Application of the flexibility influence function method in the dynamic analysis of composite beams

C. Santiuste, S. Sánchez-Sáez, E. Barbero *

Department of Continuum Mechanics and Structural Analysis, Carlos III University of Madrid, Avda. de la Universidad 30, 28911 Leganés, Madrid, Spain

Abstract

The flexibility influence function technique is validated as a method for calculating the displacements and the rotations of a laminated beam subjected to a dynamic load, using the first-order shear deformation laminate theory and comparing the results with those obtained by modal analysis and two finite element models (one-dimensional and three-dimensional). The movements (displacements and rotations) were calculated from a single-span beam subjected to a time-variable load with four boundary conditions: clamped-clamped, hinged-hinged, clamped-free, clamped-hinged. A carbon/epoxy cross-ply laminated beam was selected to avoid bending-torsion coupling. The maximum movements calculated by the flexibility influence function method differs very little from those calculated with the other two models accounted for by the first-order shear deformation laminate theory: modal analysis and the one-dimensional numerical model. The differences in the rotations between the three-dimensional numerical model and the flexibility influence function method are slightly bigger, and could be due to the warping of the cross-section of the beam, which is not included in the first-order shear deformation laminate theory.

Keywords: Flexibility influence function method; First order shear deformation theory; Composite laminates; Beam models; Bending

1. Introduction

Many structural elements, such as windmill blades, helicopter rotor blades, robot arms, transmission axes, etc., are made of composite laminates on account of the excellent mechanical properties and low weight of these materials. The elements can be modelled as beams subjected to loads that originate mainly bending moments (Demakos, 2003), and since these loads are often dynamic, the study of the dynamic flexural behaviour of beam elements should be considered in the design process.

Among the various methods used to study the behaviour of a beam under dynamic loading numerical analysis is one of the most used (the Galerkin method, the finite element method, the boundary element method or

* Corresponding author. Tel.: +34 91 624 99 65; fax: +34 91 624 94 30.
E mail address: ebarbero@ing.uc3m.es (E. Barbero).

the finite difference method). Rand (1998) studied the interlaminar shear stress that appears in composite beams of arbitrary solid cross-section subjected to several static loads. In this latter case a model that includes a complete description of the out-plane warping and solving the equations of the movement of the beam by the finite difference method was used. Khalid et al. (2005) analyzed, by the finite element method, a glass/epoxy I-beam under bending loads using shell elements, and compared the results with experimental three and four point bending tests, Sapountzakis (2005) used the boundary element method to analyze nonuniform torsional problem of composite bars of arbitrary cross-section under both free and forced torsional vibration. Subrahmanyam et al. (1987) used the Galerkin method to study the vibration of rotary isotropic beams.

Simplified models can be useful, however, to study beam elements since they provide a sufficiently accurate solution with a lower computational cost than that of the numerical methods. They are very useful in the optimization processes because they serve to evaluate in a simple way the influence of the different parameters in the global response of the structure.

In determining the global response of a laminated beam subjected to bending moments in dynamic conditions (for example, the displacement or the natural frequencies and modes of vibration) equivalent single-layer theories can be used because they are sufficiently accurate in thin laminates (Kapania and Raciti, 1989), but if a detailed interlaminar analysis of stress in specific points is required, or the laminate is very thick, it is necessary to adopt three-dimensional elasticity theories (Carvelli and Savoia, 1997). These give more accurate results than equivalent single-layer theories but their computational costs can make them unpractical (Kapania and Raciti, 1989).

The simplest equivalent single-layer theory is the classical laminated theory, based on the Kirchhoff hypothesis, and allowing an exact study of the displacements and stresses in thin laminates. Since the ratio of shear modulus to effective flexural modulus is low in composite laminates, the effect of the transverse shear deformation must be considered in their dynamic analysis (Yildirim and Kiral, 2000). High-order shear theories have been developed (Levinson, 1980; Murthy, 1981; Reddy, 1984; Rand, 1998) but usually the first-order shear deformation laminate theory is used, as it provides a similar value to that of higher order theories (Kapania and Raciti, 1989). This theory assumes a constant shear rotation through the laminate thickness so it requires the use of a shear correction factor. Several models were used to determine this factor considering its dependence on the elastic constants, different cross-section, width to depth ratio in rectangular sections (Cowper, 1966; Stephen, 1980; Hutchinson, 2001; Puchegger et al., 2003) and stacking sequence in composite materials (Dharmara and McCutchen, 1973; Madabhusi-Raman and Davalos, 1996). But often the same factor is used as for isotropic material (Reddy, 1997; Yoo et al., 2005) since it was demonstrated that in beams with a support span above a critical value, the differences in the shear factors do not influence the results (Santiuste et al., 2005).

To solve the equations of the movement of a beam by a simplified method, most researchers have used modal analysis (Banerjee, 2001a,b). This method requires the solution of a four order differential equation for the first-order shear deformation laminate theory. Miller and Adams (1975) used this technique to study natural frequencies and modes of vibration of cantilever laminated beams with first-order shear deformation laminate theory. Banerjee (2001a,b) in a similar study that included torsion, provided analytic closed solutions for cantilever beams. Dong et al. (2005) improved these studies by considering a stepped cantilever beam and comparing the results with those of a uniform beam and a finite element model. By this method Abramovich (1993) had studied the vibration of cantilever laminate beams including gravitational loads.

However, as the boundary condition becomes more complex and if the beam has a cross-section of variable bending stiffness, the differential equation is more difficult to solve by modal analysis. In these cases the transfer matrix method has been used to study the behaviour of beams, for example by Subrahmanyam and Garg (1997) analyzed the vibrations of an isotropic beam with different boundary conditions, considering effects such as the shear deformation and rotary inertia. They considered beams of variable mass and stiffness distribution. Yildirim and Kiral (2000) also used this method for the study of the out-of plane free vibration of symmetric cross-ply laminated beams, comparing the results by both the Euler–Bernoulli and the Timoshenko beam theories.

The flexibility influence function method is another alternative to calculate the displacement and rotation of a beam subjected to dynamic loads. This technique does not require the calculation of the natural frequencies and modes of vibration of the beam (Meirovitch, 1967). The flexibility influence method is independent of the boundary conditions, on the contrary, other methods (i.e., modal analysis) require a solution of the difference equations for each boundary condition. When the symmetry and isostatism conditions are not involved, the flexibility influence method is preferable. Several authors have used this model to analyze Euler–Bernoulli beams studying the free vibration of intact isotropic beams (Penny and Reed, 1971) and in cracked isotropic beams (Fernández-Sáez and Navarro, 2002). The dynamic response in Timoshenko isotropic beams has also been analyzed.

In this study, the flexibility influence function method is applied to determine the global response of a laminated beam subjected to a time-variable load. The beam was analyzed using the first-order shear deformation laminate theory. The time of application of the load was close to the characteristic time of vibration of each beam, calculated by a three-dimensional finite element model. A single-span beam with different boundary conditions (clamped-clamped, hinged-hinged, clamped-free, clamped-hinged) was selected. The displacements and the rotation were compared with those obtained in one-dimensional and three-dimensional finite element models. Two isostatic beams, cantilever and simply supported, were studied by modal analysis, and the displacements and the rotation were compared with those calculated by the flexibility influence function method. A cross-ply laminate made from a carbon/epoxy material was used to avoid any bending-torsion coupling.

2. Theory

2.1. First-order shear deformation theory

In the first-order shear deformation laminate beam theory, it is assumed that the two first Kirchhoff hypotheses hold: straight lines perpendicular to the midsurface (i.e., transverse normals) before deformation remain straight after deformation and do not experience elongation. The third hypothesis is not assumed: transverse normals do not remain perpendicular to the midsurface after deformation, Fig. 1. The displacement field is of the form (Reddy, 1997):

$$\begin{cases} u(x,y,z,t) = u_0(x,y,t) + z\cdot\phi_x(x,y,t) \\ v(x,y,z,t) = v_0(x,y,t) + z\cdot\phi_y(x,y,t) \\ w(x,y,z,t) = w_0(x,y,t) \end{cases} \tag{1}$$

where u, v, w are the displacement components along the x, y, z coordinate directions, respectively; u_0, v_0, w_0 are the displacement components of a point on the midplane, and ϕ_x, ϕ_y denote rotations of a transverse normal about the y, x axes, respectively, Fig. 1.

For the assumed displacement field the strains are of the form:

$$\begin{cases} \varepsilon_{xx} = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 + z\frac{\partial \phi_x}{\partial x} \\ \varepsilon_{yy} = \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 + z\frac{\partial \phi_y}{\partial y} \\ \varepsilon_{zz} = 0 \end{cases} \quad \begin{cases} \gamma_{xy} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right) + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \\ \gamma_{xz} = \frac{\partial w_0}{\partial x} + \phi_x \\ \gamma_{yz} = \frac{\partial w_0}{\partial y} + \phi_y \end{cases} \tag{2}$$

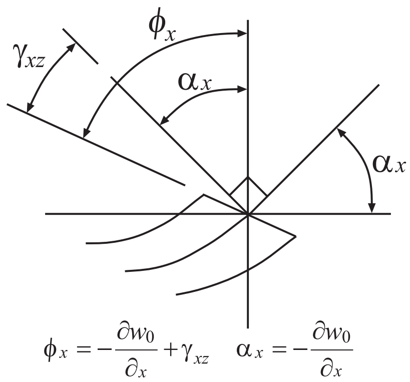

Fig. 1. Undeformed and deformed geometries of an edge of a beam.

Note that the transverse shear strains γ_{xz}, γ_{yz} are constant through the thickness of the laminate, and the strain in direction z is zero ($\varepsilon_{zz}\quad 0$).

The governing equations in this model are derived from the dynamic version of the principle of virtual displacements

$$\int_0^T (\delta U + \delta V - \delta K)\,\mathrm{d}t = 0 \tag{3}$$

where δU is the virtual strain energy, δV is the virtual work done by applied loads, and δK is the virtual kinetic energy. δU and δK are volume integral functions and δV *is a volume and boundary integral function.*

Substituting these functions into Eq. (3) yields an expression of principle of virtual displacements in function of stresses and strains. Integrating the stresses through the thickness of the laminate and substituting the strains in function of displacements, Eq. (2), an expression is obtained as a function of the forces and virtual displacements (δu_0, δv_0, δw_0, $\delta\phi_{\mathrm{x}}$ and $\delta\phi_{\mathrm{y}}$) (Reddy, 1997). The Euler–Lagrange equations are obtained by setting the coefficients of δu_0, δv_0, δw_0, $\delta\phi_{\mathrm{x}}$ and $\delta\phi_{\mathrm{y}}$ to zero separately in the volume integrals

$$\begin{aligned}
&\delta u_0: \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2} + I_1 \frac{\partial^2 \phi_x}{\partial t^2} \\
&\delta v_0: \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2} + I_1 \frac{\partial^2 \phi_y}{\partial t^2} \\
&\delta w_0: \frac{\partial Q_{xx}}{\partial x} + \frac{\partial Q_{yy}}{\partial y} + \mathsf{N}(w_0) + q_1(x,t) = I_0 \frac{\partial^2 w_0}{\partial t^2} \\
&\delta\phi_x: \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_{xx} + q_2(x,t) = I_2 \frac{\partial^2 \phi_x}{\partial t^2} + I_1 \frac{\partial^2 u_0}{\partial t^2} \\
&\delta\phi_y: \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_{yy} = I_2 \frac{\partial^2 \phi_y}{\partial t^2} + I_1 \frac{\partial^2 v_0}{\partial t^2}
\end{aligned} \tag{4}$$

where N_{ii} and N_{ij}, Q_{ii}, M_{ii} and M_{ij} are the in-plane force resultants, transverse force resultants and moment resultants per unit width (Fig. 2), ρ is the density, h the thickness, $q_1(x,t)$ the applied force and $q_2(x,t)$ the applied moment, both per unit width of the beam.

And

$$\begin{Bmatrix} I_0 \\ I_1 \\ I_2 \end{Bmatrix} = \int_{h/2}^{h/2} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} \cdot \rho \cdot \mathrm{d}z \tag{5}$$

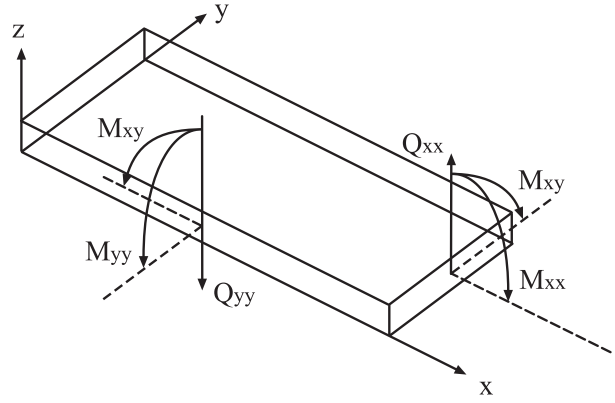

Fig. 2. Force and moment resultants on a laminate.

$$\mathsf{N}(w_0) = \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y}\right) \tag{6}$$

The natural boundary conditions are obtained by setting the coefficients of virtual displacement to zero separately in the boundary integrals.

Integrating the stresses through the thickness of the laminate, the constitutive equations are obtained in which the force and the moment resultants are related to the strains. In a symmetrical laminate the constitutive equations are of the form:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} \tag{7}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \tag{8}$$

$$\begin{Bmatrix} Q_{xx} \\ Q_{yy} \end{Bmatrix} = K \cdot \begin{bmatrix} A_{44}^s & A_{45}^s \\ A_{54}^s & A_{55}^s \end{bmatrix} \cdot \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{9}$$

where K is the shear correction factor depending on lamina properties and stacking sequence, and also on the width to depth ratio in a rectangular section.

The laminate stiffness matrices are: the extensional stiffness matrix $[A]$, the bending stiffness matrix $[D]$ and the shear stiffness matrix $[A^s]$, which are defined in terms of the components of the lamina stiffness matrix, $\overline{Q}_{ij}^{(k)}$, as

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{N} \overline{Q}_{ij}^{(k)}(z_{k+1} - z_k) \quad i,j = 1,2,6 \\ A_{ij}^s &= \sum_{k=1}^{N} \overline{Q}_{ij}^{(k)}(z_{k+1} - z_k) \quad i,j = 4,5 \\ D_{ij} &= \frac{1}{3}\sum_{k=1}^{N} \overline{Q}_{ij}^{(k)}(z_{k+1}^3 - z_k^3) \quad i,j = 1,2,6 \end{aligned} \tag{10}$$

where z_k and z_{k+1} are the k-lamina bottom and top surface z-coordinates.

In this study of a beam subjected to a load which generates bending forces, it is assumed that the forces are zero, except for the bending moment, M_{xx}, and the transverse shear force, Q_{xx}. The constitutive equations are reduced to

$$\begin{aligned} \varepsilon_{xx} &= d_{11} \cdot M_{xx} \\ \gamma_{xz} &= \frac{a_{55}^s}{K} \cdot Q_{xx} \end{aligned} \tag{11}$$

where a_{55}^s and d_{11} are the corresponding terms of the flexibility matrices, $[a^s]$ and $[d]$, inverses of the stiffness matrices, $[A^s]$ and $[D]$.

Substituting strains and forces in terms of the displacements gives two second-order coupled differential equations.

$$\frac{K}{a_{55}^c} b\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial \phi_x}{\partial x}\right)+f_1(x,t)=\rho\cdot A\frac{\partial^2 w_0}{\partial t^2}$$
$$\frac{b}{d_{11}}\frac{\partial^2 \phi_x}{\partial x^2}-\frac{K}{a_{55}^c} b\left(\frac{\partial w_0}{\partial x}+\phi_x\right)+f_2(x,t)=\rho\cdot I_{yy}\frac{\partial^2 \phi_x}{\partial t^2} \quad (12)$$

where A is the cross-section area, I_{yy} is the area moment of inertia about the y-axe, b is the width of the beam, $f_1(x,t)$ the distributed vertical load and $f_2(x,t)$ the distributed bending moment.

2.2. Flexibility influence function method

The displacements can be expressed in a generalized displacement vector by two variables, deflection and section rotation

$$\delta(x,t)=\begin{pmatrix} w_0(x,t) \\ \phi_0(x,t)\end{pmatrix} \quad (13)$$

The flexibility influence function method allows a calculation of the displacements produced in a one-dimensional continuous system by the solution of an integral equation.

The generalized displacement of a point with time, $\delta(x,t)$, in a beam of length L is obtained through the next expression

$$\delta(x,t)=\int_0^L C(x,\xi)\cdot p(\xi,t)\,\mathrm{d}\xi \quad (14)$$

where $C(x,\xi)$ is the matrix which contains the influence functions

$$C(x,\xi)=\begin{pmatrix} C^{yy}(x,\xi) & C^{yz}(x,\xi) \\ C^{zy}(x,\xi) & C^{zz}(x,\xi)\end{pmatrix} \quad (15)$$

The function $C^{yy}(x,\xi)$ is defined as the vertical deflection in direction z of the considered point x, because of a vertical unit static load applied at the point of abscissa ξ. The function $C^{yz}(x,\xi)$ is defined as the vertical deflection of a point x because of a unit static bending moment applied at point ξ. The function $C^{zy}(x,\xi)$ is defined as the rotation of the transverse section about the y-axe of a point x because of a vertical unit load applied at point ξ. The function $C^{zz}(x,\xi)$ is defined as the rotation of the transverse section about the y-axis of a point x when a unit bending moment is applied at point ξ. These influence functions are calculated in static conditions and depend on the boundary conditions.

The function $p(x,t)$ contains the dynamic forces: exterior loads and inertia forces

$$p(x,t)=-m(x)\cdot\frac{\partial^2\delta(x,t)}{\partial t^2}+\vec{f}(x,t) \quad (16)$$

where $\vec{f}(x,t)$ is the exterior load vector consisting of $f_1(x,t)$ and $f_2(x,t)$, and $m(x)$ is the mass matrix

$$\vec{f}(x,t)=\begin{pmatrix} f_1(x,t) \\ f_2(x,t)\end{pmatrix} \quad m(x)=\begin{pmatrix} \rho A & 0 \\ 0 & \rho I_{yy}\end{pmatrix} \quad (17)$$

Substituting in (14) yields

$$\int_0^L C(x,\xi)\cdot m(\xi)\cdot\frac{\partial^2\delta(\xi,t)}{\partial t^2}\,\mathrm{d}\xi+\delta(x,t)=\delta_{\mathrm{st}}(x,t) \quad (18)$$

where the displacement in static conditions, $\delta_{\mathrm{st}}(x,t)$, is defined as

$$\delta_{\mathrm{st}}(x,t)=\int_0^L C(x,\xi)\cdot\vec{f}(\xi,t)\,\mathrm{d}\xi \quad (19)$$

Eq. (18) is solved numerically. Transforming it into a second-order differential equation system. If Eq. (18) is discretized in n control points it yields

$$C \cdot R \cdot M \frac{\partial^2 \vec{\delta}(t)}{\partial t^2} + \vec{\delta}(t) = \vec{\delta}_{\mathrm{st}}(t) \tag{20}$$

where $C(2n \times 2n)$ is the influence function matrix, $R(2n \times 2n)$ is a diagonal matrix which depends on the integration algorithm, $M(2n \times 2n)$ the generalized mass matrix and $\vec{\delta}(t)(2n \times 1)$ the displacements vector, the n first components containing beam deflection and the other n components the section rotation. Eq. (20) can be transformed into

$$\frac{\partial^2 \vec{\delta}(t)}{\partial t^2} = (C \cdot R \cdot M)^{\ 1}(\vec{\delta}_{\mathrm{st}}(t) - \vec{\delta}(t)) \tag{21}$$

The problem is reduced to solving a system of $2n$ second-order differential equations. In this work the Stoerm rule was used to solve the system. To apply this method it is necessary that the first derivates of the unknown functions do not appear in the equation (Press, 1994). The numerical integration method used, which determines the matrix $[R]$, is the Gauss method.

The flexibility influence function method is independent of the boundary conditions, only changing the value of the C matrix. With other more usual methods for the study of bending of beams, such as modal analysis, it is necessary to solve the equations with each calculation of the boundary conditions.

3. Analysis

3.1. Problem description

To validate the flexibility influence function method (FIFM) in the calculation of the displacements of a composite beam, the method was applied to a specific problem; that of a rectangular cross-section laminate beam with four different boundary conditions at its ends: clamped-free, hinged-hinged, clamped-hinged and clamped-clamped.

The geometry of the beam is: thickness 1.6 mm, width 3.2 mm, support span 24 mm. A span to thickness ratio equal to 15 was selected. It is low enough to ensure the influence of transverse shear stresses in the behaviour of this orthotropic laminate as was proved in a previous work (Santiuste et al., 2005). This span to thickness ratio also ensures that the different methods to calculate the shear correction factor (Dharmara and McCutchen, 1973; Stephen, 1980; Madabhusi-Raman and Davalos, 1996) provide very similar results to 5/6 in the global response of the beam. This factor was used in this work.

The composite used is a T300 carbon fibre epoxy-matrix cross-ply laminate, $[0/90]_{2S}$. The lamina elastic properties are

$$E_1 = 114\ \mathrm{GPa} \quad E_2 = E_3 = 10\ \mathrm{GPa} \quad G_{12} = G_{13} = G_{23} = 6.2\ \mathrm{GPa} \quad \nu_{21} = 0.28 \quad \nu_{31} = \nu_{32} = 0.4$$

An impulsive load, Eq. (22), was applied on this beam whose time of application agrees with the first natural period of vibration, t_c. This load is similar to an experimental impulsive load recorded in a low velocity impact test. The first natural period of vibration was calculated for each one of the boundary conditions, Table 1, with a three-dimensional finite element model whose details are given below. The maximum applied load, F_0, has a unitary value, Fig 3.

$$F(t) = F_0 \sin^{3/2}\left(\pi \frac{t}{t_C}\right) \tag{22}$$

Table 1
Fundamental frequencies and first natural period of vibration for the different boundary conditions

	Fundamental frequency (Hz)	First natural period (s)
Clamped free	2990	3.34 E 04
Hinged hinged	8263	1.21 E 04
Clamped hinged	12319	8.12 E 05
Clamped clamped	16936	5.90 E 05

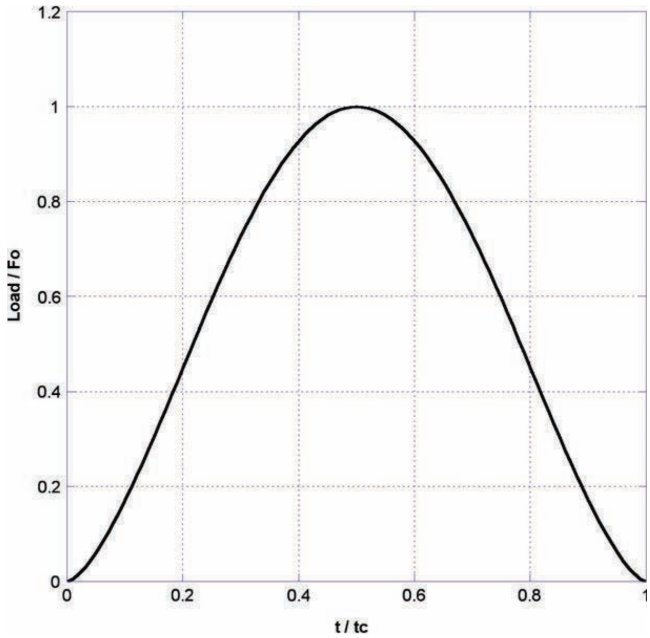

Fig. 3. Applied load.

3.2. Numerical models

To verify the accuracy of the analytical models, two finite element models were used: a one-dimensional model (FEM1D) and a three-dimensional model (FEM3D). Both were implemented in ABAQUS/Explicit (HKS, 2003).

In the FEM1D model, 3-node quadratic beam elements were used. The stiffness values used in these elements were the flexural modulus, E^f_{xx}, and shear modulus, G^f_{xz}, of the first-order shear deformation theory, Eqs. (23) and (24). The beam was divided into 30 elements and the different boundary conditions at its ends were applied.

$$E^f_{xx} = \frac{12}{h^3 \cdot d_{11}} \tag{23}$$

$$G^f_{xz} = \frac{1}{a_{55} \cdot h} \tag{24}$$

In the FEM3D model, 8-node linear brick, reduced integration, elements were used. The beam was divided into 16,000 elements of this type. The beam is composed of eight plies in which the elastic properties of the lamina were applied according to each ply orientation. The different boundary conditions at their ends were applied. Fig. 4 shows the simply supported beam under a load in its mid-span section at the time of maximum deformation. The plies oriented to 0° are seen to support greater stress than those oriented to 90°.

In the FEM1D model, the deflection and rotation of the beam were computed by ABAQUS. In the FEM3D model the deflection of the beam was computed as the middle plane deflection, and the rotation of the section was calculated manually. In an FEM3D model there is no section rotation: actually in the beam section there is a three-dimensional strain field. The value of the section rotation is an average measurement, estimated by the displacements of the upper and lower points.

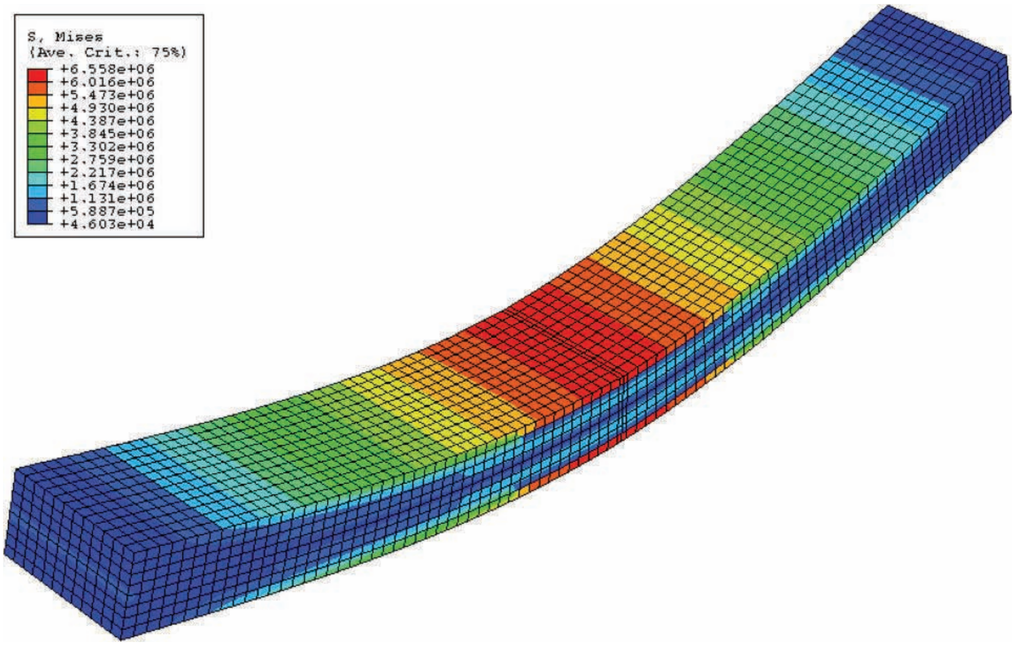

Fig. 4. Von Misses stress in FEM3D model of simply supported beam, time of maximum deformation.

3.3. Studied cases

A single-span beam was studied with four boundary conditions, Fig. 5. Case A is a cantilever beam under a load applied at its free end; case B is a simply supported beam under a load on its middle span; case C is a clamped-hinged beam under a load applied at $x \quad 2/3L$ and case D is a clamped-clamped beam under a load applied at $x \quad 2/3L$.

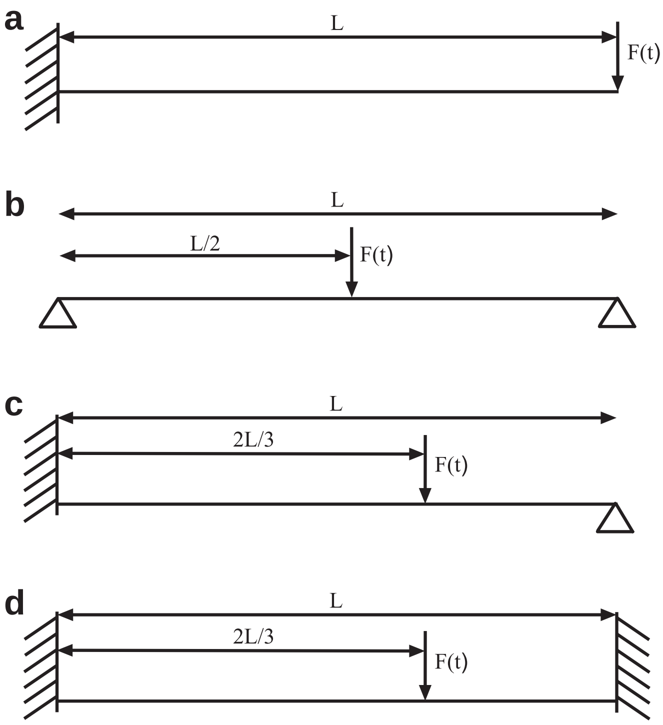

Fig. 5. (a) Case A (clamped free). (b) Case B (hinged hinged). (c) Case C (clamped hinged). (d) Case D (clamped clamped).

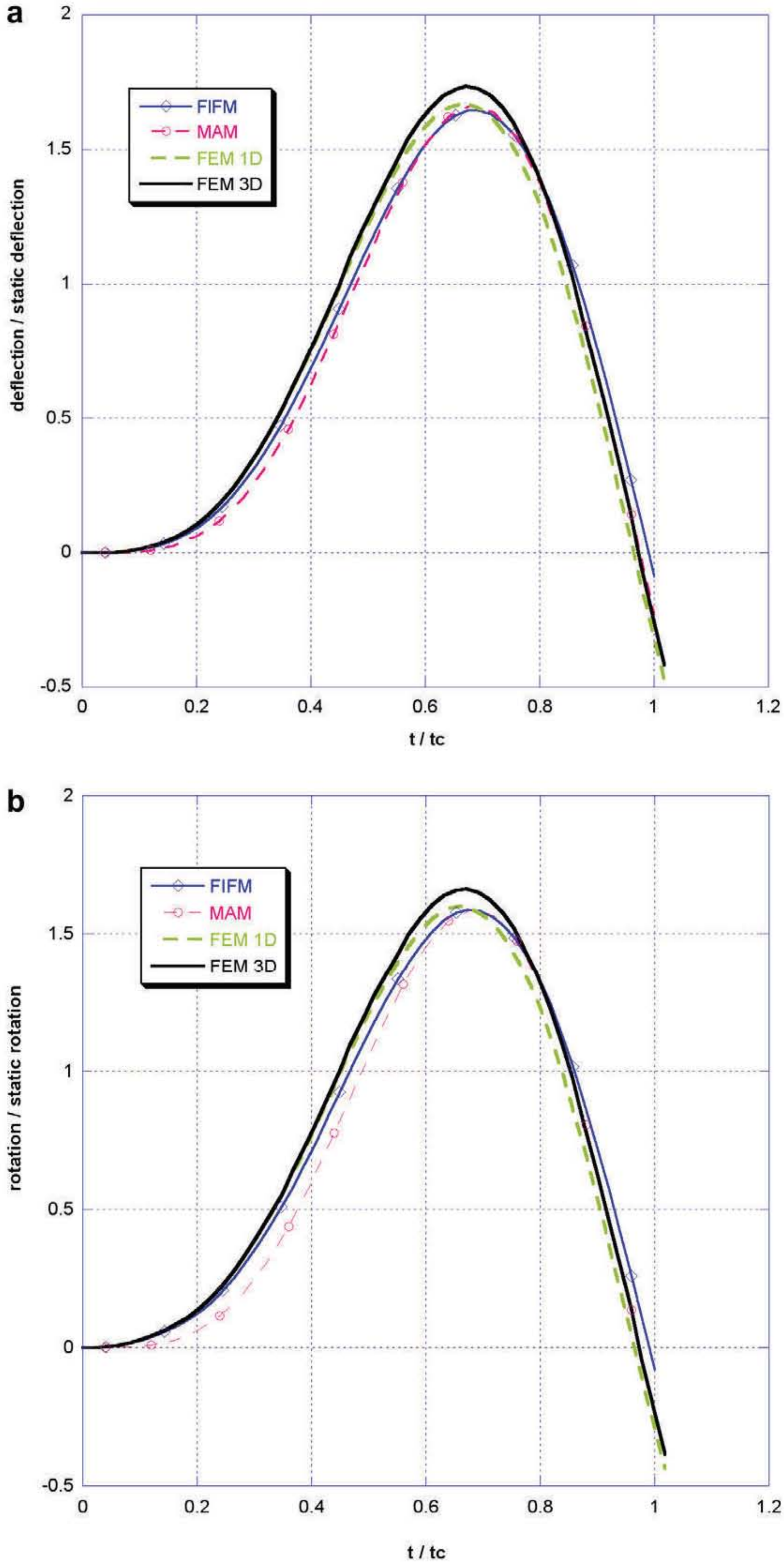

Fig. 6. Case A, cantilever beam. (a) Maximum dimensionless deflection. (b) Maximum dimensionless rotation.

The flexibility influence function method was applied to the four cases to calculate the deflections and section rotation of the beam. To verify the accuracy of the method the results were compared with the 1D and 3D numerical models.

Cases A and B are isostatic problems, the modal analysis method (MAM) is quite simple to apply in these cases so the flexibility influence function method was compared with this other method widely used for the calculation of displacements in bending beams. The cases C and D are hyperstatic problems in which the condition of symmetry is not fulfilled because the load is applied at $x \quad 2/3L$. In these cases the modal analysis presents difficulties of application since it increases its complexity. On the contrary, the flexibility influence function method can be applied as in the previous cases.

4. Results

The dynamic maximum deflection and the rotation in the four cases were calculated by the flexibility influence function method (FIFM), the numerical one-dimensional model (FEM1D) and the three-dimensional model (FEM3D). In cases A and B (isostatic problems) the dynamic maximum deflection and the rotation were calculated also by the modal analysis method (MAM).

Figs. 6–9 show, respectively, the results from cases A, B, C and D. The results obtained are dimensionless deflection and rotation in the free end section in case A and at mid-span section in cases B, C and D. In the simply supported beam (case B), because of symmetry conditions, the rotation of the mid-span section is zero and is not shown.

Deflections and rotations were made dimensionless, respectively, by dividing the dynamic results by the values of the static deflections and rotations, calculated by a 3D finite element model under a unitary load, and time was made dimensionless with the first natural period of vibration, Table 1.

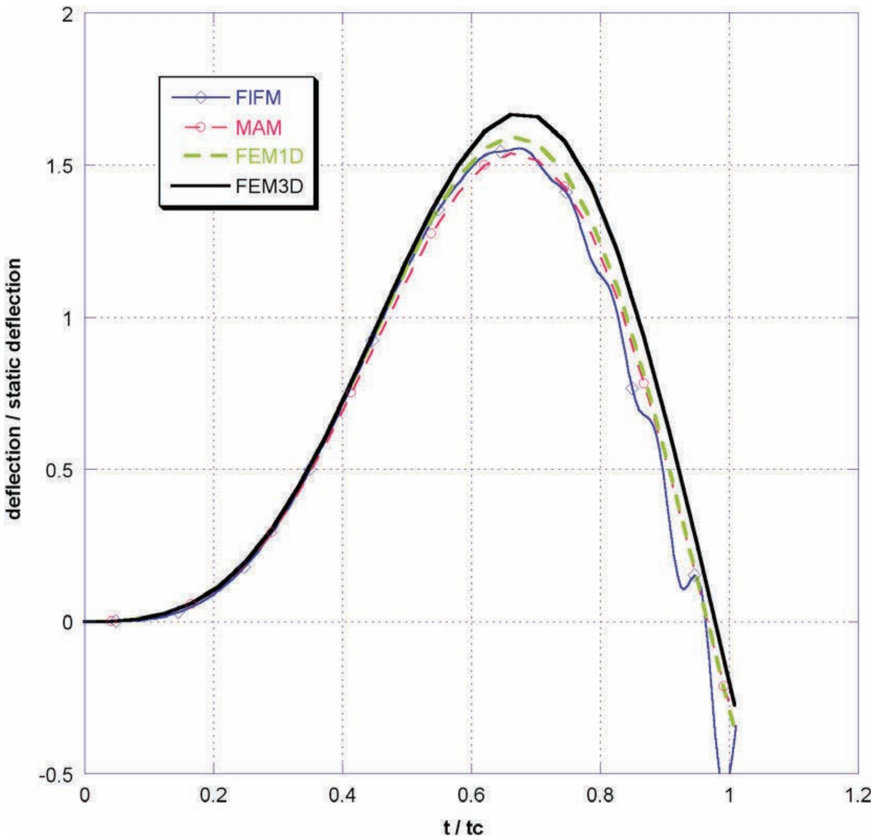

Fig. 7. Case B. Maximum dimensionless deflection in a simply supported beam.

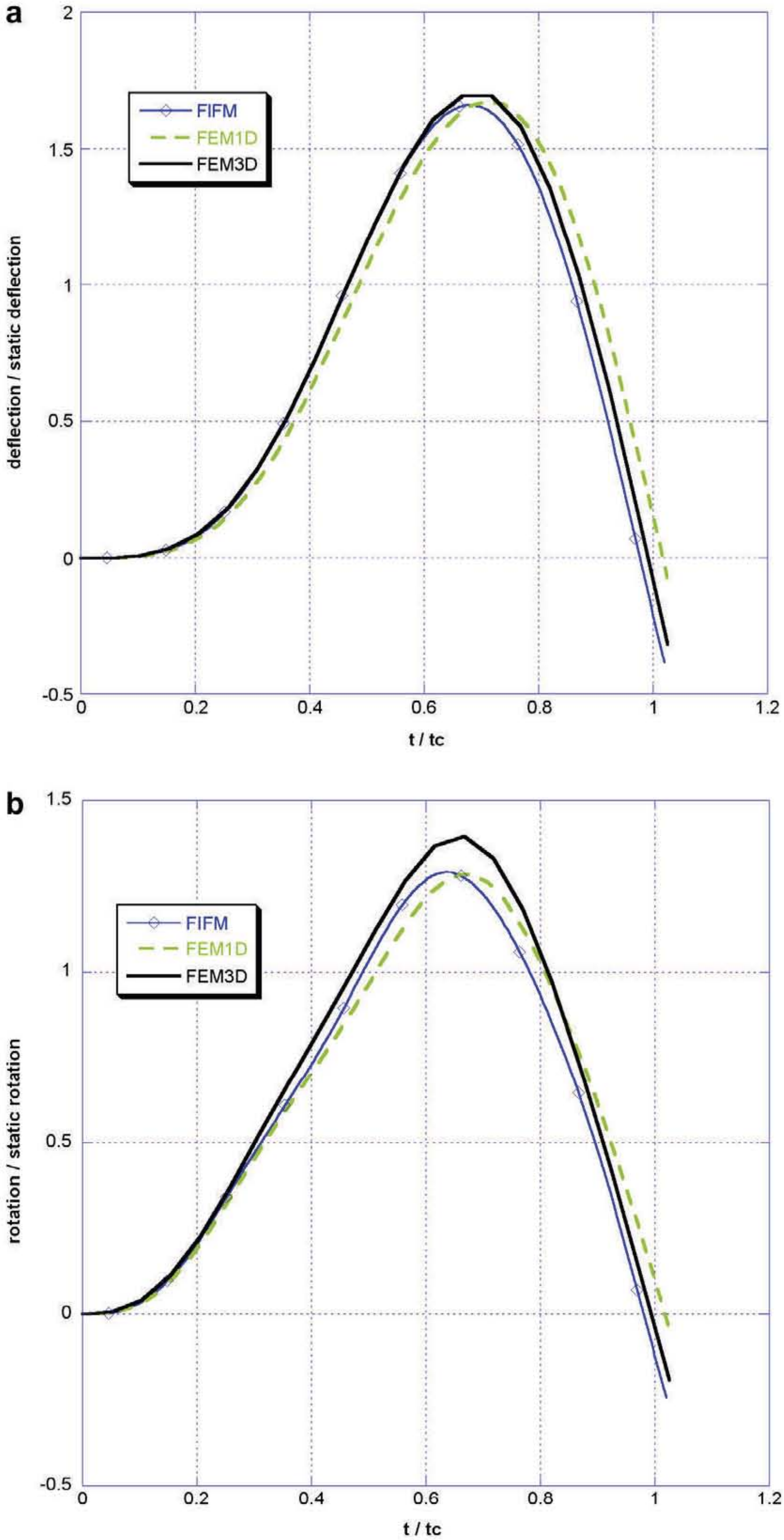

Fig. 8. Case C clamped hinged beam. (a) Mid span dimensionless deflection. (b) Mid span dimensionless rotation.

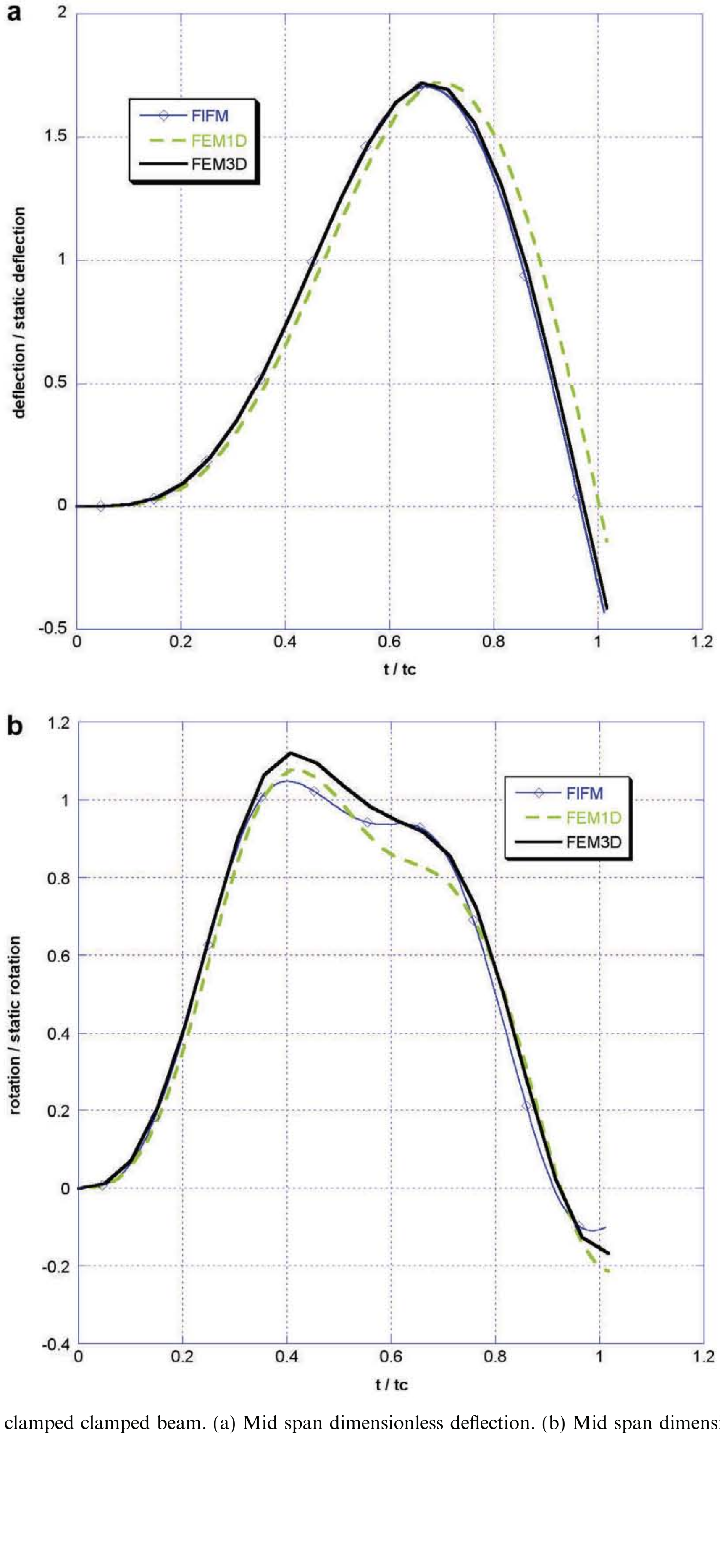

Fig. 9. Case D clamped clamped beam. (a) Mid span dimensionless deflection. (b) Mid span dimensionless rotation.

Table 2
Differences in the maximum values of deflection and rotation

	MAM		FEM1D		FEM3D	
	Defl. (%)	Rot. (%)	Defl. (%)	Rot. (%)	Defl. (%)	Rot. (%)
Case A (Clamped free)	1.09	0.01	1.35	0.90	5.42	4.79
Case B (simply supported)	0.95		2.54		7.15	
Case C (Clamped hinged)			0.74	0.62	1.91	7.40
Case D (Clamped clamped)			0.96	1.97	0.96	6.62

The results of FIFM agreed very well with those of the finite element models, as shown in Figs. 6–9. Table 2 shows the differences between FIFM and the other methods in the maximum values of beam deflection and rotation obtained in the four cases.

A significant result was the minimum difference, around 1% at the most, between the results of the two analytical models in cases A and B. Both MAM as FIFM come near to the numerical models, especially to FEM1D. In these isostatic problems, the modal analysis is used habitually and FIFM has no significant advantage. On the contrary, in a nonsymmetric and hyperstatic problem, as cases C and D, modal analysis would become difficult to apply. It is necessary to solve a different algorithm for each boundary condition and load application point. The FIFM main advantage is that it needs only one algorithm for all boundary conditions, only the matrix which contains the influence functions, $C(x, \xi)$, Eq. (15), need be modified for each boundary condition.

The differences in the comparison of the deflection are below 8% in every case and they are above 3% only in the comparison between FIFM and FEM3D model in cases A and B. Note that in these cases the results are obtained at the load application point, so in the FEM3D model there is a local compression that explains this high difference.

In the comparison of the section rotation the differences are also below 8% in every case and above 2% only in the comparison between FIFM and FEM3D model. Analytical models (FIFM and MAM) and FEM1D give the deflection and the section rotation of each section based on the abscissa of the beam, with Kirchhoff hypotheses, so any differences between them are minimum. However, the FEM3D model does not assume any of the Kirchhoff hypotheses: straight lines normal to the xy-plane before deformation remain neither straight nor normal and their length can change. Actually, the beam section experiences a three-dimensional strain field and the section rotation is an artificial measure of the medium rotation of the section. For this reason the differences between the one-dimensional theories and the FEM3D model are greater. However, the differences between the FEM3D model and the FIFM deflection values were not affected and were lower than 2% when the measured point was far enough from the load application point.

5. Conclusions

This study examines the validity of the use of the flexibility influence function method (FIFM) to determine the global response of a laminated beam subjected to dynamic loads that originate bending moments. The deflection and the rotation of a single-span beam were calculated with different boundary conditions and they were compared with the results calculated by modal analysis and numerical methods, FEM1D and 3D.

The main conclusions obtained are the following:

– The differences between the maximum deflection and rotations calculated by modal analysis and FIFM, are not significant (about 1%). Therefore the use of the FIFM is preferable when symmetry and isostatism conditions are not involved, in which the modal analysis method is difficult to apply.
– The dynamic deflections and rotations of a composite laminated beam calculated by means of FIFM are similar to those obtained with FEM1D and FEM3D and the computational cost is lower.

– The comparison of the rotation also shows a very small difference between FIFM and FEM1D. A greater difference is found between FIFM and FEM3D, because the first-order shear deformation laminate theory does not consider the warp that takes place in the section of the beam and that is reproduced in the FEM3D.

Therefore, the flexibility influence function method is effective for the calculation of the deflections and rotations in a laminated beam of high anisotropy, as it evaluates the influence of parameters of the problem in the global response of the structure, with a lower computational cost than that of the numerical methods.

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