesting because deterministic dynamics don't have that property unless there is global convergence, so ergodicity sets stochastic dynamics apart from deterministic dynamics. The processes in the papers by Foster and Young [10], Kandori, Mailath and Rob [14] and others, have ergodic distributions. The authors proceed to identify the most likely states of the population when mutation rates are small. When mutation rates are small, however, the time that is necessary for the system to wipe out the influence of the initial condition may be very long. Ellison [8] shows that changing the matching technology from random matching to more general types of interaction, can change the amount of time needed to converge to the ergodic distribution. Foster and Young point out that for applications it may be more fruitful to estimate the variances of the shocks and the size of the mutation rates rather than to obtain the limit distributions when variances and mutations go to zero.

I will give sufficient conditions for the process defined in equation (1) to have an ergodic distribution. The context will determine whether these conditions are sensible. For example, it will be important for the result that the mutation rates are bounded away from zero. If the game is played always by the same people, you cannot invoke inexperienced new players to justify mutations. The justification of mutations in terms of experimentation also becomes harder in that case. It is also important that the matrix of the variance of the noise has full rank. This implies that the sources of randomness have to be somewhat independent between the different strategies. If strategies are, say, production levels, it seems implausible to assume that a shock that affects the cost of producing

a certain amount of goods has no effect in the cost of producing a different amount. A trivial case in which the shocks are not sufficiently uncorrelated is the one in which the cost of production changes randomly for all strategies in the same amount. The differences between the payoffs to all strategies are not affected, and since the dynamics depend on the difference between the payoff to a strategy and the average population payoff, the dynamics are not affected by this type of shock. If the resulting deterministic dynamics are not globally convergent there is not a unique limiting ergodic distribution. Nontrivial cases arise when the shocks are more complicated than this simple additive one but still not sufficiently diverse in origin to generate a regular variance matrix.

The process I presented in equation (1) is ergodic when the matrix of the variance terms has a rank higher than or equal to the total number of pure strategies in the game and all the mutation rates are different from zero. The reason for this is that if the variances satisfy the rank condition the process can move in every direction when it is in the interior of the simplex, and the mutation rates move the process away from the boundaries. In other words, as long as people are myopic and each strategy is being used by somebody (which is guaranteed by the presence of mutations) a string of successes or failures for different strategies due purely to random fluctuation in payoffs, can cause the population to reach all conceivable states infinitely often.

Let  $x(t)$  be the solution to equation (1), which I will write

$$dx(t) = a(x(t))dt + B(x(t))dW(t),$$

and let

$$\Delta = \{x: 0 \leq x_j^i \leq 1 \text{ for } j=1, \dots, n_i, i=1, \dots, N, \text{ and } \sum_1^{n_i} x_j^i = 1 \text{ for all } i=1, \dots, N\}.$$

The process  $x(t)$  belongs to  $\Delta$  almost surely if  $x(0)$  belongs to  $\Delta$  since  $\sum_{j=1}^{n_i} dx_j^i(t) = 0$  for  $i=1, \dots, N$ , and  $dx_j^i = 0$  for  $x_j^i$  equal to zero and one, for all  $i$  and  $j$ . I will only consider  $x(0)$  belonging to  $\Delta$ .

Let  $P(s, x, E)$  be the probability that the process, starting at  $x$ , is at time  $s$  in the set  $E$ . Let  $\Gamma$  be the  $\left( \sum_{i=1}^N n_i \right) \times d$  matrix whose  $\left( \sum_{k=1}^{i-1} n_k + j \right)$ th row is the  $d$  vector  $\sigma_j^i$ .

**Proposition 2** If the rank of  $\Gamma$  is equal to  $\sum_{k=1}^N n_k$ , there exists an invariant measure  $\pi$  for the process  $x'(t)$ , and for all  $x \in \Delta$  and all  $E \in B_\Delta$  (the set of Borel subsets of  $\Delta$ )

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t P(s, x, E) ds = \pi(E).$$

**Proof:**

The process  $x(t)$  has an invariant measure by Theorem 21 from Skorohod [26], because it is a Markov process in a compact metric space,  $\Delta$ . To show uniqueness I will apply Theorem 5.1 in Arnold and Kliemann [1]. Once the existence of a unique invariant distribution is established the result follows by Birkhoff's ergodic theorem (see Skorohod [26] theorem 1, or Arnold and Kliemann [1] p. 54). For the details of the proof of uniqueness see the appendix.

## 6. RELATIONSHIP WITH OTHER STOCHASTIC DYNAMICS

In this section I present an example which shows that the stochastic dynamics

can have an ergodic distribution whose weight is concentrated, when both mutation rates and the variances of the stochastic shocks are small, on an equilibrium which is not the one with the largest basin of attraction for the deterministic replicator dynamics. Furthermore, the ergodic distribution would concentrate its weight on a different equilibrium for the dynamics that Kandori, Mailath and Rob study. The distinction appears only when games with more than two players are considered. With two strategies and two players the stochastic dynamics of Kandori, Mailath and Rob and Fudenberg and Harris have ergodic distributions that put most of the weight on the same equilibrium for small variances and mutation rates. Young and Foster [33] consider an example in which the equilibrium with the largest basin of attraction would not be the one to which the ergodic distribution gives the highest weight. In their example, however, the dynamics of Kandori, Mailath and Rob would have the same limiting ergodic distribution.

Suppose now that members of the population are randomly matched every period in groups of  $N$  players to play a game that has two strategies. The strategy played by player  $i$  is denoted  $x_i$  and  $x_i$  can be either 1 or 2. Payoffs are  $u_i(x_1, \dots, x_N) = a \min_j x_j - bx_i$ .

Given the random matching structure of the game, if we let  $x$  be the proportion of people in the population using strategy 2, the payoff to strategy 1 given  $x$  will be  $u(1, x) = a - b$  and the payoff to strategy 2 will be  $u(2, x) = 2ax^N + a(1 - x^N) - 2b = ax^N + a - 2b$ .

The game has two strict equilibria in pure strategies that are Pareto ranked. The deterministic replicator dynamics converge to one of them from all initial states except

from the unstable mixed strategy equilibrium. The basin of attraction of the Pareto superior equilibrium is smaller when  $N$  is large.

In the presence of mutants and random shocks to payoffs, if the changes in  $x$  are slow enough, its evolution can be modeled as,

$$dx(t) = \left[ x(t)(1 - x(t))(ax(t)^N - b) + \lambda_2(1 - x(t)) - \lambda_1 x(t) \right] dt + x(t)(1 - x(t))\sigma dW(t) \quad (10)$$

### Proposition 3

a) The process  $x(t)$  defined in equation (10) has an ergodic distribution.

b) If  $a > 2b$  the limit of the ergodic distribution puts probability one on the state  $x = 1$  where all the population is using the high effort strategy, as  $\lambda_1$ ,  $\lambda_2$  and  $\sigma$  go to zero, if  $\frac{\lambda_1}{\lambda_2}$  is bounded.

**Proof:**

See the appendix.

The equilibrium that has more weight under the ergodic distribution is the one for which the temporary shocks to payoffs that will convince the people to switch to the other equilibrium are less likely to arise. In this model the difficulty in changing from a state where most of the people are playing one strategy to one where mostly the other one is played, lies in getting the first few people to defect from the popular strategy, because it is more difficult to imitate something that almost nobody is doing. The first few defectors have to see that playing the other strategy has been good lately, and that will happen when payoffs suffer a shock that makes the strategy that is played by the majority

have a lower payoff than the alternative strategy. Then it is necessary to compare how likely are the shocks that move the dynamics from the different equilibria to know how the ergodic distribution looks like. When  $a > 2b$  the shocks necessary to move the dynamics from the Pareto dominant equilibrium to the other one are much more unlikely than the shocks that produce the opposite transition, if the variance of the shocks is small. Thus the Pareto dominant equilibrium has more weight under the ergodic distribution.

In the model of Kandori, Mailath and Rob the factor that determines which equilibrium has more weight under the ergodic distribution is the number of mutations necessary for the rest of the population to start thinking that it is a good idea to change their action. When  $N$  is large, less mutants are necessary to change from the Pareto dominant equilibrium to the Pareto inferior equilibrium than the ones necessary to do the opposite transition. Thus the Pareto dominated equilibrium has more weight under the ergodic distribution.

When there are only two players in each match the two criteria, size of the shocks and number of mutants, coincide, which is why the papers of Fudenberg and Harris and Kandori, Mailath and Rob give the same conclusions.

The game presented in this section was studied experimentally by Van Huyck, Battalio and Beil ([29], [30]). The equilibrium selected in most of the experiments was the Pareto inferior one, contrary to what Proposition 3 would suggest. This is not surprising since in the experimental setup there were no random shocks to payoffs and agents did not adjust their strategies in ways that were consistent with any of the stories I used to motivate the replicator dynamics. The model in Crawford [4] seems better adapted to

model the experimental framework. The model presented in this paper could be better suited for decisions where the payoffs to different choices are not given to the players in advance and are small compared to the cost of a careful consideration of the problem or of the difficulty of gathering information.

## 7. THE SMALLWOOD-CONLISK DYNAMICS

Section 4 showed that the result that replicator dynamics eliminates strictly dominated strategies is robust to the presence of some types of shocks. In this section I present an example which shows that this result does not necessarily hold for more general models of selection dynamics, even for some dynamics that are arbitrarily close to the replicator dynamics, in a parametric sense that I will specify later.

I will use the Smallwood-Conlisk dynamics I described in section 2. As I showed in that section the replicator dynamics are a member of that family of dynamics, when the parameter  $\alpha$  takes the value of one. Smallwood and Conlisk [27] characterize completely the set of limit points for the dynamics of their consumer choice problem. The game theoretic setup does not allow such a complete analysis as the consumer choice case, because the function that determines payoffs may depend on the proportions of the population that use every strategy in a game, but in the Smallwood and Conlisk model quality does not change with the proportion of people using a product. Nevertheless, the following can be said about the game dynamics.

### **Proposition 4**

Every pure strategy profile is a fixed point of the breakdown dynamics. a) For  $\alpha < 1$  it is locally unstable, b) for  $\alpha > 1$  it is locally stable.

**Proof:**

Rewriting the dynamics in the way Smallwood and Conlisk do,

$$x_j^i(t+1) = x_j^i(t) + \frac{x_j^i(t)^\alpha}{\sum_k x_k^i(t)^\alpha} \left\{ \sum_{r \neq j} \frac{c - u_i(j, x^{-i}(t)) - a}{b - a} x_j^i(t) \left[ 1 - \left[ \frac{c - u_i(j, x^{-i}(t)) - a}{c - u_i(r, x^{-i}(t)) - a} \right] \left[ \frac{x_r^i(t)}{x_j^i(t)} \right]^{\alpha-1} \right] \right\}.$$

If  $x_j^i(t)$  is sufficiently close to one, and  $\alpha$  is more than one then the second term is positive and therefore  $x_j^i(t+1) > x_j^i(t)$ . Iterating this argument yields the desired conclusion about local stability. A similar argument proves the local instability of pure strategy profiles when  $\alpha < 1$ .

The local stability and instability of pure strategy profiles when  $\alpha$  is greater than and less than one respectively, is independent of the precise magnitude of payoffs. And so it is possible for the dynamics to converge to a strictly dominated strategy when  $\alpha$  is greater than one and to diverge from a strict equilibrium when  $\alpha$  is less than one. This happens because if nearly everybody uses the same strategy, users of other strategies who decide to change will do it with high probability to the "leading" strategy. At the same time, many agents are ceasing to use the "leading" strategy, because even a very good strategy will sometimes fail to perform satisfactorily due to random factors. The parameter  $\alpha$  controls which of these effects dominates. When neither dominates, superior quality can overcome the effects of popularity and random failure. The elimination of strictly dominated strategies is sensitive to the formulation of the model. In fact, strictly

dominated strategies need not be eliminated even for parameter values that are arbitrarily close to one, the case of replicator dynamics.

One possible criticism to this result is that while functions with a similar  $\alpha$  parameter are close by in the sense that  $\max_{x \in \Delta} |f_\alpha(x) - f_{\alpha'}(x)|$  is small when  $\alpha - \alpha'$  is close to zero, the first derivative of  $f_1$  and  $f_\alpha$  are very different near the vertices of the simplex, even for values of  $\alpha$  very close to 1, and the result depends on the behavior near the vertices of the simplex.

Another criticism is that when the parameter is close to but greater than one the basin of attraction of the equilibrium where everybody is playing a strictly dominated strategy is small. In the presence of stochastic shocks one could conjecture that the population would get knocked very easily out of an equilibrium with a small basin of attraction, and therefore the system would spend on average very little time near that equilibrium, even if the dynamics are not precisely the replicator dynamics.

The example I will present next is intended to show that this is not necessarily the case. The reason is that for stochastic dynamics there are factors other than the size of the basin of attraction that determine the distribution of future outcomes. In particular, my example depends on the form of the variance term.

Suppose that in a game with two strategies instantaneous payoffs are determined as follows,

$$d\tilde{u}_1(t) = u_1(x_1(t), x_2(t)) + x_2(t)\sigma_1 dW_1(t),$$

$$d\tilde{u}_2(t) = u_2(x_2(t), x_1(t)) + x_1(t)\sigma_2 dW_2(t),$$

where  $\sigma_1 \geq 0$  and  $\sigma_2 > 0$ .

The variance of the shocks in this case, unlike in the model presented in section 3 depends on the number of players using strategy 2.

The SC model when there are two strategies can be written,

$$x_1(t+1) - x_1(t) = - \frac{c - \tilde{u}_1(x_1(t), x_2(t)) - a}{b - a} \frac{x_1(t)x_2(t)^\alpha}{x_1(t)^\alpha + x_2(t)^\alpha} + \frac{c - \tilde{u}_2(x_2(t), x_1(t)) - a}{b - a} \frac{x_2(t)x_1(t)^\alpha}{x_1(t)^\alpha + x_2(t)^\alpha},$$

$$\text{and } x_2(t) = 1 - x_1(t).$$

The continuous time version with shocks to payoffs and mutations will be then,

$$\begin{aligned} dx_1(t) = & \left[ - \frac{c - u_1(x_1(t), x_2(t)) - a}{b - a} \frac{x_1(t)x_2(t)^\alpha}{x_1(t)^\alpha + x_2(t)^\alpha} + \frac{c - u_2(x_2(t), x_1(t)) - a}{b - a} \frac{x_2(t)x_1(t)^\alpha}{x_1(t)^\alpha + x_2(t)^\alpha} \right] dt \\ & + \left[ \lambda_2 x_2(t) - \lambda_1 x_1(t) \right] dt + \frac{x_1(t)x_2(t)^\alpha}{x_1(t)^\alpha + x_2(t)^\alpha} \sigma_1 x_2(t) dW_1(t) - \frac{x_2(t)x_1(t)^\alpha}{x_1(t)^\alpha + x_2(t)^\alpha} \sigma_2 x_2(t) dW_2(t), \end{aligned}$$

and  $dx_1(t) = -dx_2(t)$ . If  $x_1(0) + x_2(0) = 1$ , then  $x_1(t) + x_2(t) = 1$ , for all  $t$ .

Let's define now,

$$dW(t) = \frac{1}{(x_2(t)^{2(\alpha-1)}\sigma_1^2 + x_1(t)^{2(\alpha-1)}\sigma_2^2)^{1/2}} \left[ x_2(t)^{\alpha-1}\sigma_1 dW_1(t) - x_1(t)^{\alpha-1}\sigma_2 dW_2(t) \right].$$

The process  $W(t)$  thus defined is a one dimensional Wiener process. The process  $x(t) = x_1(t) = 1 - x_2(t)$  can be studied using the theory of one-dimensional Itô processes, which allows us to know the exact form of the ergodic distribution if one exists.

Suppose that  $u_1(x, 1-x) = u$  and  $u_2(1-x, x) = U$  for all  $x \in [0, 1]$ , and  $u < U$ . Let

$$\frac{c - u - a}{b - a} = B, \quad \frac{c - U - a}{b - a} = A, \quad \beta(x) = \frac{\sigma_1^2 y^{2(\alpha-1)} + \sigma_2^2 (1-y)^{2(\alpha-1)}}{(y^\alpha + (1-y)^\alpha)^2}, \text{ and}$$

$$\delta(x) = (Ay^{\alpha-1} - B(1-y)^{\alpha-1}) \frac{y(1-y)}{y^\alpha + (1-y)^\alpha} + \lambda_2(1-y) - \lambda_1 y$$

We have then,

$$dx(t) = \delta(x(t))dt + x(t)(1-x(t))^2 \beta^{1/2} dW(t). \quad (11)$$

**Proposition 5**

- a) The process  $x(t)$  defined in equation (11) has an ergodic distribution.
- b) The limit of the ergodic distribution of  $x(t)$  as  $\lambda_1, \lambda_2, \sigma_1$  and  $\sigma_2$  go to zero gives all the weight to  $x = 1$  if  $\alpha > 1$  and  $\frac{\sigma_1}{\sigma_2}, \frac{\sigma_2}{\sigma_1}, \frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}$  are bounded.

**Proof:**

See the appendix.

In this example, for small values of the variances and the mutation rates, the process will spend very little time outside the areas where  $x$  is close to one, provided that the popularity parameter is bigger than one. This happens despite the fact that the basin of attraction of the equilibrium where  $x$  is one will be very small if  $\sigma$  is close to one. The reason for this is that the first strategy is worse on average, but it rarely fails for a lot of people at the same time, which is what you need in this framework to escape from a state in which a strategy is used because it is the most popular. The second strategy is usually better but in some periods it performs badly. If it does so for a sufficiently long time the first strategy will become very popular and from then on its steady performance will make it hard to beat.

The variance of shocks on this example could depend on  $x_1$  and  $x_2$  in a more general way. For example, instead of  $x_2(t)\sigma_1 dW_1(t)$  we could have

$(\sigma_{10} + \sigma_{11}x_1(t) + \sigma_{12}x_2(t))dW_1(t)$ . If  $\sigma_{10}$  or  $\sigma_{11}$  were different from zero the example would not be possible. The purpose of the example is to show that thinking that the stochastic dynamics will spend less time near equilibria which have small basins of attraction under deterministic dynamics than near equilibria with large basins of attraction is wrong unless additional assumptions are made.

The problem now is finding examples of situations with the required variance structure. One such situation arises when deciding whether to participate or not in a game of bingo, where each participant pays a fixed amount and the randomly selected winner receives a portion of the total amount paid by the participants. Some scientific endeavors also have the property that the value of the research done increases with the number of scientists working in the field, but only one or a few lucky researchers will receive credit for the discoveries. More precisely, suppose that  $M$  individuals have to choose between participating or not in a lottery. Denote by  $N$  the number of people who decide to participate. If they don't participate they get nothing and pay nothing. If they do, they obtain the prize, worth  $N/2$ , with a probability of  $1/N$ , and the cost of entering the contest is 1. Under these conditions the payoff for a contestant is  $-1/2 + ((N-1)/4)^{1/2}w$ , where  $w$  is a random variable with mean 0 and variance 1. If we denote the nonparticipation strategy by one and the proportion of nonparticipants by  $x$ ,  $u(1, x) = 0$  and  $u(2, x) = -1/2 + (M(N-1)/4N)^{1/2}x^{1/2}w$ .

De Long, Shleifer, Summers and Waldmann [7] have a model for the stock market where some of the agents (noise traders) have an expectation about the price of a stock that deviates from the rational expectation by a random amount. The rest of the agents have rational expectations. With this setup the payoff to both types of agents has a

variance that depends on how many noise traders there are. Another game where the variance of the payoffs depend on the number of users of a strategy would be one in which producers in a market choose between two technologies. One of those technologies produces goods with a random quality that changes over time, but is identical for all users of that technology. The quality of the goods produced with the alternative technique doesn't change. Costs are also deterministic. Demand depends on average quality in the whole market. If the proportion of users of the random technique is  $x$ , the price is  $P((1-x)+xw)$ , where  $w$  is the random quality of the technique. The variance of payoffs will depend on  $x$  through price in this case.

## 8. CONCLUSIONS

In this paper I extend to games with more than two players and strategies which are not necessarily symmetric two results found by Fudenberg and Harris [11]. First, I show that strictly dominated strategies have little asymptotic weight even in the presence of shocks to payoffs if mutation rates are small. Then I show that unique ergodic distributions exist. Nevertheless, at the present stage it doesn't seem easy to say much about the transition probabilities on large time intervals analytically, unless one assumes that the variances go to zero.

The present approach is complementary to the one Kandori, Mailath and Rob [14] or Young [32] use, because it studies very large populations, where their model is less powerful, because independent mutations are much less likely to take the process very far from the basin of attraction of a stable equilibrium, even for nonnegligible mutation rates. As Ellison [8] studies, if the matching technology were different, for example

if the chances of being matched with a few individuals were not very small, then low mutation rates wouldn't be that much of a problem. In such a case the potential for supergame effects is much larger, though.

Foster and Young introduced the study of stochastic evolutionary dynamics. As I have said, their model did not discriminate between mutations and shocks to payoffs. For their purposes, establishing the existence of a unique ergodic measure, and analyzing the limit of that measure when variances are taken to zero, this is not very important. But it becomes more relevant if one wants to distinguish what are the factors that cause ergodicity, and which ones are not essential, especially if one thinks that ergodicity is a counterintuitive property for some situations.

None of the justifications I gave for the replicator dynamics provide very strong foundations outside of the realm of biological games. But these stories show that with fairly weak assumptions on rationality one can conclude that strictly dominated strategies can be eliminated. However, the result seems to depend quite sensitively on assumptions. More research needs to be done, allowing more heterogeneity in the way agents behave, and the amount of information they process to be more confident about the force of aggregate rationality, which seems to be the basis for the belief that the behavior that eventually prevails has to be the best. Stahl [28] studies a model with agents who differ in their abilities to best respond to the present population. My paper explores a model in which payoffs are constantly changing around a central value. It would be interesting to see the results obtained when payoffs can change in more general ways, since in that case it may not be possible to always use the same strategy successfully and the definition of a strictly dominated strategy could be a dynamic one that requires agents to be more

active.

## APPENDIX

**Lemma 2**

$$\begin{aligned} \text{a) } E(\exp \beta A_{ks}^t) &\leq \exp \left[ dn^2 \beta^2 \sigma^2 (t-s) \right]. \\ \text{b) } E(\exp \beta A(p)_s^t) &\leq \exp \left[ dn^4 \beta^2 \sigma^2 (t-s) \right]. \end{aligned}$$

**Proof:**

$$\text{a) Let } Z_t(x) = \exp \left[ \sum_{l=1}^t \sum_j \int_s^t 2\beta (\delta_{kj} - x_j(u)) \sigma_{jl} dW_l(u) - \frac{1}{2} \int_s^t \sum_j \left( \sum_l 2\beta (\delta_{kj} - x_j(u)) \sigma_{jl} \right)^2 du \right].$$

By applying Itô's rule to the exponential function we have,

$$Z_t(x) = 1 + \sum_{l=1}^t \sum_j \int_s^t Z_u(x) 2\beta (\delta_{kj} - x_j(u)) \sigma_{jl} dW_l(u).$$

By Novikov's [22] sufficient condition to Girsanov's theorem  $Z_t(x)$  is a martingale if

$$E \left[ \exp \left[ \frac{1}{2} \int_s^t \sum_j \left( \sum_l 2\beta (\delta_{kj} - x_j(u)) \sigma_{jl} \right)^2 du \right] \right] < \infty \text{ for } s \leq t < \infty.$$

Which in this case is true because  $0 \leq x_j(t) \leq 1$ .If  $Z_t(x)$  is a martingale  $E(Z_t(x)) = 1$ . Using that and Hölder's inequality,

$$E(\exp \beta A_{ks}^t) = \left[ E(Z_t(x)) \right]^{1/2} \left[ E \left[ \exp \left[ \frac{1}{2} \int_s^t \sum_j \left( \sum_l 2\beta (\delta_{kj} - x_j(u)) \sigma_{jl} \right)^2 du \right] \right] \right]^{1/2} \leq \exp \left[ n^2 d \beta^2 \sigma^2 (t-s) \right].$$

The same argument applies for b).

**Lemma 3**

$$E(x_k(t)^{-c}) \leq C(M, \sigma) (\text{Min}_j [\lambda_{jk}])^{-c}.$$

**Proof:**

$$\text{Since } x_k(t) = \exp \left[ \int_0^t m_k(x(s), \lambda) ds + A_{k0}^t \right] x_k(0) + \int_0^t \exp \left[ \int_s^t m_k(x(u), \lambda) du + A_{ks}^t \right] \sum_j \lambda_{jk} x_k(s) ds,$$

and by the positivity of the exponential function,  $\lambda$  and  $x$ ;

$$E(x_k(t)^{-c}) \leq E \left[ \left[ \int_{t-1}^t \exp \left[ \int_s^t m_k(x(u), \lambda) du + \sum_{l=1}^j \sum_{s=1}^t (\delta_{jk} - x_j(u)) \sigma_{jl} dW_l(u) \right] \sum_k \lambda_{jk} x_j(s) ds \right]^{-c} \right] \\ \leq e^{cM} (\text{Min} \lambda_{jk})^{-c} E \left[ \left[ \int_{t-1}^t \exp \left[ \sum_{l=1}^j \sum_{s=1}^t (\delta_{kj} - x_j(u)) \sigma_{jl} dW_l(u) \right] ds \right]^{-c} \right],$$

which by Hölder's inequality

$$\leq e^{cM} (\text{Min} \lambda_{jk})^{-c} \left\{ E \left[ \left[ \exp \left[ \sum_{l=1}^j \sum_{t=1}^t (\delta_{kj} - x_j(u)) \sigma_{jl} dW_l(u) \right] \right]^{-2c} \right] \right\}^{1/2} \\ \left\{ E \left[ \left[ \int_{t-1}^t \exp \left[ - \sum_{l=1}^j \sum_{t=1}^s (\delta_{kj} - x_j(u)) \sigma_{jl} dW_l(u) \right] ds \right]^{-2c} \right] \right\}^{1/2} \\ \leq e^{cM} (\text{Min} \lambda_{jk})^{-c} \left\{ E \left[ \exp \left[ \sum_{l=1}^j \sum_{t=1}^t (\delta_{kj} - x_j(u)) (-2c) \sigma_{jl} dW_l(u) \right] \right] \right\}^{1/2} \\ \left\{ E \left[ \sup_{t-1 \leq s \leq t} \exp \left[ \sum_{l=1}^j \sum_{t=1}^s 2c (\delta_{kj} - x_j(u)) \sigma_{jl} dW_l(u) \right] \right] \right\}^{1/2},$$

which by lemma 2 and Hölder's inequality

$$\leq e^{cM} (\text{Min} \lambda_{jk})^{-c} \left\{ \exp \left[ dn^2 4c^2 \sigma^2 \right] \right\}^{1/2} \\ \left\{ E \left[ \left[ \sup_{t-1 \leq s \leq t} \exp \left[ \sum_{l=1}^j \sum_{t=1}^s 2c (\delta_{kj} - x_j(u)) \sigma_{jl} dW_l(u) - \frac{1}{2} \sum_{l=1}^j \sum_{t=1}^s \left[ \sum_j 2c (\delta_{kj} - x_{ju}) \sigma_{jl} \right]^2 ds \right] \right]^2 \right] \right\}^{1/4} \\ \left\{ E \left[ \left[ \sup_{t-1 \leq s \leq t} \exp \left[ \sum_{l=1}^j \sum_{t=1}^s \left[ \sum_j 2c (\delta_{kj} - x_{ju}) \sigma_{jl} \right]^2 ds \right] \right]^2 \right] \right\}^{1/2}.$$

By the proof of lemma 2 we know that,

$$\exp \left[ \sum_{l=1}^j \sum_{t=1}^s 2c (\delta_{kj} - x_j(u)) \sigma_{jl} dW_l(u) - \frac{1}{2} \sum_{l=1}^j \sum_{t=1}^s \left[ \sum_j 2c (\delta_{kj} - x_{ju}) \sigma_{jl} \right]^2 ds \right]$$

is a martingale, and so we can use Novikov's [21] martingale moment inequalities to bound the expectation of the square of its supremum.

$$E(x_k(t)^{-c}) \leq e^{cM} (\text{Min} \lambda_{jk})^{-c} \left\{ \exp \left[ dn^2 4c^2 \sigma^2 \right] \right\}^{1/2} C_1^{1/2} \exp[4c^2 dn^2 \sigma^2].$$

Since  $C_1$  is a constant independent of both the time index and the particular martingale, we are done.

**Lemma 4**

$$E(V_{2(p-p)}(t)) \leq \exp[2(m + 2\sigma^2 n^4 d)t] V_{2(p-p)}(0) + \left[ 2M_{pp} \cdot N + 2 \sum_j C(M, \sigma) (\text{Min} \lambda_{jk})^{-1} \text{Max} \lambda_{jk} \right] \left[ 2(m + 2\sigma^2 n^4 d) \right]^{-1} \left[ 1 - \exp \left[ 2(m + 2\sigma^2 n^4 d)t \right] \right].$$

**Proof:**

By Lemma 1b') we know,

$$\begin{aligned} V_{2(p-p)}(t) &= \exp \left[ \int_0^t \left( 2 \sum_j (p_j - p'_j) m_j(x(s), \lambda) - M_{pp} \cdot x_c(s) + \sigma_{2(p-p)}(s) \right) ds + A(2(p-p))_0^t \right] V_{2(p-p)}(0) \\ &+ \int_0^t \exp \left[ \int_s^t \left( 2 \sum_j (p_j - p'_j) m_j(x(u), \lambda) - M_{pp} \cdot x_c(u) + \sigma_{2(p-p)}(u) \right) du + A(2(p-p))_0^s \right] 2M_{pp} \cdot x_c(s) V_{2(p-p)}(s) ds \\ &+ \int_0^t \exp \left[ \int_s^t \left( 2 \sum_j (p_j - p'_j) m_j(x(u), \lambda) - M_{pp} \cdot x_c(u) + \sigma_{2(p-p)}(u) \right) du + A(2(p-p))_0^s \right] \sum_j \frac{2(p_j - p'_j)}{x_j(s)} \left[ \sum_k \lambda_{jk} x_k(s) \right] V_{2(p-p)}(s) ds. \end{aligned}$$

By the definition of  $m$  and  $\sigma_{2(p-p)}$ ,

$$\begin{aligned} V_{2(p-p)}(t) &\leq \exp[2(m + \sigma^2 n^4 d)t + A(2(p-p))_0^t] V_{2(p-p)}(0) \\ &+ \int_0^t \exp[2(m + \sigma^2 n^4 d)(t-s) + A(2(p-p))_s^t] \left[ 2M_{pp} \cdot N + \sum_j \frac{2}{x_j(s)} \text{Max} \lambda_{jk} \right] ds. \end{aligned}$$

Taking expectations,

$$\begin{aligned} E(V_{2(p-p)}(t)) &\leq \exp[2(m + \sigma^2 n^4 d)t] (E(\exp[A(2(p-p))_0^t])) V_{2(p-p)}(0) \\ &+ \int_0^t \exp[2(m + \sigma^2 n^4 d)(t-s)] (E(\exp[2A(p-p)_s^t]))^{1/2} \left[ 2M_{pp} \cdot N + \sum_j \left[ E \left( \frac{2}{x_j^2(s)} \right) \right]^{1/2} \text{Max} \lambda_{jk} \right] ds, \end{aligned}$$

which by Lemmas 2 and 3,

$$\begin{aligned} &\leq \exp[(2m + \sigma^2 n^4 d)t] \exp \left[ 2n^4 d \sigma^2 t \right] V_{2(p-p)}(0) \\ &+ \int_0^t \exp \left[ 2(m + \sigma^2 n^4 d)(t-s) + 2n^4 d \sigma^2 (t-s) \right] \left[ 2M_{pp} \cdot N + 2 \sum_j C(M, \sigma) (\text{Min} \lambda_{jk})^{-1} \text{Max} \lambda_{jk} \right] ds. \end{aligned}$$

The Lemma then follows by integration.

**Proposition 2** If the rank of  $\Gamma$  is equal to  $\sum_{k=1}^N n_i$ , there exists an invariant measure  $\pi$  for the process  $x'(t)$ , and for all  $x \in \Delta$  and all  $E \in B_\Delta$  (the set of Borel subsets of  $\Delta$ )

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t P(s, x, E) ds = \pi(E).$$

**Proof:**

The process  $x(t)$  has an invariant measure by Theorem 21 from Skorohod [26], because it is a Markov process in a compact metric space,  $\Delta$ . To show uniqueness I will apply Theorem 5.1 in Arnold and Kliemann [1]. Once the existence of a unique invariant distribution is established the result follows by Birkhoff's ergodic theorem (see Skorohod [26] theorem 1, or Arnold and Kliemann [1] p. 54).

To establish uniqueness I need three lemmas and some definitions.

**Lemma 5**

If  $x$  is in  $\Delta$ , the rank of  $B(x)$  is equal to  $\sum_{i=1}^N (n_i - 1)$ .

**Proof:**

Let  $B'(x)$  be the matrix formed by suppressing from  $B(x)$  the rows corresponding to the last strategy,  $n_i$ , of all players, and suppose the rows in  $B'(x)$  are not linearly independent. Then there exist  $\alpha_j^i$  for  $j = 1, \dots, n_i - 1, i = 1, \dots, N$ , such that

$$\sum_{i=1}^N \sum_{j=1}^{n_i-1} \alpha_j^i x_j^i \left( \sigma_j^i - \sum_{k=1}^{n_i} x_k^i \sigma_k^i \right) = 0,$$

and there is some  $\alpha_j^i \neq 0$ . The coefficient that multiplies  $\sigma_{n_i}^i$  in the previous expression is

equal to  $-x_{n_i}^i \left[ \sum_{j=1}^{n_i-1} \alpha_j^i x_j^i \right]$ . Since all  $x_j^i \neq 0$  by assumption for  $j = 1, \dots, n_i$ , and  $i = 1, \dots, N$

and some  $\alpha_j^i \neq 0$  also by assumption, there is an  $i$  such that  $-x_{n_i}^i \left[ \sum_{j=1}^{n_i-1} \alpha_j^i x_j^i \right] \neq 0$ , which

implies that the rank of  $\Gamma$  is not equal to  $\sum_{k=1}^N n_k$ . This is a contradiction.

### Lemma 6

Let  $x'$  be in the neighborhood of radius  $\delta$  around  $x$ ,  $N_\delta(x)$ , and let  $\varepsilon = \|x - x'\|$ . Let  $t^{x'}$  be the first exit time of the system,

$$\dot{x}(t) = a(x(t)) + B(x(t))C, \quad x(0) = x'$$

from  $N_\delta(x)$ , then  $\frac{\delta - \varepsilon}{a^M + B^M |C|} < t^{x'}$ , where  $a^M = \max_{i, u \in \Delta} \{ |a_i(u)| \}$ ,  $B^M = \max_{i, j, u \in \Delta} \{ |B_{ij}(u)| \}$

and  $|C| = \max_i |C_i|$ .

**Proof:**

Let  $t \leq \frac{\delta - \varepsilon}{a^M + B^M |C|}$ , then,

$$\delta - \varepsilon \geq (a^M + B^M |C|)t \geq \left\| \int_0^t (a(x(s)) + B(x(s))C) ds \right\|$$

so

$$\begin{aligned} \delta &\geq \left\| \int_0^t (a(x(s)) + B(x(s))C) ds \right\| + \|x' - x\| \\ &\geq \left\| x' + \int_0^t (a(x(s)) + B(x(s))C) ds - x \right\| = \|x(t) - x\| \end{aligned}$$

Thus  $\|x(t) - x\| < \delta$  for all  $t < \frac{\delta - \varepsilon}{a^M + B^M |C|}$ , therefore  $\frac{\delta - \varepsilon}{a^M + B^M |C|} < t^{x'}$ .

For a set  $U \subset \Delta$ , I denote  $\bar{U}$ ,  $\text{int}U$ ,  $U^c$  and  $\partial U$ , the closure, the interior, the complement and the boundary of  $U$  in  $\Delta$  respectively. Let the deterministic control system

$$\frac{\partial \phi(t)}{\partial t} = a(\phi(t)) + B(\phi(t))u(t), \quad (12)$$

where the admissible controls  $u: \mathbb{R}^+ \rightarrow \mathbb{R}^d$  are the piecewise constant functions. Let  $\theta^+(x)$  be the set of points reachable from  $x$  forward in time,

$$\theta^+(x) = \bigcup_{t \geq 0} \theta^+(t, x),$$

$\theta^+(t, x) = \{y: \text{there exists an admissible } u: \mathbb{R}^+ \rightarrow \mathbb{R}^d \text{ such that } y = \phi(t; u, x)\}$ , where  $\phi(t; u, x)$  = solution of (12) starting from  $x$  at time 0 and using control function  $u$ .

**DEFINITION.** A set  $D \neq \emptyset$  in  $\Delta$  is called invariant control set for (12) if

$$\overline{\theta^+(x)} = \bar{D} \quad \forall x \in D,$$

and  $D$  is maximal with respect to inclusion.

**Lemma 7.**

For all  $x \in \Delta$ ,  $\overline{\theta^+(x)} = \Delta$ .

**Proof:**

Suppose that for some  $z$ ,  $\overline{\theta^+(z)} \neq \Delta$ .

If  $z \in \text{int}\Delta$  there is  $y \in \Delta$  and  $r > 0$  such that  $N_r(y) \cap \overline{\theta^+(z)} = \emptyset$ , where  $N_r(y)$  is the ball of radius  $r$  around  $y$ . We can choose  $y$  and  $r$  such that  $\partial N_r(y) \cap \partial \theta^+(z) = x$ , for some  $x \in \text{int}\Delta$  such that  $x + \varepsilon(x-y) \in \theta^+(z)$  for all  $\varepsilon$  greater than zero and smaller than some  $\varepsilon'$ .

In the proof of Lemma 5 I showed that the rows of the matrix  $B'(x)$ , formed by suppressing from  $B(x)$  the rows corresponding to the last strategy,  $n_i$ , of all players, are independent for all  $x \in \text{int}\Delta$ . Then for all  $v \in \mathbb{R}^{\sum_{i=1}^N (n_i-1)}$  there is a  $v' \in \mathbb{R}^d$  such that  $B'(x)v' = v$ . This, together with the fact that the row of  $B(x)$  corresponding to the last strategy,  $n_i$ , of all players is equal to the sum of the preceding  $n_i - 1$  rows, and also that the element of the vector  $(x - y + a(x))$  corresponding to the last strategy,  $n_i$ , of all players is equal to the sum of the preceding  $n_i - 1$  elements, implies that there is a vector  $C$  such that,  $a(x) + B(x)C = -(x - y)$ .

By continuity of  $a(\cdot)$  and  $B(\cdot)$  there is some  $\delta > 0$ , such that for all  $x'$  with  $\|x - x'\| < \delta$

$$\|a(x) - a(x') + B(x) - B(x')C\| < \frac{r}{2} \quad (13)$$

Let  $\varepsilon < \min \left\{ \frac{\delta}{2(a^M + B^M \|C\|) + 1}, \varepsilon' \right\}$ . Let  $x'' = x + \varepsilon(x - y)$ . Since  $\varepsilon < \varepsilon'$ ,

$x'' \in \theta^+(z)$ , and since  $\varepsilon < \delta$ ,  $x'' \in N_\delta(x)$ . For the deterministic control system in (12), let  $\phi(0) = x''$ , and  $u(t) = C$ , then

$$\phi(t) = x + \varepsilon(x - y) + \int_0^t (a(\phi(s)) + B(\phi(s))C) ds.$$

Since  $a(x) + B(x)C = -(x - y)$ ,

$$\phi(t) - y = (x - y)(1 + \varepsilon) - \int_0^t (x - y) ds + \int_0^t (a(\phi(s)) - a(x) + (B(\phi(s)) - B(x))C) ds.$$

For  $t < t^{x''}$ , since  $\phi(t) \in N_\delta(x)$ , (13) holds, thus

$$\|\phi(t) - y\| \leq \|x - y\|(1 + \varepsilon - t) + \frac{r}{2}t = r \left[ 1 + \varepsilon - \frac{t}{2} \right].$$

For  $t^* = \frac{\delta - \varepsilon}{a^M + B^M |C|}$ , which by Lemma 6 is smaller than  $t^{x''}$ ,

$$\|\phi(t^*) - y\| \leq r \left[ 1 + \varepsilon \left( 1 + \frac{1}{2(a^M + B^M |C|)} \right) - \frac{\delta}{2(a^M + B^M |C|)} \right]. \quad (14)$$

Since  $\varepsilon < \frac{\delta}{2(a^M + B^M |C|) + 1}$ , (14) implies that  $\phi(t^*) \in N_r(y)$ . But since  $\phi(t^*)$  can be reached from  $x''$  by an admissible control,  $C$ , and  $x'' \in \theta^+(z)$ ,  $\phi(t^*) \in \theta^+(z)$ . So  $N_r(y) \cap \theta^+(z) \neq \emptyset$ . This is a contradiction.

Suppose instead that for some  $z \in \partial\Delta$ ,  $\theta^+(z) \neq \Delta$ . In the deterministic control system in (12) let  $\phi(0) = z$  and let  $u(t) = 0$ . Then,

$$\phi(t) = z + \int_0^t a(\phi(s)) ds$$

For  $t$  small enough  $\phi_i(t) > 0$  if  $z_i > 0$ . Since  $a_i(z) > 0$  for  $i$  such that  $z_i = 0$ , by continuity  $\phi_i(t) > 0$  for  $t$  small enough. So for  $t$  small and  $u(t) = 0$ ,  $\phi(t, u; z) \in \Delta$ . But I just showed that  $\overline{\theta^+(x)} = \Delta$  for  $x$  in the interior of  $\Delta$ . Thus  $\theta^+(z) = \Delta$ , which is a contradiction.

Lemma 7 establishes that there is only one control set in  $\Delta$ . Lemma 5 proves that  $B(x)$ 's rank is at least the dimension of  $\Delta$  if  $x$  is in the interior of  $\Delta$ . So Lemmas 5 and 7 show that the assumptions needed to apply Theorem 5.1 in Arnold and Kliemann are satisfied for our process. Thus there is a unique invariant distribution.

### Proposition 3

- a) The process  $x(t)$  defined in equation (10) has an ergodic distribution.
- b) If  $a > 2b$  the limit of the ergodic distribution puts probability one on  $x = 1$  as  $\lambda_1, \lambda_2$  and  $\sigma$  go to zero, if  $\frac{\lambda_1}{\lambda_2}$  is bounded.

**Proof:**

This proof, as well as that of the next proposition borrows heavily from the proof of propositions 3 and 4 in Fudenberg and Harris [8], so that readers familiar with their work can follow my proofs more easily.

a) Let  $\delta(x) = x(1-x)(ax^N - b) - \lambda_1 x + \lambda_2(1-x)$ . Let an arbitrary  $z \in (0, 1)$ , and

$$I_1 = \int_0^{x(0)} \exp \left\{ -2 \int_z^x \frac{\delta(y)}{y^2(1-y)^2 \sigma^2} dy \right\} dx ,$$

$$I_2 = \int_{x(0)}^1 \exp \left\{ -2 \int_z^x \frac{\delta(y)}{y^2(1-y)^2 \sigma^2} dy \right\} dx ,$$

$$D(x) = \frac{2}{x^2(1-x)^2 \sigma^2} \exp 2 \int_z^x \frac{\delta(y)}{y^2(1-y)^2 \sigma^2} dy .$$

The process  $x(t)$  is ergodic (see Theorem 1.17 of Skorohod [26]), if  $I_1$  and  $I_2$  are infinite and  $\int_0^1 D(x) dx$  is finite.

But  $\frac{\delta(y)}{y^2(1-y)^2 \sigma^2} dy$ , is of order  $\frac{\lambda}{y^2}$  around  $y = 0$  and of order  $-\frac{\lambda_1}{(1-y)^2}$  around  $y = 1$ . Thus  $I_1$  and  $I_2$  are infinite.  $D(x)$  is of order  $\exp(-\lambda_1/x)/x^2$  in a neighborhood of  $x = 0$  and of order  $\exp(-\lambda_2/(1-x))/(1-x)^2$  in the vicinity of  $x = 1$ , so  $\int_0^1 D(x) dx$  is finite.

b) The density of the ergodic distribution is proportional to,

$$D(x) = \frac{2}{x^2(1-x)^2 \sigma^2} \exp 2 \int_z^x \frac{\delta(y)}{y^2(1-y)^2 \sigma^2} dy .$$

This implies, as I showed in the last example,

$$D(x) = \frac{2}{z^2(1-z)^2\sigma^2} \exp 2 \int_z^x \frac{\delta(y) + (2y-1)y(1-y)\sigma^2}{y^2(1-y)^2\sigma^2} dy.$$

$$\text{Let } \gamma(y) = \delta(y) + (2y-1)y(1-y)\sigma^2, \text{ and } F(x) = \exp 2 \int_z^x \frac{\gamma(y)}{y^2(1-y)^2\sigma^2} dy.$$

Let  $y_1$  be the smallest  $y \in [0, 1]$  such that  $\gamma(y) = 0$ .

$$\text{Since } \gamma(y) > -by - \lambda_1 y + \lambda_2(1-y) - \sigma^2 y \text{ then } y_1 > \frac{\lambda_2}{b + \lambda_1 + \sigma^2}$$

Since  $\gamma(y) > 0$  for  $y < y_1$ ,  $F(y) < F(y_1)$ .

Choose  $y_2$  so that  $ay_2^N - b > b + k$  for some  $k > 0$ . Let  $y_3 = 1 - \frac{\lambda_1}{b+k}$ . Since

$\gamma(y) > 0$  in  $[y_2, y_3]$ , then  $F(y)$  is increasing in that interval.

Let now  $x \in [y_1, y_2)$  and  $x' \in (y_2, y_3]$ ,

$$\begin{aligned} \frac{F(x')}{F(x)} &= \exp 2 \left[ \int_{y_2}^{x'} \frac{\gamma(y)}{y^2(1-y)^2\sigma^2} dy + \int_x^{y_2} \frac{\gamma(y)}{y^2(1-y)^2\sigma^2} dy \right] \\ &\geq \exp \frac{2}{\sigma^2} \left[ \int_{y_2}^{x'} \left[ \frac{b+k}{y(1-y)} - \frac{\lambda_1}{y(1-y)^2} \right] dy - \left[ \int_x^{y_2} \left[ \frac{b+\sigma^2}{y(1-y)} + \frac{\lambda_1}{y(1-y)^2} \right] dy \right] \right] \\ &\geq \exp \frac{2}{\sigma^2} \left[ -(b+k)(\ln(1-x') - \ln(1-y_2)) + (b+\sigma^2+\lambda_1)\ln x + (b+\sigma^2)\ln(1-y_2) + \lambda_1 \ln(1-x') - \frac{\lambda_1}{(1-x')} \right] \end{aligned}$$

If  $x' \geq 1 - \lambda_1 \frac{b+k/4}{b+k}$  then given the definition of  $x'$  and  $y_1$ ,

$$\geq \exp \frac{2}{\sigma^2} \left[ -(b+k/4)\ln \lambda_1 + (b+\lambda_1+\sigma^2)\ln \frac{\lambda_2}{b+\lambda_1+\sigma^2} + (2b+k+\sigma^2)\ln(1-y_2) + \lambda_1 \ln(1-x') - \frac{\lambda_1}{(1-x')} \right]$$

$$\geq \exp 2 \left[ \int_{y_2}^{x'} \frac{A^2 y(1-y) - \lambda_1 y}{y^2(1-y)^4 \sigma_M^2} dy - \int_x^{y_2} \frac{(B + \lambda_1 + 2\sigma_M^2)y}{y^2(1-y)^4 \sigma_m^2} dy \right]$$

$$\geq \exp 2 \left[ \frac{A^2(1-x')/2 - \lambda_1/3}{(1-x')^3 \sigma_M^2} - \frac{A^2}{2(1-y_2)^3 \sigma_M^2} + \frac{(B + \lambda_1 + \sigma_M^2) \ln x}{(1-y_2)^4 2\sigma_m^2} \right].$$

Since  $\frac{\lambda_1}{\lambda_2}$  and  $\frac{\sigma_M}{\sigma_m}$  is bounded, if  $x' > 1 - \frac{\lambda_1^{1/4}}{A^2}$  and  $\lambda_1$  is small  $F(x') > F(x)$ . Given that for  $y < y_1$ ,  $F(y) < F(y_1)$  and that  $F(y)$  is increasing in the interval  $[y_2, y_3]$ , this implies that  $F(x') > F(y)$  for all  $y < x'$ .

$$\text{Let } x_1 = 1 - \frac{\lambda_1^{1/3}}{A^2}, \text{ let } x_2 = 1 - \frac{\lambda_1^{1/4}}{A^2} \text{ and } x_3 = 1 - \frac{\lambda_1^{1/2}}{A^2}.$$

Now let the ratio of probabilities under the ergodic distribution,

$$\frac{P(x > x_1)}{P(x < x_2)} \geq \frac{(x_3 - x_1) \min_{x \in [x_1, x_3]} D(x)}{\max_{x \in [0, x_2]} D(x)} \geq \frac{(x_3 - x_1) F(x_1) / \sigma_M^2}{F(x_2) / \sigma_m^2}$$

$$\geq \frac{\sigma_m^2 (x_3 - x_1)}{\sigma_M^2} \exp 2 \left[ \frac{A^2(1-x_1)/2 - \lambda_1/3}{(1-x_1)^3 \sigma_M^2} - \frac{A^2(1-x_2)/2 - \lambda_1/3}{(1-x_2)^3 \sigma_M^2} \right]$$

Since the previous expression tends to infinity as  $\lambda_1, \lambda_2, \sigma_1, \sigma_2$  tend to zero all the probability mass tends to be concentrated in the interval  $\left[ 1 - \frac{\lambda_1^{1/3}}{A^2}, 1 \right]$ . Since  $\lambda_1$  goes to zero the result follows.

## References

1. L. Arnold and W. Kliemann, "On Unique Ergodicity for Degenerate Diffusions", *Stochastics* **21** (1987), 41-61.
2. Brown, "Iterative Solution of Games by Fictitious Play", in *Activity Analysis of Production and Allocation*, John Wiley and Sons, 1951.
3. A. Cabrales, and J. Sobel, "On the Limit Points of Discrete Selection Dynamics", *Journal of Economic Theory* **57** (1992), 407-20.
4. V. P. Crawford, Adaptive dynamics in coordination games, University of California, San Diego (1993).
5. V. P. Crawford, An "evolutionary" explanation of Van Huyck, Battalio and Beil's experimental results on coordination. University of California, San Diego (1989).
6. E. Dekel and S. Scotchmer, "On the Evolution of Optimizing Behavior", *Journal of Economic Theory* **57** (1992), 392-407.
7. B.J. De Long, A. Shleifer, L.H. Summers and R.J. Waldmann, Noise trader risk in financial markets, *Journal of Political Economy* **98** (1990), 703-738.
8. G. Ellison "Learning, Local Interaction and Coordination" (1991).
9. M. J. Farrell, "Some Elementary Selection Processes in Economics", *Review of Economic Studies* **37** (1970) 305-21.
10. D. Foster and P. Young, "Stochastic Evolutionary Game Dynamics", *Theoretical*

*Population Biology* 38 (1990), 219-32.

11. D. Fudenberg and C. Harris, "Evolutionary Dynamics in Games with Aggregate Shocks", *Journal of Economic Theory* 57 (1992), 420-42.

12. D. Fudenberg and D. Kreps, "A Theory of Learning, Experimentation and Equilibrium in Games" (1988).

13. I. Gihman and A.V. Skorohod, *Stochastic Differential Equations*, Berlin: Springer Verlag, 1972.

14. M. Kandori, G. Mailath and R. Rob, "Learning, Mutation and Long-Run Equilibrium in Games", *Econometrica* 61 (1993), 29-56.

15. M. Kandori and R. Rob, "Evolution of Equilibrium in the Long-Run: a General Theory and Applications", (1991).

16. A. Matsui, "Best Response Dynamics and Socially Stable Strategies", *Journal of Economic Theory* 57 (1992), 343-63.

17. P. Milgrom and J. Roberts, "Adaptive and Sophisticated Learning in Repeated Normal Form Games", *Games and Economic Behavior* 3 (1990) 82-101.

18. J. Nachbar, Evolutionary selection dynamics in games: convergence and limit properties, *International Journal of Game Theory* 19 (1990) 59-89.

19. R. Nisbett and L. Ross, *Strategies and Shortcomings of Social Judgement*, Prentice Hall, (1980).

20. G. Noldeke, L. Samuelson, and E. Van Damme, "The Evolution of Communication", (1991).
21. A.A. Novikov, "On Moment Inequalities for Stochastic Integrals", *Theory of Probability and its Applications* 16 (1971), 538-541.
22. A.A. Novikov, "On an Identity for Stochastic Integrals", *Theory of Probability and its Applications* 17 (1972), 717-720.
23. J. Robinson, "An Iterative Method of Solving a Game", *Annals of Mathematics* 54 (1951), 296-301.
24. L. Samuelson, "How to Tremble if you Must" (1991).
25. L. Samuelson and J. Zhang, "Evolutionary Stability in Asymmetric Games", *Journal of Economic Theory* 57 (1992), 363-392.
26. A.V. Skorohod, *Asymptotic Methods in the Theory of Stochastic Differential Equations*, (Translated by H.H. Mc Faden), American Mathematical Society, (1989).
27. D. Smallwood and J. Conlisk, "Product Quality in Markets where Consumers are Imperfectly Informed", *Quarterly Journal of Economics* 93 (1979), 1-23.
28. D. Stahl, "The Evolution of Smart-n Players" (1992).
29. J. Van Huyck, R. Battalio, R. Beil, "Tacit Coordination Games, Strategic Uncertainty and Coordination Failures", *American Economic Review* 80 (1990) 234-48.
30. J. Van Huyck, R. Battalio, R. Beil, "Strategic Uncertainty, Equilibrium Selection

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31. S. Winter, "Optimization and Evolution in the Theory of the Firm", in *Adaptive Economic Models*, R. Day and T. Groves ed., Academic Press (1975).

32. P. Young, "The Evolution of Conventions", *Econometrica* **61** (1993) 57-84.

33. P. Young and D. Foster, "Cooperation in the Short and in the Long Run", *Games and Economic Behavior* **3** (1991) 145-156.

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