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PhD. Thesis

CONTROLLED MARKOV MODELS.
AN APPLICATION TO THE RUIN PROBLEM

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TESIS DOCTORAL

CONTROL DE MODELOS MARKOVIANOS. UNA APROXIMACIÓN AL PROBLEMA DE RUINA.

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To Diamaris and Paolo.

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Abstract

In this thesis the ruin probabilities in some controlled discrete-time risk processes with a Markov chain interest are studied. To reduce the risk of ruin there is a possibility to reinsure a part or the whole reserve. Recursive and integral equations for ruin probabilities are given. Generalized Lundberg inequalities for the ruin probabilities are derived given a constant stationary policy. The relationships between these inequalities are discussed. To illustrate these results some numerical examples are included.

It is shown that the problems considered can be imbedded in the framework of Markov decision problem but with some special features. We establish the dynamic programming algorithm in finite and infinite horizon cases for a general Markov Decision Process (MDP). Moreover, we provide conditions for the existences of measurable selectors.

Resumen

En este trabajo se estudia la probabilidad de ruina de algunos procesos de riesgo controlados en tiempo discreto que incluyan una cadena de Markov para las tasas de interés. Para reducir el riesgo de ruina existe la posibilidad de reasegurar parte o la totalidad del fondo de reservas. Se facilitan formulas recursivas y ecuaciones para calcular la probabilidad de ruina. Desigualdades generalizadas tipo Lundberg para la probabilidad de ruina son deducidas cuando consideramos una política estacionaria constante desde el inicio. Se analizan las relaciones entre las desigualdades halladas. Se incluyen algunos ejemplos numéricos para ilustrar estos resultados.

Se muestra que los problemas considerados pueden ser vistos en el marco de los problemas de Decisión Markovianos. Se establecen algoritmos de programación dinámica para un modelo de Decisión Markoviano general en los casos de horizonte finito e infinito. Además, se muestran

las condiciones necesarias para la existencia de selectores medibles.

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Maikol Diasparra

Introduction

The foundation of modern actuarial mathematics were laid only in 1903 by the Swedish mathematician Filip Lundberg [40, 41], and later in the 1930's by the famous Swedish probabilistic Harald Cramér [10, 11]. Insurance mathematics today is considered a part of applied probability theory, however, a major portion of it is described in term of continuous time stochastic processes.

At first view, the ruin probability is not a classical performance criterion for control problems. As is pointed out by Schäl [48] and Schmidli [54] one can write the ruin probability as some total cost without discounting where one has to pay one unit of cost when entering a ruin state. After this simple observation, the results from discrete-time dynamic programming apply. Nevertheless obtaining explicit optimal solutions is a difficult task in a general setting. An analytic method commonly used in ruin theory is to derive inequalities for ruin probabilities (see Asmussen [2], Grandell [32], Willmot, et al. [60] and Willmot and Lin [61, 62]).

Our aim is to choose the reinsurance control strategies in order to minimize the ruin probability of a controlled risk process in discrete-time. We assume statistical dependency over time for the interest rate process and following a realistic point of view as is suggested in Cai [6] and in Cai and Dickson [7].

First, for this purpose we develop generalized Lundberg inequalities for the ruin probability that depend on the decision or control strategy. Previously we derive recursive and integral equations for the ruin probability. Secondly, optimality over the admissible control set can be achieved by the monotonic property of the upper bounds that we obtain*. These results are illustrated for the Phase-type distribution case. Especially, we show that if the distribution function of claims is of any particular class of distribution (in the sense of stochastic order), we can simplify the calculation of our bounds.

Finally, we also consider a surplus process in the usual formulation in Markov decision theory following González-Hernández, López-Martínez and Pérez-Hernández [30], and Hernández-Lerma and Lasserre [33, 34, 35]. Also, we use the Hinderer's results for canonical construction [36]. Particularly, we specified how to rewrite the minimization of the ruin probability as a MDP.

The outline of the thesis is as follows. In Chapter 1 we review the main results on the classical Cramér-Lundberg model.

In Chapter 2 we briefly review the most relevant issues of stochastic orders related with the ruin problem.

In Chapter 3 we give an introduction to discrete-time dynamic programming focused to minimization of the ruin probability.

In Chapter 4 we study an insurance model where the risk process can be controlled by proportional reinsurance. The performance criterion is to choose reinsurance control strategies to bound the ruin probability of a discrete-time process with a Markov chain interest. To illustrate our results we present some numerical examples that use Matlab and Maple implementations.

In Chapter 5 we study a general Markov decision problem. Particularly, we specified how to rewrite the minimization of the ruin probability as a MDP by applying the previous results.

Finally, we present our concluding remarks.

*A major part of these results was published and there are available, see [13, 14].

Chapter 1

Risk theory

In this chapter we review the main results on the classical Cramér-Lundberg model. Most of the results can also be found in [46].

The reader should be aware that the model has to be considered as a *technical tool* only. It is used to *measure* the effect of a certain decision of the actuary on the risk. In this model the present environment of the insurer is fixed and cannot be changed in the future. Of course, in reality the environment does change.

The time t in the model has to be considered as *operational time*. On the one hand, the insured's exposure to risk is not constant over time. On the other hand, the number of persons insured is not constant over time either.

The ruin probabilities defined here are therefore not the probability that the company is ruined, even though for some claim size distribution this could be the case. Ruin means that the capital set aside for the risk considered was not enough. The ruin probability is then a measure for the risk. Ruin theory gives the actuary a tool to measure the risk in a simple way. The goal is therefore not to have the realistic model but a simple model that is able to characterise the risk connected to the business.

1.1 Ruin probabilities for the classical risk model

A sound mathematical basis for the stochastic modelling of insurance risk goes to the pioneering work by Filip Lundberg [40, 41] and Harald Cramér [10, 11].

Figure 1.1: Pioneers. Left panel: Ernest Filip Oskar Lundberg (Sweden, 1876 – 1965). Right panel: Carl Harald Cramér (Sweden, 1893 – 1985).

Their *collective risk model* was obtained as a limit of a sum of individual risk models for an increasing number of individual contracts. It turns out that many of the basic constructions like adjustment coefficient, expense loading, premium structure, etc. Needed in more general models are already present in the early model. Despite the obvious lack of reality of many in the assumptions made, one uses the *Cramér-Lundberg model* as a skeleton for many recently developed "more realistic generalisations".

In a classical risk model the surplus of a collective contract or a large portfolio is modelled as

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i \tag{1.1.1}$$

where N is a Poisson process with rate λ, the $\{Y_i\}$ are i.i.d., (strictly) positive, and independent of N, $c > 0$ is the premium rate, and x is the initial capital. We denote the claim occurrence times by $T_1 < T_2 < \cdots$, and for convenience we let $T_0 = 0$. Because the process has stationary and independent increments, the process is always in its stationary state. It does not matter whether or not there was a claim at time zero. We denote the distribution of Y_i by $F(y)$, and its moments by $\mu_n = E\left[Y_i^n\right]$. For simplicity we let $\mu = \mu_1$. This is the standard model of an insurance company, where N_t is to be interpreted as the number of claims on the company during the interval $(0, t]$. At each point of N the company has to pay out a stochastic amount of money, and the company receives (deterministically) c units of money per unit time.

For mathematical purposes, it is frequently more convenient to work with the claim surplus process $\{S_t\}_{t\geq 0}$ defined by $S_t = x - X_t$. Letting $M = \sup_{0\leq t<\infty} S_t$ and $M_T = \sup_{0\leq t\leq T} S_t$.

The main object of interest in risk theory is the ruin probability. Let $\tau(x) = \inf\{t \geq 0 : X_t < 0\}$ $= \inf\{t : S_t > x\}$ be the *time of ruin*. As usual, we let $\inf \emptyset = \infty$. The *probability of ultimate ruin* is the probability that the risk process ever drop below zero or equivalently is the probability that ruin occurs in finite time

$$\psi(x) = P\left(\tau(x) < \infty\right) = P\left(\inf_{t\geq 0} X_t < 0 | X_0 = x\right) = P\left(M > x\right). \tag{1.1.2}$$

The probability of ruin before time T is

$$\psi(x, T) = P\left(\tau(x) \leq T\right) = P\left(\inf_{0\leq t\leq T} X_t < 0 | X_0 = x\right) = P\left(M_T > x\right). \tag{1.1.3}$$

From the theory of random walks one knows that $\tau(x) < \infty$ (a.s.) if and only if $E\left[c\left(T_i - T_{i-1}\right) - Y_i\right] \leq 0$. That is, $\psi(x) = 1$ for all x if $c > \lambda\mu$. One therefore usually assumes the *net profit condition* $c > \lambda\mu$. Note that $E\left[X_t - x\right] = (c - \lambda\mu)t$, which explain the name "net profit condition". Another quantity of interest is the *relative safety loading* $\rho = \frac{c}{\lambda\mu} - 1$. The ruin probability is absolutely continuous and differentiable at all points y where $F(y)$ is

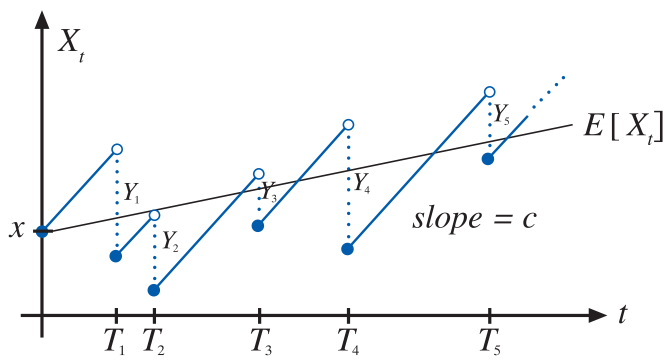

Figure 1.2: Illustration of Risk Process notation.

continuous. The density $\psi'(x)$ fulfils

Proposition 1.1.1. *Let $\Phi(x) = 1 - \psi(x)$ the non-ruin probability. Then*

1. $\Phi'(x) = \frac{\lambda}{c}\left[\Phi(x) - \int_0^x \Phi(x-y)dF(y)\right]$.
2. $\psi(0) = \frac{\lambda\mu}{c}$.
3. $\psi(x) = \frac{\lambda}{c}\left[\int_x^\infty (1-F(y))\,dy + \int_0^x \psi(x-y)(1-F(y))\,dy\right]$.
4. $c\psi'(x) + \lambda\left[\int_0^x \psi(x-y)dF(y) + 1 - F(x) - \psi(x)\right] = 0$.

See, Section A.3 for more details about the proof of this proposition.

1.1.1 Special case: Exponentially distributed claims

Consider the simple case, when Y_i is exponentially distributed. Then equation 1. in Proposition 1.1.1 is reduced to

$$\begin{aligned}\Phi'(x) &= \frac{\lambda}{c}\left[\Phi(x) - \frac{1}{\mu}\int_0^u \Phi(x-y)e^{-y/\mu}dy\right] \\ &= \frac{\lambda}{c}\left[\Phi(x) - \frac{1}{\mu}\int_0^u \Phi(y)e^{-(x-y)/\mu}dy\right].\end{aligned}$$

Differentiation lead to

$$\begin{aligned}\Phi"(x) &= \frac{\lambda}{c}\Phi'(x) + \frac{1}{\mu}\left[\frac{\lambda}{c}\Phi(x) - \Phi'(x)\right] - \frac{\lambda}{c\mu}\Phi(x) \\ &= \left[\frac{\lambda}{c} - \frac{1}{\mu}\right]\Phi'(x) = -\frac{\rho}{\mu(1+\rho)}\Phi'(x)\end{aligned}$$

and thus

$$\Phi(x) = C_1 - C_2 e^{-\frac{\rho x}{\mu(1+\rho)}}.$$

For $\rho > 0$ we have $\lim\limits_{x\to\infty}\Phi(x) = 1$ and $\Phi(0) = 1 - \frac{1}{1+\rho}$, which implies

$$\Phi(x) = 1 - \frac{1}{1+\rho}e^{-\frac{\rho x}{\mu(1+\rho)}}.$$

or

$$\psi(x) = \frac{1}{1+\rho}e^{-\frac{\rho x}{\mu(1+\rho)}} = \frac{\lambda\mu}{c}e^{-x\left(\frac{1}{\mu}-\frac{\lambda}{c}\right)}. \tag{1.1.4}$$

1.1.2 Lundberg exponent and Cramér-Lundberg approximation

Let $\Theta(r) = \int_0^\infty e^{ry} dF(y) - 1$ with $r \in \mathbb{R}^+$. We assume that there exists $r_* > 0$ such that $\Theta(r) \uparrow +\infty$ when $r \uparrow r_*$ (we allow the possibility $r_* = \infty$). It is easily seen that $\Theta(0) = 0$ and that Θ is increasing, convex and continuous on $[0, r_*)$. The important part of this assumption is that $\Theta(r) < \infty$ for some $r > 0$. This means that the tail of dF decreases at least exponentially fast*. Further, the rather pathological case when $\Theta(r_*^-) < \infty$ and $\Theta(r) = \infty$ for $r > r_*$ is excluded.

Since $\int_0^\infty \frac{\lambda}{c} [1 - F(y)] \, dy = \frac{\lambda\mu}{c} < 1$ the equation (A.3.4) is a defective renewal equation. Following Feller [23] p. 376, we assume that there exists a constant R such that

$$\frac{\lambda}{c} \int_0^\infty e^{Ry} [1 - F(y)] \, dy = 1 \tag{1.1.5}$$

or equivalently,

$$\int_0^\infty e^{Ry} dF_1(y) = 1 + \rho.^\dagger \tag{1.1.6}$$

Then $\frac{\lambda}{c} e^{Ry} [1 - F(y)]$ is the density of a proper probability distribution. Multiplication of (A.3.4) by e^{Rx} yields

$$e^{Rx}\psi(x) = \frac{\lambda}{c} e^{Rx} \int_x^\infty (1 - F(y)) \, dy + \frac{\lambda}{c} \int_0^x e^{R(x-y)} \psi(x - y) e^{Ry} (1 - F(y)) \, dy. \tag{1.1.7}$$

which is a proper renewal equation. From the renewal theorem‡, it then follows that

$$\lim_{x \to \infty} e^{Rx}\psi(x) = \frac{C_1}{C_2}, \tag{1.1.8}$$

where

$$C_1 = \frac{\lambda}{c} \int_0^\infty e^{Rx} \int_x^\infty (1 - F(y)) \, dy dx \tag{1.1.9}$$

*For example the Lognormal and the Pareto distributions are not allowed.
†Where $F_1(y)$ is the equilibrium distribution of F, see equation (2.1.4).
‡See Feller [23] p. 363.

and

$$C_2 = \frac{\lambda}{c} \int_0^{\infty} y e^{Ry} \left(1 - F(y)\right) dy \tag{1.1.10}$$

provided R, C_1 and C_2 exist in $(0, \infty)$. We get from 1.1.5,

$$\frac{c}{\lambda} = \int_0^{\infty} e^{Ry} \left[1 - F(y)\right] dy = \frac{1}{R} \left[\int_0^{\infty} e^{Ry} dF(y) - 1\right] = \frac{\Theta(R)}{R}$$

and thus R is the positive solution of

$$\Theta(r) = \frac{cr}{\lambda}, \tag{1.1.11}$$

where R is called the *Lundberg exponent*.

Moreover, we have,

$$C_1 = \frac{1}{R} \frac{\rho}{1 + \rho}$$

and

$$C_2 = \frac{1}{\mu R(1 + \rho)} \left(\Theta'(R) - \frac{c}{\lambda}\right).$$

Thus, we obtain

$$\lim_{x \to \infty} e^{Rx} \psi(x) = \frac{\rho \mu}{\Theta'(R) - \frac{c}{\lambda}}, \tag{1.1.12}$$

which is called the *Cramér-Lundberg approximation*[§].

We shall now consider a completely different approach, due to Gerber [26], which uses martingales. Before considering his approach, we shall need some basic facts about martingales.

1.2 Basic Martingale theory

The definitions and results to be given here are standard issues on martingales theory, see for instance, Elliott [18].

We shall, for future purposes, be somewhat more general than is really needed for the moment. Therefore the probability space $(\Omega, \mathcal{F}, P)$ may carry more objects than the risk process.

[§]For more details, see [32] p. 7.

Definition 1.2.1. A *filtration* $\mathbf{F} = \{\mathcal{F}_n\}_{n\geq 0}$ is a non-decreasing family of sub-σ-algebras of $\mathcal{F}$.

Definition 1.2.2. Let for any process $Y = \{Y_n\}_{n\geq 0}$, the filtration $\mathbf{F}^Y = \left\{\mathcal{F}_n^Y\right\}_{n\geq 0}$ be defined by

$$\mathcal{F}_n^Y = \sigma\left\{Y_s : 0 \leq s \leq n\right\}.$$

Thus $\mathcal{F}_n^Y$ is the σ-algebra generated by Y up to time n, and represents the *history* of Y up to time n. Y is *adapted* to $\mathbf{F}$, i.e., Y is $\mathcal{F}_n$-measurable for all $n \geq 0$, if and only if $\mathcal{F}_n^Y \subseteq \mathcal{F}_n$ for all $n \geq 0$.

Definition 1.2.3. An $\mathbf{F}$-martingale (respectively $\mathbf{F}$-submartingale, $\mathbf{F}$-supermartingale)

$$M = \{M_n\}_{n\geq 0}$$

is a real valued process such that:

1. M_n is $\mathcal{F}_n$-measurable for $n \geq 0$.
2. $E\left[|M_n|\right] < \infty$ for $n \geq 0$.
3. $E^{\mathcal{F}_s}\left[M_n\right] = E\left[M_n|\mathcal{F}_s\right] = (\geq {}^{\P}, \leq {}^{\clubsuit})\ M_s$ P- a.s. for $n \geq s$.

Definition 1.2.4. An $\mathbf{F}$-martingale or an $\mathbf{F}$-supermartingale M is called *right continuous* if

1. the trajectories M_n are right continuous;
2. the filtration $\mathbf{F}$ is right continuous, i.e.,

$$\mathcal{F}_n = \bigcap_{s>n} \mathcal{F}_s \text{ for } n \geq 0.$$

Definition 1.2.5. A random variable $T : \Omega \to [0, \infty]$, is an $\mathbf{F}$-stopping time if $\{T \leq n\} \in \mathcal{F}_n$ for each $n \geq 0$.

This means that, knowing the history up to time n, one can decide if $T \leq n$ or not. Note that outcome $T = \infty$ is allowed. If T is a stopping time, so is $n \wedge T = \min\{n, T\}$ for each n. The following simplified version of the **Optional Stopping Theorem** is essential for our applications.

Theorem 1.2.1. *Let T be a bounded stopping time, i.e., $T \leq t_0 < \infty$, and M a right continuous **F**-martingale (**F**-supermartingale). Then*

$$E^{\mathcal{F}_0}\left[M_T\right] = (\leq {}^{\clubsuit})\ M_0 \quad P - a.s. \tag{1.2.1}$$

¶Respectively **F**-submartingale.
♣Respectively **F**-supermartingale.

Now we consider the *martingale approach.*

Theorem 1.2.2. *Let Y_n be right continuous process such that:*

1. $Y_0 = 0$ *P-a.s..*
2. Y *has stationary and independent increments.*
3. $E[Y_n] = \beta \cdot n$*, where* $\beta > 0$.
4. $E[exp(-rY_n)] < \infty$ *for some* $r > 0$.

Then

$$E[exp(-rY_n)] = e^{n \cdot g(r)} \text{ for some function } g(\cdot).$$

Remark 1.2.1. If Y is a classical risk process with positive safety loading we have $\beta = c - \lambda\mu$. Further, we have

$$\begin{aligned} E[exp(-rY_t)] &= e^{-rct} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} (\Theta(r) + 1)^k \\ &= e^{-rct + \lambda t(\Theta(r)+1) - \lambda t} = e^{t(\lambda\Theta(r) - rc)} \end{aligned}$$

and thus $g(r) = \lambda\Theta(r) - rc$.

1.2.1 Lundberg inequality

In this subsection we will obtain the called Cramér-Lundberg inequality for ruin in infinite time.

Let T_x be the time of ruin, i.e.,

$$T_x = \inf\{n \geq 0 : x + Y_n < 0\}.$$

Obviously T_x is a $\mathbf{F}^Y$-stopping time and note that $\psi(x) = P(T_x < \infty)$. Put

$$M_n^x = \frac{e^{-r(x+Y_n)}}{e^{ng(r)}}$$

M^x is an $\mathbf{F}^Y$-martingale, since

$$\begin{aligned} E^{\mathcal{F}_s^Y}\left[M_n^x\right] &= E^{\mathcal{F}_s^Y}\left[\frac{e^{-r(x+Y_n)}}{e^{ng(r)}}\right] \\ &= E^{\mathcal{F}_s^Y}\left[\frac{e^{-r(x+Y_s)}}{e^{sg(r)}}\cdot\frac{e^{-r(Y_n-Y_s)}}{e^{(n-s)g(r)}}\right] \\ &= M_s^x\cdot E^{\mathcal{F}_s^Y}\left[\frac{e^{-r(Y_n-Y_s)}}{e^{(n-s)g(r)}}\right] \\ &= M_s^x. \end{aligned}$$

Choose $t_0 < \infty$ and consider $t_0 \wedge T_x$ which is a bounded $\mathbf{F}^Y$-stopping time. Since $\mathcal{F}_0^Y$ is trivial and since M^x is positive, it follows from Theorem 1.2.1 that

$$\begin{aligned} e^{-rx} = M_0^x &= E\left[M_{t_0\wedge T_x}^x\right] \\ &= E\left[M_{t_0\wedge T_x}^x|T_x \le t_0\right]P\left(T_x\le t_0\right) + E\left[M_{t_0\wedge T_x}^x|T_x > t_0\right]P\left(T_x > t_0\right) \\ &\ge E\left[M_{t_0\wedge T_x}^x|T_x \le t_0\right]P\left(T_x\le t_0\right) \\ &= E\left[M_{T_x}^x|T_x \le t_0\right]P\left(T_x\le t_0\right) \end{aligned}$$

and thus, since $x + Y_{T_x} \le 0$ on $\{T_x < \infty\}$,

$$\begin{aligned} P\left(T_x \le t_0\right) &\le \frac{e^{-rx}}{E\left[M_{T_x}^x|T_x\le t_0\right]} \le \frac{e^{-rx}}{E\left[e^{-T_x g(r)}|T_x\le t_0\right]} \\ &\le e^{-rx}\sup_{t\ge 0} e^{tg(r)}. \end{aligned}$$

In order to get this inequality as good as possible, we shall choose r as large as possible under the restriction $\sup_{t\ge0} e^{tg(r)} < \infty$. Let R denote that value. Obviously this means that

$$R = \sup\left\{r : g(r) \le 0\right\}.$$

In the classical risk process case this gives R as the positive solution of $\Theta(r) = cr/\lambda$, i.e., R is the Lundberg exponent. Thus we have

$$\psi(x) \le e^{-Rx} \tag{1.2.2}$$

which is called the **Lundberg inequality**. Comparing with (1.1.12) and (1.2.2) it is seen that R actually is the best possible exponent.

1.2.2 Cramér-Lundberg asymptotics

Under the condition (1.1.6), the Cramér-Lundberg asymptotic formula states that if

$$\int_0^\infty ye^{Ry}dF_1(y) < \infty,$$

then

$$\psi(x) \sim \frac{\rho\mu}{R\int_0^\infty ye^{Ry}\overline{F}(y)dy}e^{-Rx} \text{ as } x \to \infty. \tag{1.2.3}$$

If

$$\int_0^\infty ye^{Ry}dF_1(y) = \infty, \tag{1.2.4}$$

then

$$\psi(x) = o(e^{-Rx}) \text{ as } x \to \infty. \tag{1.2.5}$$

and meanwhile, the Lundberg inequality states that

$$\psi(x) \le e^{-Rx}, x \ge 0, \tag{1.2.6}$$

where $C_1(x) \sim C_2(x)$ as $x \to \infty$ means $\lim\limits_{x\to\infty} C_1(x)/C_2(x) = 1$.

The asymptotic formula (1.2.3) provides an exponential asymptotic estimate for the ruin probability as $x \to \infty$, while the Lundberg inequality (1.2.6) gives an exponential upper bound for the ruin probability for all $x \ge 0$. These two results constitute the well-known Cramér-Lundberg approximations for the ruin probability in the classical risk model.

When the claim sizes are exponentially distributed, that is, $\overline{F}(y) = e^{-y/\mu}$, $y \ge 0$, the ruin probability has an explicit expression given by (1.1.4).

Thus, the Cramér-Lundberg asymptotic formula is exact when the claim sizes are exponentially distributed. Further, the Lundberg upper bound can be improved so that the improved Lundberg upper bound is also exact when the claim sizes are exponentially distributed. Indeed, it can be proved under the Cramér-Lundberg condition (e.g. [8, 37, 39, 62]) that

$$\psi(x) \le \beta e^{-Rx}, x \ge 0, \tag{1.2.7}$$

where β is a constant, given by

$$\beta^{-1} = \inf_{0 \le t < \infty} \frac{\int_t^\infty e^{Ry} dF_1(y)}{e^{Rt}\overline{F}_1(t)}, \tag{1.2.8}$$

where $0 < \beta \le 1$.

This improved Lundberg upper bound (1.2.7) equals the ruin probability when the claim sizes are exponentially distributed. In fact, the constant β in (1.2.7) has an explicit expression of $\beta = 1/(1+\rho)$ if the distribution F has a **decreasing failure rate** (DFR)$^\diamond$.

The Cramér-Lundberg approximations provide an exponential description of the ruin probability in the classical risk model. They have become two standard results on ruin probabilities in risk theory.

The original proofs of the Cramér-Lundberg approximations were based on **Wiener-Hopf** methods and can be found in Cramér [10, 11] and Lundberg [40, 41]. However, these two results can be proved in different ways now. For example, the martingale approach of Gerber [26, 27], Wald's identity in [47], and the induction method in [31] have been used to prove the Lundberg inequality. Further, since the integral equation (A.3.4) can be rewritten as the following **defective renewal equation**

$$\psi(x) = \frac{1}{1+\rho}\left(\overline{F}_1(x) + \int_0^x \psi(x-y) dF_1(y)\right), \quad x \ge 0, \tag{1.2.9}$$

the Cramér-Lundberg asymptotic formula can be obtained simply from the key renewal theorem for the solution of a defective renewal equation, see, for instance [23]. All these methods are much simpler than the Wiener-Hopf methods used by Cramér and Lundberg and have been used extensively in risk theory and other disciplines. In particular, the martingale approach is a powerful tool for deriving exponential inequalities for ruin probabilities. See, for example [12], for a review on this topic. In addition, the induction method is very effective for one to improve and generalize the Lundberg inequality. The applications of the method for the generalizations and improvements of the Lundberg inequality can be found in [9, 58, 59, 61, 62].

Further, the key renewal theorem has become a standard method for deriving exponential

$^\diamond$See, Section 2.1 or [62] for more details.

asymptotic formulae for ruin probabilities and related ruin quantities, such as the distributions of the surplus just before ruin, the deficit at ruin, and the amount of claim causing ruin; see, for example [29, 62].

Moreover, the Cramér-Lundberg asymptotic formula is also available for the solution to defective renewal equation, see, for example [25, 53] for details. Also, a generalized Lundberg inequality for the solution to defective renewal equation can be found in [60].

On the other hand, the solution to the defective renewal equation (1.2.9) can be expressed as the tail of a compound geometric distribution, namely,

$$\psi(x) = \frac{\rho}{1+\rho}\sum_{n=1}^{\infty}\left(\frac{1}{1+\rho}\right)^{n}\overline{F_1^{(n)}}(x), \qquad x \geq 0, \tag{1.2.10}$$

where $F_1^{(n)}(x)$ is the n-fold convolution of the distribution function (df) $F_1(x)$. This expression is known as **Beekman's convolution series**.

Thus, the ruin probability in the classical risk model can be characterized as the tail of a compound geometric distribution. Indeed, the Cramér-Lundberg asymptotic formula and the Lundberg inequality can be stated generally for the tail of a compound geometric distribution. The tail of a compound geometric distribution is a very useful probability model arising in many applied probability fields such as risk theory, queueing, and reliability. More applications of a compound geometric distribution in risk theory can be found in [38, 62], among others.

It is clear that the Cramér-Lundberg condition plays a critical role in the Cramér-Lundberg approximations. However, there are many interesting claim size distributions that do not satisfy the Cramér-Lundberg condition. For example, when the moment generating function of a distribution does not exist or a distribution is **heavy-tailed** such as **Pareto** and **lognormal distributions**, the Cramér-Lundberg condition is not valid. Further, even if the **moment generating function** of a distribution exists, the Cramér-Lundberg condition may still fail. In fact, there exist some claim size distributions, including certain **inverse Gaussian** and generalized inverse Gaussian distributions, so that for any $r > 0$ with $\int_0^\infty e^{rx}dF_1(x) < \infty$,

$$\int_0^\infty e^{rx}dF_1(x) < 1+\rho.$$

Such distributions are said to be medium tailed; see, for example, [19] for details.

For these medium- and heavy-tailed claim size distributions, the Cramér-Lundberg approximations are not applicable. Indeed, the asymptotic behavior of the ruin probability in these cases are totally different from those when the Cramér-Lundberg condition holds. For instance, if F is a **subexponential distribution**, which means

$$\lim_{x\to\infty} \frac{\overline{F^{(2)}}(x)}{\overline{F}(x)} = 2, \tag{1.2.11}$$

where $F^{(2)}$ means the two-fold convolution, i.e.,

$$F^{(2)}(x) = \int_{-\infty}^{\infty} F\,(x-y)\,dF(y),$$

then the ruin probability $\psi(x)$ has the following asymptotic form

$$\psi(x) \sim \frac{1}{\rho}\overline{F}_1(x) \text{ as } x \to \infty, \tag{1.2.12}$$

which implies that ruin is asymptotically determined by a large claim. A review of the asymptotic behavior of the ruin probability with medium- and heavy-tailed claim size distributions can be found in [21, 22].

However, the Cramér-Lundberg condition can be generalized so that a generalized Lundberg inequality holds for more general **claim size distributions**. In doing so, we recall from the theory of **stochastic orderings** that a distribution B supported on $[0,\infty)$ is said to be **new worse than used** (NWU) if for any $x \geq 0$ and $y \geq 0$,

$$\overline{B}(x+y) \geq \overline{B}(x)\overline{B}(y). \tag{1.2.13}$$

In particular, an exponential distribution is an example of an NWU distribution when the equality holds in (1.2.13).

Willmot [58] used an NWU distribution function to replace the exponential function in the Lundberg equation (1.1.6) and assumed that there exists an NWU distribution B so that

$$\int_0^{\infty} (\overline{B}(x))^{-1} dF_1(x) = 1+\rho. \tag{1.2.14}$$

Under the condition (1.2.14), Willmot [58] derived a generalized Lundberg upper bound for the ruin probability, which states that

$$\psi(x) \leq \overline{B}(x). \tag{1.2.15}$$

The condition (1.2.14) can be satisfied by some medium and heavy-tailed claim size distributions. See, [8, 58, 59, 62] for more discussions on this aspect. However, the condition (1.2.14) still fails for some claim size distributions; see, for example, [8] for the explanation of this case.

Dickson [16] adopted a truncated Lundberg condition and assumed that for any $x > 0$ there exists a constant $r_x > 0$ so that

$$\int_0^x e^{r_x y} dF_1(y) = 1 + \rho. \tag{1.2.16}$$

Under the truncated condition (1.2.16), Dickson [16] derived an upper bound for the ruin probability, and further Cai and Garrido [9] gave an improved upper bound and a lower bound for the ruin probability, which state that

$$\frac{\rho e^{-2r_x x} + \overline{F}_1(x)}{\rho \overline{F}_1(x)} \leq \psi(x) \leq \frac{\rho e^{-r_x x} + \overline{F}_1(x)}{\rho \overline{F}_1(x)}, \quad x \geq 0. \tag{1.2.17}$$

The truncated condition (1.2.16) applies to any positive claim size distribution with a finite mean. In addition, even when the Cramér-Lundberg condition holds, the upper bound in (1.2.17) may be tighter than the Lundberg upper bound; see [9] for details.

1.2.3 More general risk models

The Cramér-Lundberg approximations are also available for ruin probabilities in some more general risk models. For instance, if the claim number process N_t in the classical risk model is assumed to be a **renewal process**, the resulting risk model is called the compound renewal risk model or the **Sparre Andersen risk model**. In this risk model, interclaim times $\{T_1, T_2, \ldots\}$ form a sequence of independent and identically distributed positive random variables with common distribution function $G(t)$ and common mean $\int_0^\infty \overline{G}(t)dt = (1/\lambda) > 0$. The ruin probability in the Sparre Andersen risk model, denoted by $\psi^0(x)$, satisfies the same **defective renewal equation** as (1.2.9) for $\psi(x)$ and is thus the tail of a compound geometric distribution. However, the underlying distribution in the defective renewal equation in this case is unknown in general; see, for example, [20, 32] for details.

Suppose that there exists a constant r^0 so that

$$E\left[e^{r^0(Y_1-cT_1)}\right] = 1. \tag{1.2.18}$$

Thus, under the condition (1.2.18), by the key renewal theorem, we have

$$\psi^0(x) \sim C_0 e^{-r_x x} \text{ as } x \to \infty; \tag{1.2.19}$$

where $C_0 > 0$ is a constant. Unfortunately, the constant $C_0 > 0$ is unknown since it depends on the unknown underlying distribution. However, the Lundberg inequality holds for the ruin probability $\psi^0(x)$, which states that

$$\psi^0(x) \leq e^{-r_x x}, x \to \infty; \tag{1.2.20}$$

see, for example, [32] for the proofs of these results.

Further, if the claim number process N_t in the classical risk model is assumed to be a **stationary renewal process**, the resulting risk model is called the compound stationary renewal risk model. In this risk model, interclaim times $\{T_1, T_2, \ldots\}$ form a sequence of independent positive random variables; $\{T_2, T_3, \ldots\}$ have a common distribution function $G(t)$ as that in the compound renewal risk model; and T_1 has an equilibrium distribution function of $G_e(t) = \lambda \int_0^t \overline{G}(s)ds$. The ruin probability in this risk model, denoted by $\psi^e(x)$, can be expressed as the function of $\psi^0(x)$, namely

$$\psi^e(x) = \frac{\lambda\mu}{c}\left(\overline{F_1}(x) + \int_0^x \psi^0(x-y)dF_1(y)\right), \tag{1.2.21}$$

which follows from conditioning on the size and time of the first claim *.

Thus, applying (1.2.19) and (1.2.20) to (1.2.21), we have

$$\psi^e(x) \sim C_e e^{-r^0 x} \text{ as } x \to \infty, \tag{1.2.22}$$

*See, for example, (40) on page 69 of [32].

and

$$\psi^e(x) \le \frac{\lambda}{cr^0}\left(M_Y(r^0) - 1\right) e^{-r^0 x}, \qquad x \ge 0, \tag{1.2.23}$$

where $C_e = \frac{\lambda}{cr^0}\left(M_Y(r^0) - 1\right) C_0$ and $M_Y(t)$ is the **moment generating function** of the claim size distribution F. Like the case in the Sparre Andersen risk model, the constant C_e in the asymptotic formula (1.2.22) is also unknown. Further, the constant $\frac{\lambda}{cr^0}\left(M_Y(r^0) - 1\right)$ in the Lundberg upper bound (1.2.23) may be greater than one.

The Cramér-Lundberg approximations to the ruin probability in a risk model when the claim number process is a **Cox process** can be found in [4, 32, 52]. For the Lundberg inequality for the ruin probability in the Poisson **shot noise** delayed-claims risk model, see [5]. Moreover, the Cramér-Lundberg approximations to ruin probabilities in **dependent risk** models can be found in [28, 42, 44].

In addition, the ruin probability in the perturbed compound Poisson risk model with diffusion also admits the Cramér-Lundberg approximations. In this risk model, the surplus process X_t satisfies

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i + W_t, \qquad t \ge 0, \tag{1.2.24}$$

where $\{W_t\}_{t\ge 0}$ is a **Wiener process**, independent of the Poisson process $\{N_t\}_{t\ge 0}$ and the claim sizes $\{Y_1, Y_2, \ldots\}$, with infinitesimal drift 0 and infinitesimal variance $2D > 0$.

Denote the ruin probability in the perturbed risk model by $\psi_p(x)$ and assume that there exists a constant $R_p > 0$ so

$$\lambda \int_0^\infty e^{R_p y} dF(y) + DR_p^2 = \lambda + cR_p. \tag{1.2.25}$$

Then Dufresne and Gerber [17] derived the following Cramér-Lundberg asymptotic formula

$$\psi^e(x) \sim C_p e^{-R_p x} \text{ as } x \to \infty, \tag{1.2.26}$$

and the following Lundberg upper bound

$$\psi^e(x) \le e^{-R_p x}, \qquad x \ge 0, \tag{1.2.27}$$

where $C_p > 0$ is a known constant. For the Cramér-Lundberg approximations to ruin probabilities in more general perturbed risk models, see [24, 51]. A review of perturbed risk models and the Cramér-Lundberg approximations to ruin probabilities in these models can be found in [50].

We point out that the Lundberg inequality is also available for ruin probabilities in risk models with interest. For example, Sundt and Teugels [57] derived the Lundberg upper bound for the ruin probability in the classical risk model with a constant force of interest; Cai and Dickson [7] gave exponential upper bounds for the ruin probability in the Sparre Andersen risk model with a constant force of interest; Yang [63] obtained exponential upper bounds for the ruin probability in a discrete time risk model with a constant rate of interest; and Cai [6] derived exponential upper bounds for ruin probabilities in generalized discrete time risk models with dependent rates of interest. A review of risk models with interest and investment and ruin probabilities in these models can be found in [43]. For more topics on the Cramér-Lundberg approximations to ruin probabilities, we refer to [2, 21, 27, 32, 46, 62], and references therein.

To sum up, the Cramér-Lundberg approximations provide an exponential asymptotic formula and an exponential upper bound for the ruin probability in the classical risk model or for the tail of a compound geometric distribution. These approximations are also available for ruin probabilities in other risk models and appear in many other applied probability models.

1.3 Premium calculation principles

In this section we study the rules how to fix an adequate price, called a premium, for a family of risks $\mathcal{R}$ to be insured. The investigation of such rules is an essential element of actuarial science. Clearly, premiums cannot be too low because this would result in unacceptably large losses for the insurer. On the other hand, premium cannot be too high either because of competition between insurers. A premium calculation principle is a rule that determines the premium as a functional, assigning a value $C(\mathcal{R}) \in \overline{\mathbb{R}}$ (we denote by $\overline{\mathbb{R}}$ the set $\mathbb{R} \cup \{\pm\infty\}$) to the risk distribution $F_{\mathcal{R}}$. Typically, the premium $C(\mathcal{R})$ dependes on certain characteristics of

$F_{\mathcal{R}}$. For easy application, a premium calculation principle should require as little as possible information on the distribution of the risk $\mathcal{R}$. For example the simplest premium principle is the (pure) *net premium principle* $C(\mathcal{R}) = E[\mathcal{R}]$. The difference $C(\mathcal{R}) - E[\mathcal{R}]$ is called the *safety loading*. Any reasonable premium C should consist of the net premium $E[\mathcal{R}]$ and some security loading, i.e., $C > E[\mathcal{R}]$. For completeness we review here some of the most popular premium calculation principles.

- Expected value principle: the premium is calculated by $C = (1+\theta)E[\mathcal{R}]$ for some safety loading $\theta > 0$.
- Variance principle: the premium is $C = E[\mathcal{R}] + \alpha Var[\mathcal{R}]$ for some $\alpha > 0$.
- Modified variance principle: the premium is $C = E[\mathcal{R}] + \alpha Var[\mathcal{R}]/E[\mathcal{R}]$ for some $\alpha > 0$.
- Standard deviation principle: the premium is $C = E[\mathcal{R}] + \alpha\sqrt{Var[\mathcal{R}]}$ for some $\alpha > 0$.
- Exponential principle: The premium is $C = \frac{1}{\alpha}\log E[exp(\alpha\mathcal{R})]$ for some $\alpha > 0$.
- Zero utility principle: let $u(x)$ be some strictly concave function. The zero utility premium is the unique solution to the equation $u(w) = E[u(w + C - \mathcal{R})]$. Here w is considered as the insurer's initial wealth. One then compares the utility of the initial wealth (no risk is taken over) with the expected utility of the wealth after the risk is taken over. The exponential premium principle is a special case with $u(x) = -exp(-\alpha x)$.
- Adjusted risk principle: denote by $F(x)$ the distribution function of the risk $\mathcal{R}$ and we assume that $\mathcal{R} \geq 0$. The premium is calculated as $C = \int_0^\infty (1 - F(x))^\theta\, dx$ for some $\theta \in (0,1)$. Note that $\theta = 1$ would give the net premium.

Unfortunately, the disadvantage of the premium calculation given before is that they are not monotone with respect to stochastic ordering.

1.4 Some aspects of reinsurance

If a risk $\mathcal{R}$ is too dangerous (for instance if $\mathcal{R}$ has large variance), the insurer may want to transfer part of the risk $\mathcal{R}$ to another insurer. This risk transfer from a first insurer to another insurance company is called *reinsurance*. The first insurer that transfers (part of) his risk is called a *cedant* (or cedent). The supervising authority will then ask for large investment from the shareholders[ℷ], i.e., a large initial capital. If several cedent constituted a pool, the portfolio[♠] would become large. And with it also the capital requirements would become smaller for each of the participating companies.

We will review here some basic reinsurance forms.

1.4.1 Reinsurance acting on individual claims

For each claim Y the part of the claim left to the insurer is $0 \leq h(Y) \leq Y$. The reinsurer pays $Y - h(Y)$. The function [**] $h(Y)$ is called the *self-insurance function* or *retention function*.

- Full reinsurance: The self-insurance function is $h(Y) = 0$, i.e., the reinsurer pays all the claim. This form is not used in practice. But it is a popular form of a contract between cedent and policyholder[††].

- Proportional reinsurance: The self-insurance function is $h(Y) = bY$ for a retention level $b \in (0, 1)$. The reinsurer pays $(1 - b)Y$.

- Excess of loss reinsurance: The self-insurance function is $h(Y) = \min\{Y, b\}$ for some deductible $b \in (0, \infty)$. The reinsurer pays $(Y - b)^+$.

- First risk deductible: The reinsurer pays $\min\{Y, b\}$ for some deductible $b \in (0, \infty)$. Thus, the self-insurance function is $h(Y) = (Y - b)^+$.

[ℷ] A mutual shareholder or stockholder is an individual or company (including a corporation) that legally owns one or more shares of stock in a joint stock company. A company's shareholders collectively own that company. Thus, the typical goal of such companies is to enhance shareholder value.

[♠] A collection of investments held by an institution or a private individual.

[**] Sometimes, we denote the retention function by $h(Y, b)$ where b represent the retention level.

[††] The owner of an insurance policy; usually, but not always, the insured.

- Proportional reinsurance a layer: The self-insurance function is $h(Y) = \min\{Y, a\} + (Y - b - \gamma)^+ + b \min\left\{(Y - a)^+, \gamma\right\}$ for some $a, \gamma > 0$ and $b \in (0, 1)$.

1.4.2 Reinsurance acting on the aggregate claim

For the aggregate sum of claims $\mathcal{S} = \sum_{i=1}^{N} Y_i$, the insurer pays the amount $h(\mathcal{S})$ with $0 \leq h(\mathcal{S}) \leq \mathcal{S}$. The reinsurer pays $\mathcal{S} - h(\mathcal{S})$.

- Proportional reinsurance: The self-insurance function is $h(\mathcal{S}) = b\mathcal{S}$ for a retention level $b \in (0, 1)$. This is the same as proportional reinsurance acting on individual claims.

- Stop-loss reinsurance: The self-insurance function is $h(\mathcal{S}) = \min\{\mathcal{S}, b\}$ for some deductible $b \in (0, \infty)$.

- First risk deductible: The self-insurance function is $h(\mathcal{S}) = (\mathcal{S} - b)^+$ for some deductible $b \in (0, \infty)$.

Chapter 2

Stochastic orders

In this chapter we briefly review the most relevant issues of stochastic orders related with the ruin problem. Stochastic orderings have found a wide field of application in probability, statistics, and statistical theory, see Shaked and Shanthikumar [49], Lin and Willmot [62], as comprehensive references. In probability theory, they are useful in deducing probability inequalities, comparing stochastic models, establishing bounds and inequalities in reliability and queueing theory, in statistics for example in hypothesis testing, simultaneous comparisons, multiple decision problems, and in economics in decisions under risk, particularly in multi-attribute utility theory. The stochastic orderings are associated with inequalities between expectations of functions with respect to the corresponding distributions or random variables.
Consider a positive random variable Y with distribution function (df) $F(y) = P(Y \leq y)$, $y \geq 0$. The random variable Y may represent the time-until-death of an individual, or in the present context the amount of the insurance loss. It is of importance to quantify and analyze the thickness of the right tail for valuation purposes. In order to do so we use some notions from theory of reliability.

Let X and Y be two random variables such that

$$P(X > x) \leq P(Y > x) \text{ for all } x \in \mathbb{R}. \tag{2.0.1}$$

Then X is said to be *smaller than* Y *in the usual stochastic order* (denoted by $X \leq_{st} Y$). Roughly speaking, (2.0.1) says that X is less likely than Y to take on large values, where *large* means any value greater than x, and that this is the case for all $x's$.

It is easy to verify (by noting that every closed interval is an infinite intersection of open intervals) that $X \leq_{st} Y$ if, and only if,

$$P(X \geq x) \leq P(Y \geq x) \text{ for all } x \in \mathbb{R}. \tag{2.0.2}$$

In fact, we can recast (2.0.1) and (2.0.2) in a seemingly more general, but actually an equivalent, way as follows *

$$P(X \in A) \leq P(Y \in A) \text{ for all upper sets } A \subset \mathbb{R}. \tag{2.0.3}$$

Another way of rewriting (2.0.3) is the following

$$E[1_A(X)] \leq E[1_A(Y)] \text{ for all upper sets } A \subseteq \mathbb{R}, \tag{2.0.4}$$

where 1_A denotes the indicator function of A. From (2.0.3) it follows that if $X \leq_{st} Y$ then

$$E\left[\sum_{i=1}^{m} a_i 1_{A_i}(X)\right] - b \leq E\left[\sum_{i=1}^{m} a_i 1_{A_i}(Y)\right] - b \text{ for all upper sets } A \subseteq \mathbb{R}, \tag{2.0.5}$$

for all $a_i \geq 0$, $i = 1, 2, \ldots, m$, $b \in \mathbb{R}$ and $m \geq 0$. Given an increasing function ϕ, it is posible, for each m, to define a sequence of $A_i's$, a sequence of $a_i's$, and a b (all of which may depend on m), such that as $m \to \infty$ then (2.0.5) converges to

$$E[\phi(X)] \leq E[\phi(Y)], \tag{2.0.6}$$

provided the expectations exist. It follows that $X \leq_{st} Y$ if, and only if, (2.0.6) holds for all increasing function ϕ for which the expectations exist.

An important characterization of the usual stochastic order is the following theorem (here $=_{st}$ denotes equality in law).

Theorem 2.0.1. *Two random variables X and Y satisfy $X \leq_{st} Y$ if, and only if, there exist two random variables $\widehat{X}$ and $\widehat{Y}$, defined on the same probability space, such that*

*In the univariate case, that is on the real line, a set A is an upper set if, and only if, it is an open or closed right half line.

$$\widehat{X} =_{st} X, \tag{2.0.7}$$
$$\widehat{Y} =_{st} Y, \tag{2.0.8}$$

and

$$P\left(\widehat{X} \leq \widehat{Y}\right) = 1. \tag{2.0.9}$$

Proof. Obviously (2.0.7), (2.0.8), and (2.0.9) imply that $X \leq_{st} Y$. In order to prove the necessity part of the Theorem 2.0.1, let F and G be, respectively, the distribution of X and Y and let F^{-1} and G^{-1} be the corresponding right continuous inverses, defined by $F^{-1}(z) = \sup\{x : F(x) \leq z\}$ and $G^{-1}(z) = \sup\{x : G(x) \leq z\}$, $z \in [0,1]$. Define $\widehat{X} = F^{-1}(V)$ and $\widehat{Y} = G^{-1}(V)$ where V is a uniform $[0,1]$ random variable. Then it is easy to see that $\widehat{X}$ and $\widehat{Y}$ satisfy (2.0.7) and (2.0.8). From (2.0.2) it is seen that (2.0.9) also holds. □

Clearly, if $X \leq_{st} Y$ then $E[X] \leq E[Y]$. However, as the following result shows, if two random variables are ordered in the usual stochastic order and have the same expected values, they must have the same distribution.

Theorem 2.0.2. 1. *If $X \leq_{st} Y$ and $E[X] = E[Y]$, then $X =_{st} Y$.*

2. *If $X \leq_{st} Y$ and if $E[h(X)] = E[h(Y)]$ for some strictly increasing function h, then $X =_{st} Y$.*

Proof. Let $\widehat{X}$ and $\widehat{Y}$ be as the Theorem 2.0.1. If $P\left(\widehat{X} < \widehat{Y}\right) > 0$ then $E[X] = E\left[\widehat{X}\right] < E\left[\widehat{Y}\right] = E[Y]$, a contradiction to the assumption in 1. Therefore $X =_{st} \widehat{X} =_{st} \widehat{Y} =_{st} Y$. The proof of 2 is similar: Observe that if $X \leq_{st} Y$ and h is as in 2 then $h(X) \leq_{st} h(Y)$ and therefore from Part 1 we have that $h(X) =_{st} h(Y)$. But the strict monotonicity of h now gives that $X =_{st} Y$. □

2.1 A property in reliability theory

If Y is a nonnegative random variable with an absolutely continuous distribution function F, then the hazard rate, failure rate, or force of mortality of Y at $y \geq 0$ are defined by

$$r(y) = \frac{d}{dy}\left(-\ln\left(\overline{F}(y)\right)\right) = \lim_{h \to 0} \frac{1 - P\left(Y > y + h | Y > y\right)}{h} = \frac{f(y)}{\overline{F}(y)} \tag{2.1.1}$$

where $\overline{F}(y) = 1 - F(y)$ is the survival function and $f(y) = \frac{d}{dy}(F(y))$ is the corresponding density function.

In many situations of practical interest, the failure rate $r(y)$ is strictly monotone nonincreasing (nondecreasing) in y, and this is associated with the situation where the distribution has thick (thin) right tail.

The distribution function $F(y)$ is said to be **decreasing failure rate** (DFR) if $\overline{F}(x+y)/\overline{F}(x)$ is nondecreasing in y for fixed $x \geq 0$, i.e. if $\overline{F}(y)$ is log-convex. It is evident from (2.1.1) that if $F(y)$ is absolutely continuous, then to have a decreasing failure rate is equivalent to $r(y)$ nonincreasing in y.

From (2.1.1),

$$\int_0^y r(x)dx = -\ln\left(\overline{F}(y)\right)$$

in other words

$$\overline{F}(y) = e^{\int_0^y r(x)dx}, y \geq 0,$$

and so $r(y)$ uniquely determines the distribution of Y.

Suppose now that the mean $E[Y]$ of Y exists, i.e. $E[Y] = \int_0^\infty ydF(y) < \infty$. Then integration by parts yields

$$\begin{aligned} \textstyle\int_0^\infty ydF(y) &= -y\overline{F}(y)|_0^\infty + \textstyle\int_0^\infty \overline{F}(y)dy \\ &= -\lim_{y\to\infty} y\overline{F}(y) + \textstyle\int_0^\infty \overline{F}(y)dy. \end{aligned}$$

But,

$$0 \leq y\overline{F}(y) = y\int_y^\infty dF(x) \leq \int_y^\infty xdF(x),$$

and since $E[Y] < \infty$, it follow that

$$0 \leq \lim_{y\to\infty} y\overline{F}(y) \leq \lim_{y\to\infty}\int_y^\infty xdF(x) = 0,$$

i.e. $\lim_{y\to\infty} y\overline{F}(y) = 0$. Thus,

$$E[Y] = \int_0^\infty \overline{F}(y)dy. \tag{2.1.2}$$

From (2.1.1) and (2.1.2), therefore, we have

$$E[Y] = \int_0^\infty \overline{F}(y)dy = \int_0^\infty \frac{f(y)}{r(y)}dy = E[1/r(y)]. \tag{2.1.3}$$

Equation (2.1.3) is in agreement with our intuition that small values of $r(y)$ are associated with large values of Y.

In many situations of practical interest, the failure rate $r(y)$ is strictly monotone nonincreasing (nondecreasing) in y, and this is associated with the situation where the distribution has a thick (thin) right tail.

The distribution function $F(y)$ is said to be decreasing failure rate (DFR) if $\overline{F}(x+y)/\overline{F}(y)$ is nondecreasing in y for fixed $x \geq 0$, i.e. if $\overline{F}(y)$ is log-convex. It is evident from (2.1.1) that if $F(y)$ is absolutely continuous, then DFR is equivalent to $r(y)$ nonincreasing in y.

2.1.1 Equilibrium distributions

Equation (2.1.2) may be divided by $E[Y]$ to given $\int_0^\infty \frac{\overline{F}(y)}{E[Y]}dy = 1$, which implies that $f_1(y) = \frac{\overline{F}(y)}{E[Y]}$ is a probability distribution function (even if $F(y)$ is not absolutely continuous). The corresponding distribution function is given by

$$F_1(y) = 1 - \overline{F}_1(y) = \int_0^y \frac{\overline{F}(x)}{E[Y]}dx, y \geq 0, \tag{2.1.4}$$

is called the **equilibrium distribution function** of $F(y)$. The n-th moment is, by integration by parts,

$$\begin{aligned} \int_0^\infty y^n \frac{\overline{F}(y)}{E[Y]}dy &= \frac{y^{n+1}\overline{F}(y)}{(n+1)E[Y]}\Big|_0^\infty + \int_0^\infty \frac{y^{n+1}dF(y)}{(n+1)E[Y]} \\ &= \lim_{y\to\infty} \frac{y^{n+1}\overline{F}(y)}{(n+1)E[Y]} + \int_0^\infty \frac{y^{n+1}dF(y)}{(n+1)E[Y]} \end{aligned}$$

Now,

$$0 \leq y^{n+1}\overline{F}(y) = y^{n+1}\int_y^\infty dF(x) \leq \int_y^\infty x^{n+1}dF(x).$$

Thus, if $E[Y^{n+1}] = \int_0^\infty x^{n+1}dF(x) < \infty$,

$$0 \leq \lim_{y\to\infty} y^{n+1}\overline{F}(y) \leq \lim_{y\to\infty} \int_y^\infty x^{n+1}dF(x)) = 0,$$

implying that $\lim_{y\to\infty} y^{n+1}\overline{F}(y) = 0$, and so for $n \geq 0$

$$\int_0^\infty y^n dF_1(y) = \frac{E\left[Y^{n+1}\right]}{(n+1)E\left[Y\right]}. \tag{2.1.5}$$

For $n = 1$, we have the equilibrium mean

$$\int_0^\infty y dF_1(y) = \frac{E\left[Y^2\right]}{2E\left[Y\right]}. \tag{2.1.6}$$

There is a useful identity involving $F(y)$ and $F_1(y)$. Integration by parts yields, for $y \geq 0$,

$$\int_y^\infty x dF(x) = -x\overline{F}(x)|_y^\infty + \int_y^\infty \overline{F}(x)dx.$$

As show in the last section, $E\left[Y\right] < \infty$ implies that $\lim_{y\to\infty} y\overline{F}(y) = 0$. Thus,

$$\int_y^\infty x dF(x) = -y\overline{F}(y)| + E\left[Y\right]\overline{F}_1(y), y \geq 0. \tag{2.1.7}$$

It is sometimes convenient to solve (2.1.7) for $\overline{F}_1(y)$, yielding

$$\overline{F}_1(y) = \frac{\int_y^\infty (x-y)dF(x)}{E\left[Y\right]}, y \geq 0. \tag{2.1.8}$$

2.1.2 The residual lifetime distribution and its mean

Consider the residual lifetime random variable $T_y = \begin{cases} Y - y|Y > y & \text{for } Y > y, \\ \\ \text{undefined} & \text{otherwise} \end{cases}.$

Then, for $y \geq 0$

$$P\left(T_y > t\right) = P\left(Y - y|Y > y\right) = \tfrac{\overline{F}(y+t)}{\overline{F}(y)}$$

$$P\left(T_y > t\right) = 1 - P\left(T_y \leq t\right)$$

$$P\left(T_y \leq t\right) = 1 - \tfrac{\overline{F}(y+t)}{\overline{F}(y)},$$

where $\overline{F}(y) = 1 - F(y)$ and $F(y) = P(Y \leq y)$.

The expected value of T_y, termed the **mean residual lifetime** (MRL), is given by

$$\overline{r}(y) = E\left[T_y\right] = \frac{\int_y^\infty (t-y)dF(t)}{\overline{F}(y)}, y \geq 0. \tag{2.1.9}$$

Either from integration by parts or from (2.1.2),

$$\overline{r}(y) = \int_0^\infty P\left(T_y > t\right) dt = \int_0^\infty \frac{\overline{F}(y+t)}{\overline{F}(y)} dt.$$

Equations (2.1.8) and (2.1.9) yield

$$\overline{r}(y) = \frac{\int_y^\infty \overline{F}(x)dx}{\overline{F}(y)} = \frac{E\left[Y\right] \overline{F}_1(y)}{\overline{F}(y)}$$

using (2.1.4). Obviously, $\overline{r}(0) = E\left[Y\right]$. The mean residual lifetime is closely related to the failure rate $r(y)$ where the latter exists, but does not require absolute continuity for its existence. It is very useful for analysis of tail thickness, and large values of $\overline{r}(y)$ are associated with a thick tail.

We have from (2.1.4) that

$$-\frac{d}{dy}\left(\ln\left(\overline{F}_1(y)\right)\right) = \frac{\overline{F}(y)/E\left[Y\right]}{\overline{F}_1(y)} = \frac{1}{\overline{r}(y)}, \tag{2.1.10}$$

which implies that the reciprocal $1/\overline{r}(y)$ of the mean residual lifetime $\overline{r}(y)$ is the failure rate associated with the equilibrium distribution function $\overline{F}_1(y)$, and from (2.1.2)

$$\overline{F}_1(y) = e^{\int_0^y (1/\overline{r}(x))dx}, y \geq 0. \tag{2.1.11}$$

Equation (2.1.11), together with $\overline{F}_1' = \overline{F}(y)/\overline{r}(0)$, shows that $F(y)$ is uniquely determined by $\overline{r}(y)$.

The df $F(y)$ is said to be **increasing mean residual lifetime** (IMRL) if $\overline{r}(y)$ is nondecreasing in y.

2.1.3 Other classes of distributions

In the last two subsection, we have introduced the notion of failure rate and mean residual lifetime, and classifications based on these notions. There are many other classes of distributions, some of which are of interest for the present application. These distributions are classified in terms of their survival function or the survival function of their equilibrium distributions.

- The **decreasing failure rate** (DFR) class is defined by $\frac{\overline{F}(x+y)}{\overline{F}(x)}$ nondecreasing in y for fixed $x \geq 0$.

- The distribution function $F(y)$ is said to be **increasing mean residual lifetime** (IMRL) if $\overline{r}(y)$ is nondecreasing in y.

- The distribution function $F(x)$ is said to be **new worse than used** (NWU) if $\overline{F}(x+y) \geq \overline{F}(x)\overline{F}(y)$ for all $x \geq 0$ and $y \geq 0$.

  The name has its origin in the fact that the inequality is a restatement of $P(T_y > x) \geq P(Y > x)$, i.e. the residual lifetime is stochastically larger than the original lifetime Y.

- The distribution function $F(y)$ is said to be **2-NWU** if its equilibrium distribution function $F_1(y)$ is NWU, i.e if $\overline{F}_1(x+y) \geq \overline{F}_1(x)\overline{F}_1(y)$ for all $x \geq 0$ and $y \geq 0$.

- Another class is the **new worse than used in convex ordering** (NWUC) class. A distribution F concentrated on $(0, \infty)$ is said to be NWUC if

$$\overline{F}_1(x+y) \geq \overline{F}_1(y)\overline{F}(x) \text{ for all } x \geq 0, y \geq 0. \tag{2.1.12}$$

  In other words,

$$\int_y^\infty \overline{F}(x)dx \leq \int_y^\infty \frac{\overline{F}(x+z)}{\overline{F}(z)}dx \Leftrightarrow \int_y^\infty P(Y > x)\,dx \leq \int_y^\infty P(T_y > x)\,dx. \tag{2.1.13}$$

  That is, the residual lifetime of the equilibrium distribution function $\overline{F}_1(y)$ is stochastically larger than Y.

Table 2.1 shows the relations between the class of distributions discussed.

The next proposition give us an important advantage of the distribution class (distributional

$$\begin{array}{ccccc} \text{DRF} & & \Rightarrow & & \text{NWU} \\ \Downarrow & & & & \Downarrow \\ \text{IMRL} & \Rightarrow & \text{2-NWU} & \Rightarrow & \text{NWUC} \end{array}$$

Table 2.1: The relation between the class of distributions.

properties) to be considered in order to derive a better bound for the ruin probability.

Proposition 2.1.1. *Suppose $t > 0, x \geq 0$, and $\int_0^\infty e^{ty} dF(y) < \infty$. Then if $F(y)$ is NWUC*

$$\inf_{0 \leq z \leq x, \overline{F}(z) > 0} \frac{\int_z^\infty e^{ty} dF(y)}{e^{tz}\overline{F}(z)} = \int_0^\infty e^{ty} dF(y) \tag{2.1.14}$$

Proof. Let

$$P\left(T_Z \leq y\right) = 1 - \frac{\overline{F}(y+z)}{\overline{F}(z)},$$

and thus

$$\frac{\int_z^\infty e^{ty} dF(y)}{e^{tz}\overline{F}(z)} = \int_0^\infty e^{ty} dP\left(T_Z \leq y\right) = E\left[e^{tT_Z}\right]$$

If $F(y)$ is NWUC,

$$\int_y^\infty P\left(Y > x\right) dx \leq \int_y^\infty P\left(T_y > x\right) dx. \tag{2.1.15}$$

Since e^{ty} is convex, it follow from Shaked and Shanthikumar [49], that $E\left(e^{tT_Z}\right) \geq E\left(e^{tY}\right)$. Thus,

$$\inf_{0 \leq z \leq x, \overline{F}(z) > 0} E\left(e^{tT_Z}\right) \geq E\left(e^{tY}\right).$$

But when $z = 0$, T_Z and Y have the same distribution function $F(y)$ and thus

$$\inf_{0 \leq z \leq x, \overline{F}(z) > 0} E\left(e^{tT_Z}\right) = E\left(e^{tY}\right).$$

Hence (2.1.14) hold. $\square$

2.2 Phase-type distribution

A **phase-type distribution** is a probability distribution that results from a system of one or more inter-related Poisson processes occurring in sequence, or phases. The sequence in which each of the phases occur may itself be a stochastic process. The distribution can be represented by a random variable describing the time until absorption of a Markov process with one absorbing state. Each of the states of the Markov process represents one of the phases.

It has a discrete time equivalent the *discrete phase-type distribution.*

The set of phase-type distributions is dense in the field of all positive-valued distributions, that is, it can be used to approximate any positive valued distribution (in the sense that for any non-negative distribution function $F(\cdot)$ a sequence of phase-type distributions can be found which pointwise converges at the points of continuity of $F(\cdot)$). The denseness of this class makes them very useful as a practical modelling tool. A proof of the denseness can be found in [55, 56].

Definition 2.2.1. Consider a continuous-time Markov process with $m+1$ states, where $m \geq 1$, such that the states $1, \ldots, m$ are transient states and state $m+1$ is an absorbing state. Further, let the process have an initial probability of starting in any of the $\mathbf{m+1}$ phases given by the probability vector $(\boldsymbol{\alpha}, \alpha_{m+1})$.

The *continuous phase-type distribution* is the distribution of time from the above process's starting until absorption in the absorbing state.

This process can be written in the form of a transition rate matrix,

$$Q = \begin{bmatrix} S & \mathbf{S}^0 \\ \mathbf{0} & 0 \end{bmatrix},$$

where S is an $m \times m$ matrix and $\mathbf{S}^0 = -S \times \mathbf{1}$. Here $\mathbf{1}$ represents an $m \times 1$ vector with every element being 1.

The interpretation of the column vector $\mathbf{S}^0$ is as the exit rate vector, i.e. the ith component $\mathbf{s}_i^0$ gives the intensity in state i for leaving $1, \ldots, m$ and going to the absorbing state $\mathbf{m+1}$. A convenient graphical representation is the phase diagram in term of the entrance probabilities α_i, the exit rates $\mathbf{s}_i^0$ and the transition rates (intensities) s_{ij}:

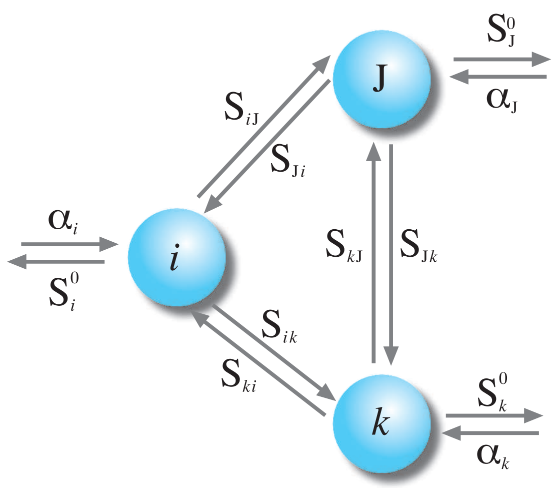

Figure 2.1: Phase diagram of a phase-type distribution with 3 phases $\{i, j, k\}$.

The basic analytical properties of phase-type distributions are given by the following result (a proof of these result can be found in [2]). Recall that the matrix-exponential $e^{\mathbf{K}}$ is defined by the standard series expansion $\sum_{n=0}^{\infty} \mathbf{K}^n/n!$.

2.2.1 Characterization

The distribution of time X until the process reaches the absorbing state is said to be phase-type distributed and is denoted $PH(\boldsymbol{\alpha}, S)$.

The distribution function of X is given by,

$$F(x) = 1 - \boldsymbol{\alpha} \exp(Sx)\mathbf{1},$$

and the density function,

$$f(x) = \boldsymbol{\alpha} \exp(Sx)\mathbf{S^0},$$

for all $x > 0$, where $\exp(\cdot)$ is the matrix exponential. It is usually assumed the probability of process starting in the absorbing state is zero. The moments of the distribution function are given by

$$E[X^n] = \int_0^\infty x^n F(dx) = (-1)^n n! \boldsymbol{\alpha} S^{-n}\mathbf{1}.$$

The moment-generating function is given by

$$M_X(t) = \int_0^\infty e^{tx} F(dx) = \alpha(-tI - S)^{-1}\mathbf{S}^0.$$

2.2.2 Special cases

The following probability distributions are all considered special cases of a continuous phase-type distribution:

- Degenerate distribution, point mass at zero or the *empty phase-type distribution* - 0 phases.
- Exponential distribution - 1 phase.
- Erlang distribution - 2 or more identical phases in sequence.
- Deterministic distribution (or constant) - The limiting case of an Erlang distribution, as the number of phases become infinite, while the time in each state becomes zero.

- Coxian distribution - 2 or more (not necessarily identical) phases in sequence, with a probability of transitioning to the terminating/absorbing state after each phase.

- Hyper-exponential distribution (also called a mixture of exponential) - 2 or more non-identical phases, that each have a probability of occurring in a mutually exclusive, or parallel, manner. (Note: The exponential distribution is the degenerate situation when all the parallel phases are identical.)

- Hypoexponential distribution - 2 or more phases in sequence, can be non-identical or a mixture of identical and non-identical phases, generalises the Erlang.

As the phase-type distribution is dense in the field of all positive-valued distributions, we can represent any positive valued distribution. However, the phase-type is a light-tailed or platikurtic distribution. So the representation of heavy-tailed or leptokurtic distribution by phase type is an approximation, even if the precision of the approximation can be as good as we want.

2.2.3 Examples

In all the following examples it is assumed that there is no probability mass at zero, that is $\alpha_{m+1} = 0$.

1. **Exponential distribution:**

   The simplest non-trivial example of a phase-type distribution is the exponential distribution of parameter λ. The phase-type distribution is the lifetime of a particle with constant failure rate λ, the parameter of the phase-type distribution are : $S = -\lambda$ and $\boldsymbol{\alpha} = 1$.

2. **Hyper-exponential or mixture of exponential distribution:**

   The hyper-exponential distribution H_k with k parallel channels is defined as a mixture of k exponential distributions with parameters $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ can be represented as a phase type distribution:

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k),$$

and

$$S = \begin{bmatrix} -\lambda_1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\lambda_k \end{bmatrix},$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is the initial vector (such that $\sum \alpha_i = 1$ and $\alpha_i > 0$ for all i). The mixture of exponential can be characterized through its density

$$f(x) = \sum_{i=1}^{k} \alpha_i \lambda_i e^{-\lambda_i x}$$

or its distribution function

$$F(x) = 1 - \sum_{i=1}^{k} \alpha_i e^{-\lambda_i x}.$$

This mixture of k exponential distributions have the following phase diagram with $k+1$ states

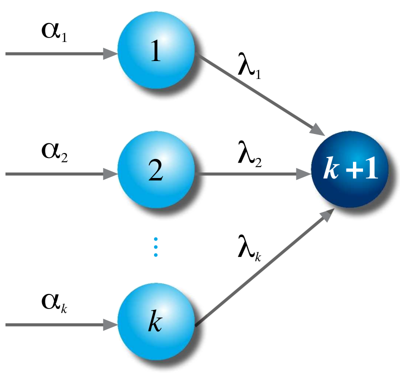

Figure 2.2: Phase diagram for the hyper-exponential distribution.

3. **Erlang distribution:**

The Erlang distribution has two parameters, the shape an integer $k > 0$ (phases) and the rate $\lambda > 0$. This is sometimes denoted $E(k, \lambda)$. The Erlang distribution with k phases is defines the Gamma distribution with integer parameter k and density

$$\lambda^k \frac{x^{k-1}}{k!} e^{-\lambda x}.$$

Since this corresponds to a convolution of k exponential densities with the same rate λ. The Erlang distribution can be written in the form of a phase-type distribution by making S a $k \times k$ matrix with diagonal elements $-\lambda$ and super-diagonal elements λ, with the probability of starting in state 1 equal to 1. For example $E(k, \lambda)$,

$$\boldsymbol{\alpha} = (1, 0, \dots, 0),$$

and

$$S = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & \lambda \\ 0 & 0 & 0 & \cdots & 0 & -\lambda \end{bmatrix}.$$

The hypoexponential distribution is a generalisation of the Erlang distribution by having different rates for each transition (the non-homogeneous case).

The Erlang distribution may be represented by the phase diagram with k phases:

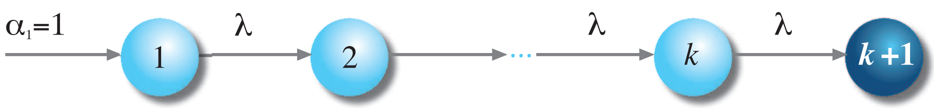

Figure 2.3: Phase diagram for the Erlang distribution.

4. **Mixture of Erlang distribution:**

The mixture of two Erlang distribution with parameter $E(3, \lambda_1)$, $E(3, \lambda_2)$ and (α_1, α_2) (such that $\lambda_1 + \lambda_2 = 1$ and for each i, $\alpha_i \geq 0$) can be represented as a phase type distribution with

$$\boldsymbol{\alpha} = (\alpha_1, 0, 0, \alpha_2, 0, 0),$$

and

$$S = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_1 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_2 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_2 & \lambda_2 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_2 \end{bmatrix}.$$

The mixture of Erlang distribution may be represented by the diagram in Figure 2.4.

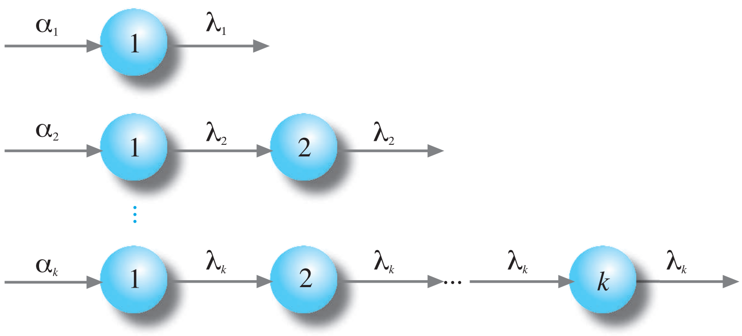

Figure 2.4: Phase diagram for the mixture of Erlang distribution.

5. **Coxian distribution:**

The Coxian distribution is a generalisation of the hypoexponential. Instead of only being able to enter the absorbing state from state k it can be reached from any phase. The phase-type representation is given by,

$$S = \begin{bmatrix} -\lambda_1 & p_1\lambda_1 & 0 & \dots & 0 & 0 \\ 0 & -\lambda_2 & p_2\lambda_2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & -\lambda_{k-2} & p_{k-2}\lambda_{k-2} & 0 \\ 0 & 0 & \dots & 0 & -\lambda_{k-1} & p_{k-1}\lambda_{k-1} \\ 0 & 0 & \dots & 0 & 0 & -\lambda_k \end{bmatrix}$$

and

$$\boldsymbol{\alpha} = (1, 0, \dots, 0),$$

where $0 < p_1, \ldots, p_{k-1} \leq 1$. In the case where all $p_i = 1$ we have the hypoexponential distribution. The Coxian distribution is extremely important as any acyclic phase-type distribution has an equivalent Coxian representation.

The *generalised Coxian* distribution relaxes the condition that requires starting in the first phase. This class of distribution is defined as the class of phase-type distributions with a phase diagram of the following form:

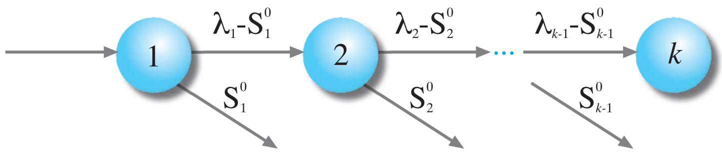

Figure 2.5: Phase diagram for the Coxian distribution.

The Erlang distribution is a special case of a Coxian distribution.

2.3 Stochastic orders and phase-type distributions

Let $F(y)$ phase-type distribution with parameters $(\boldsymbol{\alpha}, S)$, we want to see under what condition the phase-type belongs to each class of distribution in the sense of stochastic orders.

Remark 2.3.1. If $F(x)$ is a phase-type distribution with parameter $(\boldsymbol{\alpha}, S)$, always S^{-1} exist. Moreover, $E[Y] = \int_0^\infty \overline{F}(y)dy = -\alpha S^{-1}\mathbf{1}$, $F_1(y) = \dfrac{\int_0^y \overline{F}(x)dx}{E[Y]} = \dfrac{\int_0^y \alpha e^{Sx}\mathbf{1}dx}{-\alpha S^{-1}\mathbf{1}} = \dfrac{\alpha S^{-1}\{e^{Sy} - I\}\mathbf{1}}{-\alpha S^{-1}\mathbf{1}}$ and $\overline{F}_1(y) = \dfrac{\alpha S^{-1}e^{Sy}\mathbf{1}}{\alpha S^{-1}\mathbf{1}}$ with $y \geq 0$.

- The df F is DFR (decreasing failure rate): if $\frac{\overline{F}(x+y)}{\overline{F}(y)}$ is nondecreasing in y for fixed $x \geq 0$, that is equivalent to satisface that $\frac{\partial}{\partial y}\frac{\overline{F}(x+y)}{\overline{F}(y)}$ is non-negative in y for fixed $x \geq 0$.

  In this case $\frac{\overline{F}(x+y)}{\overline{F}(y)} = \frac{\alpha e^{S(x+y)}\mathbf{1}}{\alpha e^{Sy}\mathbf{1}}$, then

$$\begin{aligned}
\frac{\partial}{\partial y}\frac{\overline{F}(x+y)}{\overline{F}(y)} &=^{\dagger} \frac{\alpha S e^{S(x+y)}\widehat{\alpha} e^{Sy}\mathbf{1} - \alpha e^{S(x+y)}\widehat{\alpha} S e^{Sy}\mathbf{1}}{(\alpha e^{Sy}\mathbf{1})^2} \\
&= \frac{\alpha\{S e^{S(x+y)}\widehat{\alpha} - e^{S(x+y)}\widehat{\alpha} S\}e^{Sy}\mathbf{1}}{(\alpha e^{Sy}\mathbf{1})^2} \qquad (2.3.1) \\
&= \frac{\alpha\{SA - AS\}e^{Sy}\mathbf{1}}{(\alpha e^{Sy}\mathbf{1})^2} \geq 0;
\end{aligned}$$

where $A = e^{S(x+y)}\widehat{\alpha}$ and $\widehat{\alpha} = \mathbf{1} \times \alpha$. Therefore, (2.3.1) is non-negative, if, and only if, the numerator is non-negative, i.e, that is only posible if $SA - AS$ is non-negative definite. The equality is when S commute with $A = e^{S(x+y)}e\alpha$.

So that, The phase-type distribution is DFR if only if $SA - AS$ is non-negative definite.

Remark 2.3.2. Particularly, the exponential df $F(y) = 1 - e^{-\mu y}$, $y \geq 0$ is both decreasing and increasing failure rate since $r(y) = \mu$ for all y.

- The df F is IMRL (increasing mean residual life): the phase-type distribution is IMRL if $\overline{r}(y)$ is non-decreasing in y for fixed $x \geq 0$ (see, (2.1.9)), i.e.,

$$\begin{aligned} \overline{r}(y) &= E[T_y] = \frac{\int_y^{\infty}(t-y)dF(t)}{\overline{F}(y)}dt \\ &= \frac{E(Y)\overline{F}_1(y)}{\overline{F}(y)} \\ &= \frac{-\alpha S^{-1}e^{Sy}\mathbf{1}}{\alpha e^{Sy}\mathbf{1}} \text{ is increasing;} \end{aligned}$$

or equivalently, if $\dfrac{\partial}{\partial y}\overline{r}(y) = \dfrac{\alpha(-e^{Sy}\widehat{\alpha} + S^{-1}e^{Sy}\widehat{\alpha}S)e^{Sy}\mathbf{1}}{(\alpha e^{Sy}\mathbf{1})^2}$ is non-negative, i.e, $F(y)$ is IMRL iff $-B + S^{-1}BS$ is non-negative definite with $B = e^{Sy}\widehat{\alpha}$.

- The df F is NWU (new worse than used): if

$$\overline{F}(x+y) - \overline{F}(y)\cdot\overline{F}(x) = \alpha e^{Sx}(I - \mathbf{1}\alpha)e^{Sy}\mathbf{1} \geq 0 \quad \forall y \geq 0 \text{ and } x \text{ fixed }; \tag{2.3.2}$$

i.e, the phase-type distribution is NWU iff $I - \widehat{\alpha}$ is non-negative definite.

- The df F is 2-NWU (second new worse than used): we say that F is 2-NWU iff

$$\overline{F}_1(x+y) - \overline{F}_1(y)\cdot\overline{F}_1(x) = \frac{\alpha S^{-1}(e\alpha S^{-1}e^{Sy} - e^{Sy}e\alpha S^{-1})e^{Sx}\mathbf{1}}{(\alpha S^{-1}\mathbf{1})^2} \geq 0.$$

The phase-type distribution is 2-NWU iff $S^{-1}(Be^{Sy} - e^{Sy}B)$ is non-negative definite, where $B = \widehat{\alpha}S^{-1}$.

† Here, we use that $\dfrac{\partial}{\partial t}e^{\mathbf{B}t} = \mathbf{B}e^{\mathbf{B}t} = e^{\mathbf{B}t}\mathbf{B}$.

- The df F is NWUC (new worse than used in convex ordering): if

$$\overline{F}_1(x+y) - \overline{F}_1(y) \cdot \overline{F}(x) = \frac{\alpha S^{-1} e^{Sy}(I - \mathbf{1}\alpha) e^{Sx} \mathbf{1}}{\alpha S^{-1} \mathbf{1}} \geq 0, \quad \forall x \geq 0, \ \forall y \geq 0. \quad (2.3.3)$$

  Then, The phase-type distribution is NWUC iff S^{-1} and $S^{-1}e^{Sy}(I - \mathbf{1}\alpha)$ are non-negative or non-positive definite both simultaneously.

Chapter 3

Stochastic control and dynamic programming

In this chapter we give an introduction to discrete-time dynamic programming focused to the minimization of the ruin probability. There are many textbooks where we can found a more general introduction to this topic*. Stochastic control is one of methods being used to find optimal decision-making strategies in fields as operation research, actuarial science and mathematical finance.

We include in this chapter some results in of Schmidli's textbook [54] in order to introduce some important definitions, notations and basic notions.

We will consider processes in discrete time, i.e., the set of possible time points is $\mathbb{N}$. We will work on some Polish measurable space space $(\mathbb{X}, \mathcal{X})$, with $\mathcal{X}$ denoting the Borel-σ-algebra on $\mathbb{X}$. The Borel-σ-algebra is the smallest σ-algebra† containing all the open sets (a reader not familiar with metric spaces can just replace $\mathbb{X}$ by $\mathbb{N}^d$, $\mathbb{Z}^d$, or $\mathbb{R}^d$ endowed with the Euclidean distance). By $\mathbb{N}$ we denote the strictly positive integers.

*See, [3, 33, 34, 45] for more details.
†See, Section A.2 for more details.

3.1 Dynamic programming

3.1.1 Introduction

Let $\{Y_n\}_{n\in\mathbb{N}}$ be an independent and identically distributed (i.i.d.) sequence of random variables on some Polish space $(\mathbb{Y}, \mathcal{Y})$. These random variables model the stochastic changes over time. We work with the natural filtration $\{\mathcal{F}_n\} = \left\{\mathcal{F}_n^Y\right\}$. At each time point $n \in \mathbb{Z}^+$ a decision is made.

Definition 3.1.1. A *decision function* is a measurable function $a : \mathbb{X} \to \mathcal{B}$. A *plan (policy, strategy)* is a sequence $\pi = \{a_n\}_{n\geq 0}$.

Then $a_n(X_n)$ will represent the action chosen at the beginning of period $n + 1$. We model this decision as a variable a_n from some space $\mathcal{B}$ (which is endowed with some topology we do not mention explicitly here). The stochastic process $\pi = \{a_n\}_{n\in\mathbb{Z}^+}$ must be adapted, because the decision can only be based on the present and not on future information. We therefore only allow controls π that are adapted. We may make some restriction to the possible strategies π. Let Π denote the set of admissible strategies, i.e., the adapted strategies $\pi = \{a_n\}$ that are allowed.

The controlled stochastic process is now constructed in the following way. Let $(\mathbb{X}, \mathcal{X})$ be a Polish space, the state space of the stochastic process, and $x \in \mathbb{X}$ be the initial state. We let $X_0 = 0$ be the starting value of the process. Note that the initial value is not stochastic. The process at time $n + 1$ is

$$X_{n+1} = f\left(X_n, a_n, Y_n\right)$$

where $f : \mathbb{X} \times \mathcal{B} \times \mathbb{Y} \to \mathbb{X}$ is a measurable function. The interpretation is the following. The next state of the process X only depends on the present state and the present decision. The decisions made at earlier times and the path up to the present state do not matter. The process X is a Markov process, because the decision a_n depends on X_n only.

At each time point there is a reward, $r(X_n, a_n)$. A negative value of $r(X_n, a_n)$ can be regarded as a cost. The value connected to some strategy π is then

$$V_T^\pi(x) = E\left[\sum_{n=0}^{T} r(X_n, a_n)\, e^{-\alpha n}\right]. \tag{3.1.1}$$

The time horizon T can be finite or infinite. The parameter $\alpha \geq 0$ is a discounting parameter. If $T = \infty$, we often will have to assume that $\alpha > 0$ in order for $V_\infty^\pi(x)$ to be finite for all $\pi \in \Pi$.

Our goal will be to maximise $V_T^\pi(x)$. We therefore define the *value function*

$$V_T(x) = \sup_{\pi\in\Pi} V_T^\pi(x).$$

In the case $T = \infty$, we just write $V(x)$ and $V^\pi(x)$ instead of $V_\infty(x)$ and $V_\infty^\pi(x)$, respectively. We now assume that $V_T(x) \in \mathbb{R}$ for all x. It is clear that if there is a strategy π such that $V_T^\pi(x) \in \mathbb{R}$, then $V_T(x) > -\infty$. The property $V_T(x) < \infty$ has to be proved for every problem separately. Another (technical) problem is to be shown that $V_T(x)$ is a measurable function. In many problem it can be shown that $V_T(x)$ is increasing or continuous, and hence measurable.

It is not feasible to find $V(x)$ by calculating the value function $V_T^\pi(x)$ for each possible strategy π, particularly not if $\mathbb{X}$ and T are infinite. One therefore has to find a different way to characterise the value function $V_T(x)$. In our setup it turns out that the problem can be simplified. We next prove the *dynamic programming principle*, also called *Bellman's equation.* We allow all controls $\{a_n\}$ that are adapted. With $V_t(x)$ and $V_t^\pi(x)$ we denote the remaining value if t time units are left. For instance, $V_{t-1}(x)$ is the value if we stand at time 1 and X_1. We let $V_{-1}(x) = 0$.

Lemma 3.1.1. *Suppose that $V_T(x)$ is finite. The function $V_T(x)$ fulfils the dynamic programming principle*

$$V_T(x) = \sup_{\pi\in\Pi} \left\{r(x,a) + e^{-\alpha} E\left[V_{T-1}(f(x,a,Y))\right]\right\}, \tag{3.1.2}$$

where Y is a generic random variable with the same distribution as Y_n. If $T = \infty$, the dynamic programming principle becomes

$$V(x) = \sup_{\pi\in\Pi} \left\{r(x,a) + e^{-\alpha} E\left[V(f(x,a,Y))\right]\right\}. \tag{3.1.3}$$

Proof. Let π be an arbitrary strategy. Then $X_1 = f(x, a_0, Y_1)$ and

$$V_T^\pi(x) = E[r(x, a_0)] + e^{-\alpha} E\left[\sum_{n=0}^{T-1} r(X_{n+1}, a_{n+1}) e^{-\alpha n}\right].$$

Condition on X_1, a_0 (we allow random decision) and let $\widetilde{X}_n = X_{n+1}$, $\widetilde{a}_n = a_{n+1}$ and $\widetilde{Y}_n = Y_{n+1}$. Then

$$\widetilde{X}_{n+1} = f\left(\widetilde{X}_x, \widetilde{a}_n, \widetilde{Y}_n\right)$$

and

$$E\left[\sum_{n=0}^{T-1} r(X_{n+1}, a_{n+1}) e^{-\alpha n} | X_1, a_0\right] = E\left[\sum_{n=0}^{T-1} r(\widetilde{X}_n, \widetilde{a}_n) e^{-\alpha n} | X_1, a_0\right] \le V_{T-1}^{\widetilde{\pi}}(X_1) \le V_{T-1}(X_1).$$

Thus,

$$\begin{aligned} V_T^\pi(x) &\le E[r(x, a_0) + e^{-\alpha} V_{T-1}(X_1)] \\ &= E[r(x, a_0) + e^{-\alpha} V_{T-1}(f(x, a_0, Y_1))] \\ &\le \sup_{\pi \in \Pi} \{r(x, a) + e^{-\alpha} E[V_{T-1}(f(x, a, Y))]\}. \end{aligned}$$

Because π is arbitrary, this shows that

$$V_T(x) \le \sup_{\pi \in \Pi} \left\{r(x, a) + e^{-\alpha} E[V_{T-1}(f(x, a, Y))]\right\}.$$

Fix $\varepsilon > 0$ and $a \in \mathcal{B}$. Let us now consider a strategy $\widetilde{\pi}$ such that, conditioned on $X_1 = f(x, a, Y_1)$, $V_{T-1}(X_1) < V_{T-1}^{\widetilde{\pi}}(X_1) + \varepsilon$. Here, we do not address the problem of whether we can do that in a measurable way because this point usually is clear in the examples, particularly the examples treated in [3, 33, 34, 35, 54]. Let $a_0 = a$ and $a_n = \widetilde{a}_{n-1}$. Then

$$\begin{aligned} r(x, a) + e^{-\alpha} E[V_{T-1}(f(x, a, Y_1))] &< r(x, a) + e^{-\alpha} E\left[V_{T-1}^{\widetilde{\pi}}(X_1)\right] + \varepsilon \\ &= V_T^\pi(x) \le V_T(x) + \varepsilon. \end{aligned}$$

Because ε is arbitrary, the result follows.

The proof does not explicitly use the finiteness of T. Thus, we can replace T and $T-1$ by ∞, and (3.1.3) is proved in the same way. □

The result says that we have to maximise the present reward plus the value of the future rewards. If we do that at each time point, we end up with the optimal value. Equation (3.1.2) can be solved recursively. Moreover, equation (3.1.3) can be solve numerically.

3.1.2 The optimal strategy

We next characterise the optimal strategy.

Corollary 3.1.2. *Suppose that $T < \infty$, $V_T(x)$ is finite, and that for any $t \le T$ there exists $\varphi_t(x)$ such that $\varphi_t(x) = a$ is maximising the right-hand side of (3.1.2) for $T = t$. We assume that $\varphi_t : \mathbb{X} \to \mathcal{B}$ is measurable for each t. Let $a_n = \varphi_{T-n}(X_n)$. Then*

$$V_T(x) = V_T^\pi(x).$$

Proof. Clearly, $V_T^\pi(x) \le V_T(x)$. If $T = 0$, then for any strategy $\pi' = a_0'$

$$V_0^{\pi'}(x) = E\left[r\left(x, a_0'\right)\right] \le r\left(x, \varphi_0(x)\right) = V_0^\pi(x),$$

and $V_0(x) \le V_0^\pi(x)$ follows. We prove the assertion for $T < \infty$ by induction. Suppose that the assertion is proved for $T = n$. Let π' be an arbitrary strategy for $T = n+1$, and use the tilde sign as in the proof of Lemma 3.1.1. Then

$$\begin{aligned}
V_{n+1}^{\pi'}(x) &= E\left[r\left(x, a_0'\right) + e^{-\alpha} E\left[V_n^{\tilde{\pi}'}\left(f\left(x, a_0', Y_1\right)\right) | a_0'\right]\right] \\
&\le E\left[r\left(x, a_0'\right) + e^{-\alpha} E\left[V_n\left(f\left(x, a_0', Y_1\right)\right) | a_0'\right]\right] \\
&\le r\left(x, \varphi_{n+1}(x)\right) + e^{-\alpha} E\left[V_n\left(f\left(x, \varphi_{n+1}(x), Y_1\right)\right)\right] \\
&= r\left(x, \varphi_{n+1}(x)\right) + e^{-\alpha} E\left[V_n^{\tilde{\pi}}\left(f\left(x, \varphi_{n+1}(x), Y_1\right)\right)\right] = V_{n+1}^\pi(x).
\end{aligned}$$

This proves that $V_{n+1}(x) \le V_{n+1}^\pi(x)$. □

We can easily see from that if a_n does not maximise the Bellman equation, then it cannot be optimal. In particular, if $\varphi_n(x)$ does not exist for all $n \le T$, then an optimal strategy cannot exist.

If the time horizon is infinite, the proof of the existence of an optimal strategy is slightly more complicated. But the optimal strategy does not explicitly depend on time and is therefore simpler.

Corollary 3.1.3. *Suppose that $T = \infty$, $V(x) < \infty$, and that for every x there is a $\varphi(x)$ maximising the right-hand side of (3.1.3). Suppose further that $\varphi(x)$ is measurable and that*

$$\lim_{n\to\infty} \sup_{\pi' \in \Pi} E\left[\sum_{k=n}^{\infty} \left| r\left(X_k', a_k'\right) \right| e^{-\alpha k}\right] = 0, \tag{3.1.4}$$

where $X_{n+1}' = f\left(X_n', a_n', Y_{n+1}\right)$. Let $a_n = \varphi\left(X_n\right)$. Then $V^\pi(x) = V(x)$.

Proof. We first show that for any strategy π' with a value a_0' that does not maximise the right-hand side of (3.1.3) there exists a strategy π'' with $a_0'' = \varphi(x)$ that yields a large value. Choose $\varepsilon > 0$. For each initial value $\widetilde{x}$ there exists a strategy $\widetilde{\pi}''$ such that $V(\widetilde{x}) \le V^{\widetilde{\pi}''}(\widetilde{x}) + \varepsilon$. Also here we refrain from the technical problem of showing that $\widetilde{\pi}''$ can be chosen in a measurable way, because it is simpler to address this problem for the specific examples. Let π'' be the strategy with $a_0' = \varphi(x)$ and $a_{n+1}' = \widetilde{a}_n'$, where the initial capital is $\widetilde{x} = f(x, \varphi(x), Y_1)$. Thus,

$$\begin{aligned} V^{\pi'}(x) &= E\left[r(x, a_0') + e^{-\alpha} E\left[V^{\widetilde{\pi}'}(f(x, a_0', Y_1)) \,|a_0'\right]\right] \\ &\le E\left[r(x, a_0') + e^{-\alpha} E\left[V(f(x, a_0', Y_1)) \,|a_0'\right]\right] \\ &< r(x, \varphi(x)) + e^{-\alpha} E\left[V(f(x, \varphi(x), Y_1))\right] = V(x) \\ &\le r(x, \varphi(x)) + e^{-\alpha} E\left[V(f(x, \varphi(x), Y_1))\right] + \varepsilon = V(x) + \varepsilon. \end{aligned}$$

If $\varepsilon < V(x) - V^{\pi'}(x)$, we have that $V^{\pi'}(x) < V^{\pi''}(x)$.

Let Π_n be the set of all strategies π' with $a_k' = \varphi(X_k)$ for $0 \le k \le n$. We just have shown that $V(x) = \sup_{\pi' \in \Pi_0} V^{\pi'}(x)$. Suppose that $V(x) = \sup_{\pi' \in \Pi_n} V^{\pi'}(x)$. Let π' be a strategy such that $a_k' = \varphi(X_k)$ for $k \le n$ and π_{n+1}' does not maximise the right-hand side of (3.1.3) for $x = X_{n+1}$. Let $\widetilde{a}_k' = a_{n+1+k}'$. Then by the argument used for $n = 0$, there is a strategy $\widetilde{\pi}''$ with $\widetilde{a}_0'' = \varphi(X_{n+1})$ such that $V^{\widetilde{\pi}''}(X_{n+1}) > V^{\widetilde{\pi}'}(X_{n+1})$. Let π'' be the strategy with $a_k'' = a_k'$ and $a_{n+1+k}'' = \widetilde{a}_k''$. Because

$$\begin{aligned} V^{\pi'}(x) &= E\left[\sum_{k=0}^{n} r(X_k, \varphi(X_k)) e^{-\alpha k} + e^{-\alpha(n+1)} V^{\widetilde{\pi}'}(X_{n+1})\right] \\ &< E\left[\sum_{k=0}^{n} r(X_k, \varphi(X_k)) e^{-\alpha k} + e^{-\alpha(n+1)} V^{\widetilde{\pi}''}(X_{n+1})\right], \end{aligned}$$

we get $V(x) = \sup_{\pi' \in \Pi_{n+1}} V^{\pi'}(x)$.

Because for all n we have that $V(x) = \sup_{\pi' \in \Pi_n} V^{\pi'}(x)$, we are now able to prove that $a_n = \varphi(X_n)$ is optimal. Let $\varepsilon > 0$. There exists $n \in \mathbb{Z}^+$ such that $E\left[\sum_{k=n+1}^{\infty} |r(X_k', a_k')| e^{-\alpha k}\right] < \varepsilon$ for any strategy π'. Let π' be a strategy in Π_n such that $V(x) - V^{\pi'}(x) < \varepsilon$. Then

$$\begin{aligned} V(x) &< V^{\pi'}(x) + \varepsilon = E\left[\sum_{k=0}^{\infty} r(X_k', a_k') e^{-\alpha k}\right] + \varepsilon \\ &< E\left[\sum_{k=0}^{n} r(X_k', a_k') e^{-\alpha k}\right] + 2\varepsilon \\ &= E\left[\sum_{k=0}^{n} r(X_k, a_k) e^{-\alpha k}\right] + 2\varepsilon \le V^{\pi}(x) + 3\varepsilon. \end{aligned}$$

Because ε is arbitrary, it follows that $V(x) \le V^{\pi}(x)$. □

Remark 3.1.1. The technical condition (3.1.4) is always fulfilled if the reward function $r(x, \varphi)$ is bounded and $\alpha > 0$. Alternatively, if one knows the value function $V(x)$, one could just prove that $V^{\pi}(x) = V(x)$. Then condition (3.1.4) is not needed.

3.2 Minimizing ruin probability and optimal reinsurance

We now consider the discrete-time surplus process connected to some insurance portfolio. The insurer earns some premium. With the surplus, reinsurance is bought, and a premium has to be paid. The insurer can control how much reinsurance is bought. Our goal will be to maximize the survival probability, or, equivalently, to minimize the ruin probability. In order for the problem not to become trivial, we will have to assume that the cedent (first insurer) would have to pay more for full reinsurance than the premium he receives.

Let $Y_i \geq 0$ be the aggregate claim in period i and let $F(y)$ denote its distribution function. The sequence is assumed to be iid. We work again with filtration $\{\mathcal{F}_t^Y\}$. The insurer can at each time i choose the reinsurance for the next period. The set of possible reinsurance treaties is a compact connected subset $\mathcal{B} \subset \mathbb{R}^d$. If the reinsurance treaty $b \in \mathcal{B}$ is chosen, the insurer has to pay $h(b, Y_i)$ for the aggregate claim Y_i in the ith period, and the rest is paid by the reinsurer. Here $h : (0, \infty) \times \mathcal{B} \to [0, \infty)$ is some function with properties we will assume below. We only allow reinsurance treaties with $0 \leq h(b, y) \leq y$, i.e, the reinsurer pays at most the whole claim size. For this protection the insurer has to pay a reinsurance premium. Let $\mathcal{C}(b) \in \mathbf{R}$ denote the premium left for the insurer if reinsurance b is chosen, i.e, the original premium minus the reinsurance premium. The reinsurer can at any time change the reinsurance for the next period, i.e., he chooses an adapted strategy $\{b_n\}$. The income for the next period is then $\mathcal{C}(b)$[‡].

We assume that more reinsurance is more expensive ($h(b, y) \geq h(b', y)$ for all y implies that $\mathcal{C}(b) \geq \mathcal{C}(b')$) and that full reinsurance leads to a strictly negative income ($\mathcal{C}(b_r) < 0$). We also assume that $\mathcal{C}(b)$ and $h(b, y)$ are continuous in b. The income in case of no reinsurance is denoted by c. We then have that $\mathcal{C}(b) \leq c$. We also assume that $h(b, y)$ is increasing in y. In this case the generalised inverse $\rho(z, b)$ given by

$$\rho(z, b) := \sup\{y : h(b, y) \leq z\} \tag{3.2.1}$$

[‡]For popular reinsurance forms, see Section 1.4.

is well defined. Note that $\rho(z,b)$ is increasing and right-continuous in z. We also assume that $P\left(h(b,Y) > \mathcal{C}(b)\right) > 0$ for all $b \in \mathcal{B}$. Otherwise, ruin can be prevented by reinsurance and the problem considered in this section becomes trivial. We also assume the net profit condition $E\left[\mathcal{C}(b) - h(b,Y)\right] > 0$ for some b. Otherwise, ruin cannot be prevented because the surplus would be decreasing in time for all reinsurance treaties.

Let the initial capital x. Then the surplus process is $X_0 = x$ and

$$X_{n+1} = X_n + \mathcal{C}(b) - h\left(b_n, Y_{n+1}\right), \tag{3.2.2}$$

as long as $X_n \geq 0$. If $b \in \mathcal{B}$ is now a reinsurance treaty fulfilling the net profit condition $E\left[\mathcal{C}(b) - h(b,Y)\right] > 0$, then by the law of large numbers, $n^{-1}\sum_{k=1}^{n} \mathcal{C}(b) - h\left(b, Y_{k+1}\right) \to E\left[\mathcal{C}(b) - h(b,Y)\right]$. This implies that for the constant strategy $b_n = b$ the process X_n tends to infinity. In particular, $\inf_n X_n > -\infty$. Hence, there is an initial capital x_0 such that $P\left(\inf_n X_n \geq 0 | X_0 = x_0\right) > 0$. Because there is a strictly positive probability that from initial capital zero the set $[x_0, \infty)$ is reached before the set $(-\infty, 0)$, we get also that $P\left(\inf_n X_n \geq 0 | X_0 = 0\right) > 0$. Hence, we have a strategy such that ruin is not certain.

We introduce a cemetary state $\varkappa$. If $X_n < 0$ or $X_n = \varkappa$. This allows us to formulate the problem related with (3.1.1). We let $X_{n+1} = \varkappa$, $\alpha = 0$ and choose the reward function

$$r(X_n, b_n) = \begin{cases} 0, & \text{if } X_n \geq 0 \text{ or } X_n = \varkappa, \\ \\ -1, & \text{if } X_n < 0. \end{cases} \tag{3.2.3}$$

In this way the cost is paid at most once. The value of a reinsurance strategy is $V^b(x) = -P\left(X_n = \varkappa \text{ for some } n\right)$, and the value function is $V(x) = \sup_b V^b(x)$, where we take the supremum over all adapted reinsurance strategies b. Clearly, $V(x) \in (-1, 0)$; hence, $V : \mathbb{R}^+ \to$

$(-1, 0)$. Equation (3.1.3) then reads for $x \geq 0$

$$\begin{aligned} V(x) &= \sup_{b \in \mathcal{B}} \int_0^\infty V\left(x + \mathcal{C}(b) - h(b, y)\right) dF(y) \\ &= \sup_{b \in \mathcal{B}} \int_0^{\rho(x+\mathcal{C}(b), b)} V\left(x + \mathcal{C}(b) - h(b, y)\right) dF(y) - \left(1 - F(\rho(x + \mathcal{C}(b), b))\right), \end{aligned}$$

where we used that $V(x) = -1$ for $x < 0$.

The difficulty with solving the equation is that it does not have a unique solution within the set of real function. We have to pick the solution with $\lim_{x\to\infty} V(x) = 0$. It turns out to be simpler to consider the survival function $\delta(x) = 1 + V(x)$. Then

$$\delta(x) = \sup_{b \in \mathcal{B}} \int_0^{\rho(x+\mathcal{C}(b), b)} \delta\left(x + \mathcal{C}(b) - h(b, y)\right) dF(y).$$

Any multiple of $\delta(x)$ also solves the equation. Let us therefore consider

$$f(x) = \sup_{b \in \mathcal{B}} \int_0^\infty f\left(x + \mathcal{C}(b) - h(b, y)\right) dF(y). \tag{3.2.4}$$

with $f(0) = 1$, imposing $f(x) = 0$ for $x < 0$.

For the following results we will need that either $\tau < \infty$ or $X_n \to \infty$. For a proof we need the following lemma, taken from [54] p.22.

Lemma 3.2.1. *Let $S_n = \sum_{k=1}^n W_k - Z_k$ with $0 \leq W_k \leq w$ for some $w < \infty$ and $Z_k \geq 0$. If $\{S_n\}$ is a submartingale, then*

$$P\left(\sum_{k=1}^\infty Z_k = \infty, \sum_{k=1}^\infty W_k < \infty\right) = 0.$$

Proof. Let $\xi \in (0, \infty)$ and $N = \inf\{n : S_n > \xi\}$. Then $\{S_{n\wedge N}\}$ is a submartingale that is bounded from above by $\xi + w$. Thus, $S_{n\wedge N}$ converges to an integrable random variable S_N. In particular, if $\sup_n S_n < \xi$, then $\inf_n S_n > -\infty$. Because this holds for all $\xi > 0$, we have that $\sup_n S_n \leq \xi$, then $\inf_n S_n > -\infty$. If $\sup_n S_n = \infty$, then $\sum_{k=1}^\infty W_k = \infty$. If $\sup_n S_n < \infty$, then $\sum_{k=1}^\infty Z_k = \infty$, then, because $\inf_n S_n > -\infty$, we also have $\sum_{k=1}^\infty W_k = \infty$. □

Lemma 3.2.2. *For any strategy either ruin occurs or the capital tends to infinity.*

Proof. The function $b \mapsto P\left(h(b,Y) > \mathcal{C}(b)\right)$ is lower semi-continuous. Thus, there is b_0 where the infimum is taken. By our assumption we have $P\left(h(b,Y) > \mathcal{C}(b)\right) \geq 2\delta$ for some $\delta > 0$ For each b let

$$\varepsilon(b) = \sup\left\{\varepsilon : P\left(h(b,Y) > \mathcal{C}(b) + \varepsilon\right) \geq \delta\right\}.$$

By continuity and compactness we have that $\varepsilon(b)$ is bounded away from zero. Thus, there is $\varepsilon > 0$ such that $P\left(h(b,Y) > \mathcal{C}(b) + \varepsilon\right) \geq \delta$ for all b.

Choose $\xi > 0$. Let $W_n = \mathbf{1}_{\{X_n \leq \xi, h(b_n, Y_{n+1}) > \mathcal{C}(b_n) + \varepsilon\}}$ and $Z_n = \delta \mathbf{1}_{\{X_n \leq \xi\}}$. Then $S_n = \sum_{k=1}^n W_k - Z_k$ fulfils the assumptions of Lemma 3.2.1. Thus, $X_n \leq \xi$ infinitely often implies that $X_{n+1} \leq \xi - \varepsilon$ infinitely often. Therefore, if $\liminf X_n \leq \xi$, then $X_n \leq \xi - \varepsilon/2$. By induction, $\liminf X_n \leq -\xi/2$, so ruin occurs almost surely if $\liminf X_n < \infty$. □

Clearly, the solution we are looking for is increasing. If we use the same strategy for initial capital x and initial capital $x + h$, then ruin cannot occur for initial capital $x + h$ unless it occurs for initial capital x. We can say something about the uniqueness of the solution if we restrict to increasing functions.

Theorem 3.2.3. *Suppose that $f(x)$ is an increasing solution to (3.2.4) with $f(0) = 1$. Then $f(x)$ is bounded and $f(x) = \delta(x)/\delta(0)$.*

Proof. Denote by $f(\infty) = \lim_{x\to\infty} f(x) \in [1, \infty]$. Let $\{b_n\}$ be an arbitrary strategy. We prove that $\{f\left(X_{\tau\wedge n}\right)\}$ is a supermertingale. We have

$$\begin{aligned} E\left[f\left(X_{\tau\wedge n+1}\right)|\mathcal{F}_n\right] &= \int_0^\infty f\left(X_n + \mathcal{C}(b_n) - h(b_n, y)\right) dF(y) \mathbf{1}_{\{\tau > n\}} \\ &\leq \sup_{b\in\mathcal{B}} \int_0^\infty f\left(X_n + \mathcal{C}(b_n) - h(b_n, y)\right) dF(y) \mathbf{1}_{\{\tau > n\}} \\ &= f\left(X_n\right) \mathbf{1}_{\{\tau > n\}} = f\left(X_{\tau\wedge n}\right) \end{aligned}$$

where we used (3.2.4). Then $\{f\left(X_{\tau\wedge n}\right)\}$ is a positive supermatingale and $\lim_{n\to\infty} f\left(X_{\tau\wedge n}\right)$ exists by the martingale convergence theorem. This limit must be $f(\infty)$ or zero by Lemma 3.2.2. For an arbitrary strategy $\{b_t\}$ we let $\delta^b(x)$ be the survival probability that ruin does not occur. Because it is possible to choose a strategy such that $\delta^b(x) > 0$ and $\lim_{n\to\infty} f\left(X_{\tau\wedge n}\right)$ must be integrable, we have that $f(\infty) < \infty$. By bounded convergence, $f(x) \geq f(\infty)\delta^b(x)$. Choose $\varepsilon > 0$. We now choose a strategy $\{b_n\}$ such that

$$f\left(X_n\right) < \int_0^\infty f\left(X_n + \mathcal{C}(b_n) - h(b_n, y)\right) dF(y) + \frac{\varepsilon}{(n+1)^2}.$$

Then $\left\{f\left(X_{\tau\wedge n}\right) + \sum_{k=0}^{n-1} \frac{\varepsilon}{(k+1)^2}\right\}$ is bounded submartingale. Letting $n \to \infty$ shows that $f(x) < \delta(x) f(\infty) + \sum_{k=0}^{n-1} \frac{\varepsilon}{(n+1)^2}$. Thus, $f(x) = \delta(x) f(\infty)$. □

We next show how the solution can be calculated. Let $f_0(x) = \mathbf{1}_{\{x\geq 0\}}$ and define recursively

$$f_{n+1}(x) = \left(\sup_{b\in\mathcal{B}} \int_0^\infty f_n(x + \mathcal{C}(b) - h(b,y))\, dF(y)\right) \mathbf{1}_{\{x\geq 0\}}.$$

Then the following proposition holds.

Proposition 3.2.4. *The functions $f_n(x)$ converge to $\delta(x)$.*

Proof. By induction it follows that $0 \leq f_n(x) \leq 1$. We next show by induction that the functions $f_n(x)$ are increasing in x. Let $\kappa > 0$ and $b \in \mathcal{B}$ be arbitrary. Then because $f_n(x)$ is increasing,

$$\int_0^\infty f_n(x + \mathcal{C}(b) - h(b,y))\, dF(y) \leq \int_0^\infty f_n(x + \kappa + \mathcal{C}(b) - h(b,y))\, dF(y) \leq f_{n+1}(x+\kappa).$$

Taking the supremum over all b gives $f_{n+1}(x) \leq f_{n+1}(x+\kappa)$. We show that $\{f_n(x)\}_{n\in\mathbb{Z}^+}$ is monotone in n. Clearly, $f_0(x) = 1 \geq f_1(x)$. Suppose that $f_{n-1}(x) \geq f_n(x)$ for all x. Let $\varepsilon > 0$ and fix x. Denote by b an argument such that

$$\int_0^\infty f_n(x + \mathcal{C}(b) - h(b,y))\, dF(y) > f_{n+1}(x) - \varepsilon.$$

Then

$$f_n(x) - f_{n+1}(x) \geq \int_0^\infty (f_{n-1}(x + \mathcal{C}(b) - h(b,y)) - f_n(x + \mathcal{C}(b) - h(b,y)))\, dF(y) - \varepsilon.$$

By our assumption $f_n(x) - f_{n+1}(x) > -\varepsilon$. Because ε was arbitrary, we get that $\{f_n(x)\}$ is a decreasing sequence in n. In particular, it converges point-wise to a function $f(x)$. By monotone convergence the limit has to fulfil (3.2.4). Moreover, $f(x)$ is increasing because $f_n(x)$ is increasing for each n. By Theorem 3.2.3 we have $f(x) = \delta(x)f(\infty)$. In order to show that $f(\infty) = 1$, we show that $f_n(x) \geq \delta(x)$ because $\delta(x)$ solves (3.2.4). Thus, we also have $f(x) \geq \delta(x)$. Because $1 \geq f(\infty) \geq \delta(\infty) = 1$, the result is proved. □

Remark 3.2.1. Schäl [48] introduced a similar model that considered optimal reinsurance and investment. The existence of an optimal strategy is proved from general results on dynamic programming in infinite time. In particular, the case of exponentially distributed claim size is discussed.

Chapter 4

Inequalities for the ruin probability

The preceding chapters have briefly reviewed the most relevant theoretic tools used in this chapter.

We concentrate now ourselves in the analysis of an insurance model, for which risk theory, stochastic orders and stochastic control theory become necessary issues. Previous versions of the results obtained in this chapter have appeared in [13] and [14].

4.1 Introduction

This Chapter studies an insurance model where the risk process can be controlled by proportional reinsurance. The performance criterion is to choose reinsurance control strategies to bound the ruin probability of a discrete-time process with a Markov chain interest. Controlling a risk process is a very active area of research, particularly in the last decade; see [33, 34, 48, 54], for instance. Nevertheless obtaining explicit optimal solutions is a difficult task in a general setting. Hence, an alternative method commonly used in ruin theory is to derive *inequalities* for ruin probabilities (see Asmussen [2], Grandell [32], Schmidli [54], and Willmot and Lin [62]). Following Cai [6] and Cai and Dickson [7], we model the interest rate process as a denumerable state Markov chain. This model can be in fact a discrete counterpart of the most frequently occurring effect observed in continuous interest rate process, e.g., mean-reverting effect. Stochastic inequalities for the ruin probabilities are derived by martingales and inductive techniques (see Section 1.2). The inequalities can be used to obtain upper bounds for the ruin probabilities. For the sake of simplicity, we restrict ourselves to use stationary control policies. Explicit condition are obtained for the optimality of employing no reinsurance.

4.2 Deterministic length of periods and aggregate claims

The outline of this section is as follows. In Subsection 4.2.1 the risk model is formulated. Some important special cases of this model are briefly discussed. In Subsection 4.2.2 we derive recursive equations for finite-horizon ruin probabilities, and integral equations for the ultimate ruin probability. In Subsection 4.2.3 we obtain upper bounds for the ultimate probability of ruin. An analysis of the new bounds and a comparison with Lundberg's inequality is also included (see Subsection 1.2.1). Finally, in Subsection 4.2.4 we illustrate our results on the ruin probability in a risk process with a heavy tail claims distribution under proportional reinsurance and a Markov interest rate process.

4.2.1 The model

We consider a discrete-time insurance risk process in which the surplus X_n varies according to the equation

$$X_n = X_{n-1}\,(1 + I_n) + \mathcal{C}(b_{n-1}) - h(b_{n-1}, Y_n), \text{ for } n \geq 1 \tag{4.2.1}$$

with $X_0 = x \geq 0$. Following Schmidli [54] p. 21, we introduce an absorbing (cemetery) state $\varkappa$, such that if $X_n < 0$ or $X_n = \varkappa$, then $X_{n+1} = \varkappa$. We denote the state space by $\mathbb{X} = \mathbb{R} \cup \varkappa$. Let Y_n be the *total claims* during the n-th period (from time *n-1* to time n), which we assume to form a sequence of i.i.d. random variables with common probability distribution function (p.d.f.) F. The process can be controlled by reinsurance, that is, by choosing the *retention level* (or proportionality factor or risk exposure) $b \in \mathcal{B}$ of a reinsurance contract for one period, where $\mathcal{B} := [b_{\min}, 1]$, and $b_{\min} \in (0, 1]$ will be introduced below. Let $\{I_n\}_{n\geq 0}$ be the *interest rate process*; we suppose that I_n evolves as a Markov chain with a denumerable (possibly finite) state space $\mathbb{I}$ consisting of nonnegative integers.

The function $h(b, y)$ with values in $[0, y]$ specifies the fraction of the claim y paid by the insurer, and it also depends on the retention level b at the beginning of the period. Hence $y - h(b, y)$ is the part paid by the reinsurer. The retention level $b = 1$ stands for the control action *no reinsurance* (see Section 1.4). In this section, we consider the case of *proportional reinsurance*, which means that

$$h(b, y) := b \cdot y, \text{ with retention level } b \in \mathcal{B}. \tag{4.2.2}$$

The premium (income) rate c is fixed. Since the insurer pays to the reinsurer a premium rate, which depends on the retention level b, we denote by $\mathcal{C}(b)$ the premium left for the insurer if the retention level b is chosen, where

$$0 \leq \mathcal{C}(b) \leq c, \ \ b \in \mathcal{B}.$$

We define $b_{\min} := \min\{b \in (0, 1] | \mathcal{C}(b) \geq 0\}$. Moreover, $\mathcal{C}(b)$ is an increasing function that we will calculate according to the *expected value principle* with added safety loading θ from the reinsurer:

$$\mathcal{C}(b) = c - (1 + \theta) \cdot (1 - b) E[Y], \tag{4.2.3}$$

where Y is a generic random variable with p.d.f. F.

We consider Markovian control policies $\pi = \{a_n\}_{n \geq 1}$, which at each time n depend only on the current state, that is, $a_n(X_n) := b_n$ for $n \geq 0$. Abusing notation, we will identify functions $a : \mathbb{X} \to \mathcal{B}$ with stationary strategies, where $\mathcal{B} = [b_{\min}, 1]$, the decision space (see Section 3.1). Consider an arbitrary initial state $X_0 = x \geq 0$ (note that the initial value is not stochastic) and a control policy $\pi = \{a_n\}_{n \geq 1}$. Then, by iteration of (4.2.1), and assuming (4.2.2) and (4.2.3), it follows that for $n \geq 1$, X_n satisfies

$$X_n = x \prod_{l=1}^{n} (1 + I_l) + \sum_{l=1}^{n} \left(\mathcal{C}(b_{l-1}) - b_{l-1} \cdot Y_l \prod_{m=l+1}^{n} (1 + I_m) \right). \tag{4.2.4}$$

Let (p_{ij}) be the matrix of transition probabilities of $\{I_n\}$, i.e.,

$$p_{ij} := P\left(I_{n+1} = j | I_n = i\right), \tag{4.2.5}$$

where $p_{ij} \geq 0$ and $\sum_j p_{ij} = 1$ for all $i, j \in \mathbb{I}$. The ruin probability when using the policy π, given the initial surplus x, and the initial interest rate $I_0 = i$ is defined as

$$\psi^{\pi}(x,i) := P^{\pi}\left(\bigcup_{k=1}^{\infty}\{X_k < 0\}\,|X_0 = x, I_0 = i\right), \tag{4.2.6}$$

which we can also express as

$$\psi^{\pi}(x,i) = P^{\pi}\left(X_k < 0 \text{ for some } k \geq 1 | X_0 = x, I_0 = i\right). \tag{4.2.7}$$

Similarly, the ruin probabilities in the finite horizon case are given by

$$\psi_n^{\pi}(x,i) := P^{\pi}\left(\bigcup_{k=1}^{n}\{X_k < 0\}\,|X_0 = x, I_0 = i\right). \tag{4.2.8}$$

Thus,

$$\psi_1^{\pi}(x,i) \leq \psi_2^{\pi}(x,i) \leq \cdots \leq \psi_n^{\pi}(x,i) \leq \cdots,$$

and

$$\lim_{n\to\infty} \psi_n^{\pi}(x,i) = \psi^{\pi}(x,i).$$

The following Lemma is used below to simplify some calculations.

Lemma 4.2.1. *For any given policy π, there is a function $\psi^{\pi}(x)$ such that*

$$\psi^{\pi}(x,i) \leq \psi^{\pi}(x)$$

for every initial state $x > 0$ and initial interest rate $I_0 = i$.

Proof. By (4.2.1) and (4.2.2), the risk model is given by

$$X_n = X_{n-1}(1 + I_n) + \mathcal{C}(b_{n-1}) - b_{n-1}Y_n.$$

Since $I_n \geq 0$, we have

$$X_n = X_{n-1}(1 + I_n) + \mathcal{C}(b_{n-1}) - b_{n-1}Y_n \geq X_{n-1} + \mathcal{C}(b_{n-1}) - b_{n-1}Y_n. \tag{4.2.9}$$

Define recursively

$$\widetilde{X}_n := \widetilde{X}_{n-1} + \mathcal{C}(b_{n-1}) - b_{n-1}Y_n, \tag{4.2.10}$$

with $X_0 = \widetilde{X}_0 = x$. Hence, $X_n \geq \widetilde{X}_n$ for all $n \in \mathbb{N}$. Clearly, if $X_n < 0$, then $\widetilde{X}_n < 0$.
Let

$$\mathcal{E}_1 := \left\{\omega \in \Omega | \bigcup_{n=1}^{\infty}\{X_n(\omega) < 0\}\right\} \text{ and } \mathcal{E}_2 := \left\{\omega \in \Omega | \bigcup_{n=1}^{\infty}\left\{\widetilde{X_n}(\omega) < 0\right\}\right\},$$

and note that $\mathcal{E}_1 \subset \mathcal{E}_2$. Therefore,

$$P^\pi \left(\bigcup_{n=1}^{\infty} \{X_n < 0\} | I_0 = i \right) \le P^\pi \left(\bigcup_{n=1}^{\infty} \left\{ \widetilde{X}_n < 0 \right\} | I_0 = i \right),$$

and since the $\widetilde{X}_n$ do not depend on I_n, we obtain from (4.2.7)

$$\psi^\pi(x,i) = P^\pi \left(\bigcup_{n=1}^{\infty} \{X_n < 0\} | I_0 = i \right) \le P^\pi \left(\bigcup_{n=1}^{\infty} \left\{ \widetilde{X}_n < 0 \right\} \right) =: \psi^\pi(x). \qquad \square$$

$\square$

We denote by Π the policy space. A control policy π^* is said to be optimal if for any initial values $(X_0, I_0) = (x, i)$, we have

$$\psi^{\pi^*}(x,i) \le \psi^\pi(x,i)$$

For all $\pi \in \Pi$. Schmidli [54] and Schäl [48] show that existence of an optimal control policy for some special cases of the model risk (4.2.1). However, even in these special cases it is extremely difficult to obtain closed expressions for $\psi^{\pi^*}(x,i)$. We are thus led to consider bounds for the ruin probabilities, which we do in Subsection 4.2.2, 4.2.3, and 4.2.4, below. First, we note that (4.2.1) includes some interesting ruin models.

Special cases. To conclude this subsection we note the following subcases of the risk model (4.2.1).

- If $I_n = 0$ and $b_n = 1$ for all $n \ge 1$, then (4.2.1) reduces to the classical discrete-time risk model without investment and reinsurance:

$$X_k = x - \sum_{t=1}^{k} (Y_t - c).$$

  This is the well-known Cramér-Lundberg model, for which there are several bounds for the ruin probability, see Section 1.1 and [2, 32, 62].

- If $I_n = 0$ and $b_n \in \mathcal{B}$ for $n \ge 1$, then the risk model reduces to the discrete-time risk model with proportional reinsurance:

$$X_k = x - \sum_{t=1}^{k} (b_{t-1} Y_t - \mathcal{C}(b_{t-1})). \qquad (4.2.11)$$

Let $\psi^\pi(x) := P^\pi\left(\bigcup_{k=1}^{\infty}\{X_k < 0\}\,|X_0 = x\right)$ be the corresponding ruin probability. More explicitly, by (4.2.11),

$$\psi^\pi(x) \;\; = P^\pi\left(\bigcup_{k=1}^{\infty}\left\{\sum_{t=1}^{k}\left[b_{t-1}Y_t - \mathcal{C}(b_{t-1})\right] > x\right\}|X_0 = x\right)$$

If we assume constant stationary strategies, say $b_n = b_0$ for all $n \geq 1$, and in addition $E[b_0 Y] < \mathcal{C}(b_0)$, then there exists a constant $R_0 > 0$ satisfying

$$e^{-R_0\mathcal{C}(b_0)} \cdot E[e^{R_0(b_0 Y)}] = 1. \tag{4.2.12}$$

Therefore, by the classical Lundberg inequality for ruin probabilities (see Subsection 1.2.1 and [2, 32, 62])

$$\psi^\pi(x) \leq e^{-R_0 x}, \text{ for } x \geq 0. \tag{4.2.13}$$

- Let d_n be the constant, short-term dividend rate in the $n-th$ period (the dividends are payments made by a corporation to its shareholder members). Then the discrete-time risk model with stochastic interest rate and dividends is given by

$$X_n = X_{n-1}\,(1 + I_n) + \mathcal{C}(b_{n-1}) - h(b_{n-1}, Y_n) - d_n X_n,$$

with $h(b, y)$ as in (4.2.2). Thus, rearranging terms,

$$X_n = X_{n-1}\left(\frac{1 + I_n}{1 + d_n}\right) + \frac{\mathcal{C}(b_{n-1})}{(1 + d_n)} - \frac{h(b_{n-1}, Y_n)}{(1 + d_n)}.$$

Let $Y_n' := \frac{Y_n}{(1+d_n)}$ and $I_n{}' := \frac{I_n - d_n}{(1+d_n)}$. Since $\{I_n\}$ and $\{Y_n\}$ are independent, then so are $\{I_n{}'\}$ and $\{Y_n'\}$. Let $\mathcal{C}'(b_{n-1}) := \frac{\mathcal{C}(b_{n-1})}{(1+d_n)}$. Then the model becomes

$$X_n = X_{n-1}\,(1 + I_n{}') + \mathcal{C}'(b_{n-1}) - h(b_{n-1}, Y_n'),$$

which from a statistical viewpoint is essentially the same as the model without dividends (4.2.1) and can be analyzed in a similar way.

- As an extension of the latter case, some companies have dividend reinvestment plans (or DRIPs). These plans allow shareholders to use dividends to systematically buy small amounts of stock. Let $\widetilde{d}_n$ be the short term dividend reinvestment rate in the n-th period, $\widetilde{d}_n \in [0,1)$. Then, the discrete-time risk model with stochastic interest rate and dividends reinvestment is given by

$$X_n = X_{n-1}(1+I_n) + \mathcal{C}(b_{n-1}) - h(b_{n-1}, Y_n) + \widetilde{d}_n X_n.$$

Hence, rearranging terms, we obtain

$$X_n = X_{n-1}\left(\frac{1+I_n}{1-\widetilde{d}_n}\right) + \frac{\mathcal{C}(b_{n-1})}{(1-\widetilde{d}_n)} - \frac{h(b_{n-1}, Y_n)}{(1-\widetilde{d}_n)}.$$

Let $Y_n^{"} := \frac{Y_n}{(1-\widetilde{d}_n)}$, $I_n^{"} := \frac{I_n - \widetilde{d}_n}{(1-\widetilde{d}_n)}$, and $\mathcal{C}^{"}(b_{n-1}) := \frac{\mathcal{C}(b_{n-1})}{(1-\widetilde{d}_n)}$. It follows that

$$X_n = X_{n-1}\left(1+I_n^{"}\right) + \mathcal{C}^{"}(b_{n-1}) - h(b_{n-1}, Y_n^{"}),$$

which, again, is essentially the same as the model (4.2.1).

Let us go back to the original risk model (4.2.1). Since determining ruin probabilities is essentially an infinite-horizon problem, it suffices to consider stationary strategies [54]. In the next subsection, we will derive recursive equations for the ruin probabilities and integral equations for the ultimate ruin probability associated to the model (4.2.1).

Remark 4.2.1. Given a p.d.f. G, we denote the tail of G by $\overline{G}$, that is, $\overline{G}(x) := 1 - G(x)$.

4.2.2 Recursive and integral equations for ruin probabilities

In this subsection, we first derive a recursive equation for $\psi_n^\pi(x,i)$. Secondly, we give an integral equation for $\psi^\pi(x,i)$. Finally, we obtain an equation for the ruin probability with horizon $n=1$ given $I_0 = i$, $X_0 = x$ and a stationary policy π. These results, which are valid for any initial interest rate, are summarized in the following Lemma.

Lemma 4.2.2. *Let $u(y) := b_0 y - \mathcal{C}(b_0)$, where b_0 is the initial retention level. Let $\tau_j := (x(1+j) + \mathcal{C}(b_0))/b_0$, $X_0 = x \geq 0$, and p_{ij} as in (4.2.5). Then*

$$\psi_1^\pi(x,i) = \sum_{j\in\mathbb{I}} p_{ij}\overline{F}(\tau_j), \tag{4.2.14}$$

and for $n = 1, 2, \ldots$

$$\psi_{n+1}^\pi(x,i) = \sum_{j\in\mathbb{I}} p_{ij} \int_0^{\tau_j} \psi_n^\pi(x(1+j) - u(y), j)\,dF(y) + \sum_{j\in\mathbb{I}} p_{ij}\overline{F}(\tau_j). \tag{4.2.15}$$

Moreover,

$$\psi^\pi(x,i) = \sum_{j\in\mathbb{I}} p_{ij} \int_0^{\tau_j} \psi^\pi(x(1+j)-u(y),j)dF(y) + \sum_{j\in\mathbb{I}} p_{ij}\overline{F}(\tau_j). \tag{4.2.16}$$

Proof. Let $U_k := u(Y_k) = b_0Y_k - \mathcal{C}(b_0)$. Given $Y_1 = y$, the control strategy π, and $I_1 = j$, from (4.2.4) we have $U_1 = u(y)$. Therefore,

$$X_1 = x(1+I_1) - U_1 = h_1 - u(y), \text{ where } h_1 = x(1+j)$$

Thus, if $u(y) > h_1$ then

$$P^\pi\left(X_1 < 0 | Y_1 = y, I_1 = j, X_0 = x, I_0 = i\right) = 1.$$

This implies that for $u(y) > h_1$

$$P^\pi\left(\bigcup_{k=1}^{n+1}\{X_k < 0\}|Y_1 = y, I_1 = j, X_0 = x, I_0 = i\right) = 1,$$

while if $0 \le u(y) \le h_1$, then

$$P^\pi\left(X_1 < 0 | Y_1 = y, I_1 = j, X_0 = x, I_0 = i\right) = 0. \tag{4.2.17}$$

Let $\{\widetilde{Y}_n\}_{n\ge1}$ and $\{\widetilde{I}_n\}_{n\ge0}$ be independent copies* of $\{Y_n\}_{n\ge1}$, and $\{I_n\}_{n\ge0}$, respectively. Let $\widetilde{U}_k := b_0\widetilde{Y}_k - \mathcal{C}(b_0)$. Thus, (4.2.17) and (4.2.4) yield that for $0 \le u(y) \le h_1$,

$$\begin{aligned}
& P^\pi\left(\bigcup_{k=1}^{n+1}\{X_k < 0\}|Y_1 = y, I_1 = j, X_0 = x, I_0 = i\right) \\
= \;& P^\pi\left(\bigcup_{k=2}^{n+1}\{X_k < 0\}|Y_1 = y, I_1 = j, X_0 = x, I_0 = i\right) \\
= \;& P^\pi\left(\bigcup_{k=2}^{n+1}\left\{(h_1 - u(y))\prod_{l=1}^{k}(1+I_l) - \sum_{l=1}^{k} U_l \prod_{m=l+1}^{k}(1+I_m) < 0\right\}|X_0 = x, I_1 = j\right) \\
= \;& P^\pi\left(\bigcup_{k=1}^{n}\left\{(h_1 - u(y))\prod_{l=1}^{k}(1+\widetilde{I}_l) - \sum_{l=1}^{k} \widetilde{U}_l \prod_{m=l+1}^{k}(1+\widetilde{I}_m) < 0\right\}|X_0 = x, \widetilde{I}_0 = j\right) \\
= \;& \psi_n^\pi(h_1 - u(y), j) = \psi_n^\pi(x(1+j) - u(y), j)
\end{aligned}$$

where the second equality follows from the Markov property of $\{I_n\}_{n\ge0}$, and the independence of $\{Y_n\}_{n\ge1}$ and $\{I_n\}_{n\ge0}$.

*An independent copy of a process X is an independent process Y with the same distributional properties.

Let us now consider the event $\mathcal{A} = \{Y_1 = y, I_1 = j, X_0 = x, I_0 = i\}$, and recall that $F(y) = P(Y \leq y)$. From (4.2.8) and (4.2.4) we obtain

$$\begin{aligned}\psi^{\pi}_{n+1}(x,i) &= P^{\pi}\left(\bigcup_{k=1}^{n+1}\{X_k < 0\}\,|X_0 = x, I_0 = i\right)\\ &= \sum_{j\in\mathbb{I}} p_{ij}\int_0^{\infty} P^{\pi}\left(\bigcup_{k=1}^{n+1}\{X_k < 0\}\,|\mathcal{A}\right) dF(y).\end{aligned}$$

Then, recalling that $\tau_j = \frac{x(1+j)+\mathcal{C}(b_0)}{b_0}$,

$$\begin{aligned}\psi^{\pi}_{n+1}(x,i) &= \sum_{j\in\mathbb{I}} p_{ij}\left\{\int_0^{\tau_j} P^{\pi}\left(\bigcup_{k=1}^{n+1}\{X_k < 0\}\,|\mathcal{A}\right) dF(y)\right.\\ &+ \left.\int_{\tau_j}^{\infty} P^{\pi}\left(\bigcup_{k=1}^{n+1}\{X_k < 0\}\,|\mathcal{A}\right) dF(y)\right\}\\ &= \sum_{j\in\mathbb{I}} p_{ij}\left\{\int_0^{\tau_j}\psi^{\pi}_n(x(1+j)-u(y),j)dF(y) + \int_{\tau_j}^{\infty} dF(y)\right\}\\ &= \sum_{j\in\mathbb{I}} p_{ij}\left\{\int_0^{\tau_j}\psi^{\pi}_n(x(1+j)-u(y),j)dF(y) + \overline{F}(\tau_j))\right\}.\end{aligned} \tag{4.2.18}$$

This gives (4.2.15). In particular,

$$\psi^{\pi}_1(x,i) = \sum_{j\in\mathbb{I}} p_{ij}\overline{F}(\tau_j).$$

Finally, letting $n \to \infty$ in (4.2.18) and using dominated convergence we obtain $\lim_{n\to\infty}\psi^{\pi}_{n+1}(x,i) = \psi^{\pi}(x,i)$, and (4.2.16) follows. □

Remark 4.2.2. If we consider the risk model without reinsurance, that is, $b = 1$, we obtain similar results to those in Cai and Dickson [7].

4.2.3 Bounds for ruin probabilities

We will use the results obtained in Subection 4.2.2 to find upper bounds for the ruin probabilities with infinite horizon taking into account the information contributed by the Markov chain of the interest rate process. We derive a functional for the ultimate ruin probability in terms of the new worse than used in convex (NWUC) ordering (see Subsection 2.1.3. This idea was first introduced by Willmot and Lin [62] and has been generalized by other authors.

We will present two upper bounds for the ruin probabilities. The first bound is obtained by an inductive approach, and the second by a martingale approach. These bounds are discussed in Remark 4.2.3, at the end of this subsection.

4.2.3.1 Bounds obtained by the inductive approach

Theorem 4.2.3. *Let $R_0 > 0$, be the constant satisfying (4.2.12). Then, for all $x \geq 0$ and $i \in \mathbb{I}$,*

$$\begin{aligned} \psi^{\pi}(x,i) &\leq \beta \sum_{j\in\mathbb{I}} p_{ij} E^{\pi}[e^{-R_0 x(1+j)}] \\ &= \beta E^{\pi}[e^{-R_0[x(1+I_1)]} | I_0 = i], \end{aligned} \tag{4.2.19}$$

where $\beta \equiv \beta(b_0)$ and is given by

$$\beta^{-1} = \inf_{t\geq 0} \frac{\int_t^{\infty} e^{R_0 b_0 y} dF(y)}{e^{R_0 b_0 t}\overline{F}(t)}.$$

Proof. It suffices to show that the rightmost term in (4.2.19) is an upper bound for $\Psi_n^{\pi}(x,i)$, for all $n \geq 1$. We will prove this by induction. First note that

$$\begin{aligned} \overline{F}(\theta) &= \left(\frac{\int_\theta^{\infty} e^{R_0 b_0 y} dF(y)}{e^{R_0 b_0 \theta}\overline{F}(\theta)}\right)^{-1} e^{-R_0 b_0 \theta} \int_\theta^{\infty} e^{R_0 b_0 y} dF(y) \\ &\leq \beta e^{-R_0 b_0 \theta} \int_\theta^{\infty} e^{R_0 b_0 y} dF(y) \leq \beta e^{-R_0 b_0 \theta} E^{\pi}[e^{R_0 b Y_1}] \end{aligned} \tag{4.2.20}$$

for any $\theta \geq 0$. This implies that for every $x \geq 0$, $i \geq 0$, and $b_0 \in \mathcal{B}$, by (4.2.14) and (4.2.20) we have

$$\begin{aligned} \psi_1^{\pi}(x,i) &= \sum_{j\in\mathbb{I}} p_{ij} \overline{F}(\tau_j) \\ &\leq \sum_{j\in\mathbb{I}} p_{ij} \left(\beta E^{\pi}[e^{R_0 b Y_1}] \cdot e^{-R_0 b_0 \left(\frac{x(1+j)+\mathcal{C}(b_0)}{b_0}\right)}\right) \\ &= \beta E^{\pi}[e^{R_0 b Y_1}] \sum_{j\in\mathbb{I}} p_{ij} e^{-R_0[x(1+j)+\mathcal{C}(b_0)]} \\ &= \beta E^{\pi}[e^{R_0 b Y_1}] \cdot E^{\pi}[e^{-R_0[x(1+I_1)+\mathcal{C}(b)]} | I_0 = i] \\ &= \beta E^{\pi}[e^{R_0 b Y_1}] E^{\pi}[e^{-R_0 \mathcal{C}(b)}] E^{\pi}[e^{-R_0 x(1+I_1)} | I_0 = i] \end{aligned}$$

$$\begin{aligned} &= \beta E^{\pi}[e^{-R_0[\mathcal{C}(b)-bY_1]}] \cdot E^{\pi}[e^{-R_0 x(1+I_1)} | I_0 = i] \\ &= \beta E^{\pi}[e^{-R_0 x(1+I_1)} | I_0 = i] \qquad \text{(by (4.2.12))}. \end{aligned}$$

This shows that the desired result holds for $n = 1$. To prove the result for general $n \geq 1$, the induction hypothesis is that, for some $n \geq 1$, and every $x \geq 0$ and $i \in \mathbb{I}$,

$$\psi_n^\pi(x,i) \leq \beta E^\pi[e^{-R_0 x(1+I_1)}|I_0 = i]. \tag{4.2.21}$$

Now, let $0 \leq y \leq \tau_j$, with τ_j as in Lemma 4.2.2. Further, in (4.2.21) replace x and i by $x(1+j) + \mathcal{C}(b_0) - b_0 y$ and j, respectively to obtain

$$\begin{aligned} \psi_n^\pi(x(1+j) + \mathcal{C}(b_0) - b_0 y, j) &\leq \beta E^\pi[e^{-R_0[x(1+j)+\mathcal{C}(b)-by](1+I_1)}|I_0 = j] \\ &\leq \beta e^{-R_0[x(1+j)+\mathcal{C}(b_0)-b_0 y]}. \end{aligned} \tag{4.2.22}$$

Therefore, replacing (4.2.22) in (4.2.15), we get

$$\begin{aligned} \psi_{n+1}^\pi(x,i) &\leq \sum_{j\in\mathbb{I}} p_{ij} \left(\beta e^{-R_0[x(1+j)+\mathcal{C}(b_0)]} \textstyle\int_{\tau_j}^\infty e^{R_0 b_0 y} dF(y)\right) \\ &+ \sum_{j\in\mathbb{I}} p_{ij} \left(\beta e^{-R_0[x(1+j)+\mathcal{C}(b_0)]} \textstyle\int_0^{\tau_j} e^{R_0 b_0 y} dF(y)\right) \\ &= \sum_{j\in\mathbb{I}} p_{ij} \left(\beta e^{-R_0[x(1+j)+\mathcal{C}(b_0)]} \textstyle\int_0^\infty e^{R_0 b_0 y} dF(y)\right) \\ &= \beta E^\pi[e^{R_0 b Y_1}] \sum_{j\in\mathbb{I}} p_{ij} e^{-R_0[x(1+j)+\mathcal{C}(b_0)]} \\ &= \beta E^\pi[e^{R_0 b Y_1}] \cdot E^\pi[e^{-R_0 \mathcal{C}(b)}] \cdot E^\pi[e^{-R_0 x(1+I_1)}|I_0 = i] \\ &= \beta E^\pi[e^{-R_0 x(1+I_1)}|I_0 = i]. \end{aligned}$$

Hence, (4.2.21) holds for any $n = 1, 2, \ldots$, and letting $n \to \infty$ in (4.2.21) we obtain (4.2.19). $\square$

As an application of Theorem 4.2.3, we next consider the special case in which the claim distribution is in the class of NWUC distributions[†].

Corollary 4.2.4. *Under the hypotheses of Theorem 4.2.3, and assuming that $E^\pi[e^{R_0 b Y_1}] < \infty$ for all $b \in \mathcal{B}$, and that, in addition, F is a NWUC distribution, we have*

$$\psi^\pi(x,i) \leq (E^\pi[e^{R_0 b Y_1}])^{-1} E^\pi[e^{-R_0 x(1+I_1)}|I_0 = i]. \tag{4.2.23}$$

Proof. Following the proof of proposition 2.1.1, let $r := R_0 b > 0$. Therefore

$$\beta^{-1} := \inf_{t\geq 0} \frac{\int_t^\infty e^{ry} dF(y)}{e^{rt}\overline{F}(t)} = \int_0^\infty e^{ry} dF(y),$$

that is, $\beta^{-1} = E^\pi[e^{R_0 b Y_1}]$. Finally, replacing this equality in (4.2.19), we obtain (4.2.23). $\square$

[†]See Subsection 2.1.3 equation (2.1.12).

4.2.3.2 Bounds by means of the martingale approach

Another way for deriving upper bounds for ruin probabilities is the martingale approach. To this end, let $V_n := X_n \prod_{l=1}^{n}(1+I_l)^{-1}$ with $n \geq 1$, be the so-called discounted risk process. The ruin probabilities ψ_n^π in (4.2.8) associated to the process $\{V_n, n = 1, 2 \ldots\}$ are

$$\psi_n^\pi(x,i) \quad = \quad P^\pi\left(\bigcup_{k=1}^{n}(V_k < 0)|X_0 = x, I_0 = i\right)$$

In the classical risk model, $\{e^{-R_0 X_n}\}_{n\geq 1}$ is a martingale. However, for our model (4.2.4), there is no constant $r > 0$ such that $\{e^{-rX_n}\}_{n\geq 1}$ is a martingale. Still, there exists a constant $r > 0$ such that $\{e^{-rV_n}\}_{n\geq 1}$ is a *supermartingale*, which allows us to derive probability inequalities by the optional stopping theorem. Such a constant is defined in the following proposition.

Proposition 4.2.5. *Assume that* $E^\pi\left[\mathcal{C}(b) - bY_1\right] > 0$. *In addition we suppose that for each* $i \in \mathbb{I}$, *there exists* $\rho_i > 0$ *satisfying that*

$$E^\pi\left[e^{-\rho_i[\mathcal{C}(b)-bY_1](1+I_1)^{-1}}|I_0 = i\right] = 1. \tag{4.2.24}$$

Then

$$R_1 := \min_{i\in\mathbb{I}} \rho_i \geq R_0 \tag{4.2.25}$$

and, furthermore, for all $i \in \mathbb{I}$

$$E^\pi\left[e^{-R_1[\mathcal{C}(b)-bY_1](1+I_1)^{-1}}|I_0 = i\right] \leq 1. \tag{4.2.26}$$

Proof. For each $i \in \mathbb{I}$, let

$$l_i(r) := E^\pi\left[e^{-r[\mathcal{C}(b)-bY](1+I_1)^{-1}}|I_0 = i\right] - 1, \quad \text{for } r > 0.$$

Then the first derivative of $l_i(r)$ at $r = 0$ is

$$l_i^{'}(0) = E^\pi\left[-(\mathcal{C}(b) - bY)\right] \cdot E^\pi\left[(1+I_1)^{-1}|I_0 = i\right] < 0 \quad \text{(by independence)},$$

and the second derivative is

$$l_i^{''}(r) = E^\pi\left[\left((\mathcal{C}(b) - bY)(1+I_1)^{-1}\right)^2 \cdot e^{-r[\mathcal{C}(b)-bY](1+I_1)^{-1}}|I_0 = i\right] > 0.$$

This shows that $l_i(r)$ is a convex function. Let ρ_i be the unique positive root of the equation $l_i(r) = 0$ on $(0, \infty)$. Further, if $0 < \rho \leq \rho_i$, then $l_i(\rho) \leq 0$. However,

$$\begin{aligned} E^\pi\left[e^{-R_0[\mathcal{C}(b)-bY](1+I_1)^{-1}}|I_0 = i\right] &= \sum_{j\in\mathbb{I}} p_{ij} E\left[e^{-R_0[\mathcal{C}(b_0)-b_0Y](1+j)^{-1}}\right] \\ \text{(by Jensen's inequality)} \qquad &\leq \sum_{j\in\mathbb{I}} p_{ij} E\left[e^{-R_0[\mathcal{C}(b_0)-b_0Y_1]}\right]^{(1+j)^{-1}}. \end{aligned}$$

Consequently, by (4.2.12), we have that $E\left[e^{-R_0[\mathcal{C}(b_0)-b_0Y_1]}\right] = 1$. Hence, since $\sum\limits_{j\in\mathbb{I}} p_{ij} = 1$,

$$E^{\pi}\left[e^{-R_0[\mathcal{C}(b)-bY](1+I_1)^{-1}}|I_0 = i\right] \;\leq\; 1.$$

This implies that $l_i(R_0) \leq 0$. Moreover, $R_0 \leq \rho_i$ for all i, and so

$$R_1 := \min_{i\in\mathbb{I}} \rho_i \geq R_0.$$

Thus, (4.2.25) holds. In addition $R_1 \leq \rho_i$ for all $i \in \mathbb{I}$, which implies that $l_i(R_1) \leq 0$. This yields (4.2.26). □

Theorem 4.2.6. *Under the hypotheses of Proposition* 1*, for all $i \in \mathbb{I}$ and $x \geq 0$,*

$$\psi^{\pi}(x, i) \leq e^{-R_1 x}. \tag{4.2.27}$$

Proof. By (4.2.4), the discounted risk process $V_k := X_k \prod\limits_{l=1}^{k}(1 + I_l)^{-1}$ satisfies that

$$V_k := x + \sum_{l=1}^{k}\left((\mathcal{C}(b_0) - b_0Y_l)\prod_{t=1}^{l}(1 + I_t)^{-1}\right). \tag{4.2.28}$$

Let $S_n = e^{-R_1 V_n}$. Then

$$S_{n+1} = S_n e^{-R_1(\mathcal{C}(b_0)-b_0Y_{n+1})\prod\limits_{t=1}^{n+1}(1+I_t)^{-1}}.$$

Thus, for any $n \geq 1$,

$$\begin{aligned}
&E^{\pi}[S_{n+1} \mid Y_1, \dots Y_n, I_1, \dots I_n] \\
&= S_n E\left[e^{-R_1(\mathcal{C}(b_0)-b_0Y_{n+1})\prod\limits_{t=1}^{n+1}(1+I_t)^{-1}} \mid Y_1, \dots Y_n, I_1, \dots I_n\right] \\
&= S_n E\left[e^{-R_1(\mathcal{C}(b_0)-b_0Y_{n+1})(1+I_{n+1})^{-1}\prod\limits_{t=1}^{n}(1+I_t)^{-1}} \mid I_1, \dots I_n\right] \\
&\leq S_n E\left(\left[e^{-R_1(\mathcal{C}(b_0)-b_0Y_{n+1})(1+I_{n+1})^{-1}} \mid I_1, \dots I_n\right]\right)^{\prod\limits_{t=1}^{n}(1+I_t)^{-1}} \\
&\leq S_n.
\end{aligned}$$

This implies that $\{S_n\}_{n\geq 1}$ is a supermartingale.
Let $T_i = \min\{n : V_n < 0 \mid I_0 = i\}$, where V_n is given by (4.2.28). Then T_i is a stopping time and $n \wedge T_i := \min\{n, T_i\}$ is a finite stopping time. Thus, by the optional stopping theorem for martingales, we get

$$E^{\pi}(S_{n\wedge T_i}) \leq E^{\pi}(S_0) = e^{-R_1 x}.$$

Hence,

$$\begin{aligned} e^{-R_1 x} &\geq E^{\pi}(S_{n\wedge T_i}) \geq E^{\pi}((S_{n\wedge T_i}) I_{(T_i \leq n)}) \geq E^{\pi}((S_{T_i}) I_{(T_i \leq n)}) \\ &= E^{\pi}(e^{-R_1 V_{T_i}} I_{(T_i \leq n)}) \geq E^{\pi}(I_{(T_i \leq n)}) = \psi_n^{\pi}(x,i), \end{aligned} \tag{4.2.29}$$

where (4.2.29) follows because $V_{T_i} < 0$. Thus, by letting $n \to \infty$ in (4.2.29) we obtain (4.2.27). □

Remark 4.2.3. Summarizing, we have three upper bounds for the ruin probabilities with infinite horizon. First, the Lundberg bound, which only depends on R_0, the Lundberg exponential in (4.2.12), (4.2.13). Second, the inductive bound (4.2.19) which depends on R_0 and also on the interest rate process. Third, the martingale bound in (4.2.27), which depends on R_1. Note that the last two bounds are sharper than the Lundberg bound. Observe also that the number of operations to get R_1 in (4.2.27) is higher than that to get R_0 in (4.2.19).

In the next subsection we present some numerical results.

4.2.4 Numerical results

To illustrate the bounds obtained in Theorems 4.2.3 and 4.2.6 we present a numerical example that uses Matlab and Maple implementations for different values of the retention level b.

We assume that the claim amount in year k is Y_k, which has a gamma density $Gamma(\frac{1}{2}, 2)$. Since this distribution is NWUC, we will use (4.2.23).

The annual premium is $c = 1.1$; namely, there is a loading of 10% given by the reinsurer. In this example, $\mathcal{C}(b) = 1.1 - (1.1) \cdot (1 - b) > 0$ if $b \in (0, 1]$.

Consider an interest model with three possible interest rates:

$$\mathbb{I} = \{6\%, 8\%, 10\%\}.$$

The transition matrix (see (4.2.5)) is given by

$$\begin{pmatrix} 0.2 & 0.8 & 0 \\ 0.15 & 0.7 & 0.15 \\ 0 & 0.8 & 0.2 \end{pmatrix}.$$

Thus, our interest rate model incorporates mean reversion to a level of 8%. In this example $\mathcal{B} = (0, 1]$.

In figure 1, we show that the relation between R_l and b, with $l \in \{0, 1\}$, is inversely proportional. Remember that $b = 1$ means no reinsurance, and so we will hope to have a small value of R_l. Analogously, when b is close to zero (reinsure almost everything), R_l becomes extremely large.

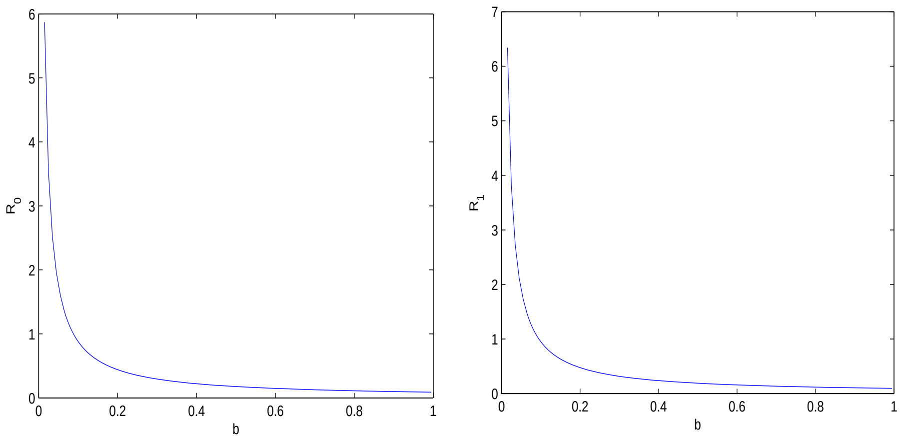

Figure 4.1: In the left side: The relation between R_0 and b. In the right side: The relation between R_1 and b.

The numerical results in Table 4.2.4 show that the upper bound in (4.2.19) can be tighter than that in (4.2.27). This suggests that the upper bounds derived by the inductive approach are tighter than the upper bounds obtained by supermartingales. In addition, Table 4.2.4 shows that the upper bounds derived in this section are sharper than the Lundberg upper bound.

Retention level b	Lundberg	Induction	Martingale	R_0	R_1
0.01	$0.752e-19$	$0.226e-20$	$0.224e-20$	8.8067	9.5091
0.25	0.171	0.135	0.149	0.352	0.380
0.5	0.414	0.350	0.386	0.176	0.190
0.75	0.555	0.481	0.530	0.117	0.126
1	0.643	0.564	0.621	0.0880	0.0950

Table 4.1: Table of upper bounds for ruin probabilities with $x = 5$ and $i = 8\%$

Remark 4.2.4. In the case $b = 1$, our results are the same as in Cai and Dickson [7].

4.3 Random length of periods and individual claim

The outline of the section is as follows. In Subsection 4.3.1 the risk model is formulated. Some important special cases of this model are briefly discussed. In Subsection 4.3.2 we derive recursive equations for finite-horizon ruin probabilities and integral equations for the ultimate ruin probability. In Subsection 4.3.3 we obtain upper bounds for the ultimate probability of ruin. An analysis of the new bounds and a comparison with the Lundberg's inequality is also included. Finally, in Subsection 4.3.4 we illustrate our results on the ruin probability in a risk process with a heavy tail claims distribution under proportional reinsurance and a Markov interest rate process‡.

4.3.1 The model

We consider a discrete-time insurance risk process in which the surplus X_n varies according to the equation

$$X_n = X_{n-1}\,(1 + I_n) + \mathcal{C}(b_{n-1}) \cdot Z_n - h(b_{n-1}, Y_n), \tag{4.3.1}$$

for $n \geq 1$, with $X_0 = x \geq 0$. As Subsection 4.2.1, we introduce an absorbing (cemetery) state $\varkappa$, such that if $X_n < 0$ or $X_n = \varkappa$, then $X_{n+1} = \varkappa$. We denote the state space by $\mathbb{X} = \mathbb{R} \cup \varkappa$. Let Y_n be the *n-th claim* payment, which we assume to form a sequence of i.i.d. random variables with common probability distribution function (p.d.f.) F. The random variable Z_n stands for the *length of the n-th period*, that is, the time between the ocurrence of the claims Y_{n-1} and Y_n. We assume that $\{Z_n\}$ is a sequence of i.i.d. random variables with p.d.f. G. This case includes a controlled version of the Cramér-Lundberg model if we assume that the claims occur as a Poisson process. Of course, we can also think of the case where $Z_n = 1$ is deterministic. In addition, we suppose that $\{Y_n\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 1}$ are independent.

The process can be controlled by reinsurance, that is, by choosing the *retention level* (or proportionality factor or risk exposure) $b \in \mathcal{B}$ of a reinsurance contract for one period, where

‡This section was submitted, see [15].

$\mathcal{B} := [b_{\min}, 1]$, and $b_{\min} \in (0,1]$ will be introduced below. Let $\{I_n\}_{n\geq 0}$ be the *interest rate process*; we suppose that I_n evolves as a Markov chain with a denumerable (possibly finite) state space $\mathbb{I}$ consisting of nonnegative integers.

We consider the case of proportional reinsurance, then the function $h(b, y)$ is given by (4.2.2). The premium (income) rate c is fixed. Since the insurer pays to the reinsurer a premium rate, which depends on the retention level b, we denote by $\mathcal{C}(b)$ the premium left for the insurer if the retention level b is chosen, where

$$0 \leq \mathcal{C}(b) \leq c, \qquad b \in \mathcal{B}.$$

As Section 4.2, we define $b_{\min} := \min\{b \in (0,1] | \mathcal{C}(b) \geq 0\}$. Moreover, $\mathcal{C}(b)$ is an increasing function that we will calculate according to the *expected value principle* with added safety loading θ from the reinsurer:

$$\mathcal{C}(b) = c - (1+\theta)(1-b)\frac{E[Y]}{E[Z]}, \tag{4.3.2}$$

where Y and Z are generic random variables with p.d.f. F and G, respectively.

We consider Markovian control policies $\pi = \{a_n\}_{n\geq 1}$, which at each time n depend only on the current state, that is, $a_n(X_n) := b_n$ for $n \geq 0$. Abusing notation, we will identify functions $a : \mathbb{X} \to \mathcal{B}$ with stationary strategies, where $\mathcal{B} = [b_{\min}, 1]$, the decision space. Consider an arbitrary initial state $X_0 = x \geq 0$ (note that the initial value is not stochastic) and a control policy $\pi = \{a_n\}_{n\geq 1}$. Then, by iteration of (4.3.1) and assuming (4.2.2), and (4.3.2), it follows that for $n \geq 1$, X_n satisfies

$$X_n = x\prod_{l=1}^{n}(1+I_l) + \sum_{l=1}^{n}\left(\mathcal{C}(b_{l-1})Z_l - b_{l-1}\cdot Y_l \prod_{m=l+1}^{n}(1+I_m)\right). \tag{4.3.3}$$

Let (p_{ij}) be the matrix of transition probabilities of $\{I_n\}$, i.e.,

$$p_{ij} := P\left(I_{n+1} = j | I_n = i\right), \tag{4.3.4}$$

where $p_{ij} \geq 0$ and $\sum_j p_{ij} = 1$ for all $i, j \in \mathbb{I}$. The ruin probability when using the policy π, given the initial surplus x, and the initial interest rate $I_0 = i$ is defined as

$$\psi^\pi(x,i) := P^\pi\left(\bigcup_{k=1}^{\infty}\{X_k < 0\}\,|X_0 = x, I_0 = i\right), \tag{4.3.5}$$

which we can also express as

$$\psi^\pi(x,i) = P^\pi\left(X_k < 0 \text{ for some } k \geq 1|X_0 = x, I_0 = i\right). \tag{4.3.6}$$

Similarly, the ruin probabilities in the finite horizon case are given by

$$\psi_n^\pi(x,i) := P^\pi\left(\bigcup_{k=1}^{n}\{X_k < 0\}\,|X_0 = x, I_0 = i\right). \tag{4.3.7}$$

Thus,

$$\psi_1^\pi(x,i) \leq \psi_2^\pi(x,i) \leq \cdots \leq \psi_n^\pi(x,i) \leq \cdots,$$

and

$$\lim_{n\to\infty} \psi_n^\pi(x,i) = \psi^\pi(x,i).$$

The following lemma is an extension of Lemma 4.2.1 and it is used below to simplify some calculations.

Lemma 4.3.1. *For any given policy π, there is a function $\psi^\pi(x)$ such that*

$$\psi^\pi(x,i) \leq \psi^\pi(x)$$

for every initial state $x > 0$ and initial interest rate $I_0 = i$.

Proof. By (4.3.1) and (4.2.2), the risk model is given by

$$X_n = X_{n-1}\left(1 + I_n\right) + \mathcal{C}(b_{n-1})Z_n - b_{n-1}Y_n.$$

Since $I_n \geq 0$, we have

$$\begin{aligned} X_n &= X_{n-1}\left(1 + I_n\right) + \mathcal{C}(b_{n-1})Z_n - b_{n-1}Y_n \\ &\geq X_{n-1} + \mathcal{C}(b_{n-1})Z_n - b_{n-1}Y_n. \end{aligned} \tag{4.3.8}$$

Define recursively

$$\widetilde{X}_n := \widetilde{X}_{n-1} + \mathcal{C}(b_{n-1})Z_n - b_{n-1}Y_n, \tag{4.3.9}$$

with $X_0 = \widetilde{X}_0 = x$. Hence, $X_n \geq \widetilde{X}_n$ for all $n \in \mathbb{N}$. Clearly, if $X_n < 0$, then $\widetilde{X}_n < 0$. Let

$$\mathcal{E}_1 := \left\{ \omega \in \Omega | \bigcup_{n=1}^{\infty} \{X_n(\omega) < 0\} \right\} \text{ and } \mathcal{E}_2 := \left\{ \omega \in \Omega | \bigcup_{n=1}^{\infty} \left\{ \widetilde{X_n}(\omega) < 0 \right\} \right\},$$

and note that $\mathcal{E}_1 \subset \mathcal{E}_2$. Therefore,

$$P^\pi \left(\bigcup_{n=1}^{\infty} \{X_n < 0\} | I_0 = i \right) \leq P^\pi \left(\bigcup_{n=1}^{\infty} \left\{ \widetilde{X}_n < 0 \right\} | I_0 = i \right),$$

and since the $\widetilde{X}_n$ do not depend on I_n, we obtain from (4.3.5)

$$\begin{aligned} \psi^\pi(x,i) &= P^\pi \left(\textstyle\bigcup_{n=1}^{\infty} \{X_n < 0\} | X_0 = x, I_0 = i \right) \\ &\leq P^\pi \left(\textstyle\bigcup_{n=1}^{\infty} \left\{ \widetilde{X}_n < 0 \right\} | X_0 = x \right) =: \psi^\pi(x). \end{aligned}$$

□

We denote by Π the policy space. A control policy π^* is said to be optimal if for any initial values $(X_0, I_0) = (x, i)$, we have

$$\psi^{\pi^*}(x,i) \leq \psi^\pi(x,i)$$

for all $\pi \in \Pi$. Schmidli [54] and Schäl [48] show the existence of an optimal control policy for some special cases of the model risk (4.3.1). However, even in these special cases it is extremely difficult to obtain closed expressions for $\psi^{\pi^*}(x,i)$. We are thus led to consider bounds for the ruin probabilities, which we do in Subsections 4.3.2, 4.3.3, and 4.3.4, below. First, we note that (4.3.1) includes some interesting ruin models.

Special cases. To conclude this section we note the following subcases of the risk model (4.3.1).

1. If $I_n = 0$ for all $n \geq 1$, then the risk model (4.3.3) reduces to the discrete-time risk model with proportional reinsurance:

$$X_n = x - \sum_{t=1}^{n} (b_{t-1} Y_t - \mathcal{C}(b_{t-1}) Z_t). \tag{4.3.10}$$

or equivalently,

$$X_n = X_{n-1} + \mathcal{C}(b_{n-1})Z_n - b_{n-1}Y_n.$$

The corresponding ruin probability is

$$\psi^{\pi}(x) := P^{\pi}\left(\bigcup_{n=1}^{\infty}\{X_n < 0\}\,|X_0 = x\right).$$

Assuming, constant stationary strategies, say $b_n = b_0$, then, by (4.3.10),

$$\psi^{\pi}(x) \quad = P^{\pi}\left(\bigcup_{n=1}^{\infty}\left\{\sum_{t=1}^{n}\left[b_0Y_t - \mathcal{C}(b_0)Z_t\right] > x\right\}|X_0 = x\right).$$

Moreover, if we assume that $b_0E[Y] < \mathcal{C}(b_0)E[Z]$, then there exists a constant $R_0 \equiv R_0(b_0) > 0$ satisfying

$$E\left[e^{-R_0(\mathcal{C}(b_0)Z - b_0Y)}\right] = 1. \tag{4.3.11}$$

Therefore, by the classical Lundberg inequality for ruin probabilities (see [2, 32, 62]), for $x \geq 0$

$$\psi^{\pi}(x) \leq e^{-R_0x}. \tag{4.3.12}$$

Since determining ruin probabilities is essentially an infinite-horizon problem, it suffices to consider stationary strategies.

Remark 4.3.1. It is enough to consider constant stationary strategies in this section, i.e, $b_n = b$ for all $n \geq 1$ and we will to argue: first, we assume that $P\left(bY > \mathcal{C}(b)Z\right) > 0$ for all $b \in \mathcal{B}$. Because, if there is some $b_e \in \mathcal{B}$ such that $P\left(b_eY > \mathcal{C}(b_e)Z\right) = 0$, the ruin can be prevented by retention level b_e and the risk process considered in this case becomes trivial. Secondly, we assume the net profit condition $E^{\pi}\left[\mathcal{C}(b)Z - bY\right] > 0$ for some $\pi \in \Pi$. Otherwise, ruin cannot be prevented because the surplus would be decreasing in time for all reinsurance treaties. Therefore, using the law of large numbers, we have

$$\frac{1}{n}\sum_{i=1}^{n}\left[\mathcal{C}(b)Z_i - bY_i\right] \to E^{\pi}\left[\mathcal{C}(b)Z - bY\right],$$

this implies that for the stationary strategy $b_n = b$ the process X_n tends to infinity (in particular, $\inf_n X_n > -\infty$). Hence, there is an initial capital $X_0 = x$ such that $P\left(\inf_n X_n \geq 0|X_0 = x\right) > 0$. Because there is a strictly positive probability that from initial capital zero the set $[x, \infty)$ is reached before the set $(-\infty, 0)$, we get also that $P\left(\inf_n X_n \geq 0|X_0 = 0\right) > 0$. Finally, we have a stationary strategy for which ruin is not certain.

2. **Exponentially distributed length periods**. Our process (4.3.1) a controlled version of Cramér-Lundberg model if the claims occur as a Poisson process, in which case the Z_n are exponentially distributed, say $Z_n \simeq Exp(\lambda)$. Suppose that, in addition, $I_n = 0$ for all $n \geq 1$, and that the single claims have expectation μ and moment generating function $M_Y(s)$. Thus, Y_n has a compound distribution with expectation $\lambda\mu$ and moment generation function $e^{[\lambda(M_Y(s)-1)]}$. Let $M_Y(b;r) := \int_0^\infty e^{bry} dF(y)$ be the moment-generating function of the part of the claim the insurer has to pay if the retention level b is chosen. We assume constant stationary strategies, say $b_n = b_0$ for all $n \geq 1$. Moreover, we assume that $\mathcal{C}(b_0) > b_0\lambda\mu$ and $M_Y(b_0;r) < \infty$ for some $r > 0$ and $b_0 \in \mathcal{B}$. It is clear that the risk process $X_n - x = \sum_{k=1}^n (\mathcal{C}(b)Z_n - bY_k)$ satisfies all the hypotheses of theorem 1.2.2. Then

$$E^\pi \left[e^{-R_0[\mathcal{C}(b)-bY_n]}\right] = e^{-R_0\mathcal{C}(b_0)} \cdot e^{[\lambda(M_Y(b_0;R_0)-1)]}.$$

Then, by (4.3.11), we have that the adjustment coefficient $R_0 = R_0(b_0)$ fulfils

$$-R_0\mathcal{C}(b_0) + \lambda(M_Y(b_0;R_0) - 1) = 0. \tag{4.3.13}$$

By Lemma 4.1 Schmidli [54], R_0 is unimodal and it attains its maximum value at a point $b_0^* \in \mathcal{B}$. Then, it is easy to see that it is optimal to have no reinsurance ($b_0^* = 1$) if and only if the safety loading θ is too high in the sense that

$$1 + \theta \geq \frac{M_Y'(1, R_0)}{\mu}. \tag{4.3.14}$$

3. Let d_n be the constant, short-term dividend rate in the $n-th$ period (the dividends are payments made by a corporation to its shareholder members). Then the discrete-time risk model with stochastic interest rate and dividends is given by

$$X_n = X_{n-1}(1 + I_n) + \mathcal{C}(b_{n-1})Z_n - h(b_{n-1}, Y_n) - d_nX_n,$$

with $h(b,y)$ as in (4.2.2). Thus, rearranging terms,

$$X_n = X_{n-1}\left(\frac{1 + I_n}{1 + d_n}\right) + \frac{\mathcal{C}(b_{n-1})}{(1 + d_n)}Z_n - \frac{h(b_{n-1}, Y_n)}{(1 + d_n)}.$$

Let $Y_n' := \frac{Y_n}{(1+d_n)}$ and $I_n{}' := \frac{I_n - d_n}{(1+d_n)}$. Since $\{I_n\}$ and $\{Y_n\}$ are independent, then so are $\{I_n{}'\}$ and $\{Y_n'\}$. Let $\mathcal{C}'(b_{n-1}) := \frac{\mathcal{C}(b_{n-1})}{(1+d_n)}$. Then the model becomes

$$X_n = X_{n-1}\left(1 + I_n{}'\right) + \mathcal{C}'(b_{n-1})Z_n - h(b_{n-1}, Y_n'),$$

which from a statistical viewpoint is essentially the same as the model without dividends (4.3.1) and can be analyzed in a similar way.

4. As an extension of the latter case, some companies have dividend reinvestment plans (or DRIPs). These plans allow shareholders to use dividends to systematically buy small amounts of stock. Let $\widetilde{d}_n$ be the short term dividend reinvestment rate in the n-th period, $\widetilde{d}_n \in [0, 1)$. Then, the discrete-time risk model with stochastic interest rate and dividends reinvestment is given by

$$X_n = X_{n-1}\left(1 + I_n\right) + \mathcal{C}(b_{n-1})Z_n - h(b_{n-1}, Y_n) + \widetilde{d}_n X_n.$$

Hence, rearranging terms, we obtain

$$X_n = X_{n-1}\left(\frac{1 + I_n}{1 - \widetilde{d}_n}\right) + \frac{\mathcal{C}(b_{n-1})}{(1 - \widetilde{d}_n)} Z_n - \frac{h(b_{n-1}, Y_n)}{(1 - \widetilde{d}_n)}.$$

Let $Y_n^{"} := \frac{Y_n}{(1-\widetilde{d}_n)}$, $I_n{}^{"} := \frac{I_n - \widetilde{d}_n}{(1-\widetilde{d}_n)}$, and $\mathcal{C}^{"}(b_{n-1}) := \frac{\mathcal{C}(b_{n-1})}{(1-\widetilde{d}_n)}$. It follows that

$$X_n = X_{n-1}\left(1 + I_n{}^{"}\right) + \mathcal{C}^{"}(b_{n-1})Z_n - h(b_{n-1}, Y_n^{"}),$$

which, again, is essentially the same as the model (4.3.1).

Let us go back to the original risk model (4.3.1). In the next subsection, we will derive recursive equations for the ruin probabilities and integral equations for the ultimate ruin probability associated to the model (4.3.1).

4.3.2 Recursive and integral equations for ruin probabilities

In this subsection, we first derive a recursive equation for $\psi_n^\pi(x, i)$. Secondly, we give an integral equation for $\psi^\pi(x, i)$. Finally, we obtain an equation for the ruin probability with horizon $n = 1$

given $I_0 = i$, $X_0 = x$ and a stationary policy π. These results, which are valid for any initial interest rate, are summarized in the following lemma.

Lemma 4.3.2. *Let $u(y,z) := b_0 y - \mathcal{C}(b_0)z$, where b_0 is the initial retention level. Let $\tau_j(z) := \left(x(1+j) + \mathcal{C}(b_0)z\right)/b_0$, $X_0 = x \geq 0$, and p_{ij} as in (4.3.4). Then*

$$\psi_1^\pi(x,i) = \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty \overline{F}(\tau_j(z))dG(z), \tag{4.3.15}$$

and for $n = 1, 2, \ldots$

$$\begin{aligned} \psi_{n+1}^\pi(x,i) &= \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty \int_0^{\tau_j(z)} \psi_n^\pi(x(1+j) - u(y,z), j)dF(y)dG(z) \\ &+ \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty \overline{F}(\tau_j(z))dG(z). \end{aligned} \tag{4.3.16}$$

Moreover,

$$\begin{aligned} \psi^\pi(x,i) &= \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty \int_0^{\tau_j(z)} \psi^\pi(x(1+j) - u(y,z), j)dF(y)dG(z) \\ &+ \sum_{j \in \mathbb{I}} p_{ij} \int_0^\infty \overline{F}(\tau_j(z))dG(z). \end{aligned} \tag{4.3.17}$$

Proof. Let $U_k := u(Y_k, Z_k) = b_0 Y_k - \mathcal{C}(b_0)Z_k$. Given $Y_1 = y$, $Z_1 = z$, the control strategy π, and $I_1 = j$, from (4.3.3) we have $U_1 = u(y,z)$. Therefore,

$$X_1 = x(1 + I_1) - U_1 = h_1 - u(y,z), \text{ where } h_1 = x(1+j)$$

Thus, if $u(y,z) > h_1$ then

$$P^\pi\left(X_1 < 0 | Y_1 = y, Z_1 = z, I_1 = j, X_0 = x, I_0 = i\right) = 1.$$

This implies that for $u(y,z) > h_1$

$$P^\pi\left(\bigcup_{k=1}^{n+1} \{X_k < 0\} | Y_1 = y, Z_1 = z, I_1 = j, X_0 = x, I_0 = i\right) = 1,$$

while if $0 \leq u(y,z) \leq h_1$, then

$$P^\pi\left(X_1 < 0 | Y_1 = y, Z_1 = z, I_1 = j, X_0 = x, I_0 = i\right) = 0. \tag{4.3.18}$$

Let $\{\widetilde{Y}_n\}_{n\geq 1}$, $\{\widetilde{Z}_n\}_{n\geq 1}$, and $\{\widetilde{I}_n\}_{n\geq 0}$ be independent copies of $\{Y_n\}_{n\geq 1}$, $\{Z_n\}_{n\geq 1}$, and $\{I_n\}_{n\geq 0}$, respectively.
Let $\widetilde{U}_k := b_0\widetilde{Y}_k - \mathcal{C}(b_0)\widetilde{Z}_k$. Thus, (4.3.18) and (4.3.3) yield that for $0 \leq u(y,z) \leq h_1$,

$$\begin{aligned}
& P^{\pi}\left(\bigcup_{k=1}^{n+1}\{X_k<0\}\,|Y_1=y,Z_1=z,I_1=j,X_0=x,I_0=i\right)\\
= \;& P^{\pi}\left(\bigcup_{k=2}^{n+1}\{X_k<0\}\,|Y_1=y,Z_1=z,I_1=j,X_0=x,I_0=i\right)\\
= \;& P^{\pi}\left(\bigcup_{k=2}^{n+1}\left\{(h_1-u(y,z))\prod_{l=1}^{k}(1+I_l)\right.\right.\\
& \left.\left.-\sum_{l=1}^{k}U_l\prod_{m=l+1}^{k}(1+I_m)<0\right\}|X_0=x,I_1=j\right)\\
= \;& P^{\pi}\left(\bigcup_{k=1}^{n}\left\{(h_1-u(y,z))\prod_{l=1}^{k}(1+\widetilde{I}_l)\right.\right.\\
& \left.\left.-\sum_{l=1}^{k}\widetilde{U}_l\prod_{m=l+1}^{k}(1+\widetilde{I}_m)<0\right\}|X_0=x,\widetilde{I}_0=j\right)\\
= \;& \psi_n^{\pi}(h_1-u(y,z),j)=\psi_n^{\pi}(x(1+j)-u(y,z),j)
\end{aligned}$$

where the second equality follows from the Markov property of $\{I_n\}_{n\geq 0}$, and the independence of $\{Y_n\}_{n\geq 1}$, $\{Z_n\}_{n\geq 1}$ and $\{I_n\}_{n\geq 0}$.

Let us now consider the event $\mathcal{A} = \{Y_1 = y, Z_1 = z, I_1 = j, X_0 = x, I_0 = i\}$, and recall that $F(y) = P(Y \leq y)$ and $G(z) = P(Z \leq z)$. From (4.3.7) and (4.3.3) we obtain

$$\begin{aligned}
\psi_{n+1}^{\pi}(x,i) &= P^{\pi}\left(\bigcup_{k=1}^{n+1}\{X_k<0\}\,|X_0=x,I_0=i\right)\\
&= \sum_{j\in\mathbb{I}}p_{ij}\int_0^{\infty}\int_0^{\infty}P^{\pi}\left(\bigcup_{k=1}^{n+1}\{X_k<0\}\,|\mathcal{A}\right)dF(y)dG(z).
\end{aligned}$$

Then, recalling that $\tau_j(z) = \frac{x(1+j)+\mathcal{C}(b_0)z}{b_0}$,

$$
\begin{aligned}
\psi^{\pi}_{n+1}(x,i) &= \sum_{j\in\mathbb{I}} p_{ij} \left\{ \int_0^{\infty} \int_0^{\tau_j(z)} P^{\pi} \left(\bigcup_{k=1}^{n+1} \{X_k < 0\} \,|\mathcal{A} \right) dF(y)dG(z) \right. \\
&+ \left. \int_0^{\infty} \int_{\tau_j(z)}^{\infty} P^{\pi} \left(\bigcup_{k=1}^{n+1} \{X_k < 0\} \,|\mathcal{A} \right) dF(y)dG(z) \right\} \\
&= \sum_{j\in\mathbb{I}} p_{ij} \int_0^{\infty} \left\{ \int_0^{\tau_j(z)} \psi^{\pi}_n(x(1+j) - u(y,z), j) dF(y)dG(z) \right. \\
&+ \left. \textstyle\int_0^{\infty} \int \_\tau_j(z)^{\infty} dF(y)dG(z) \right\} \\
&= \sum_{j\in\mathbb{I}} p_{ij} \left\{ \int_0^{\infty} \int_0^{\tau_j(z)} \psi^{\pi}_n(x(1+j) - u(y,z), j) dF(y)dG(z) \right. \\
&+ \left. \textstyle\int_0^{\infty} \overline{F}(\tau_j(z)) dG(z) \right\}.
\end{aligned} \tag{4.3.19}
$$

This gives (4.3.16). In particular,

$$
\psi^{\pi}_1(x,i) = \sum_{j\in\mathbb{I}} p_{ij} \int_0^{\infty} \overline{F}(\tau_j(z)) dG(z).
$$

Finally, letting $n \to \infty$ in (4.3.19) and using dominated convergence we obtain $\lim_{n\to\infty} \psi^{\pi}_{n+1}(x,i) = \psi^{\pi}(x,i)$, and (4.3.17) follows. □

Remark 4.3.2. If we consider the risk model without reinsurance, that is, $b = 1$, we obtain similar results to those in Cai and Dickson [7].

4.3.3 Bounds for ruin probabilities

We will use the results obtained in Subsection 4.3.2 to find upper bounds for the ruin probabilities with infinite horizon taking into account the information contributed by the Markov chain of the interest rate process. We derive a functional for the ultimate ruin probability in terms of the new worse than used in convex (NWUC) ordering; see Remark 4.3.3, below. This idea was first introduced by Willmot and Lin [62] and has been generalized by other authors.

We will present two upper bounds for the ruin probabilities. The first bound is obtained by an inductive approach, and the second by a martingale approach. These bounds are discussed in Remark 4.3.4, at the end of this subsection.

4.3.3.1 Bounds obtained by the inductive approach

Theorem 4.3.3. *Let $R_0 > 0$ be the constant satisfying (4.3.11). Then, for all $x \geq 0$ and $i \in \mathbb{I}$,*

$$\begin{aligned} \psi^\pi(x,i) &\leq \beta \sum_{j\in\mathbb{I}} p_{ij} E^\pi[e^{-R_0 x(1+j)}] \\ &= \beta E^\pi[e^{-R_0[x(1+I_1)]}|I_0 = i], \end{aligned} \tag{4.3.20}$$

where $\beta \equiv \beta(b_0)$ and is given by

$$\beta^{-1} = \inf_{t\geq 0} \frac{\int_t^\infty e^{R_0 b_0 y} dF(y)}{e^{R_0 b_0 t}\overline{F}(t)}.$$

Proof. It suffices to show that the rightmost term in (4.3.20) is an upper bound for $\psi_n^\pi(x,i)$, for all $n \geq 1$. We will prove this by induction. First note that

$$\begin{aligned} \overline{F}(\vartheta) &= \left(\frac{\int_\vartheta^\infty e^{R_0 b_0 y} dF(y)}{e^{R_0 b_0 \vartheta}\overline{F}(\vartheta)}\right)^{-1} e^{-R_0 b_0 \vartheta} \int_\vartheta^\infty e^{R_0 b_0 y} dF(y) \\ &\leq \beta e^{-R_0 b_0 \vartheta} \int_\vartheta^\infty e^{R_0 b_0 y} dF(y) \leq \beta e^{-R_0 b_0 \vartheta} E^\pi[e^{R_0 b Y_1}] \end{aligned} \tag{4.3.21}$$

for any $\vartheta \geq 0$. This implies that for every $x \geq 0$, $i \geq 0$, and $b_0 \in \mathcal{B}$, by (4.3.15) and (4.3.21) we have

$$\begin{aligned} \psi_1^\pi(x,i) &= \sum_{j\in\mathbb{I}} p_{ij} \int_0^\infty \overline{F}(\tau_j(z)) dG(z) \\ &\leq \sum_{j\in\mathbb{I}} p_{ij} \left(\beta E^\pi[e^{R_0 b Y_1}] \cdot \int_0^\infty e^{-R_0 b_0 \left(\frac{x(1+j)+\mathcal{C}(b_0)z}{b_0}\right)} dG(z)\right) \\ &= \beta E^\pi[e^{R_0 b Y_1}] \sum_{j\in\mathbb{I}} p_{ij} \int_0^\infty e^{-R_0[x(1+j)+\mathcal{C}(b_0)z]} dG(z) \\ &= \beta E^\pi[e^{R_0 b Y_1}] \sum_{j\in\mathbb{I}} p_{ij} E^\pi\left[e^{-R_0[x(1+j)+\mathcal{C}(b_0)Z_1]}|I_0 = i\right] \\ &= \beta E^\pi[e^{R_0 b Y_1}] \cdot E^\pi[e^{-R_0[x(1+I_1)+\mathcal{C}(b)Z_1]}|I_0 = i] \\ &= \beta E^\pi[e^{R_0 b Y_1}] E^\pi[e^{-R_0 \mathcal{C}(b) Z_1}] E^\pi[e^{-R_0 x(1+I_1)}|I_0 = i] \\ &= \beta E^\pi[e^{-R_0[\mathcal{C}(b)Z_1 - bY_1]}] \cdot E^\pi[e^{-R_0 x(1+I_1)}|I_0 = i] \\ &= \beta E^\pi[e^{-R_0 x(1+I_1)}|I_0 = i] \text{ (by (4.3.11))}. \end{aligned}$$

This shows that the desired result holds for $n = 1$. To prove the result for general $n \geq 1$, the induction hypothesis is that, for some $n \geq 1$, and every $x \geq 0$ and $i \in \mathbb{I}$,

$$\psi_n^\pi(x,i) \leq \beta E^\pi[e^{-R_0 x(1+I_1)}|I_0 = i]. \tag{4.3.22}$$

Now, let $0 \leq y \leq \tau_j(z)$, with $\tau_j(z)$ as in Lemma 4.3.2. Further, in (4.3.22) replace x and i by $x(1+j) + \mathcal{C}(b_0)z - b_0 y$ and j, respectively, to obtain

$$\begin{aligned} \psi_n^\pi(x(1+j) + \mathcal{C}(b_0)z - b_0 y, j) &\leq \beta E^\pi[e^{-R_0[x(1+j)+\mathcal{C}(b)z-by](1+I_1)}|I_0 = j] \\ &\leq \beta e^{-R_0[x(1+j)+\mathcal{C}(b_0)z-b_0 y]}. \end{aligned} \tag{4.3.23}$$

Therefore, replacing (4.3.23) in (4.3.16), we get

$$\begin{aligned} \psi_{n+1}^\pi(x,i) &\leq \sum_{j\in\mathbb{I}} p_{ij} \left(\beta \int_0^\infty e^{-R_0[x(1+j)+\mathcal{C}(b_0)z]} \int_{\tau_j(z)}^\infty e^{R_0 b_0 y} dF(y) dG(z)\right) \\ &+ \sum_{j\in\mathbb{I}} p_{ij} \left(\beta \int_0^\infty e^{-R_0[x(1+j)+\mathcal{C}(b_0)z]} \int_0^{\tau_j(z)} e^{R_0 b_0 y} dF(y) dG(z)\right) \\ &= \sum_{j\in\mathbb{I}} p_{ij} \left(\beta \int_0^\infty e^{-R_0[x(1+j)+\mathcal{C}(b_0)z]} \int_0^\infty e^{R_0 b_0 y} dF(y) dG(z)\right) \\ &= \beta E^\pi[e^{R_0 b Y_1}] \sum_{j\in\mathbb{I}} p_{ij} \int_0^\infty e^{-R_0[x(1+j)+\mathcal{C}(b_0)z]} dG(z) \\ &= \beta E^\pi[e^{R_0 b Y_1}] \cdot E^\pi[e^{-R_0 \mathcal{C}(b) Z_1}] \cdot E^\pi[e^{-R_0 x(1+I_1)}|I_0 = i] \\ &= \beta E^\pi[e^{-R_0 x(1+I_1)}|I_0 = i]. \end{aligned}$$

Hence, (4.3.22) holds for any $n = 1, 2, \ldots$. Finally, letting $n \to \infty$ in (4.3.22) we obtain (4.3.20). $\square$

As an application of Theorem 4.3.3.1, we next consider the special case in which the claim distribution is in the class of NWUC distributions.

Remark 4.3.3. For example, let F a phase-type distribution with parameters (α, T) (see 2.3.3). Then F is NWUC if and only if T^{-1} and $T^{-1}e^{Ty}(\mathcal{I} - \overrightarrow{\mathbf{1}}\alpha)$ are both non-negative or non-positive definite simultaneously for all $y \geq 0$ (where $\mathcal{I}$ represent the identity matrix and $\overrightarrow{\mathbf{1}}$ is the column vector of ones).

Corollary 4.3.4. *Under the hypotheses of Theorem 4.3.3.1, and assuming that $E^\pi[e^{R_0 b Y_1}] < \infty$ for all $b \in \mathcal{B}$, and that, in addition, F is a NWUC distribution, we have*

$$\psi^\pi(x,i) \leq (E^\pi[e^{R_0 b Y_1}])^{-1} E^\pi[e^{-R_0 x(1+I_1)}|I_0 = i]. \tag{4.3.24}$$

Proof. Following proof of proposition 2.1.1, let $r := R_0 b > 0$. Therefore

$$\beta^{-1} := \inf_{t\geq 0} \frac{\int_t^\infty e^{ry} dF(y)}{e^{rt}\overline{F}(t)} = \int_0^\infty e^{ry} dF(y),$$

that is, $\beta^{-1} = E^\pi[e^{R_0 b Y_1}]$. Finally, replacing this equality in (4.3.20), we obtain (4.3.24). $\square$

4.3.3.2 Bounds by means of the martingale approach

Another way for deriving upper bounds for ruin probabilities is the martingale approach. To this end, let $V_n := X_n \prod_{l=1}^{n}(1+I_l)^{-1}$ with $n \geq 1$, be the so-called discounted risk process. The ruin probabilities ψ_n^π in (4.3.7) associated to the process $\{V_n, n = 1, 2\ldots\}$ are

$$\psi_n^\pi(x,i) \quad = \quad P^\pi\left(\bigcup_{k=1}^{n}(V_k < 0)|X_0 = x, I_0 = i\right).$$

In the classical risk model, $\{e^{-R_0X_n}\}_{n\geq 1}$ is a martingale. However, for our model (4.3.3), there is no constant $r > 0$ such that $\{e^{-rX_n}\}_{n\geq 1}$ is a martingale. Still, there exists a constant $r > 0$ such that $\{e^{-rV_n}\}_{n\geq 1}$ is a *supermartingale*, which allows us to derive probability inequalities by the optional stopping theorem. Such a constant is defined in the following proposition.

Proposition 4.3.5. *Assume that for each $i \in \mathbb{I}$, there exists $\rho_i > 0$ satisfying that*

$$E^\pi\left[e^{-\rho_i[\mathcal{C}(b)Z_1 - bY_1](1+I_1)^{-1}}|I_0 = i\right] = 1. \tag{4.3.25}$$

Then

$$R_1 := \min_{i\in\mathbb{I}} \rho_i \geq R_0 \tag{4.3.26}$$

and, furthermore, for all $i \in \mathbb{I}$

$$E^\pi\left[e^{-R_1[\mathcal{C}(b)Z_1 - bY_1](1+I_1)^{-1}}|I_0 = i\right] \leq 1. \tag{4.3.27}$$

Proof. For each $i \in \mathbb{I}$ and $r > 0$, let

$$l_i(r) := E^\pi\left[e^{-r[\mathcal{C}(b)Z - bY](1+I_1)^{-1}}|I_0 = i\right] - 1.$$

Then the first derivative of $l_i(r)$ at $r = 0$ is

$$l_i'(0) = E^\pi\left[-(\mathcal{C}(b)Z - bY)\right] \cdot E\left[(1+I_1)^{-1}|I_0 = i\right] < 0 \text{ (by independence)},$$

and the second derivative is

$$l_i''(r) = E^\pi\left[\left((\mathcal{C}(b)Z - bY)(1+I_1)^{-1}\right)^2 \cdot e^{-r[\mathcal{C}(b)Z - bY](1+I_1)^{-1}}|I_0 = i\right] > 0.$$

This shows that $l_i(r)$ is a convex function. Let ρ_i be the unique positive root of the equation $l_i(r) = 0$ on $(0, \infty)$. Further, if $0 < \rho \leq \rho_i$, then $l_i(\rho) \leq 0$. However,

$$\begin{aligned} E^\pi\left[e^{-R_0[\mathcal{C}(b)Z - bY](1+I_1)^{-1}}|I_0 = i\right] \quad &= \quad \sum_{j\in\mathbb{I}} p_{ij} E\left[e^{-R_0[\mathcal{C}(b_0)Z - b_0Y](1+j)^{-1}}\right] \\ \text{(by Jensen's inequality)} \qquad &\leq \quad \sum_{j\in\mathbb{I}} p_{ij} E\left[e^{-R_0[\mathcal{C}(b_0)Z_1 - b_0Y_1]}\right]^{(1+j)^{-1}}. \end{aligned}$$

Consequently, by (4.3.11), we have $E\left[e^{-R_0[\mathcal{C}(b_0)Z_1-b_0Y_1]}\right]=1$. Hence, since $\sum_{j\in\mathbb{I}} p_{ij}=1$,

$$E^{\pi}\left[e^{-R_0[\mathcal{C}(b)Z-bY](1+I_1)^{-1}}|I_0=i\right] \;\leq\; 1.$$

This implies that $l_i(R_0)\leq 0$. Moreover, $R_0\leq\rho_i$ for all i, and so

$$R_1:=\min_{i\in\mathbb{I}}\rho_i\geq R_0.$$

Thus, (4.3.26) holds. In addition $R_1\leq\rho_i$ for all $i\in\mathbb{I}$, which implies that $l_i(R_1)\leq 0$. This yields (4.3.27). □

Theorem 4.3.6. *Under the hypotheses of Proposition 4.3.3.2, for all $i\in\mathbb{I}$ and $x\geq 0$,*

$$\psi^{\pi}(x,i)\leq e^{-R_1x}. \tag{4.3.28}$$

Proof. By (4.3.3), the discounted risk process $V_k:=X_k\prod_{l=1}^{k}(1+I_l)^{-1}$ satisfies that

$$V_k:=x+\sum_{l=1}^{k}\left((\mathcal{C}(b_0)Z_1-b_0Y_l)\prod_{t=1}^{l}(1+I_t)^{-1}\right). \tag{4.3.29}$$

Let $S_n=e^{-R_1V_n}$. Then

$$S_{n+1}=S_ne^{-R_1(\mathcal{C}(b_0)Z_{n+1}-b_0Y_{n+1})\prod_{t=1}^{n+1}(1+I_t)^{-1}}.$$

Thus, for any $n\geq 1$,

$$\begin{aligned}
&E^{\pi}[S_{n+1}\mid Y_1,\dots Y_n,Z_1,\dots Z_n,I_1,\dots I_n]\\
&=S_nE\left[e^{-R_1(\mathcal{C}(b_0)Z_{n+1}-b_0Y_{n+1})\prod_{t=1}^{n+1}(1+I_t)^{-1}}\mid Y_1,\dots Y_n,Z_1,\dots Z_n,I_1,\dots I_n\right]\\
&=S_nE\left[e^{-R_1(\mathcal{C}(b_0)Z_{n+1}-b_0Y_{n+1})(1+I_{n+1})^{-1}\prod_{t=1}^{n}(1+I_t)^{-1}}\mid I_1,\dots I_n\right]\\
&\leq S_nE\left(\left[e^{-R_1(\mathcal{C}(b_0)Z_{n+1}-b_0Y_{n+1})(1+I_{n+1})^{-1}}\mid I_1,\dots I_n\right]\right)^{\prod_{t=1}^{n}(1+I_t)^{-1}}\\
&\leq S_n.
\end{aligned}$$

This implies that $\{S_n\}_{n\geq 1}$ is a supermartingale.
Let $T_i=\min\{n:V_n<0\mid I_0=i\}$, where V_n is given by (4.3.29). Then T_i is a stopping time

and $n \wedge T_i := \min\{n, T_i\}$ is a finite stopping time. Thus, by the optional stopping theorem for martingales, we get

$$E^\pi(S_{n\wedge T_i}) \leq E^\pi(S_0) = e^{-R_1 x}.$$

Hence,

$$\begin{aligned} e^{-R_1 x} &\geq E^\pi(S_{n\wedge T_i}) \geq E^\pi((S_{n\wedge T_i}) I_{(T_i \leq n)}) \geq E^\pi((S_{T_i}) I_{(T_i \leq n)}) \\ &= E^\pi(e^{-R_1 V_{T_i}} I_{(T_i \leq n)}) \geq E^\pi(I_{(T_i \leq n)}) = \psi_n^\pi(x, i), \end{aligned} \tag{4.3.30}$$

where (4.3.30) follows because $V_{T_i} < 0$. Thus, by letting $n \to \infty$ in (4.3.30) we obtain (4.3.28). □

Remark 4.3.4. Summarizing, we have three upper bounds for the ruin probabilities with infinite horizon. First, the Lundberg bound, which only depends on R_0, the Lundberg exponential in (4.3.11), (4.3.12). Second, the inductive bound (4.3.20) which depends on R_0 and also on the interest rate process. Third, the martingale bound in (4.3.28), which depends on R_1. Note that the last two bounds are sharper than the Lundberg bound. Observe also that the number of operations to get R_1 in (4.3.28) is higher than that to get R_0 in (4.3.20).

In the next subsection we present some numerical results.

4.3.4 Numerical results

To illustrate the bounds given by Theorems 4.3.3.1 and 4.3.3.2 we present two numerical examples that use Matlab and Maple implementations. Without loss of generality we can work in monetary units equal to $E[Y]$ in all examples.

4.3.4.1 Exponentially distributed claims

Let consider the special case 2 in Subsection 4.3.1, in which Z_n and Y_n are exponentially distributed with parameters λ and $1/\mu$, respectively. In addition, we will consider an interest model with three possible interest rates:

$$\mathbb{I} = \{6\%, 8\%, 10\%\}.$$

The transition matrix (see (4.3.4)) is given by

$$\begin{pmatrix} 0.2 & 0.8 & 0 \\ 0.15 & 0.7 & 0.15 \\ 0 & 0.8 & 0.2 \end{pmatrix}.$$

Thus, our interest rate model incorporates mean reversion to a level of 8%. If θ is too high, in the sense that

$$1+\theta \geq (1-\mu R_0)^{-2},$$

then the optimal policy is given by $\pi^* = \{a_n^*\}_{n\geq 1}$ with $a_n^* = 1$ for all n. If we assume that $c > \lambda\mu$, then we have that the ruin probability for the Cramér-Lundberg model is [§]

$$\psi^{\pi^*}(x) = \left(\frac{\lambda\mu}{c}\right) e^{-x\left(\frac{1}{\mu}-\frac{\lambda}{c}\right)}.$$

Recalling (4.3.13), the Cramér-Lundberg exponent R_0 is the solution of equation

$$\lambda + cR_0 = \lambda (1-\mu R_0)^{-1}.$$

By Lemma 4.3.1 and (4.3.12), we have that $\psi^{\pi^*}(x,i) \leq \psi^{\pi^*}(x) \leq e^{-R_0 x}$. In the case that Y has **DFR** distribution[¶] [♣], then the inductive bound is given by (4.3.24). The martingale bound can be obtained from Theorem 4.3.3.2. The Table 4.3.4.1 shows the numerical results when $\lambda = 1$, $\mu = 2$, $\theta = 3$, $c = 4$, and $x = 1$.

Lundberg	$\psi^{\pi^*}(x)$	Inductive	Martingale	R_0	R_1
0.7788	0.3894	0.3817	0.4366109286	0.25	0.8287128040

Table 4.2: Table of upper bounds for ruin probabilities, with $x = 1$ and $i = 8\%$.

Note that

$$\psi^{\pi^*}(x,i) \leq 0.3817 < \psi^{\pi^*}(x).$$

The numerical results in Table 4.3.4.1 show that the upper bound in (4.3.20) can be tighter than that in (4.3.28). This suggests that the upper bounds derived by the inductive approach are tighter than the upper bounds obtained by supermartingales. In addition, the upper bounds derived by the inductive approach are tighter than the ruin probability without interest rate. Moreover, Table 4.3.4.1 shows that the upper bounds derived in this chapter are sharper than the Lundberg upper bound.

[§] See equation (1.1.4).
[¶] See Remark 2.3.2.
[♣] If f has **DFR**, this imply that df F is **NWUC**.

4.3.4.2 Claims with phase-type distribution

We consider claim distributions of the phase-type because this class is a generalization of the exponential distribution such that they and their moments can be written in a closed form, various quantities of interest can be evaluated with relative ease, and furthermore, the set of phase-type distributions is dense in the set of all distributions with support in $[0, \infty)$ (see [2]).

Suppose that the claim size Y has a phase-type density with parameters (α, T) where

$$T = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \text{ and } \alpha = (1/2, 1/2).$$

Let

$$\mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \overrightarrow{1} = (1,1), \text{ and } \qquad t = -T \cdot \overrightarrow{1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

In this case,

$$M_Y(s) = E[e^{s \cdot Y}] = \alpha \, (-s\mathcal{I} - T)^{-1} \, t.$$

Thus, $E[Y] = \frac{d}{ds} M_Y(s) \mid_{s=0} = \alpha(T)^{-2} t = 0.75$, and Y has NWUC distribution. Let $Z \backsimeq Exp(1)$, $E[Z] = 1$, and $M_Z(s) = E[e^{s \cdot Z}] = (1-s)^{-1}$.

We consider an interest model with three possible interest rates: $\mathbb{I} = \{6\%, 8\%, 10\%\}$. We would like to have an idea of the dependence of our bounds on the transition probability matrix of the interest rate process. To this end, we consider two transition probability matrices, namely,

$$P_{\mathbf{1}} = \begin{pmatrix} 0 & 0.9 & 0.1 \\ 0.8 & 0.2 & 0 \\ 0.9 & 0.1 & 0 \end{pmatrix} \text{ and } P_{\mathbf{2}} = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0 & 0.2 & 0.8 \\ 0 & 0.1 & 0.9 \end{pmatrix}.$$

We fix the premium income rate $c = 0.975$ and the safety loading $\theta = 0.1$ of the reinsurer. In addition, $\mathcal{B} = (0, 1]$. In this case (4.3.14) is not satisfied.

The Lundberg bound: In this example we can guarantee that the Lundberg bound (4.3.12) holds for each $b \in \mathcal{B}$. Then there exists a constant R_0 such that (4.3.11) is achieved. Moreover, solving

$$E^{\pi}[e^{R_0 b Y_1}] \cdot E^{\pi}[e^{-R_0 \mathcal{C}(b) Z_1}] = 1,$$

is equivalent to find the Cramér-Lundberg adjustment coefficient such that

$$1 + \mathcal{C}(b) R_0 = \alpha(-bR_0\mathcal{I} - T)^{-1} t.$$

Then the Lundberg bound for the ruin probability is

$$\psi^b(x) \leq e^{-R_0 x}, \text{ for } x \geq 0.$$

Figure 4.2 shows the relation between R_0 and b in this inequality is inversely proportional. Table 4.3.4.2 presents numerical values of the bounds obtained for several admissible decision policies.

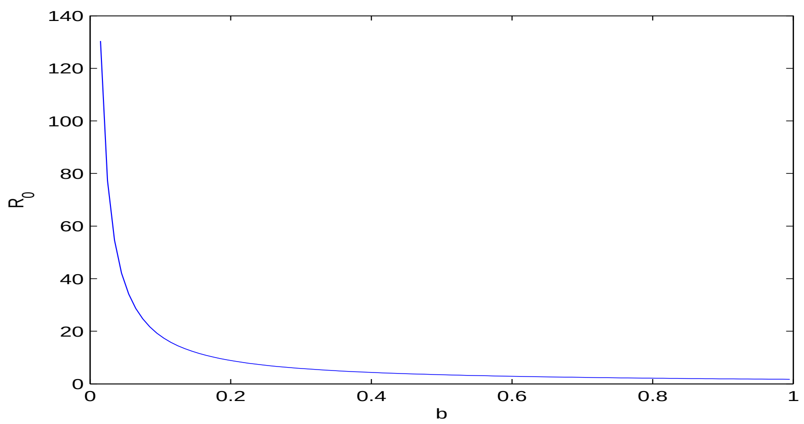

Figure 4.2: Relation between R_0 and b.

The Induction bound: Here, the claim distribution is a NWUC (see [62], page 24) and such that $E^{\pi}[e^{R_0 b Y_1}] = M_Y(R_0 b) < \infty$ for each $b \in \mathcal{B}$. Then Corollary 4.3.3.1 applies and for each $i \in \mathbb{I}$ and $x \geq 0$, we have

$$\begin{aligned}\psi^{\pi}(x,i) &\leq (E^{\pi}[e^{R_0 b Y_1}])^{-1} E^{\pi}[e^{-R_0 x(1+I_1)} | I_0 = i] \\ &\leq \left[\alpha \left(-bR_0\mathcal{I} - T\right)^{-1} t\right]^{-1} \sum_{k \in \mathbb{I}} p_{ik} e^{-R_0 x(1+k)}.\end{aligned}$$

See Table 4.3.4.2 for numerical values of this bound obtained for several admissible decision policies. As it is to be expected we get induction bounds smaller than the Lundberg bounds for the same decision policies.

The Martingale bound:By the condition (4.3.25) of Proposition 4.3.3.2 and Theorem 4.3.3.1, we get the martingale bound (4.3.28). Observe that

$$E\left[e^{-\rho_i(\mathcal{C}(b)Z_1-bY_1)(1+I_1)^{-1}}|I_0=i\right]=1$$

which is equivalent to the following condition for each $i \in \mathbb{I}$:

$$\sum_{k\in\mathbb{I}} p_{ik}e^{\rho_i(1+k)^{-1}}M_Y\left(\frac{b\rho_i}{1+k}\right)M_Z\left(-\frac{\mathcal{C}(b)\rho_i}{1+k}\right)=1.$$

In our example we solve $R_1 = \min_{i\in\mathbb{I}} \rho_i \geq R_0$, and then we obtain $\psi^\pi(x, i_1) \leq e^{-R_1 x}$ for $x \geq 0$. Numerical results of this bound are reported in Table 4.3.4.2. It is obvious that this martingale bound improves the results of the induction bound.

We run numerical experiments to compare, for a fixed retention level b, the ruin probability bounds that could be achieved. Figure 4.3 shows the upper bounds of ruin probability from

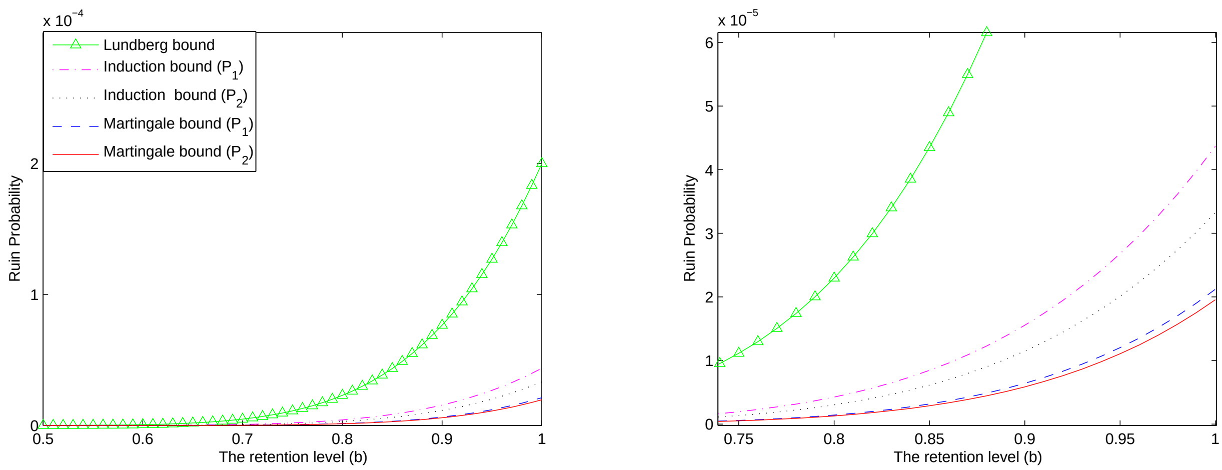

Figure 4.3: Bounds for the ruin probabilities. Left panel: $b \in [0.5, 1]$. Right panel: $b \in [0.75, 1]$.

different approaches with the initial state $x = 5$ and $i = 8\%$.

Finally, we find of special interest the case of small reinsurers for which the retention level could be restricted by economic considerations. Table 4.3.4.2 shows the numerical values of the bounds from different values of b when b is increasing towards 1. Recall, that $b = 1$ stands for the control action *no reinsurance*. Clearly, the best results are obtained in the case where the transition interest rate matrix is P_2.

The numerical results in Table 4.3.4.2 show that the upper bound in (4.3.28) can be tighter

P_κ	b	*Lundberg*	*Induction*	*Martingale*	R_0	R_1
P_1	0.5	$0.323e-7$	$0.369e-8$	$0.448e-9$	3.4491	4.3048
P_1	0.75	$0.111e-4$	$0.196e-5$	$0.586e-6$	2.2810	2.8697
P_1	0.85	$0.434e-4$	$0.846e-5$	$0.317e-5$	2.0086	2.5321
P_1	0.95	$0.126e-3$	$0.268e-4$	$0.120e-4$	1.7943	2.2655
P_1	1	$0.2e-3$	$0.436e-4$	$0.212e-4$	1.7034	2.1522
P_2	0.5	$0.323e-7$	$0.213e-8$	$0.382e-9$	3.4491	4.3368
P_2	0.75	$0.111e-4$	$0.136e-5$	$0.527e-6$	2.2810	2.8911
P_2	0.85	$0.434e-4$	$0.614e-5$	$0.288e-5$	2.0086	2.5509
P_2	0.95	$0.126e-3$	$0.201e-4$	$0.110e-4$	1.7943	2.2824
P_2	1	$0.2e-3$	$0.333e-4$	$0.195e-4$	1.7034	2.1683

Table 4.3: Numerical bounds of ruin probability.

than that in (4.3.20). This suggests that the upper bounds derived by the martingale approach are tighter than the upper bounds obtained by induction. In addition, Table 4.3.4.2 also shows that the upper bounds derived in this chapter are sharper than the Lundberg upper bound.

Chapter 5

Markov control processes

In the last chapter, we considered a similar controlled risk process $\{X_n\}$, given by

$$x_{n+1} = G\left(x_n, i_{n+1}, a_n, w_n\right), \tag{5.0.1}$$

where the sequence $\{w_n\}$ consists of i.i.d random variables with values in a Borel space $\mathbb{W}$ and $\{w_n\}$ is independent of the initial state (x_0, i_0). The common distribution of the w_n is denoted by F. Moreover, we will look on w_n as the disturbance for period n. The sequence $\{w_n\}_{n\geq 1}$ forms the source of randomness of the model. Let $\{i_n\}$ the interest rate process and $\{a_n\}$ be a sequence of decision functions. Furthermore, the interest rate process $\{i_n\}$ in (4.2.1) and (4.3.1) is supposed to be a Markov chain, i.e., i_n evolves as

$$i_{n+1} = \mathcal{H}\left(i_n, \varpi_n\right), \tag{5.0.2}$$

where $\{\varpi_n\}$ is a sequence i.i.d. random variables and independent of initial state (x_0, i_0) with a common distribution Υ. Also, we assume that $\{w_n\}$ and $\{\varpi_n\}$ are independent sequences.
In addition, the risk process may be controlled by reinsurance and the cost that will be to pay it is defined by

$$c\left(x, i, a\right) := \mathbf{1}_{(-\infty,0)}(x). \tag{5.0.3}$$

Clearly, the model (5.0.1) is a generalization of model (3.2.2) given in Chapter 3 and the cost function (5.0.3) is similar to reward function considered in (3.2.3).
Following the idea of Shäl [48] and Schmidli [54] we introduce a cemetary state. Once the

system is in state $x \in (-\infty, 0)$, then it shall move to $-\infty$ in the next step, i.e. we set $G(x, i, a, w) = -\infty$ for $x \in [-\infty, 0)$. Thus the cost of 1 unit has to be paid at most once.

We assume that the functions $\mathcal{C}(b)$* and $h(b, y) = b \cdot y$† defined in Chapter 4 are continuous in b for each y. Also, we assume that $P(\mathcal{C}(b)z < h(b, y))$ for all $b = a(x)$.

The state space $\mathbb{X}$ of our decision problem can be not countable and we have to take into account that in general we cannot take the system of all subsets of $\mathbb{X}$ as the domain of the probability measures we are interested in. Thus, consequently the measurability of functions defined on $\mathbb{X}$ is no longer a triviality‡.

Then, under certain conditions the stochastic process $(\Omega, \mathcal{F}, P_v^\pi, \{(x_n, i_n)\})$ is a discrete-time Markov control process§.

In this chapter, we will study a general Markov decision problem following the ideas of González-Hernández, López-Martínez and Pérez-Hernández [30], and Hernández-Lerma and Lasserre [33, 34, 35]. Also, we use the Hinderer's results [36]. Particularly, we will specify how to rewrite the minimization of the ruin probability as a MDP applying the results to consider a fixed discount factor $\alpha = 0$.

In spite of focused in this thesis to the special case of minimization the ruin probability, we can apply the results of this chapter to maximization the exponential utility as well. However, we do not consider a random discount factor.

We begin Section 5.1 by defining a Markov decision model with criterion the expected discounted cost criterion, where the state and action spaces are Borel spaces. We finish this

*See, equations (4.2.3) and (4.3.2).

†See, equation (4.2.2).

‡See, A.2 for definition.

§Or equivalently, a Markov decision process.

section with the canonical construction. In Section 5.2 we study finite MDPs and by using the dynamic programming algorithm we prove the existence of optimal policies, under the assumption of the existence of measurable selectors that satisfy optimality equations. In Section 5.3 we give conditions that ensure the existence of such selectors. Finally, in Section 5.4 we present infinite horizon MDPs, we obtain the optimality equation and we prove the existence of optimal policies.

5.1 Markov control model

A **Markov decision model** (also, called Markov control model (MCM)) is a tuple

$$\mathcal{M} := (\mathbb{X}, \mathcal{A}, \{\mathcal{A}(x,i) | (x,i) \in \mathbb{X}\}, Q, c), \tag{5.1.1}$$

of the following meaning:

1. $\mathbb{X} = \mathbf{X}' \times (0, \infty)$ is *state space* and $\mathbf{X}'$ is Borel space.
2. $\mathcal{A}$ is *action space* (or *control space*).
3. a family $\{\mathcal{A}(x,i) | (x,i) \in \mathbb{X}\}$ of nonempty measurable subset $\mathcal{A}(x,i)$ of $\mathcal{A}$, where $\mathcal{A}(x,i)$ denotes the *set of feasible control or actions* when the system is in state $(x,i) \in \mathbb{X}$. The set of feasible state-action pairs, namely

   $$\mathbb{K} := \{(x,i,a) | a \in \mathcal{A}(x,i), (x,i) \in \mathbb{X}\}, \tag{5.1.2}$$

   is assumed to be a measurable subset of $\mathbb{X} \times \mathcal{A}$.
4. a stochastic kernel Q on $\mathbb{X}$ given $\mathbb{K}$ called *transition law*.
5. a stochastic function $c : \mathbb{K} \to \mathbb{R}$ called the *cost-per-stage function*.

Assumption 5.1. The set $\mathbb{K}$ contains the graph of a measurable function from $\mathbb{X}$ to $\mathcal{A}$; that is, there is a measurable function $f : \mathbb{X} \to \mathcal{A}$ such that $f(x,i) \in \mathcal{A}(x,i)$ $\forall (x,i) \in \mathbb{X}$ (the family of such functions will be denoted by $\mathbb{F}$).

This assumption ensures that the sets in Definition 5.1.3 are nonempty.

Definition 5.1.1. For each $t = 0, 1, \ldots,$ define the space H_t of admissible histories up to time t as $H_0 = \mathbb{X}$, and

$$H_t := \left[\prod_{j=0}^{t-1} \mathbb{K}\right] \times \mathbb{X} \text{ for } t \in \mathbb{N}^{\P} \tag{5.1.3}$$

where $\mathbb{K}$ is the set in (5.1.2). A generic element h_t of H_t, which is called an admissible t-history, is a vector of the form

$$h_t = (x_0, i_0, a_0, \ldots, x_{t-1}, i_{t-1}, a_{t-1}, x_t, i_t) \tag{5.1.4}$$

with $(x_j, i_j, a_j) \in \mathbb{K}$ for $j = 0, \ldots, t-1$, and $(x_t, i_t) \in \mathbb{X}$. Note that, for each t, H_t is a subset of

$$\overline{H}_t := \left[\prod_{j=0}^{t-1} \mathbb{X} \times \mathcal{A}\right] \times \mathbb{X} \text{ for } t \in \mathbb{N} \tag{5.1.5}$$

and $\overline{H}_0 = H_0 = \mathbb{X}$.

A policy is a sequence of actions that is taken by the controller, that is.

Definition 5.1.2. Φ denotes the set of all stochastic kernels φ in $P(\mathcal{A}|\mathbb{X})$ such that $\varphi(\mathcal{A}(x,i)|(x,i)) = 1$ for all $(x,i) \in \mathbb{X}$, and $\mathbb{F}$ stands for the set of all measurable functions $f : \mathbb{X} \to \mathcal{A}$ satisfying that $f(x,i) \in \mathcal{A}(x,i)$ for all $(x,i) \in \mathbb{X}$. The functions in $\mathbb{F}$ are called *selectors* of the multifunction (or set-valued mapping) $(x,i) \mapsto \mathcal{A}(x,i)$.

Remark 5.1.1. Assumption 5.1 ensures that $\mathbb{F}$ is nonempty.

A function f in $\mathbb{F}$ may be identified with the stochastic kernel $\varphi \in \Phi$, for which $\varphi(\cdot|(x,i))$ is the Dirac measure at $f(x,i)$ for all $(x,i) \in \mathbb{X}$, i.e.,

$$\varphi(C|(x,i)) = \mathbf{1}_C(f(x,i)) \qquad \forall (x,i) \in \mathbb{X}, C \in \mathcal{B}(\mathcal{A}),$$

we may regard $\mathbb{F}$ as a subset of Φ, i.e.,

$$\mathbb{F} \subset \Phi.$$

Definition 5.1.3. A randomized control policy is a sequence $\pi = \{\pi_t\}_{t\in\mathbb{Z}^+}$ of stochastic kernels π_t on the control set $\mathcal{A}$ given H_t, satisfying the constraint

$$\pi_t(\mathcal{A}(x_t, i_t)|h_t) = 1 \text{ for all } h_t \in H_t, \quad t \in \mathbb{Z}^+. \tag{5.1.6}$$

The set of all policies is denoted by Π. As usual, we will identify Φ with the family of randomized stationary policies, and $\mathbb{F}$ with the subfamily of deterministic stationary policies. In this way, we have that $\mathbb{F} \subset \Phi \subset \Pi$.

With these elements we can define the next stochastic processes.

¶We denote by $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$.

5.1.1 The canonical construction

Let $(\Omega, \mathcal{F})$ be the measurable space consisting of the (canonical) sample space $\Omega := \overline{H}_\infty = [\prod_{t=1}^{\infty} \mathbb{X} \times \mathcal{A}]$ and $\mathcal{F}$ is the corresponding product σ-algebra. The element of Ω are sequence of the form $w = (x_0, i_0, a_0, x_1, i_1, a_1, \ldots)$ with $(x_t, i_t) \in \mathbb{X}$ and $a_t \in \mathcal{A}$ for all $t \in \mathbb{Z}^+$; the projections (x_t, i_t) and a_t from Ω to the sets $\mathbb{X}$ and $\mathcal{A}$ are called called state and control (or action) variables, respectively. Observe that Ω contains the space $H_\infty = \prod_{t=0}^{\infty} \mathbb{K}$ of admissible histories $(x_0, i_0, a_0, x_1, i_1, a_1, \ldots)$ with $(x_t, i_t, a_t) \in \mathbb{K}$ for all $t \in \mathbb{Z}^+$.

Let $\pi = \{\pi_t\}$ be an arbitrary control policy and υ an arbitrary probability measure on $\mathbb{X}$, referred to as the *initial distribution*. Then, by the theorem of C. Ionescu-Tulcea♣ there exists a unique probability measure P_υ^π on $(\Omega, \mathcal{F})$ which, by (5.1.6), is supported on H_∞, namely $P_\upsilon^\pi (H_\infty) = 1$, and, moreover, for all $D \in \mathcal{B}(\mathbb{X})^\diamondsuit$, $E \in \mathcal{B}(\mathcal{A})$ and $h_t \in H_t$ as in (5.1.4), $t \in \mathbb{Z}^+$ we have

$$P_\upsilon^\pi ((x_0, i_0) \in D) = \nu(D), \tag{5.1.7}$$

$$P_\upsilon^\pi (a_t \in E|h_t) = \pi_t (E|h_t), \tag{5.1.8}$$

$$P_\upsilon^\pi ((x_{t+1}, i_{t+1}) \in D|h_t, a_t) = Q (x_t, i_t, a_t). \tag{5.1.9}$$

The stochastic process $(\Omega, \mathcal{F}, P_\upsilon^\pi, \{(x_n, i_n)\})$ is a discrete-time Markov control process.

Remark 5.1.2. Particularly, in case of minimizing the ruin probability; we assume that $G : \mathbb{K} \times \mathbb{W} \to \mathbb{X}$ and $\mathcal{H} : (0, \infty) \times \Delta \to (0, \infty)$ are measurable functions, where $\mathbb{K} \subset \mathbb{X} \times \mathcal{A}$ is the set defined in (5.1.2). We define a measure $\Sigma \to \mu(\Sigma)$ on $\mathbb{W} \times \Delta$ as $\mu(w, \varpi) := F(w) \times \Upsilon(\varpi)$. Then, by assumption in (5.0.1), the variables (x_t, i_t, a_t) and w_t are independent for each $t = 0, 1, \ldots$. Thus the controlled process transition law Q is given by

$$Q(x, i, a) = \int_{\mathbb{W} \times \Delta} \mathbf{1}_D \left[G (x, i, a, \mathbf{w}), \mathcal{H}(i, \boldsymbol{\varpi})\right] \mu (d(\mathbf{w}, \boldsymbol{\varpi})). \tag{5.1.10}$$

Notation. The following notation will be useful for us.

♣See, [36] and [33] Appendix C.

$^\diamondsuit$Denotes the Borel space.

The expectation operator with respect to P_v^π is denoted by E_v^π. If ν is concentrated at the *initial state* $(x,i) \in \mathbb{X}$, then we write P_v^π and E_v^π as $P_{(x,i)}^\pi$ and $E_{(x,i)}^\pi$, respectively.

Let $\varphi \in \Phi, \quad g : \mathbb{X} \times \mathcal{A} \to \mathbb{R}$ a measurable function, and Q a stochastic kernel on $\mathbb{X}$ given $\mathbb{K}$. Then we define

$$g\left(x,i,\varphi\right) = \int_{\mathcal{A}} g\left(x,i,a\right)\varphi\left(da|x,i\right)$$

and

$$Q\left(\cdot|x,i,\varphi\right) = \int_{\mathcal{A}} Q\left(\cdot|x,i,a\right)\varphi\left(da|x,i\right).$$

In particular, for a function $f \in \mathbb{F}$, we have $g\left(x,i,f\right) = g\left(x,i,f(x,i)\right)$ and $Q\left(D|x,i,f\right) = Q\left(D|x,i,f(x,i)\right)$. Note that each of these functions is measurable.

Interpretation. We observe the system in discrete time (days, months, years,...). The system starts at the state (x_0, i_0) and we apply a policy $\pi = \{\pi_t\}$ in the following way: we choose an action a_0 with distribution law $\pi_0\left(\cdot|h_0\right)$, which incurs the immediate cost $c\left(x_0, i_0, a_0\right)$. Then, the system evolves to a new state (x_1, i_1) according to the transition law $Q\left(\cdot|x_0, i_0, a_0\right)$. Now, we choose an action a_1 with distribution law $\pi_1\left(\cdot|h_1\right)$, which generates a new cost $c(x_1, i_1, a_1)$ and the system moves to another state (x_2, a_2) according to transition law $Q\left(\cdot|x_1, i_1, a_1\right)$. The process is repeated at each time t within the problem's planning horizon.

5.2 Finite-horizon problems

In this section we consider the Markov control model (5.1.1) with a finite planning horizon N. The present value of the current cost in stage t given by

$$e^{-\alpha}c(x_t, i_t, a_t) \text{ for } t = 1, \ldots, N-1,$$

where $\alpha \geq 0$ is a **fixed** discounting parameter. Finally, we consider that at stage N there is a terminal cost $c(x_N)$. That is, the control problem we are interested in is to minimize the finite horizon performance criterion

$$J(\pi, x, i) := E^{\pi}_{(x,i)}\left[\sum_{t=0}^{N-1} e^{-\alpha t} c(x_t, i_t, a_t) + e^{-\alpha} c_N(x_N)\right]. \tag{5.2.1}$$

Thus, denoting by J^* the value function, i.e.,

$$J^*(x, i) := \inf_{\Pi} J(\pi, x, i), \qquad (x, i) \in \mathbb{X}. \tag{5.2.2}$$

the problem is to find policy $\pi^* \in \Pi$ such that

$$J(\pi^*, x, i) = \inf J^*(x, i), \qquad \text{for all } (x, i) \in \mathbb{X}. \tag{5.2.3}$$

Our main result in this section is the following *Dynamical Programming* (DP) theorem, which provides an algorithm for finding both the value function J^* and a deterministic optimal policy π^*.

Remark 5.2.1. We will consider in the case of ruin probability $\alpha = 0$.

Theorem 5.2.1. *Let $J_0, J_1, \ldots, J_N$ be the functions on $\mathbb{X}$ defined (backward, from $t = N$ to $t = 0$) by*

$$J_N(x, i) := c_N(x) \tag{5.2.4}$$

and for $t = N-1, N-2, \ldots, 0$,

$$J_t(x, i) := \min_{\mathcal{A}(x,i)}\left[c(x, i, a) + e^{-\alpha}\int_{\mathbb{X}} J_{t+1}(s, \iota) Q(d(s, \iota)|x, i, a)\right]. \tag{5.2.5}$$

Suppose that these functions are measurable and that, for each $t = 0, \ldots, N-1$, there is a selector $f_t \in \mathbb{F}$ such that $f_t(x, i) \in \mathcal{A}(x, i)$ attains the minimum in (5.2.5) for all $(x, i) \in \mathbb{X}$; that is, for all (x, i) in $\mathbb{X}$ and $t = 0, \ldots, N-1$,

$$J_t(x, i) := c(x, i, f_t) + e^{-\alpha}\int_{\mathbb{X}} J_{t+1}(s, \iota) Q(d(s, \iota)|x, i, f_t). \tag{5.2.6}$$

Then the (deterministic Markov) policy $\pi^* = \{f_0, \ldots, f_{N-1}\}$ is optimal and the value function J^* equals J_0, i.e.,

$$J^*(x, i) = J_0(x, i) = J(\pi^*, x, i) \qquad \forall (x, i) \in \mathbb{X}. \tag{5.2.7}$$

Proof. Let $\pi = \{\pi_t\}$ be an arbitrary policy, and let $C_t(\pi, x, i)$ be the corresponding expected total cost from time t to terminal time N, given the state $(x_t, i_t) = (x, i)$ at time t, i.e.,

$$C_t(\pi, x, i) := E^{\pi}\left[\sum_{n=t}^{N-1} e^{(t-n)\alpha} c(x_n, i_n, a_n) + e^{(t-N)\alpha} c_N(x_N, i_N) | x_t = x, i_t = i\right] \tag{5.2.8}$$

for $t = 0, 1, \ldots, N-1$

$$C_N(\pi, x, i) := E^{\pi}\left[c_N(x_N)\,|x_N = x, i_N = i\right] = c_N(x).$$

$C_t(\pi, x, i)$ is called the "cost-to-go" or cost from time t onwards when using the policy π and $(x_t, i_t) = (x, i)$. In particular note that, from (5.2.1) and (5.2.8)

$$J(\pi, x, i) = C_0(x, i)\,. \tag{5.2.9}$$

To prove the theorem, we shall show that, for all $(x, i) \in \mathbb{X}$ and $t = 0, \ldots, N$,

$$C_t(\pi, x, i) \geq J_t(x, i) \tag{5.2.10}$$

with equality if $\pi = \pi^*$, i.e.,

$$C_t(\pi^*, x, i) = J_t(x, i)\,. \tag{5.2.11}$$

In particular for $t = 0$,

$$J(\pi, x, i) \geq J_0(x, i) \text{ with } J(\pi^*, x, i) = J_0(x, i) \qquad \forall (x, i),$$

which yields the desired conclusion (5.2.7), as $J(\pi, \cdot, \cdot) \geq J_0(\cdot, \cdot)$ for arbitrary π implies $J^*(\cdot, \cdot) \geq J_0(\cdot, \cdot)$.
The proof of (5.2.10) and (5.2.11) is by backward induction. Observe that (5.2.10) and (5.2.11) trivially hold for $t = N$, since, from (5.2.9) and (5.2.4),

$$C_N(\pi, x, i) = J_N(x, i) = c_N(x).$$

Let us now assume (the induction hypothesis) that for some $t = N-1, \ldots, 0$,

$$C_{t+1}(\pi, x, i) \geq J_{t+1}(x, i) \qquad \forall (x, i) \in \mathbb{X}. \tag{5.2.12}$$

Then

$$\begin{aligned} C_t(\pi, x, i) &= E^{\pi}\left[\sum_{j=t}^{N-1} e^{(t-j)\alpha} c(x_j, i_j, a_j) + e^{(t-N)\alpha} c_N(x_N, i_N)\,|x_t = x, i_t = i\right] \\ &= \int_{\mathcal{A}} \left[c(x, i, a) + e^{-\alpha} \int_{\mathbb{X}} C_{t+1}(\pi, s, \iota)\, Q(ds, d\iota | x, i, a)\right] \pi_t(da | x, i) \end{aligned}$$

hence,

$$\begin{aligned} C_t(\pi, x, i) &\geq \min_{\{\mathcal{A}(x,i)\}} \left[c(x, i, a) + e^{-\alpha} \int_{\mathbb{X}} J_{t+1}(s, \iota)\, Q(ds, d\iota | x, i, a)\right] \\ &= J_t(x, i)\,. \end{aligned}$$

This prove (5.2.10). On the other hand, if equality holds in (5.2.12) with $\pi = \pi^*$ so that $\pi_t(\cdot | h_t)$ is the Dirac measure concentrated at $f(x_t, i_t)$, then equality holds throughout the previous calculations which yields (5.2.11). $\square$

5.3 The measurable selection condition

In this section we give conditions on the MCM (5.1.1) that assure the existence of selectors.

Assumption 5.2. For a given measurable function $u : \mathbb{X} \to \mathbb{R}$, the function u^* from $\mathbb{X}$ to $\mathbb{R}$ defined,

$$u^*(x,i) := \inf_{A(x,i)} \left[c(x,i,a) + e^{-\alpha} \int_{\mathbb{X}} u(s,\iota)\, Q(ds, d\iota | x,i,a) \right] \tag{5.3.1}$$

is measurable and there exists a selector $f \in \mathbb{F}$ such that the function within brackets attains its minimum at $f(x,i) \in \mathcal{A}(x,i)$ for all $(x,i) \in \mathbb{X}$, i.e.,

$$u^*(x,i) := c(x,i,f) + e^{-\alpha} \int_{\mathbb{X}} u(s,\iota)\, Q(ds, d\iota | x,i,f) \qquad \forall (x,i) \in \mathbb{X}.$$

We recall some definitions that will be used in the condition below.

Let Ξ be a metric space and v a function from Ξ to $\mathbb{R} \cup \{\infty\}$ such that $v(s) < \infty$ for at least one point $s \in \Xi$.

- The function v is said to be *lower semicontinuous* (l.s.c) at $s \in \Xi$, if $\liminf v(s_n) \geq v(s)$ for any sequence $\{s_n\}$ in Ξ that converges to s. The function v is called lower semicontinuous if its is l.s.c at every point of Ξ.

- A function $v : \mathbb{K} \to \mathbb{R}$ is said to be *inf-compact* on $\mathbb{K}$ if, for every $(x,i) \in \mathbb{X}$ and $r \in \mathbb{R}$, the set $\{a \in \mathcal{A}(x,i) | v(x,i,a) \leq r\}$ is compact. A multifunction ψ from $\mathbb{X}$ to $\mathcal{A}$ is said to be *upper semicontinuous* (u.s.c) if $\psi^{-1}[\widehat{\mathrm{F}}]$ is closed in $\mathbb{X}$ for every closed set $\widehat{\mathrm{F}} \subset \mathcal{A}$. Let $\mathbb{B}(\mathbb{X})$ be the family of measurable bounded functions on $\mathbb{X}$, and $\mathbb{C}(\mathbb{X}) \subset \mathbb{B}(\mathbb{X})$ the subfamily of continuous functions.

We now consider the three conditions under which, in particular, Assumption 5.2 is satisfied

Condition 5.1.
1. The control sets $\mathcal{A}(x,i)$ are compact all $(x,i) \in \mathbb{X}$.
2. The one-stage cost c is such that $c(x,i,\cdot)$ is l.s.c on $\mathcal{A}(x,i)$ for every $(x,i) \in \mathbb{X}$.
3. The function

$$v'(x,i,a) := \int_{\mathbb{X}} v(s,\iota)\, Q(ds, d\iota | x,i,a) \tag{5.3.2}$$

on $\mathbb{K}$ satisfies one of the two following conditions:

(a) $v'(x, i, \cdot)$ is l.s.c. on $\mathcal{A}(x, i)$ for every $(x, i) \in \mathbb{X}$ and every $v \in \mathbb{C}(\mathbb{X})$.

(b) $v'(x, i, \cdot)$ is l.s.c. on $\mathcal{A}(x, i)$ for every $(x, i) \in \mathbb{X}$ and every $v \in \mathbb{B}(\mathbb{X})$.

Condition 5.2. 1. $\mathcal{A}(x, i)$ is compact for all $(x, i) \in \mathbb{X}$ and the multifunction $(x, i) \mapsto A(x, i)$ is u.s.c.

2. The one-stage cost c is l.s.c. and bounded below.

3. The transition law Q is either:

   (a) weakly continuous, i.e., for every function $v \in \mathbb{C}(\mathbb{X})$, the function v' in (5.3.2) is continuous and bounded on $\mathbb{K}$.

   (b) strongly continuous, i.e., v' is continuous and bounded on $\mathbb{K}$ for every $v \in \mathbb{B}(\mathbb{X})$.

Condition 5.3. 1. The one-stage cost c is l.s.c. bounded below and inf-compact on $\mathbb{K}$;

2. Same as 5.2 (3), i.e., Q is either

   (a) weakly continuous, or

   (b) strongly continuous.

We next show how the last three conditions relate to Assumption 5.2.

Theorem 5.3.1. *1. Each of Condition 5.1 and 5.2 implies Assumption 5.2 for any nonnegative measurable function $u : \mathbb{X} \to \mathbb{R}$. Moreover, under 5.1(3a) or 5.2(3a), it suffices to take u nonnegative and l.s.c. in which case, under 5.2(1,2,3a) the function u^* in (5.3.1) is l.s.c.*

2. Condition 5.3 implies Assumption 5.2 if, under 5.3(2a), u is nonnegative and l.s.c. or, under 5.3(2b), if u is a nonnegative measurable function. If in addition the multifunction $(x, i) \mapsto \mathcal{A}^*(x, i)$ with $\mathcal{A}^*(x, i)$ equal to

$$\left\{ a \in \mathcal{A}(x, i) | u^*(x, i) := c(x, i, a) + e^{-\alpha} \int u(s, \iota) Q\left(ds, d\iota | x, i, a\right) \right\}$$

is l.s.c., then u^* is l.s.c..

Remark 5.3.1. In Theorem 5.3.1, we suppose that u is nonnegative, but it is easily seen that it suffices to take a bounded below.

Proof. To follow [30] and [33] pp. 29.

1. Let $u \geq 0$ be a measurable function on $\mathbb{X}$.
   To prove the first statement (1), it clearly suffices to consider Conditions 5.1 (1), (2) and (3b). Moreover, note that given l.s.c. functions v_1, v_2, then $v_1 + e^{-\alpha} v_2$ is also l.s.c.. Hence the desired conclusion follows from Proposition D.5 in Hernández-Lerma and Lasserre [33] provided that the function

$$a \mapsto \int_{\mathbb{X}} u(s, \iota) Q\left(ds, d\iota | x, i, a\right) \text{ is l.s.c. on } \mathcal{A}(x, i) \text{ for every } (x, i) \in \mathbb{X}. \tag{5.3.3}$$

To prove this, let $\{u_n\}$ be a sequence in $\mathbb{B}(\mathbb{X})$ such that $u_n \uparrow u$, and let $\{a^l\}$ be a sequence in $\mathcal{A}(x,i)$. Converging to $a \in \mathcal{A}(x,i)$. Then, for each n we have

$$\begin{aligned}\liminf_{l\to\infty}\int u(s,\iota)Q\left(ds,d\iota|x,i,a^l\right) &\geq \liminf_{l\to\infty}\int u^n(s,\iota)Q\left(ds,d\iota|x,i,a^l\right)\\ &\geq \int u^n(s,\iota)Q\left(ds,d\iota|x,i,a\right).\end{aligned}$$

Letting n tend to infinity we obtain (by the Monotone Convergence Theorem)

$$\liminf_{l\to\infty}\int u(s,\iota)Q\left(ds,d\iota|x,i,a^l\right) \geq \int u(s,\iota)Q\left(ds,d\iota|x,i,a\right)$$

which proves (5.3.3). Thus, as was already mentioned, we obtain Assumption 5.2 from Proposition D.5 in [33].
Let us now suppose the Condition 5.1 (3a) and 5.2 (3a) hold. Then the second statement in (1) follows from the same argument above, but now based on the fact that $u \geq 0$ is l.s.c., then it is the limit of an increasing sequence in $\mathbb{C}(\mathbb{X})$ (see Proposition A.2 in [33]). The last statement in (1) follows from the above arguments and Proposition D.5(b) in [33].

2. Suppose that Condition 5.3 holds with (2a), and that $u \geq 0$ is measurable. Then, as in the proof of part (1), but now approximating u from below by functions in $\mathbb{B}(\mathbb{X})$, one can show that the function

$$u'(x,i,a) := c(x,i,a) + e^{-\alpha}\int_{\mathbb{X}} u(s,\iota)Q\left(ds,d\iota|x,i,a\right), \qquad (x,i,a)\in\mathbb{K}$$

is l.s.c. and bounded below. Thus we may obtain Assumption 5.2 from Proposition D.6(a) in [33], if u' is inf-compact on $\mathbb{K}$, that is, if for every $(x,i) \in \mathbb{X}$ and $r \in \mathbb{R}$, the set $\{a \in \mathcal{A}(x,i)|u'(x,i,a) \leq r\} := D$ is compact. But this is obviously true, since (by lower semicontinuity *) D is closed and, since $u \geq 0$, it is contained in the set $\{a \in \mathcal{A}(x,i)|c(x,i,a) \leq r\}$, which, by the inf-compactness of c (see Condition 5.3(1)), is compact. The proof under (2a) is similar.
The last statement (2) follows from Proposition D.6(b) in [33].

□

Remark 5.3.2. Particularly, in case of minimizing the ruin probability; consider the equation model (5.0.1), i.e.,

$$\begin{aligned} x_{n+1} &= G\left(x_n, i_{n+1}, a_n, w_n\right),\\ i_{n+1} &= \mathcal{H}\left(i_n, \varpi_n\right) \qquad n=0,1,\ldots,\end{aligned} \tag{5.3.4}$$

*See Proposition A.1(c) in [33].

where $\{w_n\}$ and $\{\varpi_n\}$ are sequence of i.i.d. random disturbances, independent of the inicial state (x_0, i_0). The corresponding transition law Q can be written, for all $D \in \mathcal{B}(\mathbb{X})$ and $(x, i, a) \in \mathbb{K}$, as

$$Q(x,i,a) = \int_{\Delta} \int_{\mathbb{W}} \mathbf{1}_D \left[G\left(x,i,a,w\right), \mathcal{H}(i,\varpi) \right] F(dw) \Upsilon(d\varpi)$$

In this case, by the "change of variables" formula for integrals[1], for any measurable function v on $\mathbb{X}$ we have

$$\begin{aligned} E\left[v\left(x_{t+1}, i_{t+1}\right)|x_t = x, i_t = i, a_t = a\right] &= \int_{\mathbb{X}} v\left(s,\iota\right) Q\left(ds, d\iota | x,i,a\right) \\ &= \int_{\mathbb{W}\times\Delta} v\left(G\left(x,i,a,\mathbf{w}\right), \mathcal{H}(i,\boldsymbol{\varpi})\right) \mu\left(d(\mathbf{w},\boldsymbol{\varpi})\right) \\ &= Ev\left[G\left(x,i,a,\mathbf{w}_0\right), \mathcal{H}(i,\boldsymbol{\varpi}_0)\right] \end{aligned} \tag{5.3.5}$$

in the sense that, if one of the integrals exists, so does the other, and they are equal.
Thus, for the system (5.3.4), the DP equations in Theorem 5.2.1 can be rewritten, using (5.3.5), as

$$\begin{aligned} J_N(x,i) &= c_N(x) \\ J_t\left(x,i\right) &= \min_{\mathcal{A}(x,i)} \left[c(x,i,a) + \int_{\mathbb{X}} J_{t+1}(s,\iota) Q\left(d(s,\iota)|x,i,a\right) \right] \\ &= \min_{\mathcal{A}(x,i)} \left[c(x,i,a) + \int_{\mathbb{W}\times\Delta} J_{t+1}\left(G\left(x,i,a,\mathbf{w}\right), \mathcal{H}(i,\boldsymbol{\varpi})\right) \mu\left(d(\mathbf{w},\boldsymbol{\varpi})\right) \right] \\ &= \min_{\mathcal{A}(x,i)} \left[c(x,i,a) + EJ_{t+1}\left[G\left(x,i,a,w_t\right), \mathcal{H}(i,\varpi_t)\right]\right] \end{aligned} \tag{5.3.6}$$

for all $(x, i) \in \mathbb{X}$ and $t = N-1, N-2, \ldots, 0$.
One might add that we can consider w_n as in model (4.3.1), $\{w_n\} = \{(z_n, y_n)\}$ with $\{z_n\}$ and $\{y_n\}$ are i.i.d. random variables independent each other.

5.4 Infinite-horizon cost problem

The motivation to study discounted cost problem is mainly economic. In this section, we considered finite-horizon problems, but for many proposes it is convenient to introduce the fiction that the optimization horizon is infinite. Certainly, for instance, processes of capital accumulation for an economy do not necessarily have a natural stopping time in the definable

[1] See Ash [1] p.225.

future.

Given a stationary control model as (5.1.1) and the performance criterion to be minimized is

$$V(\pi, x, i) := E^{\pi}_{(x,i)}\left[\sum_{t=0}^{\infty} e^{-\alpha t} c(x_t, i_t, a_t)\right], \qquad \pi \in \Pi, (x,i) \in \mathbb{X}. \tag{5.4.1}$$

A policy π^* satisfying

$$V(\pi^*, x, i) = \inf_{\pi} V(x_t, i_t, a_t) =: V^*(x,i), \qquad \forall (x,i) \in \mathbb{X}. \tag{5.4.2}$$

is said to be *optimal* and V^* is called the *value function*.

Throughout the following, we suppose that the one-stage cost c is nonnegative$^\spadesuit$. Moreover, we will use V_n to denote the n-stage cost

$$V_n(\pi, x, i) := E^{\pi}_{(x,i)}\left[\sum_{t=0}^{n-1} e^{-\alpha t} c(x_t, i_t, a_t)\right]. \tag{5.4.3}$$

Hence (by the Monotone Convergence Theorem) we may write $V(\pi, x, i)$ in (5.4.1) as

$$V(\pi, x, i) = \lim_{n\to\infty} V_n(\pi, x, i). \tag{5.4.4}$$

A measurable function $v : \mathbb{X} \to \mathbb{R}$ is said to be a *solution of optimality equation* (OE) if it satisfies

$$V(x,i) = \min_{\mathcal{A}(x,i)} \left\{ c(x,i,a) + e^{-\alpha} \int_{\mathbb{X}} V(s,\iota) Q(ds, d\iota | x, i, a) \right\} \qquad \forall (x,i) \in \mathbb{X}. \tag{5.4.5}$$

In Theorem 5.4.1, we prove that the value function V^* in (5.4.2) is solution to the OE. To this end, we begin with the DP Theorem 5.2.1 for *finite*-horizon problems and with suitable change of indices we obtain the *forward form of the dynamic programming algorithm*, that is, the value iteration functions defined as

$$v_n(x,i) = \min_{\mathcal{A}(x,i)} \left\{ c(x,i,a) + e^{-\alpha} \int_{\mathbb{X}} v_{n-1}(s,\iota) Q(ds, d\iota | x, i, a) \right\} \qquad \forall (x,i) \in \mathbb{X}. \tag{5.4.6}$$

$^\spadesuit$It suffices to assume that c is bounded below.

and $n = 1, 2, \ldots$, with $v_0(\cdot) := 0$. The idea then is to show that

$$V^*(x,i) = \lim_{n\to\infty} v_n(x,i) \qquad \forall (x,i) \in \mathbb{X}. \tag{5.4.7}$$

this result is to be expected since v_n is the value function of the n-stage cost V_n in (5.4.3) with zero terminal cost, namely

$$v_n(x,i) = \inf_{\pi} V_n(\pi, x, i) \qquad \forall (x,i) \in \mathbb{X}. \tag{5.4.8}$$

This, letting $n \to \infty$ in (5.4.6) we anticipate to obtain (5.4.9), if we can justify the interchange of limits and minima. This approach, requires first of all, the measurable selection condition in Assumption 5.2 for (5.4.6) and (5.4.9) to be well defined. We also impose the follow requirements.

Assumption 5.3. 1. The one-stage cost c is l.s.c., nonnegative, and inf-compact on $\mathbb{K}$.

2. Q is strongly continuous.

Assumption 5.4. There exists a policy π such that $V(\pi, x, i) < \infty$ for each $(x,i) \in \mathbb{X}$.

We shall denote by Π^0 the family of policies for which Assumption 5.4 holds. We now state our main result in this section.

Theorem 5.4.1. *Suppose that Assumptions 5.3 and 5.4 hold. Then*

1. *The value function V^* is the minimal solution to the OE. i.e.,*

$$V^*(x,i) = \min_{\mathcal{A}(x,i)} \left\{ c(x,i,a) + e^{-\alpha} \int_{\mathbb{X}} V^*(s,\iota) Q(ds, d\iota | x, i, a) \right\} \qquad \forall (x,i) \in \mathbb{X}. \tag{5.4.9}$$

   for all $(x,i) \in \mathbb{X}$ and if u is another solution to the OE, then $u(\cdot) \geq V^*(\cdot)$.

2. *There exists a selector $f^* \in \mathbb{F}$ such that $f^*(x,i) \in \mathcal{A}(x,i)$ attains the minimum in (5.4.9), i.e.,*

$$V^*(x,i) = c(x,i,f^*) + e^{-\alpha} \int_{\mathbb{X}} V^*(s,\iota) Q(ds, d\iota | x, i, f^*) \qquad \forall (x,i) \in \mathbb{X}. \tag{5.4.10}$$

   and deterministic stationary policy f^*_∞ is optimal. Conversely, if f^*_∞ is a stationary deterministic optimal policy, then it satisfies (5.4.10).

3. *If π^* is a policy such that $V(\pi^*, \cdot, \cdot)$ is a solution to the OE and satisfies*

$$\lim_{n\to\infty} E^{\pi}_{(x,i)} \left[e^{-\alpha n} V^*(x_n, i_n) \right] = 0 \quad \forall \pi \in \Pi^0 \text{ and } (x,i) \in \mathbb{X}, \tag{5.4.11}$$

   then $V(\pi^*, \cdot, \cdot) = V^*(\cdot, \cdot)$, and so π^* is α-discounted optimal. In other word, if (5.4.11) holds, then π^* is optimal if and only if $V(\pi^*, \cdot, \cdot)$ satisfies the OE.

4. If an optimal policy exists, then there exists one that is deterministic stationary.

The proof of this theorem requires several lemmas.

Lemma 5.4.2. *Let u and u_n $(n = 1, 2, \ldots)$ be l.s.c. functions, bounded below, and inf-compact on $\mathbb{K}$. If $u_n \uparrow u$. Then*

$$\lim_{n\to\infty} \min_{\mathcal{A}(x,i)} u_n(x, i, a) = \min_{\mathcal{A}(x,i)} u(x, i, a) \quad \forall\, (x, i) \in \mathbb{X}, \tag{5.4.12}$$

Proof. The proof is similar to that of Lemma 4.2.4 in Hernández-Lerma and Lasserre [33] p.47, and, therefore is omitted. □

We need also in this case the existence of measurable selectors that satisfy the DP equation.

To do this we use Theorem 5.3.1 and the following definition.

Definition 5.4.1. $M(\mathbb{X})^+$ denotes the cone of nonnegative measurable function on $\mathbb{X}$, and, for every $u \in M(\mathbb{X})^+$, Tu is the function on $\mathbb{X}$ defined as

$$Tu(x, i) = \min_{\mathcal{A}(x,i)} \left\{ c(x, i, a) + e^{-\alpha} \int_{\mathbb{X}} u(s, \iota) Q(ds, d\iota | x, i, a) \right\}. \tag{5.4.13}$$

Lemma 5.4.3. *Under Assumption 5.3, T maps $M(\mathbb{X})^+$ into itself, i.e., for every u in $M(\mathbb{X})^+$, Tu is also in $M(\mathbb{X})^+$, and moreover, there exists a selector $f \in \mathbb{F}$ such that*

$$Tu(x, i) = c(x, i, f) + e^{-\alpha} \int_{\mathbb{X}} u(s, \iota) Q(ds, d\iota | x, i, f) \qquad \forall (x, i) \in \mathbb{X}.$$

Notice also that, using the operator T, we may rewrite the OE (5.4.9) and the functions in (5.4.6) as

$$V^* = TV^* \text{ and } v_n = Tv_{n-1} \text{ for } n \geq 1$$

$v_0 = 0$, respectively. We shall next relate V^* to the functions u that satisfy $u \geq Tu$ or $u \leq Tu$.

Lemma 5.4.4. *Suppose that Assumption 5.3 and 5.4 hold:*

1. *If $u \in M(\mathbb{X})^+$ is such that $u \geq Tu$, then $u \geq V^*$.*
2. *If $u : \mathbb{X} \to \mathbb{R}$ is a measurable function such that Tu is well defined and, in addition, $u \leq Tu$ and*

$$\lim_{n\to\infty} E^{\pi}_{(x,i)} \left[e^{-\alpha n} u(x_n, i_n) \right] = 0 \quad \forall \pi \in \Pi^0 \text{ and } (x, i) \in \mathbb{X}, \tag{5.4.14}$$

then $u \leq V^*$.

Proof. 1. Let $u \in M(\mathbb{X})^{+}$ such that $u \geq Tu$, then, by Lemma 5.4.3, we may choose $f \in \mathbb{F}$ such that

$$u(x,i) \geq c(x,i,f) + e^{-\alpha} \int_{\mathbb{X}} u(s,\iota) Q(ds, d\iota | x, i, f).$$

Iterations of this inequality give us

$$u(x,i) \geq E^{\pi}_{(x,i)} \left[\sum_{t=0}^{N-1} e^{-\alpha t} c(x_t, i_t, a_t) \right] + E^{\pi}_{(x,i)} \left[e^{-\alpha N} u(x_N, i_N) \right], \tag{5.4.15}$$

where $\pi = (f, f, \ldots) = f_\infty$ and $E^{\pi}_{(x,i)} \left[e^{-\alpha N} u(x_N, i_N) \right] = \int u(s,\iota) Q^n(ds, d\iota | x, i, f)$. Since $u \geq 0$, we have that

$$u(x,i) \geq E^{\pi}_{(x,i)} \left[\sum_{t=0}^{N-1} e^{-\alpha t} c(x_t, i_t, a_t) \right].$$

Letting $N \to \infty$, we get

$$u(x,i) \geq V(\pi, x, i) \geq V^{*}(x,i) \qquad \forall (x,i) \in \mathbb{X}.$$

This proves (1).

2. Let $\pi \in \Pi$ and $(x,i) \in \mathbb{X}$ be arbitrary. From the Markov-like property

$$P^{\pi}_{(x,i)}((x_{t+1}, i_{t+1}) \in D | h_t, a_t) = Q(x_t, i_t, a_t)$$

and the assumption $Tu \geq u$,

$$\begin{aligned}
E^{\pi}_{(x,i)} \left[e^{-(t+1)\alpha} u(x_{t+1}, i_{t+1}) | h_t, a_t \right] &= e^{-(t+1)\alpha} E^{\pi}_{(x,i)} \left[u(x_{t+1}, i_{t+1}) | x_t, i_t, a_t \right] \\
&= e^{-(t+1)\alpha} \left[\int_{\mathbb{X}} u(s,\iota) Q(ds, d\iota | x_t, i_t, a_t) \right] \\
&= e^{-t\alpha} \left[e^{-\alpha} \int_{\mathbb{X}} u(s,\iota) Q(ds, d\iota | x_t, i_t, a_t) \right],
\end{aligned}$$

obviously,

$$\begin{aligned}
e^{-t\alpha} \left[c(x_t, i_t, a_t) + e^{-\alpha} \int_{\mathbb{X}} u(s,\iota) Q(ds, d\iota | x_t, i_t, a_t) - c(x_t, i_t, a_t) \right] \\
\geq e^{-\alpha t} \left[u(x_t, i_t) - c(x_t, i_t, a_t) \right].
\end{aligned}$$

Hence

$$\begin{aligned}
e^{-\alpha t} c(x_t, i_t, a_t) &\geq -E^{\pi}_{(x,i)} \left[e^{-(t+1)\alpha} u(x_{t+1}, i_{t+1}) | h_t, a_t \right] + e^{-\alpha t} u(x_t, i_t) \\
&= -E^{\pi}_{(x,i)} \left[e^{-(t+1)\alpha} u(x_{t+1}, i_{t+1}) - e^{-\alpha t} u(x_t, i_t) | h_t, a_t \right].
\end{aligned}$$

Thus, taking expectations $E^{\pi}_{(x,i)}$ and summing over $t = 0, \ldots, N-1$, we have

$$\begin{aligned} E^{\pi}_{(x,i)}\left[\sum_{t=0}^{N-1} e^{-t\alpha} c\left(x_t, i_t, a_t\right)\right] &\geq \sum_{t=0}^{N-1}\left(-E^{\pi}_{(x,i)}\left[e^{-(t+1)\alpha} u(x_{t+1}, i_{t+1}) + e^{-\alpha t} u\left(x_t, i_t\right)|h_t, a_t\right]\right) \\ &= -E^{\pi}_{(x,i)}\left[e^{-N\alpha} u(x_N, i_N)\right] + u\left(x, i\right) \qquad \forall N. \end{aligned}$$

Finally, letting $N \to \infty$ and using the hypothesis (5.4.14), it follows that $V\left(\pi, x, i\right) \geq u\left(x, i\right)$, as π and (x, i) were arbitrary.

□

We shall now use Lemmas 5.4.2 and 5.4.4 to prove the limit (5.4.7).

Lemma 5.4.5. *Suppose that Assumptions 5.3 and 5.3 hold. Then $v_n \uparrow V^*$ satisfies the OE.*

Proof. To begin, note that, from (5.4.9), (5.4.1) and the assumption that $c \geq 0$,

$$v_n\left(x, i\right) \leq V_n\left(\pi, x, i\right) \leq V\left(\pi, x, i\right) \qquad n, \pi, \left(x, i\right).$$

Therefore,

$$v_n\left(x, i\right) \leq V^*\left(x, i\right) \qquad n, \left(x, i\right) \in \mathbb{X}. \tag{5.4.16}$$

Now, the operator T in (5.4.13) is monotone. Therefore, since $v_0 := 0$ and $v_n := Tv_{n-1}$ for $n \geq$, the functions from a nondecreasing sequences in $M\left(\mathbb{X}\right)^+$. This, in turn (by the Monotone Convergence Theorem), implies $u_n \uparrow u$, where

$$\begin{aligned} u_n\left(x, i\right) &= c\left(x, i, a\right) + e^{-\alpha} \int_{\mathbb{X}} v_n\left(s, \iota\right) Q\left(ds, d\iota | x, i, a\right), \\ u\left(x, i\right) &= c\left(x, i, a\right) + e^{-\alpha} \int_{\mathbb{X}} v^*\left(s, \iota\right) Q\left(ds, d\iota | x, i, a\right). \end{aligned}$$

On the other hand, as the proof of Theorem 5.3.1(2), one can show that the nonnegative functions u and u_n $(n \geq 1)$ are l.s.c. and inf-compact on $\mathbb{K}$. Thus, from Lemma 5.4.2,

$$v^* = \lim_n v_n = \lim_n Tv_{n-1} = Tv^*; \tag{5.4.17}$$

that is, v^* satisfies the OE $v^* = Tv^*$.
Hence, to complete the proof of the lemma, it only remains to show that $v^* = V^*$. But this is immediate because, by Lemma 5.4.4(2), $v^* = Tv^*$ implies $v^* \geq V^*$, and the reverse inequality follows from (5.4.16) and the already established fact that $v_n \uparrow v^*$ □

Finally, we prove Theorem 5.4.1.

Proof of Theorem 5.4.1.

1. From Lemma 5.4.5. V^* is a solution to the OE, and the fact that V^* is the minimal solution follows from Lemma 5.4.41 if $u = Tu$, then $u \geq V^*$.

2. The existence of a selector $f^* \in \mathbb{F}$ satisfying (5.4.10) is ensured by Lemma 5.4.3. Now iteration (5.4.10) shows [as in (5.4.15)] that

$$\begin{aligned} V^* &= E_{(x,i)}^{f_\infty^*}\left[\sum_{t=0}^{n-1} e^{-t\alpha} c\left(x_t, i_t, f^*\right)\right] + E_{(x,i)}^{f_\infty^*}\left[e^{-t\alpha} V^*\left(x_n, i_n\right)\right] \\ &\geq E_{(x,i)}^{f_\infty^*}\left[\sum_{t=0}^{n-1} e^{-t\alpha} c\left(x_t, i_t, f^*\right)\right] \qquad n \geq 1 \quad \forall (x,i) \in \mathbb{X}. \end{aligned}$$

This implies, letting $n \to \infty$, $V^*(x,i) \geq V\left(f_\infty^*, x, i\right) \quad \forall (x,i) \in \mathbb{X}$, whereas, from (5.4.2), $V^*(\cdot) \leq V\left(f_\infty^*, \cdot\right)$ and, therefore, f_∞^* is optimal.

To prove the converse, we note first the import fact that for any deterministic stationary policy f_∞, the cost $V(f_\infty, \cdot)$ satisfies

$$V\left(f_\infty, x, i\right) = c(x,i,f) + e^{-\alpha}\int_{\mathbb{X}} V\left(f_\infty, s, \iota\right) Q\left(ds, d\iota | x, i, f\right) \qquad \forall (x,i) \in \mathbb{X}.$$

Indeed,

$$\begin{aligned} V\left(f_\infty, x, i\right) &= E_{(x,i)}^{f_\infty}\left[\sum_{t=0}^{\infty} e^{-t\alpha} c\left(x_t, i_t, f\right)\right] \\ &= E_{(x,i)}^{f_\infty}\left[c\left(x_0, i_0, f\right) + \sum_{t=1}^{\infty} e^{-t\alpha} c\left(x_t, i_t, f\right)\right] \\ &= c(x,i,f) + e^{-\alpha} E_{(x,i)}^{f_\infty}\left[\sum_{t=1}^{\infty} e^{-(t-1)\alpha} c\left(x_t, i_t, f\right)\right] \\ &= c(x,i,f) + e^{-\alpha} E_{(x,i)}^{f_\infty} E_{(x_1,i_1)}^{f_\infty}\left[\sum_{t=1}^{\infty} e^{-(t-1)\alpha} c\left(x_t, i_t, f\right)\right] \\ &= c(x,i,f) + e^{-\alpha} E_{(x,i)}^{f_\infty}\left[V\left(f_\infty, x, i\right)\right] \\ &= c(x,i,f) + e^{-\alpha}\int_{\mathbb{X}} V\left(f_\infty, s, \iota\right) Q\left(ds, d\iota | x, i, f\right). \end{aligned} \tag{5.4.18}$$

In particular, if f^* is stationary deterministic optimal, then $V\left(f_\infty^*, \cdot\right) = V^*(\cdot)$, in which case (5.4.18), with $f = f^*$, reduces to (5.4.10).

3. if $V(\pi^*, \cdot)$ satisfies the OE, then from part (1) or Lemma 5.4.4(1) we get $V(\pi^*, \cdot) \geq V^*(\cdot)$.

   The reverse inequality follows from (5.4.11) and Lemma 5.4.4(2).

   Finally,

4. is a consequence of 1 and 2. □

Concluding remarks

Our main results in this thesis, Theorems 4.2.3, 4.2.6, 4.3.3.1 and 4.3.3.2, give upper bounds for the probability of ruin of a certain risk process, which (as shown in Subsections 4.2.1 and 4.3.1) includes as special cases several relevant models. To obtain these results, first, we present an important preliminary result, Lemmas 4.2.2 and 4.3.2, which gives recursive equations for finite-horizon ruin probabilities and an integral equation for the ultimate ruin probability. We illustrate our results with an application to the ruin probability in a risk process with a heavy tail claims distribution under proportional reinsurance and a Markov interest rate process. This application suggests that the upper bounds derived by inductive approach are tighter than the ruin probability without interest rate (the function considered in Lemmas 4.2.1 and 4.3.1). In addition, the upper bounds derived in this article are sharper than the Lundberg upper bound.

Since $\{I_n\}$ in (4.2.1) and (4.3.1) is supposed to be a Markov chain, we can rewrite the minimization of the ruin probability as a *Markov decision problem.* In chapter 3, we prove the existence of optimal policies in finite MDPs and by using the dynamic programming algorithm, under the assumption of the existence of measurable selectors that satisfy optimality equations. We give conditions that assure the existence of such selectors. Finally, we show the optimality equation in infinite horizon MDPs and we prove the existence of optimal policies.

Our paper leaves, of course, many open issues. For instance:

1. Is it possible to obtain bounds tighter than those in Theorems 4.2.3, 4.2.6, 4.3.3.1 and 4.3.3.2 ?.
2. Actually, what do we need to obtain the ruin probabilities in closed form ?.

3. Let $\tau := \inf \{k \geq 1 | X_k < 0\}$ be the *time of ruin*.

   Can we calculate or estimate quantities such as $E[\tau]$, or $P(\tau \leq T)$ for given $T > 0$?.

   These are just a few of the many questions that we can ask ourselves. But two immediate queries are:

   (a) Suppose that in (4.2.1) and (4.3.1) we include an investment process. What can we say about these models?

   (b) Can we rewrite the exponential utility or minimization of the ruin probability of controlled risk process with investment process as a *Markov decision problem*? ([33, 34, 54], for instance).

Further research in some of these directions is in progress.

Appendix A

Miscellaneous

A.1 σ-Algebra

Let X be a set. Then a σ-algebra $\mathcal{F}$ is a nonempty collection of subsets of X such that the following hold:

1. X is in $\mathcal{F}$.

2. If A is in $\mathcal{F}$, then so is the complement of A.

3. If A_n is a sequence of elements of $\mathcal{F}$, then the union of the A_n's is in $\mathcal{F}$.

If S is any collection of subsets of X, then we can always find a σ-algebra containing S, namely the power set of X. By taking the intersection of all σ-algebras containing S, we obtain the smallest such σ-algebra. We call the smallest o-algebra containing S the σ-algebra generated by S.

A.2 Borel-measurable

Given a Borel space X (i.e., a Borel subset of a complete and separable metric space), its Borel σ-algebra is denoted by $\mathcal{B}(X)$. By convention, when referring to sets or functions, "measurable" means "Borel-measurable". If X and Y are Borel spaces, a *stochastic kernel* on X given Y is a function $P(\cdot|\cdot)$ such that $P(\cdot|y)$ is a probability measure on X for each fixed $y \in Y$, and $P(D|\cdot)$ is a measurable function on Y for each fixed $D \in \mathcal{B}(X)$. The family of all stochastic kernels on X given Y is denoted by $P(X|Y)$.

A.3 Proof of Proposition 1.1.1

Proof. By using renewal arguments and conditioning on the time and size of the first claim, we have

$$\begin{aligned}
\Phi(x) &= P\,(\text{ “non-ruin in } [0,\infty)\text{”} | X_0 = x) \\
&= \textstyle\int_0^\infty \int_0^\infty P\,(\text{ “non-ruin in } [0,\infty)\text{”} | T_1 = t, Y_1 = y)\, dF(y) dF_{T_1}(t) \\
&= \textstyle\int_0^\infty \int_0^{x+ct} P\,(\text{ “non-ruin in } [0,\infty)\text{”} | T_1 = t, Y_1 = y)\, dF(y) dF_{T_1}(t) \\
&= \textstyle\int_0^\infty \lambda e^{-\lambda t} \int_0^{x+ct} P\,(\text{ “non-ruin in } [0,\infty)\text{”} | T_1 = t, Y_1 = y)\, dF(y) dt \\
&= \textstyle\int_0^\infty \lambda e^{-\lambda t} \int_0^{x+ct} \Phi\,(x+ct-y)\, dF(y) dt.
\end{aligned}$$

The change of variables $s = x + ct$ leads to

$$\Phi(x) = \frac{1}{c}\int_x^\infty \lambda e^{-\lambda(s-x)/c} \int_0^s \Phi\,(s-y)\, dF(y) ds = \frac{\lambda}{c} e^{\lambda x/c} \int_x^\infty \lambda e^{-\lambda s/c} \int_0^s \Phi\,(s-y)\, dF(y) ds.$$

Consequently Φ is differentiable and differentiation lead to

$$\Phi^{'}(x) = \frac{\lambda}{c}\Phi(x) - \frac{\lambda}{c}\int_0^x \Phi(x-y) dF(y) = \frac{\lambda}{c}\left[\Phi(x) + \int_0^x \Phi(x-y) d\,(1-F(y))\right]. \quad \text{(A.3.1)}$$

Integrating over $(0, z)$ yields

$$\begin{aligned}
\Phi(z) - \Phi(0) &= \tfrac{\lambda}{c}\left[\textstyle\int_0^z \Phi(x)dx + \int_0^z \int_0^x \Phi(x-y) d\,(1-F(y))\, dx\right] \\
&= \tfrac{\lambda}{c}\left[\textstyle\int_0^z \Phi(x)dx + \int_0^z \left[\Phi(0)\,(1-F(x)) - \Phi(x) + \int_0^x \Phi'(x-y)\,(1-F(y))\, dy\right] dx\right]^{*} \\
&= \tfrac{\lambda}{c}\left[\Phi(0) \textstyle\int_0^z (1-F(x))\, dx + \int_0^x (1-F(y))\, dy \int_y^z \Phi'(x-y) dx\right] \\
&= \tfrac{\lambda}{c}\left[\Phi(0) \textstyle\int_0^z (1-F(x))\, dx + \int_0^z (1-F(y))\,(\Phi(z-y) - \Phi(0))\, dy\right].
\end{aligned}$$

Thus we have

$$\Phi(x) = \Phi(0) + \frac{\lambda}{c}\int_0^x \Phi(x-y)\,(1-F(y))\, dy. \quad \text{(A.3.2)}$$

By monotone convergence it follows from (A.3.2), as $x \to \infty$, that

$$\lim_{x\to\infty} \Phi(x) = \Phi(0) + \frac{\lambda\mu}{c} \lim_{x\to\infty} \Phi(x).$$

It follows from the law of large numbers that $\lim\limits_{t\to\infty} X_t/t = c - \lambda\mu$ with probability one. In the case of positive safety loading, $c > \lambda\mu$, there exists a random variable T, i.e., a function of N

*Integration by parts.

and $\{Y_i\}$, such that $X_t > 0$ for all $t > T$. Since only finitely many claims can occur before T it follows that $\inf_{t>0} X_t$ is finite with probability one and thus $\lim_{x\to\infty} \Phi(x) = 1$. Thus

$$\Phi(0) = 1 - \frac{\lambda\mu}{c} \Rightarrow \psi(0) = \frac{\lambda\mu}{c} = \frac{1}{1+\rho}^{\dagger} \text{ when } c > \lambda\mu. \tag{A.3.3}$$

From (A.3.2) and (A.3.3), we have

$$\psi(x) = \frac{\lambda}{c}\left[\mu - \int_0^x (1-\psi(x-y))\,(1-F(y))\,dy\right]$$

$$\psi(x) = \frac{\lambda}{c}\left[\int_x^\infty (1-F(y))\,dy + \int_0^x \psi(x-y)\,(1-F(y))\,dy\right]. \tag{A.3.4}$$

Finally, to replace $\Phi(x) = 1 - \psi(x)$ and $\Phi'(x) = -\psi'(x)$ in (A.3.1) we have

$$c\psi^{'}(x) + \lambda\left[\int_0^x \psi(x-y)dF(y) + 1 - F(x) - \psi(x)\right] = 0.$$

□

[†]This is an *insensitivity* or *robustness* result, since $\psi(0)$ only dependes on ρ and thus on F only through its mean.

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List of principal notation

X_n	The n-th surplus process.
$X_0 = x \geq 0$	The initial surplus.
$\varkappa$	An absorbing (cemetery) state.
$\mathbb{X} := \mathbb{R} \cup \varkappa$	The surplus state space.
Y_n	The n-th claim payment.
$\mathbb{Y}$	The claim space.
Z_n	Length of the n-th period.
$\{I_n\}$	The interest rate process.
$\mathbb{I}$	Interest rate state space.
b	The retention level or proportionality factor or risk exposure.
c	Premium (income) rate.
$\mathcal{C}(b)$	The premium left for the insurer if the retention level b is chosen.
$b_{\min}$	$\min\{b \in (0,1] \vert \mathcal{C}(b) \geq 0\}$.
$\mathcal{B} := [b_{\min}, 1]$	The decision space.
θ	Safety loading from the reinsurer.
$a_n(x_n) = b_n$	The decision function or strategies.
$\pi = \{a_n\}$	Control policy.
Π	The policy space / set of all control policies.
Ω	Event space on which probabilities are defined.
1_A	The indicator function of A.
$\mathcal{M}$	Markov control model.
$\mathbb{K}$	Set of feasible state-action pairs.
$\mathbb{F}$	Set of decision functions (or selectors).
H_t	Family of admissible histories up to time t.
$\mathbb{F}$	Subfamily of deterministic stationary policies.
Φ	Family of randomized stationary policies.
P_v^π	p.m. determined by π and the initial distribution v.
E_v^π	Expectation with respect to P_v^π.

Abbreviations

a.s.	Almost surely.
DFR	Decreasing failure rate.
df	Distribution function.
DP	Dynamical programming.
IMRL	Increasing mean residual lifetime.
i.i.d.	Independent and identically distributed.
l.s.c	Lower semicontinuous.
MCM	Markov control model.
MCP	Markov control process.
MDP	Markov decision process.
MRL	Mean residual lifetime.
NWU	New worse than used.
NWUC	New worse than used in convex ordering.
OE	Optimality equation.
pdf	Probability distribution function.
p.m.	Probability measure.
sd	Survival distribution.
u.s.c	Upper semicontinuous.