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TESIS DOCTORAL

On some nonlocal elliptic problems

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路遥知馬力

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Notations

Symbol	**Meaning**
$\mathbb{R}_0^{N0\,2}\ \mathrm{T}\ \mathbb{R}^N * \mathbb{R}_0$	The upper half space of $\mathbb{R}^{N0\,2}$
$x\ \mathrm{T}\ )x_2, x_3, ..., x_N+$	An element of the euclidean space $\mathbb{R}^N$
$X\ \mathrm{T}\ )x, y+\mathrm{T}\ )x_2, x_3, ..., x_N, y+$	An element of the euclidean space $\mathbb{R}_0^{N0\,2}$
$r\ \mathrm{T}\ \|x\|\mathrm{T}\ \overline{x_2^3\ 0\ x_3^3\ 0\ \times\times\times 0\ x_N^3}$	Module of x
Λu	Laplacian of u
$)\ \ \Lambda+^{\alpha/3}u$	Fractional Laplacian of u
$\mathrm{E}_\alpha)u+$	α-harmonic extension of u
$3_\alpha^{\leq}\ \mathrm{T}\ \dfrac{3N}{N\quad\alpha}$	Critical fractional Sobolev exponent
$\|´\|$	Lebesgue measure of the domain ´
$\partial´$	Boundary of ´
$\mathcal{F}$	$´\ *\ ]1, \in +$
$\partial_L\mathcal{F}$	$\partial´\ *\ ]1, \in +$
$\mathcal{C}_R)X_1+$	Ball in $\mathbb{R}_0^{N0\,2}$ of radius R centered at X_1
$\mathcal{C}_R$	Ball in $\mathbb{R}_0^{N0\,2}$ of radius R centered at the origin
$\rangle\times,\times$	Inner product in $\mathbb{R}^N$
$´\infty\lll´$	$´\infty$open subset of ´ with $\overline{´\infty}\ll´$
δ_{x_0}	Dirac delta at x_1
$a.e.$	Almost everywhere
v^0	Positive part of v, $v^0\ \ \mathrm{T\ n\ \bar{d}\tilde{}}\}v, 1\langle$
v	Negative part of v, $v\ \ \ \mathrm{T\ n\ \bar{d}\tilde{}}\}\ \ v, 1\langle$
$C)´+$	The space of continuous functions defined in ´
$C_1)´+$	The space of functions in $C)´+$with compact support
$C^k)´+$	The space of functions with k continuous derivatives in ´

Symbol	**Meaning**
$C_1^k)´+$	The space of functions in $C^k)´+$with compact support
$C^{\in})´+$	The space of infinitely differentiable functions in $´$
$C_1^{\in})´+$	The space of functions in $C^{\in})´+$with compact support
$C^{1,\gamma})´+, C^{\gamma})´+$	$\}u\,;\,´\nearrow\mathbb{R}\,,u$ continuous $\|\underset{x,y/^-,x\not C y}{\mathrm{tvr}}\frac{\|u)x+\;u)y+\|}{\|x\;y\|^{\gamma}}<\in\langle$
$C^{k,\gamma})´+$	Hölder space of functions with k derivatives in $C^{\gamma})´+$
$L^p)´+$	$\}u\,;\,´\nearrow\mathbb{R}\,\|u$ measurable , $\bigcap_- \|u\|^p<\in\langle, 2\geq p<\in$
$\backslash\times\backslash_p$	Norm in $L^p)´+$
$L^{\in})´+$	$\}u\,;\,´\nearrow\mathbb{R}\,\|u$ measurable and $\mathcal{B}C$ such that $\|u)x+\|\geq C$ a.e. $x\,/\,´\langle$
$H_1^{\alpha/3})´+$	Completion of $C_1^{\in})´+$with respect to the norm $\backslash u\backslash_{H_0^{\alpha/2})^-+}\subseteq\Big)\bigcap_-\|\mathbb{D}\;\Lambda+^{\alpha/=}u\|^{\beta}\,dx\Big\{^{2/3}$
$X_1^{\alpha})\mathcal{F}+$	Completion of $C_1^{\in})\mathcal{F}+$with respect to the norm $\backslash w\backslash_{X^{\alpha})\mathcal{F}_\Omega+}\subseteq\Big)\bigcap_{\mathcal{F}_\Omega} y^{2\;\;\alpha}\|\;\;w\|^{\beta}\,dxdy\Big\{^{2/3}$

Índice general

Introduction and thesis contents

In the past decades the elliptic problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

has been widely investigated. See for example the survey [3] and also the list (far from complete) [4, 5, 24, 50, 51, 60, 71, 81] for more specific problems, where different nonlinearities and different classes of domains, bounded or not, are considered. Other different diffusion operators, like the p–Laplacian, fully nonlinear operators, etc, have also been treated, see for example [13, 29, 35, 48] and the references therein. This work is devoted to study a nonlocal version of the problem (1) involving the so-called fractional Laplacian, $(-\Delta)^{\alpha/2}$, for some specific nonlinearities.

A brief introduction to the fractional Laplacian

Non local operators, like the fractional Laplacian, arise in a great variety of fields like elasticity problems [69], combustion [30], crystal dislocation [82], quasi-geostrophic flows [32, 61] and others. Problems involving the fractional Laplacian include fractional porous medium equation [62, 63], blow up problems [12], obstacle problem [70], etc. On the other hand, from a probabilistic approach, the fractional Laplacian operator, defined in the whole space, can be interpreted as the generator of a α-stable Levy process, see [11, 14, 15, 16, 17]. This kind of stochastic processes appeared in some finance models, see for instance [7, 18, 57].

There exist different equivalent definitions of the fractional Laplacian when it is defined in the whole space $\mathbb{R}^N$, see Section 1.1. When one try to extend those equivalent definitions in the case of bounded domains, different operators are obtained, see Section 1.2. In this work we are interested in looking at the fractional Laplacian as fractional powers of the classical Laplacian, which is a positive self-adjoint operator, both in the whole space or in a bounded domain with appropriate boundary conditions.

In [31], L. Caffarelli and L. Silvestre develop an extension tool that allows to transform a nonlocal problem involving the fractional Laplacian into an equivalent local problem. As we will see, this tool, inspired in the classical Dirichlet to Neumann operator, implies the use of an extra variable as well as a linear operator with a degenerate/singular weight. On the other hand, the fractional powers of a linear positive self-adjoint operator in a bounded domain ´ are defined by means of its spectral decomposition. In [28], the authors consider the fractional operator $)\ \ \Lambda +^{2/3}$ defined using the Dirichlet to Neumann operator, restricted to the cylinder $\mathcal{F}\ \ \mathrm{T}\ \ ´\ *\ \mathbb{R}_0\ \ll\mathbb{R}_0^{N0\,2}$, and show that this definition coincides with the spectral one. This technique has been extended to deal with the case $\alpha\ \mathrm{T}\ 2$ in [19], see also [33, 76]. We will use this approach along this work. We recall that this is not the unique possibility of defining the fractional Laplacian in a bounded domain, see Section 1.4.

After this preliminary work, the subsequent chapters are devoted to study the fractional Laplacian problem associated to the classical problem (1),

$$\left.\begin{array}{rcll})\ \ \Lambda +^{\alpha/3}u & \mathrm{T} & g)x,u+ & \text{in } ´ \ll\mathbb{R}^N, \\ u & \mathrm{T} & 1 & \text{on } \partial ´ , \end{array}\right\} \qquad (2)$$

with $1<\alpha<3$, $N>\alpha$ and ´ a regular bounded domain.

In particular, we study the case $g)x,u+\mathrm{T}\ u^p\ 0\ \ \lambda u^q$ where $\lambda\ /\ \mathbb{R}$, $1<q<p\geq \frac{N0\ \alpha}{N\ \ \alpha}\ \mathrm{T}\ 3^{\leq}_{\alpha}\ \ \ 2$, and $2<p$. The number $3^{\leq}_{\alpha}\ \mathrm{T}\ \frac{3N}{N\ \ \alpha}$ is the critical exponent with respect to some fractional Sobolev embedding. For the critical power, we also consider a zero order perturbation, that is, $g)x,u+\mathrm{T}\ u^{\frac{N+\alpha}{N-\alpha}}\ 0\ \ f)x+$, with f small in some sense.

Thesis contents

This work is organized as follows: In Chapter 1 we establish a series of characterizations of the fractional Laplacian that we will use along the work. We describe also in this chapter the proper functional framework to be used with the fractional Laplacian as well as some useful inequalities. We extend to $\alpha\ \mathrm{T}\ 2$ known results for the square root of the Laplacian. We finish the chapter showing some alternative definitions for fractional operators in bounded domains.

Chapter 2 is devoted to study the fractional subcritical concave-convex problem

$$)P_\lambda+\left.\begin{array}{rclll})\ \ \Lambda +^{\alpha/3}u & \mathrm{T} & \lambda u^q\ 0\ \ u^p, & u>1 & \text{in } ´ , \\ u & \mathrm{T} & 1 & & \text{on } \partial ´ , \end{array}\right\} \qquad (3)$$

with $1<\alpha<3$, $1<q<2<p<\frac{N0\ \alpha}{N\ \ \alpha}$, $N>\alpha$, $\lambda>1$ and $´\ \ll\mathbb{R}^N$ a smooth bounded domain. For this problem we prove the following.

Theorem 1. *There exists $\Sigma>1$ such that for Problem $)P_\lambda+$ there holds:*

1. *If $1<\lambda<\Sigma$ there is a minimal solution. Moreover, the family of minimal solutions is increasing with respect to λ.*

2. *If* $\lambda \mathrm{T} \Sigma$ *there is at least one solution.*

3. *If* $\lambda > \Sigma$ *there is no solution.*

4. *For any* $1 < \lambda < \Sigma$ *there exist at least two solutions.*

For $\alpha \ / \ ]2, 3+$we also prove uniform a priori $L^{\in}$ estimates. We use the classical rescaling approach in [51] which usually yields to problems defined in unbounded domains. We therefore prove some related Liouville-type results, see Section 2.2.

In Chapter 3 we extend the study of the problem $)P_\lambda+$to the critical case $p \ \mathrm{T} \ 3^{\leq}_{\alpha} \quad 2$. We add also the cases $q \ \mathrm{T} \ 2$ and $2 < q < 3^{\leq}_{\alpha} \quad 2$. That is, we study the problem

$$)P^{\leq}_{\lambda}+\Big\} \begin{array}{llll}) \ \ \Lambda +^{\alpha/3}u & \mathrm{T} & \lambda u^q \, 0 \ u^{\frac{N+\alpha}{N-\alpha}}, \qquad u > 1 & \text{in } ´ , \\ u & \mathrm{T} & 1 & \text{on } \partial ´ , \end{array} \tag{4}$$

with $1 < \alpha < 3$, $1 < q < \frac{N0 \ \alpha}{N \quad \alpha}$, $N > \alpha$, $\lambda > 1$ and $´ \ \ll \mathbb{R}^N$ a smooth bounded domain. Due to the different methodology used with respect to the perturbation of the critical power, we divide this chapter in the three cases: sublinear ($1 < q < 2$), linear ($q \ \mathrm{T} \ 2$) and superlinear ($2 < q < 3^{\leq}_{\alpha} \quad 2$) perturbation, motivated by the works [4, 24] for the classical Laplacian operator. We prove respectively the following results.

Theorem 2. *Let* $1 < q < 2$*. There exists* $\Sigma > 1$ *such that for Problem* $)P^{\leq}_{\lambda}+$*there holds:*

1. *If* $1 < \lambda < \Sigma$ *there is a minimal solution. Moreover, the family of minimal solutions is increasing with respect to* λ*.*

2. *If* $\lambda \mathrm{T} \Sigma$ *there is at least one solution.*

3. *If* $\lambda > \Sigma$ *there is no solution.*

4. *If* $\alpha \sim 2$*, for any* $1 < \lambda < \Sigma$ *there exist at least two solutions.*

Theorem 3. *Let* $q \ \mathrm{T} \ 2$*,* $1 < \alpha < 3$ *and* $N \sim 3\alpha$*. Let* λ_2 *be the first eigenvalue of* $) \ \ \Lambda +^{\alpha/3}$ *on* $´$ *under Dirichlet boundary conditions. Then Problem* $)P^{\leq}_{\lambda}+$

1. *has at least one positive solution if* $1 < \lambda < \lambda_2$*.*

2. *has no solution if* $\lambda \sim \lambda_2$*.*

Theorem 4. *Let* $2 < q < \frac{N0 \ \alpha}{N \quad \alpha}$*,* $1 < \alpha < 3$ *and* $N > \alpha)20 \ 2/q+$ *Then Problem* $)P^{\leq}_{\lambda}+$ *has at least one positive solution for any* $\lambda > 1$*.*

Finally, in Chapter 4 we study a perturbation of order zero of a critical pure-power fractional problem . Namely, we study the problem

$$)P+ \quad \Big\} \begin{array}{ll}) \ \ \Lambda +^{\alpha/3}u \ \mathrm{T} \ \|u\|^{\frac{2\alpha}{N-\alpha}} u \ 0 \ \ f)x+ & \text{in } ´ , \\ u \ \mathrm{T} \ 1 & \text{on } \partial ´ , \end{array}$$

where $1 < \alpha < 3$, $N > \alpha$ and f belongs to the dual fractional Sobolev space $H^{\ \alpha/3})' +$, and is small in the sense

$$\bigcap_{-} f\varphi \geq c\backslash\varphi\backslash^{)N0\ \alpha+/\alpha}_{H_0^{\alpha/2}}, \qquad \exists\varphi\ /\ H_1^{\alpha/3})' + \text{with } \backslash\varphi\backslash_{\frac{2N}{N+\alpha}}\ \mathrm{T}\ 2. \tag{5}$$

This problem was previously studied in [81] with the classical Laplacian operator.

Theorem 5. *In the above hypotheses, Problem* $)P+$*has at least one solution. If moreover the inequality* (10) *is strict, then* $)P+$*has at least two solutions.*

The content of this work can be found in the publications [9, 19, 39].

Introducción y contenido de la tesis

El problema elíptico

$$\left.\begin{array}{rcll} \Lambda u & \mathrm{T} & g)x, u+ & \text{en } ´ \ll \mathbb{R}^N, \\ u & \mathrm{T} & 1 & \text{en } \partial ´ , \end{array}\right\} \qquad (6)$$

ha sido ampliamente investigado en las últimas décadas . Véase por ejemplo [3] así como la lista [4, 5, 24, 50, 51, 60, 71, 81] para problemas más específicos. En estos trabajos, se consideran diferentes no linealidades así como diferentes clases de dominios, acotados o no acotados. Otros operadores de difusión, como el p–Laplaciano, operadores completamente no lineales, etc, han sido también tratados, véase por ejemplo [13, 29, 35, 48] y las referencias allí incluidas. Este trabajo está dedicado al estudio de una versión no local del problema (6) con el llamado Laplaciano fraccionario, $) \quad \Lambda +^{\alpha/3}$.

Una breve introducción al Laplaciano fraccionario

Los operadores no locales, como el Laplaciano fraccionario, surgen en gran variedad de campos como por ejemplo en modelos de combustión [30], dislocación de cristales [82], problemas de elasticidad [69], fluidos quasi-geostróficos [32, 61] y otros. Algunos problemas que involucran el Laplaciano fraccionario incluyen la ecuación fraccionaria de los medios porosos [62, 63], problemas de explosión [12], problema del obstáculo [70], etc. Por otro lado, desde un punto de vista probabilístico, el operador Laplaciano fraccionario definido en todo el espacio puede ser interpretado como el generador de un proceso de Levy α-estable, véase [11, 14, 15, 16, 17]. Este tipo de procesos estocásticos aparecen en modelos financieros, [7, 18, 57].

Existen varias definiciones equivalentes del Laplaciano fraccionario en todo el espacio $\mathbb{R}^N$, véase la Sección 1.1. Al intentar extender dichas definiciones al Laplaciano fraccionario en dominios acotados se obtienen diferentes operadores no equivalentes, véase Sección 1.2. En el presente trabajo estamos interesados en el Laplaciano fraccionario que se entiende como potencia fraccionaria del operador Laplaciano clásico.

En [31], L. Caffarelli y L. Silvestre desarrollaron una herramienta que permite transformar un problema no local involucrando al Laplaciano fraccionario en otro problema local equivalente. Como veremos, esta herramienta, inspirada en el operador clásico de Dirichlet-Neumann, implica el uso de una variable extra así como un operador lineal en forma de divergencia con un peso degenerado/singular. Por otro lado, las potencias fraccionarias de un operador lineal positivo autoadjunto en un dominio acotado ´ se definen a través de su descomposición espectral. En [28], los autores consideran el operador fraccionario $)\ \Lambda+^{2/3}$ definido a través del operador Dirichlet-Neumann, restringido al cilindro infinito $\mathcal{F}\ \mathrm{T}\ ´ \ast \mathbb{R}_0 \ll \mathbb{R}_0^{N0\,2}$, y muestran que esta definición coincide con la definición espectral. Esta técnica se extiende al caso α ⫪T 2 en [19], véase también [33, 76]. Usaremos esta aproximación a lo largo de este trabajo. Hacemos notar que esta no es la única posibilidad de definir el Laplaciano fraccionario en dominios acotados, véase la Sección 1.4.

Después de este trabajo preliminar, los siguientes capítulos estarán dedicados al estudio de problemas que involucren al Laplaciano fraccionario asociados al problema clásico (6), es decir, problemas del tipo

$$\left\}\begin{array}{lll})\ \Lambda+^{\alpha/3}u & \mathrm{T} & g)x,u+ \qquad \text{en } ´ \ll \mathbb{R}^N, \\ u & \mathrm{T} & 1 \qquad\qquad \text{en } \partial´, \end{array}\right. \tag{7}$$

con $1 < \alpha < 3$, $N > \alpha$ y ´ un dominio acotado regular.

En particular, estudiaremos el caso $g)x,u+\mathrm{T}\ u^p\ 0\ \lambda u^q$ donde $\lambda\ /\ \mathbb{R}$, $1 < q < p \geq \frac{N0\ \alpha}{N\ \ \alpha}\ \mathrm{T}\ 3^{\leq}_{\alpha}\ \ 2$ y $2 < p$. El número $3^{\leq}_{\alpha}\ \mathrm{T}\ \frac{3N}{N\ \ \alpha}$ se corresponde con el exponente crítico respecto de las inclusiones fraccionarias de Sobolev. Consideramos también perturbaciones de orden cero para la potencia crítica, es decir, $g)x,u+\mathrm{T}\ u^{\frac{N+\alpha}{N-\alpha}}\ 0\ f)x+$, con f pequeña en algún sentido específico.

Contenido de la tesis

Este trabajo está organizado como sigue: En el Capítulo 1 establecemos una serie de caracterizaciones del Laplaciano fraccionario que serán usadas a lo largo de la tesis. Describimos en este capítulo también el marco funcional necesario para trabajar con el Laplaciano fraccionario así como algunas desigualdades útiles. Extendemos al caso α ⫪T 2 resultados previamente demostrados para la raíz cuadrada del Laplaciano. Concluimos el capítulo mostrando algunas definiciones alternativas del Laplaciano fraccionario en dominios acotados.

El Capítulo 2 está dedicado al estudio del problema cóncavo-convexo subcrítico siguiente

$$)P_\lambda+\left\}\begin{array}{lllll})\ \Lambda+^{\alpha/3}u & \mathrm{T} & \lambda u^q\ 0\ u^p, & u > 1 & \text{in } ´, \\ u & \mathrm{T} & 1 & & \text{on } \partial´, \end{array}\right. \tag{8}$$

con $1 < \alpha < 3$, $1 < q < 2 < p < \frac{N0\,\alpha}{N\ \ \alpha}$, $N > \alpha$, $\lambda > 1$ y $\acute{}\ \ll \mathbb{R}^N$ un dominio acotado regular. Para este problema probamos el siguiente resultado.

Teorema 1. *Existe* $\Sigma > 1$ *tal que para el problema* $)P_\lambda+$*se cumple:*

1. *Si* $1 < \lambda < \Sigma$ *existe una solución minimal. Además, la familia de soluciones es creciente con respecto a* λ.
2. *Si* $\lambda \ \mathrm{T}\ \Sigma$ *existe al menos una solución.*
3. *Si* $\lambda > \Sigma$ *no existe solución.*
4. *Para cada* $1 < \lambda < \Sigma$ *existen al menos dos soluciones.*

Para $\alpha\ /\ ]2, 3+$probamos además estimaciones uniformes en $L^{\in}$ de la soluciones. Utilizaremos una técnica clásica de cambio de escala desarrollada en [51], generalmente implica estudiar problemas en dominios no acotados. Probamos para ello algunos resultados de tipo Liouville, véase la Sección 2.2.

En el Capítulo 3 extendemos el estudio del problema $)P_\lambda+$al caso crítico $p\ \mathrm{T}\ 3^{\leq}_{\alpha}$ 2. Incluimos en el estudio también los casos $q\ \mathrm{T}\ 2$ y $2 < q < 3^{\leq}_{\alpha}$ 2. Resumiendo, estudiamos el problema

$$)P^{\leq}_{\lambda}+\left\}\ \begin{array}{llll})\ \ \Lambda +^{\alpha/3}u & \mathrm{T} & \lambda u^q\, 0\ \ u^{\frac{N+\alpha}{N-\alpha}}, \quad u > 1 & \text{in } \acute{}\,, \\ u & \mathrm{T} & 1 & \text{on } \partial\acute{}\,, \end{array}\right. \tag{9}$$

con $1 < \alpha < 3$, $1 < q < \frac{N0\,\alpha}{N\ \ \alpha}$, $N > \alpha$, $\lambda > 1$ y $\acute{}\ \ll \mathbb{R}^N$ un dominio acotado regular. Debido a la diferente metodología utilizada respecto a cada perturbación del problema crítico puro fraccionario, dividimos el capítulo en tres casos: perturbación sublineal ($1 < q < 2$), lineal ($q\ \mathrm{T}\ 2$) y superlineal ($2 < q < 3^{\leq}_{\alpha}$ 2), motivado por los trabajos [4, 24] sobre el Laplaciano clásico. Probaremos los siguientes resultados respectivamente.

Teorema 2. *Sea* $1 < q < 2$. *Entonces, existe* $1 < \Sigma < \in$ *tal que para el problema* $)P^{\leq}_{\lambda}+$*se cumple:*

1. *Si* $1 < \lambda < \Sigma$ *existe una solución minimal. Además, la familia de soluciones es creciente con respecto a* λ.
2. *Si* $\lambda \ \mathrm{T}\ \Sigma$ *existe al menos una solución.*
3. *Si* $\lambda > \Sigma$ *no existe solución.*
4. *Si* $\alpha \sim 2$, *para cada* $1 < \lambda < \Sigma$ *existen al menos dos soluciones.*

Teorema 3. *Sea* $q\ \mathrm{T}\ 2$, $1 < \alpha < 3$ *y* $N \sim 3\alpha$. *Sea* λ_2 *el primer autovalor de* $)\ \ \Lambda +^{\alpha/3}$ *en* $\acute{}$ *bajo condiciones Dirichlet en la frontera. Entonces el problema* $)P^{\leq}_{\lambda}+$

1. *tiene al menos una solución si* $1 < \lambda < \lambda_2$.

2. *no tiene solución si* $\lambda \sim \lambda_2$.

Teorema 4. *Sea* $2 < q < \frac{N 0 \alpha}{N \quad \alpha}$, $1 < \alpha < 3$ *y* $N > \alpha)2\, 0 \;\; 2/q +$ *Entonces el problema* $)P^{\leq}_{\lambda}+$ *tiene al menos una solución positiva para* $\lambda > 1$.

Finalmente, en el Capítulo 4 estudiamos una perturbación de orden cero del problema crítico. A saber, estudiamos al problema

$$)P+ \quad \left\} \begin{array}{ll}) \quad \Lambda +^{\alpha/3} u \;\mathrm{T}\; \|u\|^{\frac{2\alpha}{N-\alpha}} u \; 0 \;\; f)x+ & \text{in } ´\,, \\ u \;\mathrm{T}\; 1 & \text{on } \partial ´\,, \end{array} \right.$$

donde $1 < \alpha < 3$, $N > \alpha$ y f pertenece al espacio de Sobolev fraccionario dual $H \quad ^{\alpha/3})´+$ y cumple

$$\bigcap_{-} f\varphi \geq c \backslash \varphi \backslash^{)N 0 \alpha +/\alpha}_{H_0^{\alpha/2}}, \qquad \exists \varphi \;/\; H_1^{\alpha/3})´ + \text{with } \backslash \varphi \backslash_{\frac{2N}{N+\alpha}} \;\mathrm{T}\; 2. \tag{10}$$

Este problema ha sido estudiado previamente en [81] con el operador Laplaciano (α T 3).

Teorema 5. *Bajo estas hipótesis, el problema* $)P+$ *tiene al menos una solución. Si además la desigualdad* (10) *es estricta, entonces* $)P+$ *tiene al menos dos soluciones.*

El contenido de este trabajo puede encontrarse en las publicaciones [9, 19, 39].

1

The fractional Laplacian operator

The fractional Laplacian defined on $\mathbb{R}^N$ can be found in the literature as a functional operator related to the so-called α stable Lèvy processes. In the framework of the partial differential equations. These operators can be defined in several ways in both $\mathbb{R}^N$ and bounded domains. This chapter is devoted to explore some of these definitions and their equivalences. Furthermore, we will give a brief introduction to the functional spaces framework required to work with the fractional Laplacian.

1.1. Fractional Laplacian in $\mathbb{R}^N$

This work will be focused, mostly, on a bounded domain setting. However, the fractional Laplacian in $\mathbb{R}^N$ is fundamental to understand its homologous in bounded domains. We begin with the definition of the fractional Laplacian in $\mathbb{R}^N$ via its Fourier transform.

Fourier transform

Given a function u in the Schwartz class $\mathcal{U})\mathbb{R}^N+$ we define its Fourier transform as

$$\mathcal{H}]u\hat{})\xi+\mathrm{T}\bigcap_{\mathbb{R}^N} e^{\ 3\pi i x \cdot \xi} u)x+dx.$$

Let α be a real number in $(0,2)$. We define the fractional Laplacian of u in $\mathbb{R}^N$ as

$$(-\Delta)^{\alpha/2}u(x) = \mathcal{F}^{-1}(|2\pi\xi|^{\alpha}\,\hat{u}(\xi))(x). \tag{1.1}$$

This definition can be found in the literature under the name *pseudo-differential operator of symbol* $|2\pi\xi|^{\alpha}$. Notice that $(-\Delta)^{\alpha/2}u$ does not necessarily belong to $\mathcal{S}(\mathbb{R}^N)$ since $|2\pi\xi|^{\alpha}$ introduces a singularity at the origin in its Fourier transform. Observe also that, using the definition (1.1), one can easily check the following properties

$$(-\Delta)^{\alpha/2} \to -\Delta, \qquad \text{as } \alpha \to 2^{-},$$

$$(-\Delta)^{\alpha/2} \to I, \qquad \text{as } \alpha \to 0^{+}.$$

This definition can be extended to $\alpha \in (-N, 2]$. For $\alpha \leq -N$, $|2\pi\xi|^{\alpha}$ is no longer a tempered distribution and (1.1) makes no sense.

Integral representation

A second definition for the fractional Laplacian that we can find, see [55, 73, 76], is the one referring to its integral form. Given a function $u \in \mathcal{S}(\mathbb{R}^N)$ we have

$$\begin{aligned}(-\Delta)^{\alpha/2}u(x) &= \mu_{N,\alpha}\,\text{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(\bar{x})}{|x-\bar{x}|^{N+\alpha}}\,d\bar{x} \\ &= \mu_{N,\alpha}\lim_{\varepsilon\to 0^{+}}\int_{|x-\bar{x}|>\varepsilon} \frac{u(x)-u(\bar{x})}{|x-\bar{x}|^{N+\alpha}}\,d\bar{x}\end{aligned} \tag{1.2}$$

where $\mu_{N,\alpha}$ stands for a normalizing constant to ensure the equivalence with (1.1). Its exact value can be computed,

$$\mu_{N,\alpha} = \frac{2^{\alpha-1}\alpha\,\Gamma((N+\alpha)/2)}{\pi^{N/2}\,\Gamma(1-\alpha/2)}.$$

Notice that $\mu_{N,\alpha} \sim \alpha$ as $\alpha \to 0$ and $\mu_{N,\alpha} \sim 2-\alpha$ as $\alpha \to 2$. Here we can see the nonlocal behaviour of the operator as follows: consider, for instance, a regular function $\theta(x)$ positive and with compact support in B_1. For every point x_0 of B_1^c one clearly has $-\Delta\theta(x_0) = 0$ while $(-\Delta)^{\alpha/2}\theta(x_0) < 0$. Using the definition (1.2) it can be proved, see [70], that given a $\phi \in \mathcal{S}(\mathbb{R}^N)$

$$|(-\Delta)^{\alpha/2}\phi(x)| \leq \frac{C}{(1+|x|)^{N+\alpha}}.$$

This allows us, by duality, to define the fractional Laplacian in the space

$$\mathcal{L}_{\alpha}(\mathbb{R}^N) = \left\{ f \in L^1_{loc}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|f(x)|}{(1+|x|)^{N+\alpha}} < \infty \right\}$$

where $\mathcal{S}'(\mathbb{R}^N)$ refers to the dual space of $\mathcal{S}(\mathbb{R}^N)$. Additionally, in order to have the integral (1.2) convergent, we can require $u \in C^2(\mathbb{R}^N)$. Therefore, we can avoid the principal value as follows

$$\text{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(\bar{x})}{|x-\bar{x}|^{N+\alpha}}\,d\bar{x} = \frac{1}{2}\int_{\mathbb{R}^N} \frac{2u(x)-u(x+\bar{x})-u(x-\bar{x})}{|\bar{x}|^{N+\alpha}}\,d\bar{x} =$$
$$\frac{1}{2}\int_{B_1} \frac{2u(x)-u(x+\bar{x})-u(x-\bar{x})}{|\bar{x}|^{N+\alpha}}\,d\bar{x} + \frac{1}{2}\int_{B_1^c} \frac{2u(x)-u(x+\bar{x})-u(x-\bar{x})}{|\bar{x}|^{N+\alpha}}\,d\bar{x}.$$

Thus, the second integral converges since $u \in \mathcal{L}_\alpha(\mathbb{R}^N)$. The first integral converges since the numerator is bounded by $|\bar{x}|^2$. In fact, the definition can be extended to functions in $C^{\alpha+\varepsilon}(\mathbb{R}^N)$ with $\varepsilon > 0$, see [70]. In our context, we will focus on functions that live in the following functional spaces:

Given $\alpha \in (0,2)$ we define the homogeneous fractional Sobolev space $\dot{H}^{\alpha/2} = \dot{H}^{\alpha/2}(\mathbb{R}^N)$ as the completion of $\mathcal{D}_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{\dot{H}^{\alpha/2}} := \|(-\Delta)^{\alpha/4}u\|_2 = \left(\int_{\mathbb{R}^N} |2\pi\xi|^\alpha |\hat{u}(\xi)|^2\,d\xi\right)^{1/2}. \tag{1.3}$$

Localization

The nonlocal behaviour of the operator will play an important role along this work. Since every value of $(-\Delta)^{\alpha/2}u$ depends on the entire space, some of the tradicional variational techniques cannot be used. On the other hand, simple operations like composition or multiplication turn complex when using the fractional Laplacian on them. A way to avoid, in some cases, these difficulties is to use the so-called *Caffarelli-Silvestre extension* [31]. In order to motivate it, one considers u a bounded regular enough function in $\mathbb{R}^N$ and its harmonic extension

$$\begin{cases} -\Delta w(x,y) = 0 & (x,y) \in \mathbb{R}^{N+1}_+ \\ w(x,y) = u(x) & (x,y) \in \mathbb{R}^N \end{cases}$$

where $\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0,\infty)$. Let us consider now the Dirichlet-Neumann operator $\Sigma :$ $u \mapsto -w_y(x,0)$. Applying the operator twice to u we have $\Sigma^2(u) = \Sigma(-w_y(x,0)) = w_{yy}(x,0) = -\Delta_x u$. That is $\Sigma = (-\Delta)^{1/2}$. The Caffarelli-Silvestre procedure extends this result to every power $\alpha \in (0,2)$ of the Laplacian as follows: Given a bounded u regular enough function in $\mathbb{R}^N$ we define its $\alpha-$harmonic extension, denoted by $\mathrm{E}_\alpha(u)$, as the unique solution to the problem

$$\begin{cases} \operatorname{div}(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ w = u & \text{on } \mathbb{R}^N \end{cases} \tag{1.4}$$

Then, in [31] the authors prove that the fractional Laplacian of u can be defined by the formula

$$)\ \ \Lambda +^{\alpha/3} u)x+ \mathrm{T}\ \frac{\partial w}{\partial \nu^\alpha})x,y+;\mathrm{T}\quad \kappa_\alpha \lim_{y\prime\ \ 1+} y^{2\ \ \alpha}\frac{\partial w}{\partial y})x,y+ \qquad (1.5)$$

with $\kappa_\alpha\ \mathrm{T}\ \frac{)\alpha/3+}{3^{1-\alpha}\ \ )2\ \ \alpha/3+}$.

The proof of (1.5) is based on the following proposition where it is proved that one can write the solution of (1.4) as a convolution of u with a convenient Poisson kernel.

Proposition 1.1.1 ([31]). *Given $\alpha\ /\ )1,3+$ the function*

$$P^\alpha)x,y+ \mathrm{T}\ d_{\alpha,N}\frac{y^\alpha}{)\|x\|^\beta\ 0\ \ \|y\|^\beta+^{\frac{N+\alpha}{2}}} \qquad (1.6)$$

is the Poisson kernel for (1.4)*, that is, for every $u\ /\ C)\mathbb{R}^N+\wedge L^{\in}\ )\mathbb{R}^N+$ the function*

$$w)x,y+ \mathrm{T}\ P^\alpha \circ u\ \mathrm{T}\ d_{\alpha,N}\bigcap_{\mathbb{R}^N}\frac{y^\alpha}{)\|x\quad s\|^\beta\ 0\ \ \|y\|^\beta+^{\frac{N+\alpha}{2}}}u)s+ds \qquad (1.7)$$

is the unique solution of (1.4)*. The constant $d_{\alpha,N}$ is chosen in order to have*

$$\bigcap_{\mathbb{R}^N} P^\alpha)x,y+dx\ \mathrm{T}\ 2 \qquad \exists y>1,$$

and satisfies $\alpha\kappa_\alpha d_{\alpha,N}\ \mathrm{T}\ \mu_{\alpha,N}$.

For functions defined in $\mathbb{R}_0^{N0\ 2}$ we will work in the space $X^\alpha)\mathbb{R}_0^{N0\ 2}+$ defined as the completion of $\mathcal{F}_1^{\in}\ )\mathbb{R}_0^{N0\ 2}+$ under the norm

$$\backslash\Psi\backslash^3_{X^\alpha}\ \mathrm{T}\ \kappa_\alpha\bigcap_{\mathbb{R}^{N+1}_+} y^{2\ \ \alpha}\|\ \ \Psi)x,y+\|^\beta\,dxdy.$$

The operator $)\ \ \Lambda+^{\alpha/3}\ ;\ H^{\alpha/3})\mathbb{R}^N+\nearrow H\ \ ^{\alpha/3})\mathbb{R}^N+$ defines an isometric isomorphism between $H^{\alpha/3})\mathbb{R}^N+$ and its topological dual $H\ \ ^{\alpha/3})\mathbb{R}^N+$. Besides, the operator E_α is an isometry between $X^\alpha)\mathbb{R}_0^{N0\ 2}+$ and $H^{\alpha/3})\mathbb{R}^N+$, that is,

$$\backslash\mathrm{E}_\alpha)u+\backslash_{X^\alpha)\mathbb{R}^{N+1}_++}\mathrm{T}\ \backslash u\backslash_{H^{\alpha/2})\mathbb{R}^N+},\quad \exists u\ /\ H^{\alpha/3})\mathbb{R}^N+, \qquad (1.8)$$

see Remark 1.3.1. On the other hand, if $[\ \mathrm{s}\ ;\ X^\alpha)\mathbb{R}_0^{N0\ 2}+\nearrow H^{\alpha/3})\mathbb{R}^N+$ stands for the trace operator over $\mathbb{R}^N$, we have

$$\backslash[\ \mathrm{s})z+\backslash_{H^{\alpha/2})\mathbb{R}^N+}\geq\backslash z\backslash_{X^\alpha)\mathbb{R}^{N+1}_++},\quad \exists z\ /\ X^\alpha)\mathbb{R}_0^{N0\ 2}+. \qquad (1.9)$$

Even more, if $z\ /\ X^\alpha)\mathbb{R}_0^{N0\ 2}+$ and $w\ \mathrm{T}\ \mathrm{E}_\alpha)[\ \mathrm{s})z++$ then

$$\backslash z\backslash^3_{X^\alpha}\ \mathrm{T}\ \backslash w\backslash^3_{X^\alpha}\ 0\ \ \backslash z\quad w\backslash^3_{X^\alpha}. \qquad (1.10)$$

In particular, given $u \in H^{\alpha/2}(\mathbb{R}^N)$ we have

$$\|E_\alpha(u)\|_{X^\alpha(\mathbb{R}_+^{N+1})} = \min\left\{\|w\|_{X^\alpha(\mathbb{R}_+^{N+1})} : Tr(w) = u\right\}. \tag{1.11}$$

We define now the operator L_α from problem (1.4),

$$L_\alpha(w) := y^{\alpha-1}\operatorname{div}(y^{1-\alpha}\nabla w) = \Delta w + \frac{1-\alpha}{y} w_{yy}. \tag{1.12}$$

The next properties will be useful.

Lemma 1.1.2. *Let $\alpha \in (0,2)$ and Ω, Ψ, ϑ regular enough functions defined in $\mathbb{R}_+^{N+1}$. Then*

$$L_\alpha(\Omega\Psi) = \Omega L_\alpha \Psi + \Psi L_\alpha \Omega + 2\langle\nabla\Omega, \nabla\Psi\rangle, \tag{1.13}$$

$$L_\alpha(\vartheta(\Omega)) = \vartheta'(\Omega) L_\alpha \Omega + \vartheta''(\Omega)|\nabla\Omega|^2, \tag{1.14}$$

$$L_\alpha(|X|^\gamma) = \gamma(\gamma + N - \alpha)|X|^{\gamma-2}, \quad X \neq 0. \tag{1.15}$$

where $X = (x,y) \in \mathbb{R}_+^{N+1}$. Moreover, if Ω is radial, $\Omega = \Omega(r)$ with $r = |X|$, then

$$L_\alpha \Omega = \Omega'' + \frac{N+1-\alpha}{r}\Omega'. \tag{1.16}$$

Note that in the special case $\alpha = 1$ we have $L_1 = \Delta$. Furthermore, the operador L_α can be understood, formally, as the standard Laplacian acting in $N + (2-\alpha)$ dimensions. Notice that, in fact, equations (1.13) and (1.14), which are dimension-independent, mimic the behaviour of their homologous of the standard Laplacian. However, equations (1.15) and (1.16), which are dimension-dependent, replace $N+1$ with $N + (2-\alpha)$ with respect to the case of the standard Laplacian.

The Caffarelli-Silvestre extension transforms nonlocal problems into local problems that involve the operator L_α. Roughly speaking, a local operator in divergence form will be more convenient than one non-local in integral form in what concerns to computations. However, the weight $\sigma(x,y) = y^{1-\alpha}$ is singular and degenerated if $\alpha \neq 1$. In this case, the Caffarelli-Silvestre extension can be studied from the perspective of the differential equations with A_2 weights, see [45, 46] for further information.

1.2. Fractional Laplacian in bounded domains

Given a bounded domain Ω, a natural way to define the fractional Laplacian in that domain is to extend the previous definitions substituting $\mathbb{R}^N$ by Ω. Nevertheless, depending on how we proceed, this can lead to different and no equivalent definitions. Some examples of this fact can be checked in the Section 1.4. This section is devoted to define the fractional Laplacian in bounded domains by means of the definitions of the operator in $\mathbb{R}^N$ but keeping the equivalence between the different characterizations.

Localization in bounded domains

We start defining the fractional Laplacian in bounded domains through the Caffarelli-Silvestre extension, adapting it to this new context. This approach has been taken before in [28] for $\alpha = 1$ and afterwards in, for instance, [33], for the general case. Let Ω be a bounded domain and consider the infinity cylinder $\mathcal{C}_\Omega = \Omega \times (0,\infty)$. Let us denote its parabolic boundary as $\partial_L \mathcal{C}_\Omega = \partial\Omega \times [0,\infty)$. Let u be a regular function defined in Ω. Let us define its α-harmonic extension, $\mathrm{E}_\alpha(u)$, as the unique solution to the problem

$$\begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla w)(x,y) = 0, & (x,y)\in \mathcal{C}_\Omega, \\ w(x,y) = 0, & (x,y)\in \partial_L\mathcal{C}_\Omega, \\ w(x,y) = u(x), & (x,y)\in\Omega. \end{cases} \tag{1.17}$$

We will define the fractional Laplacian of u in Ω as

$$(-\Delta)^{\alpha/2}u(x) = \frac{\partial w}{\partial \nu^\alpha}(x,y) = -\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}(x,y). \tag{1.18}$$

Spectral decomposition

It is classical that the powers of a positive operator are defined through the spectral decomposition using the powers of the eigenvalues of the original operator. We show next that in this case this is coherent with the Dirichlet-Neumann operator defined above. Let (φ_j, ρ_j) be the eigenfunctions and eigenvalues of $-\Delta$ in Ω with zero Dirichlet boundary data. Define the space of functions $H_0^{\alpha/2}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\|^2_{H_0^{\alpha/2}(\Omega)} := \left(\sum u_j^2 \rho_j^{\alpha/2}\right)^{1/2} \tag{1.19}$$

and also the energy space $X_0^\alpha(\mathcal{C}_\Omega)$ as the completion of $C_0^\infty(\overline{\mathcal{C}_\Omega})$ under the norm

$$\|w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} = \kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla w(x,y)|^2\, dxdy.$$

Next we establish a spectral characterization of the fractional Laplacian. See also [26, 76].

Theorem 1.2.1. *Let $\alpha \in (0,2)$. Let $u = \sum u_j\varphi_j \in H_0^{\alpha/2}(\Omega)$ where u_j are the coeficients of u on the base $\{\varphi_j\}$ of $L^2(\Omega)$, then*

$$\sum u_j \rho_j^{\alpha/2}\varphi_j = (-\Delta)^{\alpha/2}u. \tag{1.20}$$

Moreover, if $E_\alpha)u+$*stands for the extension defined in* (1.17)*, we have* $E_\alpha)u+/ X_1^\alpha)\mathcal{F}+$ *and*

$$E_\alpha)u+)x,y+\mathrm{T}\int u_j\varphi_j)x+\psi)\rho_j^{2/3}y+, \tag{1.21}$$

where $\psi)s+$*is the unique solution to the problem*

$$\begin{cases} \psi^{\infty}0\ \dfrac{)2\quad\alpha+}{s}\psi^{\infty}\ \mathrm{T}\ \psi, & s>1, \\ \kappa_\alpha \lim\limits_{s\prime\ 1+} s^2\ \ ^\alpha\psi^{\infty})s+\ \mathrm{T}\ 2, & \\ \psi)1+\ \mathrm{T}\ 2. & \end{cases} \tag{1.22}$$

Proof. Let

$$z)x,y+\mathrm{T}\int u_j\varphi_j)x+\psi)\rho_j^{2/3}y+.$$

On one hand,

$$\begin{aligned} &\kappa_\alpha \bigcap_{\mathcal{F}_\Omega} y^2\ \ ^\alpha\|\ z)x,y+\|^3\,dxdy \\ &\mathrm{T}\bigcap_1^{\in} y^2\ \ ^\alpha\bigcap_{-}\Big)\int u_j^3\|\ \varphi_j)x+\|^3\psi)\rho_j^{2/3}y+^3\,0\ \ u_j^3\rho_j\varphi_j)x+^3)\psi^{\infty})\rho_j^{2/3}y+^3\Big(dxdy \\ &\mathrm{T}\bigcap_1^{\in} y^2\ \ ^\alpha\int u_j^3\rho_j\Big)\psi)\rho_j^{2/3}y+^3\,0\ \ )\psi^{\infty})\rho_j^{2/3}y+^3\Big(dy \\ &\mathrm{T}\int u_j^3\rho_j^{\alpha/3}\bigcap_1^{\in} s^2\ \ ^\alpha\Big)\psi)s+^3\,0\ \ )\psi^{\infty})s+^3\Big(ds\ \mathrm{T}\int u_j^3\rho_j^{\alpha/3}. \end{aligned} \tag{1.23}$$

Thus $z\ /\ X_1^\alpha)\mathcal{F}+$and we obtain the norm equivalence. It is easy to see that z verifies (1.17). Since the α-harmonic extension is unique in $X_1^\alpha)\mathcal{F}+$we have $\mathrm{E}_\alpha)u+\mathrm{T}$ z.

On the other hand, notice that for every $k\sim 2$, via the change of variables $s\ \mathrm{T}\ \rho_k^{2/3}y$ in (1.22) we have

$$\kappa_\alpha \lim_{y\prime\ 1+} y^2\ \ ^\alpha\frac{\partial}{\partial y})\psi)\rho_k^{2/3}y++\mathrm{T}\quad \rho_k^{\alpha/3}\kappa_\alpha\lim_{s\prime\ 1+} s^2\ \ ^\alpha\psi^{\infty})s+\mathrm{T}\ \rho_k^{\alpha/3}.$$

Therefore,

$$)\ \ \Lambda+^{\alpha/3}u\ \mathrm{T}\quad \kappa_\alpha\lim_{y\prime\ 1+} y^2\ \ ^\alpha\frac{\partial\mathrm{E}_\alpha)u+}{\partial y}\mathrm{T}\int u_j\varphi_j\rho_j^{\alpha/3}$$

$\square$

Heat semigroup

Our next step will be to establish, by means of the heat semigroup of Λ, a definition that connects the fractional Laplacian in bounded domains and in $\mathbb{R}^N$. This definition is motivated by the following identities

$$
\begin{aligned}
&a^{\ \ p} \mathrm{T} \frac{2}{)p+} \bigcap_{1}^{\in} e^{\ \ at} \frac{dt}{t^{2\ \ p}}, && p > 1 \\
&a^{p} \mathrm{T} \frac{2}{)\ \ p+} \bigcap_{1}^{\in})e^{\ \ at} \quad 2+\frac{dt}{t^{20\ p}}, && 1 < p < 2.
\end{aligned}
\tag{1.24}
$$

Moreover, this approach will allow us to define the fractional powers of a general class of operators: Let L be a linear, positive and self-adjoint operator. Let $e^{\ \ tL}$ be the heat semigroup of L, that is, for every function u in a proper space, the function $v \mathrm{T}\ e^{\ \ tL}u$ is the unique solution to the problem

$$
\begin{rcases}
v_t\, 0\ \ Lv \mathrm{T}\ 1, \quad \}t > 1\langle, \\
v)1+\mathrm{T}\ u.
\end{rcases}
\tag{1.25}
$$

We define then the fractional powers of L as

$$
\begin{aligned}
&L^{\ \ \gamma} \mathrm{T} \frac{2}{)\gamma+} \bigcap_{1}^{\in} e^{\ \ tL} \frac{dt}{t^{2\ \ \gamma}}, && \gamma > 1 \\
&L^{\gamma} \mathrm{T} \frac{2}{)\ \ \gamma+} \bigcap_{1}^{\in})e^{\ \ tL} \quad I+\frac{dt}{t^{20\ \gamma}}, && 1 < \gamma < 2.
\end{aligned}
\tag{1.26}
$$

In particular, for the fractional Laplacian we have

Proposition 1.2.2. *Consider* $\alpha\ /\ )1, 3+$ *´ a bounded domain or ´* $\mathrm{T}\ \mathbb{R}^N$ *and* $u\ /$ $\mathcal{U})$*´ + Then the following identity holds*

$$
)\ \ \Lambda+^{\alpha/3}u)x+\mathrm{T} \frac{2}{)\ \ \alpha/3+} \bigcap_{1}^{\in})e^{t\Lambda}u)x+\ \ u)x++\frac{dt}{t^{20\ \alpha/3}}, \quad x\ /\ ´. \tag{1.27}
$$

Proof. Assume first that ´ is a bounded domain. Consider the operator

$$
A)u+\mathrm{T} \frac{2}{)\ \ \alpha/3+} \bigcap_{1}^{\in})e^{t\Lambda}u \quad u+\frac{dt}{t^{20\ \alpha/3}} \tag{1.28}
$$

and the equation (1.25) defined in ´. Let $\}\rho_j, \varphi_j\langle$ as before and $u\ \mathrm{T} \int\ u_j\varphi_j$. Then, the solution of (1.25) is

$$
v)x, t+\mathrm{T}\ e^{t\Lambda}u)x+\mathrm{T} \int\ e^{\ \ \rho_j t}u_j\varphi_j)x+ \tag{1.29}
$$

Substituting (1.29) into (1.28) we have

$$\begin{aligned} A)u+)x+ \;\mathrm{T}\; & \frac{2}{)\;\;\alpha/3+} \bigcap_{1}^{\in})e^{t\Lambda} u)x+ \quad u)x++\frac{dt}{t^{20\;\alpha/3}} \\ \mathrm{T}\; & \frac{2}{)\;\;\alpha/3+} \int \; u_j\varphi_j)x+\bigcap_{1}^{\in})e^{t\rho_j} \quad 2+\frac{dt}{t^{20\;\alpha/3}} \\ \mathrm{T}\; & \int \; u_j\rho_j^{\alpha/3}\varphi_j)x+\mathrm{T}\;) \quad \Lambda+^{\alpha/3}u)x+. \end{aligned}$$

Let now $\acute{}\;\;\mathrm{T}\;\mathbb{R}^N$. Recall that the unique solution to (1.25) can be expressed as a convolution with the hear kernel, that is,

$$e^{t\Lambda} u \;\mathrm{T}\; K)\times t+\circ\, u)x+$$

where $K)\times t+$holds

$$\widetilde{K})\xi, t+\mathrm{T}\; e \quad ^{t\|\beta\pi\xi\|^{\beta}}.$$

In particular

$$e \quad ^{t\Lambda}u)x, t+\mathrm{T}\;)8\pi t+\;^{N/3}\bigcap_{\mathbb{R}^N} e \quad ^{\frac{|x-y|^2}{4t}} u)y+dy.$$

Therefore, applying the Fourier transform to (1.27) and using (1.24) we have

$$)\;\;\widehat{\Lambda+^{\alpha/3}}\;\mathrm{T}\;\frac{2}{)\;\;\alpha/3+}\bigcap_{1}^{\in})e \quad ^{\|\beta\pi\xi\|^{\beta}t} \quad 2+\frac{dt}{t^{20\;\alpha/3}}\;\mathrm{T}\;\|\beta\pi\xi\|^{\alpha}.$$

□

1.3. Fractional Sobolev and trace inequalities

In this section we prove two useful and long used inequalities that will be fundamental along this work.

Theorem 1.3.1 (Fractional trace inequality)**.** *Given N, α, r such that $N > \alpha$, $1 < \alpha < 3$ and $2 \geq r \geq \frac{3N}{N\;\;\alpha}$ there exists a constant $S)\alpha, N, r, \acute{}\;+> 1$ such that*

$$S)\alpha, N, r, \acute{}\;+\Big)\bigcap_{\bar{}} \|v)x+\|^{r}\, dx\Big\{^{3/r} \geq \kappa_{\alpha}\bigcap_{\mathcal{F}_{\Omega}} y^{2\quad\alpha}\|\quad z)x, y+\|^{\beta}\, dxdy \tag{1.30}$$

for every $z\;\;/\;\;X_1^{\alpha})\mathcal{F}\;+$where $v\;\mathrm{T}\;[\;\mathrm{s})z+$. If $r\;\mathrm{T}\;\frac{3N}{N\;\;\alpha}$, the constant $S)\alpha, N, r, \acute{}\;+$is independent of $\acute{}$ and takes the exact value

$$S)\alpha, N+\mathrm{T}\;3^{\alpha}\pi^{\frac{\alpha}{2}}\frac{)\frac{N0\;\alpha}{3}+\;)\frac{N}{3}+^{\frac{\alpha}{N}}}{)\frac{N\;\;\alpha}{3}+\;)N+^{\frac{\alpha}{N}}}. \tag{1.31}$$

Moreover, if $\Omega = \mathbb{R}^N$, the constant $S(\alpha, N)$ is achieved only by the biparametric family of functions $w_\varepsilon = E_\alpha(u_\varepsilon)$ where

$$u_\varepsilon(x) = \varepsilon^{\frac{N-\alpha}{2}} \left(|x - x_0|^2 + \varepsilon^2\right)^{-\frac{N-\alpha}{2}}, \tag{1.32}$$

for some $x_0 \in \mathbb{R}^N$, $\varepsilon > 0$.

As a consequence, by (1.8) and (1.11), we have

Corollary 1.3.2 (Fractional Sobolev inequality)**.** *Under the same assumptions than in the previous theorem we have*

$$S(\alpha, N, r, \Omega)\left(\int_\Omega |\varphi(x)|^r \, dx\right)^{2/r} \leq \int_\Omega |(-\Delta)^{\alpha/4}\varphi(x)|^2 \, dx \tag{1.33}$$

for every $\varphi \in H_0^{\alpha/2}(\Omega)$.

The classical case ($\alpha = 2$) was proven first in [68] for $N = 3$ and afterwards generalized to all dimensions in [8] and [78], see also [44, 60].

In order to prove Theorem 1.3.1, we will prove some previous technical lemmas.

Lemma 1.3.3. *Consider $v \in H^{\alpha/2}(\mathbb{R}^N)$ and set $z = E_\beta(v)$ its β-harmonic extension, $\beta \in (\alpha/2, 2)$. Then $z \in X^\alpha(\mathbb{R}_+^{N+1})$ and moreover there exist an universal constant $c(\alpha, \beta)$ such that*

$$\|v\|_{H^{\alpha/2}} = c(\alpha, \beta)\|z\|_{X^\alpha}. \tag{1.34}$$

Inequality (1.3.1) needs only the case $\beta = \alpha$, which is deduced directly from the proof of the local characterization of $(-\Delta)^{\alpha/2}$ in [31]. The calculations performed in [31] can be extended to cover the range $\alpha/2 < \beta < 2$ and in particular includes the case $\beta = 1$ proved in [83].

Proof. Since $z = E_\beta(v)$, by definition z solves $\operatorname{div}(y^{1-\beta}\nabla z) = 0$, which is equivalent to

$$\Delta_x z + \frac{1-\beta}{y}\frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2} = 0.$$

Taking Fourier transform in $x \in \mathbb{R}^N$ for $y > 0$ fixed, we have

$$-4\pi^2|\xi|^2\hat{z} + \frac{1-\beta}{y}\frac{\partial \hat{z}}{\partial y} + \frac{\partial^2 \hat{z}}{\partial y^2} = 0,$$

and $\hat{z}(\xi, 0) = \hat{v}(\xi)$. Therefore $\hat{z}(\xi, y) = \hat{v}(\xi)\phi_\beta(2\pi|\xi|y)$, where ϕ_β solves the problem

$$-\phi + \frac{1-\beta}{s}\phi' + \phi'' = 0, \qquad \phi(0) = 1, \qquad \lim_{s \to \infty} \phi(s) = 0. \tag{1.35}$$

In fact, ϕ_β minimizes the functional

$$H_\beta)\phi+\mathrm{T}\bigcap_1^{\in})|\phi)s+|^\beta\, 0\ |\phi^\infty)s+|^\beta+s^{2\ \ \beta}\, ds,$$

and it can be shown that it is a combination of Bessel functions, see [56]. More precisely, ϕ_β satisfies the following asymptotic behaviour

$$\begin{aligned} &\phi_\beta)s+\approx 2\quad c_2 s^\beta, &&\text{for } s\nearrow 1,\\ &\phi_\beta)s+\approx c_3 s^{\frac{\beta-1}{2}}e^{\ \ s}, &&\text{for } s\nearrow \in\,, \end{aligned} \tag{1.36}$$

where

$$c_2)\beta+\mathrm{T}\ \frac{3^{2\ \ \beta}\ )2\quad \frac{\beta}{3}+}{\beta\ \ )\frac{\beta}{3}+},\quad c_3)\beta+\mathrm{T}\ \frac{3^{\frac{1-\beta}{2}}\pi^{2/3}}{)\frac{\beta}{3}+}.$$

Now we observe that

$$\begin{aligned}\bigcap_{\mathbb{R}^N} \|\ \ z)x,y+|^\beta\, dx\quad &\mathrm{T}\bigcap_{\mathbb{R}^N}\Big)\|\ \ _x z)x,y+|^\beta\ 0\ \Big(\frac{\partial z}{\partial y})x,y+\Big(^\beta\Big\{\ dx\\ &\mathrm{T}\bigcap_{\mathbb{R}^N}\Big)8\pi^3|\xi|^\beta|\hat z)\xi,y+|^\beta\ 0\ \Big(\frac{\partial \hat z}{\partial y})\xi,y+\Big(^\beta\Big\{\ d\xi.\end{aligned}$$

Then, multiplying by $y^{2\ \ \alpha}$ and integrating in y,

$$\begin{aligned}&\bigcap_1^{\in}\bigcap_{\mathbb{R}^N} y^{2\ \ \alpha}\|\ \ z)x,y+|^\beta\, dxdy\\ &\mathrm{T}\bigcap_1^{\in}\bigcap_{\mathbb{R}^N} 8\pi^3|\xi|^\beta|\hat v)\xi+|^\beta\ \ |\phi_\beta)3\pi|\xi|y+|^\beta\ 0\ |\phi_\beta^\infty)3\pi|\xi|y+|^\beta\{y^{2\ \ \alpha}\, d\xi dy\\ &\mathrm{T}\bigcap_1^{\in})|\phi_\beta)s+|^\beta\ 0\ |\phi_\beta^\infty)s+|^\beta+s^{2\ \ \alpha}\, ds\bigcap_{\mathbb{R}^N}|3\pi\xi|^\alpha|\hat v)\xi+|^\beta\, d\xi.\end{aligned}$$

Using (1.36) we see that the integral $\sum^{\in})|\phi_\beta|^\beta 0\ |\phi_\beta^\infty|^\beta+s^{2\ \ \alpha}\, ds$ is convergent provided $\beta>\alpha/3$. This proves (1.34) with $c)\alpha,\beta+\mathrm{T}\ )\kappa_\alpha H_\alpha)\phi_\beta++^{2/3}$. □

Remark 1.3.1. *If* $\beta\ \mathrm{T}\ 2$ *we have* $\phi_2)s+\mathrm{T}\ e^{\ \ s}$, *y* $H_\alpha)\phi_2+\mathrm{T}\ 3^{\alpha\ \ 2}\ )3\quad\alpha+$, *see [83]. Moreover, when* $\beta\ \mathrm{T}\ \alpha$, *integrating by parts and using the equation in* (1.35), *and* (1.36), *we obtain*

$$H_\alpha)\phi_\alpha+\mathrm{T}\bigcap_1^{\in}]\phi_\alpha^3)s+0\ )\phi_\alpha^\infty+^3)s+\hat{}s^{2\ \ \alpha}\, ds\ \mathrm{T}\quad \lim_{s\nearrow\ 1} s^{2\ \ \alpha}\phi_\alpha^\infty)s+\mathrm{T}\ \alpha c_2)\alpha+\mathrm{T}\ 2/\kappa_\alpha. \tag{1.37}$$

In particular, if $\beta\ \mathrm{T}\ \alpha$ *we have that* $c)\alpha,\alpha+\mathrm{T}\ 2$ *and* (1.8) *holds.*

Lemma 1.3.4. *If $g \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ and $f \in H^{\alpha/2}(\mathbb{R}^N)$, then there exists a constant $C(\alpha, N) > 0$ such that*

$$\left|\int f(x)g(x)dx\right| \le C(\alpha, N)\|f\|_{H^{\alpha/2}}\|g\|_{\frac{2N}{N+\alpha}}. \tag{1.38}$$

Moreover, the equality in (1.38) *with the best constant holds when f and g take the form* (1.32).

The proof follows by an standard argument that can be found, for instance in [41, 83].

Proof. By Parçeval's identity and Cauchy-Schwarz's inequality, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^N} f(x)g(x)dx\right)^2 &= \left(\int_{\mathbb{R}^N} \widetilde{f}(\xi)\widetilde{g}(\xi)d\xi\right)^2 \\ &\le \left(\int_{\mathbb{R}^N} |2\pi\xi|^{\alpha}|\widetilde{f}(\xi)|^2 d\xi\right)\left(\int_{\mathbb{R}^N} |2\pi\xi|^{-\alpha}|\widetilde{g}(\xi)|^2 d\xi\right).\end{aligned}$$

The second term can be written using [59] as

$$\int_{\mathbb{R}^N} |2\pi\xi|^{-\alpha}|\widetilde{g}(\xi)|^2 d\xi = b(\alpha, N)\int_{\mathbb{R}^{2N}} \frac{g(x)g(x')}{|x - x'|^{N-\alpha}}dxdx',$$

where

$$b(\alpha, N) = \frac{\Gamma(\frac{N-\alpha}{2})}{2^{\alpha}\pi^{N/2}\Gamma(\frac{\alpha}{2})}.$$

We now use the following Hardy-Littlewood-Sobolev inequality, see again [59],

$$\int_{\mathbb{R}^{2N}} \frac{g(x)g(x')}{|x - x'|^{N-\alpha}}dxdx' \le d(\alpha, N)\|g\|^2_{\frac{2N}{N+\alpha}},$$

where

$$d(\alpha, N) = \frac{\pi^{\frac{N-\alpha}{2}}\Gamma(\frac{\alpha}{2})\Gamma(N)^{\frac{\alpha}{N}}}{\Gamma(\frac{N+\alpha}{2})\Gamma(\frac{N}{2})^{\frac{\alpha}{N}}},$$

with equality if g takes the form (1.32). From this we obtain the desired estimate (1.38) with the constant $C(\alpha, N) = \sqrt{b(\alpha, N)d(\alpha, N)}$.

When applying Cauchy-Schwarz's inequality, we obtain an identity provided the functions $|\xi|^{\alpha/2}\widetilde{f}(\xi)$ and $|\xi|^{-\alpha/2}\widetilde{g}(\xi)$ are proportional. This means

$$\widetilde{g}(\xi) = c|\xi|^{\alpha}\widetilde{f}(\xi) = c[(-\Delta)^{\alpha/2}f]^{\wedge}(\xi).$$

We end by observing that if g takes the form (1.32) and $g = c(-\Delta)^{\alpha/2}f$ then f also takes the form (1.32). $\square$

Proof of Theorem 1.3.1. Applying Lemma 1.3.4 with $g \ \mathrm{T} \ \|f\|^{\frac{N+\alpha}{N-\alpha} \quad 2} f$ we have

$$\backslash f \backslash^{\frac{2N}{N+\alpha}}_{\frac{2N}{N+\alpha}} \geq \ell)\alpha, N+\backslash f\backslash_{H^{\alpha/2}} \backslash f \backslash^{\frac{N-\alpha}{N+\alpha}}_{\frac{2N}{N+\alpha}}.$$

Then, using Lemma 1.3.3 we obtain

$$\backslash f \backslash_{\frac{2N}{N+\alpha}} \geq \ell)\alpha, N+\backslash z \backslash_{X^{\alpha}}.$$

with $z \ \mathrm{T} \ \mathrm{E}_{\alpha})f+$ We conclude using Lemma 1.10. The best constant is $S)\alpha, N+\mathrm{T}$ $2/\ell^{3})\alpha, N+$ To obtain the result in bounded domains note that if u is defined in $H^{\alpha/3})\,'\ +$ it can be approximated by regular functions that are zero outside $'$. □

Remark 1.3.2. *If we let α tend to 2, when $N > 3$, we recover the classical Sobolev inequality for a function in $H^{2})\mathbb{R}^{N}+$ with the same constant, see [78]. In order to pass to the limit in the right-hand side of* (1.30)*, at least formally, we observe that* $)3 \quad \alpha+y^{2 \quad \alpha}\, dy$ *is a measure on compact sets of $\mathbb{R}_0$ converging (in the weak-* sense) to a Dirac delta. Hence*

$$\lim_{\alpha \prime \ 3^-} \bigcap_{1}^{2} \Big) \bigcap_{\mathbb{R}^N} \| \ \ z)x, y+\|^{\beta}\, dx \Big\{)3 \quad \alpha+y^{2 \quad \alpha}\, dy \ \mathrm{T} \bigcap_{\mathbb{R}^N} \| \ \ v)x+\|^{\beta}\, dx.$$

We then obtain

$$\Big) \bigcap_{\mathbb{R}^N} \|v)x+\|^{\frac{2N}{N-2}}\, dx \Big\{^{\frac{N-2}{N}} \geq S)N+\bigcap_{\mathbb{R}^N} \| \ \ v)x+\|^{\beta}\, dx,$$

with the best constant $S)N+\mathrm{T} \lim_{\alpha \prime \ 3^-} \frac{S)\alpha, N+}{3 \quad \alpha} \ \mathrm{T} \ \frac{2}{\pi N)N \quad 3+}\Big) \frac{)N+}{)\frac{N}{2}+}\Big(^{\frac{2}{N}}$. *It is achieved when v takes the form* (1.32) *with α replaced by 2.*

Remark 1.3.3. *The uniqueness of the minimizing functions* (1.32) *is deduced directly from [36]. There the authors prove the unique solutions to the problem* $) \quad \Lambda+^{\alpha/3} f \ \mathrm{T}$ $c f^{\frac{N+\alpha}{N-\alpha}}$ *take the form* (1.32).

Remark 1.3.4. *The constant $S)\alpha, N+$cant be achieved in any $'$ different from $\mathbb{R}^N$. To see this, let us suppose $' \subsetneq \mathbb{R}^N$ and assume $S)\alpha, N+$is achieved for a function u_1. Then, as before, approximating u_1 by functions that are zero out of $'$ we would have a function defined in $\mathbb{R}^N$ that achieves $S)\alpha, N+$and it is not in the form* (1.32) *leading to a contradiction.*

1.4. Other fractional operators

Even when our focus will be the fractional Laplacian as defined in the previous sections, in this section we give a small review over other fractional operators in bounded domains.

1.4.1. Global fractional Laplacian

A natural way to extend the definition $(-\Delta)^{\alpha/2}$ to bounded domains consist on extending by zero functions defined in Ω. This method leads to the so-called *global fractional Laplacian*.

Definition 1.4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let f be a function regular enough defined in Ω. Let $\widetilde{f}$ be its extension by zero to $\mathbb{R}^N$, that is, $\widetilde{f}(x) = f(x)$ if $x \in \Omega$ and $\widetilde{f}(x) = 0$ if $x \in \Omega^c$. Then, we define the global fractional Laplacian as*

$$(-\Delta)_G^{\alpha/2} f = (-\Delta)^{\alpha/2}\widetilde{f}.$$

The operator is well defined in the space

$$\mathcal{H}^{\alpha/2}(\Omega) = \left\{ f \; ; \; \widetilde{f} \in H^{\alpha/2}(\mathbb{R}^N) \right\}$$

endowed with the norm

$$\|f\|_{\mathcal{H}^{\alpha/2}(\Omega)} = \|\widetilde{f}\|_{H^{\alpha/2}(\mathbb{R}^N)}.$$

First, note that given $f, g \in \mathcal{H}^{\alpha/2}(\Omega)$, we have

$$\int_\Omega f(-\Delta)_G^{\alpha/2} g = \int_{\mathbb{R}^N} \widetilde{f}(-\Delta)^{\alpha/2}\widetilde{g} = \int_{\mathbb{R}^N} \widetilde{g}(-\Delta)^{\alpha/2}\widetilde{f} = \int_\Omega g(-\Delta)_G^{\alpha/2} f.$$

However,

$$\int_{\mathbb{R}^N} \widetilde{g}(-\Delta)^{\alpha/2}\widetilde{f} = \int_{\mathbb{R}^N} (-\Delta)^{\alpha/4}\widetilde{g}(-\Delta)^{\alpha/4}\widetilde{f} \neq \int_\Omega (-\Delta)^{\alpha/4} g(-\Delta)^{\alpha/4} f \qquad (1.39)$$

since $(-\Delta)^{\alpha/4}\widetilde{g}$ and $(-\Delta)^{\alpha/4}\widetilde{f}$ may not be null out of Ω. In particular we have

$$\|f\|^2_{\mathcal{H}^{\alpha/2}(\Omega)} = \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/4}\widetilde{f}|^2 \neq \|(-\Delta)_G^{\alpha/2} f\|_2^2.$$

Note that the second term of (1.39) defines a scalar product in $\mathcal{H}^{\alpha/2}(\Omega)$.

1.4.2. Regional fractional Laplacian

The second operator arise when restricting the integral in (1.2) to the integral on bounded domains.

Definition 1.4.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let f be a function regular enough defined in Ω. We define the regional fractional Laplacian as*

$$(-\Delta)_R^{\alpha/2} f = \mu_{N,\alpha} P.V. \int_\Omega \frac{f(x) - f(\overline{x})}{|x - \overline{x}|^{N+\alpha}} d\overline{x}$$

Exploring again the integration by parts we have that, given ψ, ϕ regular enough,

$$\begin{aligned} \bigcap_{-} \phi) \quad \Lambda +_{R}^{\alpha/3}\psi \quad & \mathrm{T}\ \mu_{N,\alpha}\bigcap_{-}\text{P.V.}\bigcap_{-}\phi)x+\frac{\psi)x+ \quad \psi)\overline{x}+}{\|x \quad \overline{x}\|^{N0\;\alpha}}d\overline{x}dx \\ & \mathrm{T}\ \frac{\mu_{N,\alpha}}{3}\bigcap_{-}\bigcap_{-}\frac{)\phi)x+ \quad \phi)\overline{x}++)\psi)x+ \quad \psi)\overline{x}++}{\|x \quad \overline{x}\|^{N0\;\alpha}}d\overline{x}dx \\ & \mathrm{T}\bigcap_{-}\psi) \quad \Lambda +_{R}^{\alpha/3}\phi. \end{aligned} \tag{1.40}$$

However, as in the previous case

$$\bigcap_{-}\phi) \quad \Lambda +_{R}^{\alpha/3}\psi\ \mathrm{T}\bigcap_{-}) \quad \Lambda +_{R}^{\alpha/=}\phi) \quad \Lambda +_{R}^{\alpha/=}\psi.$$

The terms in (1.40) define a scalar product in

$$\mathcal{I}_{\leq}^{\alpha/3})' + \mathrm{T}\ \Big\}f \ ; \backslash f\backslash_{\mathcal{N}_{*}^{\alpha/2})^{-}+} < \in \surd$$

where

$$\backslash f\backslash^{3}_{\mathcal{N}_{*}^{\alpha/2})^{-}+}\mathrm{T}\ \frac{\mu_{N,\alpha}}{3}\bigcap_{-}\bigcap_{-}\frac{)f)x+ \quad f)\overline{x}++^{3}}{\|x \quad \overline{x}\|^{N0\;\alpha}}d\overline{x}dx$$

is the well known Gagliardo norm. The global fractional Laplacian and the regional fractional Laplacian are connected by the formula

$$) \quad \Lambda +_{G}^{\alpha/3}f\ \mathrm{T}\ ) \quad \Lambda +_{R}^{\alpha/3}f\ 0\ \ \mu_{N,\alpha}f)x+\bigcap_{-\,c}\frac{2}{\|x \quad y\|^{N0\;\alpha}}dy.$$

2

A concave-convex elliptic problem involving the fractional Laplacian

2.1. Introduction

This chapter is devoted to study the following concave-convex problem involving the fractional Laplacian operator

$$)P_\lambda+ \left\{ \begin{array}{lllll}) & \Lambda+^{\alpha/3}u & \mathrm{T} & \lambda u^q\,0\,u^p, \qquad u>1 & \text{in } ´\,, \\ & u & \mathrm{T} & 1 & \text{on } \partial ´\,, \end{array}\right. \tag{2.1}$$

with $1 < \alpha < 3$, $1 < q < 2 < p < \frac{N0\,\alpha}{N\ \ \alpha}$, $N > \alpha$, $\lambda > 1$ and $´\ \ll \mathbb{R}^N$ a smooth bounded domain.

As to the problems with concave-convex nonlinearities like the above, there is a huge amount of results involving different (local) operators, see for instance [1, 4, 13, 35, 40, 48]. We quoted the work [4] from where some ideas are used in the present chapter. In most of the problems considered in those papers a critical exponent appears, (in the fully nonlinear case the situation is slightly different, but still a critical exponent appears, [35]). In our case, the critical exponent with respect to the corresponding Sobolev embedding will be $3^{\leq}_{\alpha}\ \mathrm{T}\ \frac{3N}{N\ \ \alpha}$. This is a reason why problem $)P_\lambda+$is studied in the subcritical case $p < 3^{\leq}_{\alpha}\quad 2\ \mathrm{T}\ \frac{N0\,\alpha}{N\ \ \alpha}$; see also the Pohozaev-type nonexistence result for supercritical nonlinearities in Corollary 2.4.5.

The main results that we prove characterize the existence of solutions of $)P_\lambda+$in terms of the parameter λ. A competition between the sublinear and superlinear powers plays a role, which leads to different results concerning existence and multiplicity of solutions, among others. By a solution we mean an energy solution, see the precise definition in Section 2.4.

Theorem 2.1.1. *There exists $\Sigma > 1$ such that for Problem $)P_\lambda+$there holds:*

1. *If $1 < \lambda < \Sigma$ there is a minimal solution. Moreover, the family of minimal solutions is increasing with respect to λ.*
2. *If $\lambda \mathrm{T} \Sigma$ there is at least one solution.*
3. *If $\lambda > \Sigma$ there is no solution.*
4. *For any $1 < \lambda < \Sigma$ there exist at least two solutions.*

For $\alpha \ / \ ]2, 3+$ we also prove that there exists a universal $L^{\in}$-bound for every solution to Problem $)P_\lambda+$independently of λ.

Theorem 2.1.2. *Let $\alpha \sim 2$. Then there exists a constant $C > 1$ such that, for any $1 \geq \lambda \geq \Sigma$, every solution to Problem $)P_\lambda+$satisfies*

$$\backslash u \backslash_{\in} \ \geq C.$$

The prove of this result uses the classical argument of rescaling introduced in [51] leading to problems on unbounded domains. Therefore some Liouville-type results are required, and this is the point where the restriction $\alpha \sim 2$ appears.

2.2. Some non-existence results in unbounded domains

We prove in this section two Liouville-type results in the half space $\mathbb{R}_0^{N0\ 2}$ and the quarter space $\mathbb{R}_{0\ 0}^{N0\ 2}$ that will be useful in Section 2.4.3 in order to obtain uniform a priori bounds for the solutions to Problem $)P_\lambda+$. These results have a corresponding formulation for the fractional Laplacian operator.

2.2.1. A problem in the half-space

Theorem 2.2.1. *Let $2 \geq \alpha < 3$. Then the problem in the half-space $\mathbb{R}_0^{N0\ 2}$,*

$$\begin{cases} \mathrm{f\,lx})y^{2\ \ \alpha}\ \ w+ \ \mathrm{T} \ 1 & \text{in } \mathbb{R}_0^{N0\ 2} \\ \dfrac{\partial w}{\partial \nu^\alpha} \ \mathrm{T} \ w^p & \text{on } \partial\mathbb{R}_0^{N0\ 2} \ \mathrm{T} \ \mathbb{R}^N \end{cases} \tag{2.2}$$

has no positive bounded solution in $C^{\alpha 0\ \gamma})\mathbb{R}_0^{N0\ 2} + \wedge\, C)\overline{\mathbb{R}_0^{N0\ 2}}+$with $\gamma > 1$ provided $2 < p < \frac{N0\ \alpha}{N\ \ \alpha}$.

Theorem 2.2.1 is proved in the case α T 2 by [54]. See also [37, 79, 47] for other approaches to the general case.

The proof that we present here is based on the well known *method of moving planes*, introduced by A.D. Alexandrov and firstly used in the context of PDE's by [71] and [50], among others. Recall that the problem (2.2) can be written as

$$\begin{cases} L_\alpha w \quad \mathrm{T} \quad 1 & \text{in } \mathbb{R}_0^{N0\,2} \\ \dfrac{\partial w}{\partial \nu^\alpha} \quad \mathrm{T} \quad w^p & \text{for } y \ \mathrm{T}\ 1. \end{cases} \tag{2.3}$$

where L_α is defined in (1.12).

We begin then by establishing some useful notation in order to apply the moving planes method. The points of the upper half-space $\mathbb{R}_0^{N0\,2}$ are denoted by X T $)x, y+$, where x T $)x_2, \times\!\times\!\times, x_N+$and $y > 1$. Fix $\rho > 1$ and consider the sets

$$\Phi_\rho \ \mathrm{T} \ \}X \ / \ \mathbb{R}_0^{N0\,2} = x_N > \rho\langle\, , \qquad T_\rho \ \mathrm{T} \ \Big\}X \ / \ \overline{\mathbb{R}_0^{N0\,2}} = x_N \ \mathrm{T} \ \rho\surd . \tag{2.4}$$

For every X T $)x, y+ / \mathbb{R}_0^{N0\,2}$ we define the reflection across the hyperplane T_ρ by X^ρ T $)x^\rho, y+$T X 0 $3)\rho \quad x_N + e_N$ T $)x_2, \times\!\times\!\times, 3\rho \quad x_N, y+$. Let us also consider the point P_ρ T $)1, ..., 1, 3\rho, 1+ / \Phi_\rho$, whose reflection is the origin, and the set $\widetilde{\Phi_\rho}$ T $\overline{\Phi_\rho}\,\nabla\}P_\rho\langle$. Let B_r^0 denote the half-ball B_r^0 T $\}\|X\| < r,\ y > 1\langle$ $(B_r^0)X_1+$when the center X_1 T $)x_1, 1+$is not the origin), and let its non flat part of the boundary be denoted by S_r^0 T $\}\|X\|$T $r,\ y > 1\langle$ (resp. $S_r^0)X_1+$).

Finally, also the so-called *fractional Kelvin transform* will be useful. We consider, for a function f defined in $\mathbb{R}^N$, its fractional Kelvin transform as $K_\alpha)f+)x+$T $\|x\|^{\alpha \quad N} f)x/\|x\|^\beta+$. It is well known that this transform behaves under the action of the fractional Laplacian in a similar way as the standard Kelvin transform does with the Laplacian, $) \quad \Lambda+^{\alpha/3} K_\alpha)f+)x+$T $\|x\|^{\ \alpha \quad N}) \quad \Lambda+^{\alpha/3} f)x/\|x\|^\beta+$. We are interested in defining the analogous fractional Kelvin transform for the function w and the operator L_α. Let $z)X+$T $\|X\|^\gamma w)\xi+$, ξ T $X/\|X\|^\beta$. It is a calculus matter to see that

$$L_\alpha z)X+ \mathrm{T}\ \|X\|^{\gamma} \ \ {=}\Big) L_\alpha w)\xi+0 \ )\gamma \ 0 \ N \quad \alpha+\|X\|^\beta)\gamma w)\xi+ \quad 3\rangle\xi, \quad w)\xi++\Big(.$$

Therefore, if we choose γ T $\alpha \quad N$, and w is α-harmonic, we get that z is also α-harmonic, and so it turns to be the α-harmonic extension of $K_\alpha)f+$if w is the α-harmonic extension of f. In other words, $\mathrm{E}_\alpha \equiv K_\alpha$ T $K_\alpha \equiv \mathrm{E}_\alpha$.

Let now w be any solution to problem (2.3), and put μ T $\mathrm{tvr}_{B_1^+} w$. Then there exists $\varepsilon > 1$ such that $w)X+ \sim \varepsilon\|X\|^{\alpha \quad N}$ for $\|X\| \sim 2$, $y > 1$. To see this observe that by the Harnack inequality, Lemma 4.8 in [26], we have ε T $\log_{S_1^+} w \sim c\mu > 1$. We conclude by comparison, using Lemma 1.1.2 and Proposition 4.10 of [26]. Let

$v \text{ T } K_\alpha)w+$ We have that v satisfies analogous properties as w, but for the inversion variable:

$$
\begin{aligned}
&v)X+\sim \varepsilon && \text{in } B_2^0\ , \\
&v)X+\geq \mu\|X\|^{\alpha \quad N} && \text{in } \mathbb{R}_0^{N0\,2}\nabla B_2^0\ ,
\end{aligned}
\tag{2.5}
$$

as well as it is a solution to the problem

$$
\begin{cases}
L_\alpha v \text{ T } 1 & \text{in } \mathbb{R}_0^{N0\,2}, \\
\dfrac{\partial v}{\partial \nu^\alpha} \text{ T } \|x\|^{\ \gamma} v^p & \text{for } y \text{ T } 1,\ \|x\|\text{T } 1,
\end{cases}
\tag{2.6}
$$

where $\gamma \text{ T })N\ 0\ \ \alpha+\ \ \ )N\ \ \ \ \alpha+p > 1$.

We now proceed with the reflection. Let

$$
\psi_\rho)X+\text{T } v)X^\rho+\ \ \ v)X+. \tag{2.7}
$$

Clearly $L_\alpha)\psi_\rho+\text{T }\ 1$ in $\mathbb{R}_0^{N0\,2}$. We want to prove that $\psi_\rho \sim 1$ in $\widetilde{\Phi}_\rho$. Recall that v may have a singularity at the origin, and therefore ψ_ρ may have a singularity at P_ρ. We begin with the following result.

Lemma 2.2.2. *With the above notation, we have $\psi_\rho \sim 1$ in $\widetilde{\Phi_\rho}$, provided $\rho > 1$ is large enough.*

Proof. Let $\beta > 1$ be some constant to be chosen later, and let

$$
\varphi_\rho)X+\text{T } \|Z\|^\beta \psi_\rho)X+, \qquad Z \text{ T } X\ 0\ \ e_{N0\,2} \text{ T })x, y\ 0\ \ 2+. \tag{2.8}
$$

From the equation (2.6), we get

$$
L_\alpha)\varphi_\rho+\ \ \ \beta y^{2\quad \alpha}\|Z\|^{\ \ 3}])\ \ \ \beta\ 0\ \ N\ \ \ \ \alpha+\varphi_\rho\ 0\ \ 3\rangle Z,\ \ \ \varphi_\rho|\ \hat{}\text{T } 1. \tag{2.9}
$$

Assume by contradiction that there exists $\delta > 1$ such that

$$
\underset{\widetilde{\Omega_\rho}}{\log}\varphi_\rho \text{ T }\ \ \ \ \delta < 1. \tag{2.10}
$$

First of all we observe that (2.5) implies

$$
\|\varphi_\rho\|\geq c\|X\|^{\beta 0\ \alpha\quad N} \nearrow 1 \qquad \text{for } \|X\|\nearrow \in\ ,
$$

if we take $\beta < N\ \ \ \ \alpha$. On the other hand, close to the possible singularity P_ρ, we have $\varphi_\rho > 1$. In fact, if $X\ \ /\ \ B_2^0\ )P_\rho+$, then $X^\rho\ \ /\ \ B_2^0$, and then $v)X^\rho+\sim \varepsilon$. Since $v)X+\geq \mu\|X\|^{\alpha\quad N} \geq \mu\rho^{\alpha\quad N}$, we get

$$
\varphi_\rho)X+\sim \|Z\|^\beta)\varepsilon\ \ \ \ \mu\|\rho\|^{\alpha\quad N}+> 1 \quad \text{in } B_2^0\ )P_\rho+,
$$

provided ρ is large enough. Therefore the infimum in (4.14) is achieved in a point of regularity of φ_ρ. As to the interior points, the above choice of β gives that equation (2.9) does not allow for interior minima to exist. Finally, the fact that $\varphi_\rho \mathrm{T} 1$ on T_ρ, leads to the only possibility for the infimum to be achieved, namely on the part of the boundary $\Phi_\rho \wedge \}y \mathrm{T} 1\langle$. Let then $)x_1, 1+/ \Phi_\rho \wedge \}y \mathrm{T} 1\langle$ be such that $\varphi_\rho)x_1, 1+\mathrm{T} \quad \delta$.

We claim that the boundary condition in (2.6) implies

$$\frac{\partial \varphi_\rho}{\partial \nu^\alpha})x_1+> 1, \tag{2.11}$$

which will give the desired contradiction. It is at this point where the condition $\alpha \sim 2$ enters.

By Leibniz's rule, we have

$$\frac{\partial \varphi_\rho}{\partial \nu^\alpha})x_1+\mathrm{T} \ \|)x_1, 2+\|^\beta \frac{\partial \psi_\rho}{\partial \nu^\alpha})x_1+0 \ \ \psi_\rho)x_1, 1+\frac{\partial \|Z\|^\beta}{\partial \nu^\alpha})x_1+.$$

The first term is bounded below, since by using (2.6), (2.5), and the Mean Value Theorem, we get

$$\begin{aligned} \frac{\partial \psi_\rho}{\partial \nu^\alpha})x_1+ \ \ \mathrm{T} \ \|x_1^\rho\| \ ^\gamma v^p)x_1^\rho, 1+ \ \ \|x_1\| \ ^\gamma v^p)x_1, 1+\sim \|x_1\| \ ^\gamma)v^p)x_1^\rho, 1+ \ \ v^p)x_1, 1++ \\ \sim p\|x_1\| \ ^\gamma v^p \ \ ^2)x_1, 1+\psi_\rho)x_1, 1+, \end{aligned} \tag{2.12}$$

and thus

$$\|)x_1, 2+\|^\beta \frac{\partial \psi_\rho}{\partial \nu^\alpha})x_1+\sim \quad p\delta\|x_1\| \ ^\gamma \ )p \ \ 2+)N \ \ \alpha+\sim \quad c\rho \ \ ^3.$$

As to the second term,

$$\frac{\partial \|Z\|^\beta}{\partial \nu^\alpha})x_1+\mathrm{T} \begin{cases} 1 & \text{if } \alpha < 2, \\ \quad \beta\|)x_1, 2+\|^\beta \ \ ^3 & \text{if } \alpha \mathrm{T} 2, \\ \in & \text{if } \alpha > 2. \end{cases}$$

We conclude in our case $\alpha > 2$ that $\frac{\partial \varphi_\rho}{\partial \nu^\alpha})x_1+\mathrm{T} \ 0 \in$. In the case $\alpha \ \mathrm{T} \ 2$ a sharp control of the above terms gives (2.11); this is done in [54]. In the case $\alpha < 2$ the condition (2.11) is not necessarily true. □

The moving planes method begins with a plane in which we find some kind of symmetry and then we see how far this plane can be moved keeping that symmetry. The above lemma, instrumental in unbounded domains, provides a "starting plane". The following lemma establishes that we can move that plane up to the origin.

Lemma 2.2.3. *Let ρ_1 be defined as*

$$\rho_1 \mathrm{T} \ inf\}\rho > 1 = \varphi_\mu \sim 1 \ in \ \widetilde{\Phi_\mu} \ for \ all \ \rho < \mu < \in \langle. \tag{2.13}$$

Then $\rho_1 \mathrm{T} 1$.

Proof. By Lemma 2.2.2 ρ_1 is finite. Suppose that $\rho_1 > 1$. By continuity we have φ_{ρ_0} T $\|Z\|^\beta \psi_{\rho_0} \sim 1$ in Φ_{ρ_0}. Since $\gamma > 1$ and $\rho_1 > 1$ we have by the boundary condition that $\psi_{\rho_0} \subseteq 1$ in Φ_{ρ_0}. Also, by (2.12), $\frac{\partial \psi_{\rho_0}}{\partial \nu^\alpha} \sim 1$ on $\}y$ T $1\langle \wedge \overline{\Phi_{\rho_0}}$. Clearly $L_\alpha)\psi_{\rho_0}+$T 1 in $\mathbb{R}_0^{N0\ 2}$ and in particular in the set R_1 T $\}\|X \quad P_{\rho_0}\|$ T $\|\rho_1\|/3,\ y \sim 1\langle$. Therefore, by Proposition 4.10 of [26] we have $\psi_{\rho_0} > 1$ in R_1. Let δ T $\underset{R_0}{\log} \psi_{\rho_0} > 1$. The function ψ_{ρ_0} may have a singularity at P_{ρ_0}, so we construct the following auxiliary function. Let h_ε be the solution to the problem

$$\begin{cases} L_\alpha)h_\varepsilon+)X+\text{T } 1, & \varepsilon < \|X \quad P_{\rho_0}\| < \|\rho_1\|/3,\ y > 1, \\ h_\varepsilon)X+\text{T } \delta, & \|X \quad P_{\rho_0}\| \text{T } \|\rho_1\|/3,\ y \sim 1, \\ h_\varepsilon)X+\text{T } 1, & \|X \quad P_{\rho_0}\| \text{T } \varepsilon,\ y \sim 1, \\ \dfrac{\partial h_\varepsilon}{\partial \nu^\alpha})X+\text{T } 1, & \varepsilon < \|X \quad P_{\rho_0}\| < \|\rho_1\|/3,\ y \text{ T } 1. \end{cases} \tag{2.14}$$

Then Lemma 4.11 of [26] implies

$$\psi_{\rho_0} \sim h_\varepsilon \qquad \text{in } \varepsilon \geq \|X \quad P_{\rho_0}\| \geq \|\rho_1\|/3,\ y \sim 1. \tag{2.15}$$

Letting $\varepsilon \nearrow 1^0$ we have $\text{ılm}_{\ \varepsilon\prime \ \ 1+} h_\varepsilon \subseteq \delta$ by the uniqueness of the α-harmonic extension. Therefore

$$\psi_{\rho_0} \sim \delta \qquad \text{in } 1 < \|X \quad P_{\rho_0}\| \geq \|\rho_1\|/3,\ y \sim 1. \tag{2.16}$$

Since $\varphi_{\rho_0} \sim \psi_{\rho_0}$ in Φ_{ρ_0}, we have

$$\underset{\rho\prime \quad \rho_0}{\text{ılm}} \ \underset{R_0}{\log} \varphi_\rho \sim \underset{R_0}{\log} \varphi_{\rho_0} \sim \delta. \tag{2.17}$$

Being ρ_1 the infimum, there exists a sequence $\rho_k \searrow \rho_1$ such that

$$\underset{\Omega_{\rho_k}}{\log} \varphi_{\rho_k} < 1. \tag{2.18}$$

Clearly $\text{ılm}_{\ \|X\|\prime \ \ \in} \varphi_{\rho_k}$ T 1. Recalling (2.17) the infimum in (2.18) must be attained at some finite point $X^k \ / \ \overline{\Phi_{\rho_k}} \nabla B_{\|\rho_0\|/3)} P_{\rho_0}+$with $\|\rho_k \quad \rho_1\|$small enough. On the other hand $X^k \ \imath/ \ T_{\rho_k}$ since $\varphi_{\rho_k} \subseteq 1$ in T_{ρ_k}. Therefore X^k must belong to the set

$$\}X \ / \ \mathbb{R}^{N0\ 2} = y \text{ T } 1, x_N > 1,\ \|X \quad P_{\rho_0}\|^\beta \sim \|\rho_1\|^\beta/8\langle. \tag{2.19}$$

Reasoning like in Lemma 2.2.2 this leads to the desired contradiction. □

Now we can deal with the proof of the main theorem in this subsection.

Proof of Theorem 2.2.1. Let w be any solution to Problem (2.2) and consider its fractional Kelvin transform v T $K_\alpha)w+$ By Lemma 2.2.3 we have $v)x_2, ..., x_N, y+\sim$

$v)x_2, ..., \quad x_N, y+$for $x_N > 1$. The same argument fits for negative x_N giving the reverse inequality. Therefore $v)X+$is symmetric with respect to the x_N-axis. Obviously we can apply this argument in every direction perpendicular to y-axis. Hence $v)X+$is a two-variables function and so it is $w)X+$. Indeed,

$$w)X+\mathrm{T}\ \phi)\|x\|, y+ \tag{2.20}$$

for some function ϕ. Hence setting $\|x\|$ as the origin w is independent of $)x_2, ..., x_N+$ and therefore $w)X+\mathrm{T}\ w)y+$.

To end the proof we are reduced to consider the problem in one dimension.

$$\begin{cases})y^{2\quad\alpha}w^{\infty}+^{\infty}\mathrm{T}\ 1, & \text{for } y > 1, \\ \lim\limits_{y/\ \ 1+} y^{2\quad\alpha}w^{\infty})y+\mathrm{T}\ w^{p})1+. \end{cases} \tag{2.21}$$

The solutions of this problem are of the form $w)y+\mathrm{T}\ c \quad \frac{c^p}{\alpha}y^{\alpha}$ with $c \sim 1$, which implies that the only nonnegative solution is $w \subseteq 1$. □

2.2.2. A problem in a quarter-space

Let us denote the quarter space as

$$\mathbb{R}^{N0\ 2}_{0\ 0}\ \mathrm{T}\ \}X\ \mathrm{T}\ )x^{\infty}, x_N, y+|x^{\infty}/\ \mathbb{R}^{N\quad 2},\ x_N > 1,\ y > 1\langle,$$

.

Theorem 2.2.4. *Let $2 \geq \alpha < 3$. Then the problem in the first quarter*

$$\begin{cases} L_\alpha w \quad \mathrm{T} \quad 1, & \mathbb{R}^{N0\ 2}_{0\ 0}, \\ \dfrac{\partial w}{\partial \nu^\alpha})x^{\infty}, x_N+ \quad \mathrm{T} \quad w^p)x^{\infty}, x_N, 1+, & \\ w)x^{\infty}, 1, y+ \quad \mathrm{T} \quad 1, & \end{cases} \tag{2.22}$$

has no positive bounded solution in $C^{\alpha 0\ \gamma})\mathbb{R}^{N0\ 2}_{0\ 0}+\wedge C)\overline{\mathbb{R}^{N0\ 2}_{0\ 0}}+$with $\gamma > 1$ provided $2 < p < \frac{N0\ \alpha}{N\quad \alpha}$.

Theorem 2.2.4 is proved in the case $\alpha\ \mathrm{T}\ 2$ in [28]. We begin with a generalization of Proposition 6.1 of [38]. Let $N\ \mathrm{T}\ 3$.

Lemma 2.2.5. *Suppose w is a solution of the following problem*

$$\begin{cases} L_\alpha w \sim 1, \quad w \sim 1 & \text{in } \mathbb{R}^3_0\ , \\ \dfrac{\partial w}{\partial \nu^\alpha} \sim 1, & \text{for } y\ \mathrm{T}\ 1\,. \end{cases} \tag{2.23}$$

Then w is a constant.

Proof. Let X_1 T $)x_1, y_1+/ \overline{\mathbb{R}^3_0}$. Given ε, δ / $)1, 2+$we define the function

$$\psi)X+\text{T}\ \varepsilon w)X_1+\text{rpi}\ \Big)\frac{\|X \quad X_1\|^\beta}{\delta^3}\Big\{\ 0\ \ C_\delta, \tag{2.24}$$

where

$$C_\delta\ \text{T}\ \underset{S^+_\delta)X_0+}{\text{n d}\tilde{\ }})w)X_1+\quad w)X++,$$

where $\overline{S^0_\delta)X_1+}$T $\}\|X \quad X_1\|$ T δ, $y \sim 1\langle$. Its clear that $\psi)X+\subseteq C_\delta$ on $\overline{S^0_\delta)X_1+}$and taking δ small enough we have

$$\psi)X+\sim w)X_1+\sim w)X_1+\quad w)X+ \qquad \text{on}\ \ \overline{S^0_{e^{1/\varepsilon}})X_1+}. \tag{2.25}$$

A direct calculation shows that, if α / $)2, 3+$, then

$$\begin{cases} L_\alpha\psi \geq 1, & \text{in } \mathbb{R}^3_0, \\ \dfrac{\partial\psi}{\partial\nu^\alpha}\ \text{T}\ \in, & \text{for } y \text{ T } 1. \end{cases}$$

Thus by the maximum principle

$$\psi)X+\sim w)X_1+\quad w)X+ \qquad \text{for}\ \ X\ /\ \overline{\mathbb{R}^3_0}, \delta < \|X \quad X_1\| < e^{2/\varepsilon}$$

Letting $\varepsilon \nearrow 1$ and then $\delta \nearrow 1$, we have $w)X_1+\quad w)X+\geq 1$ for any X_1, X / $\overline{\mathbb{R}^3_0}$. □

Lemma 2.2.6. *Let $p \sim 1$ and let C be a positive constant. Then there is no solution to the problem*

$$\begin{cases} L_\alpha w \text{ T } 1, \quad 1 < w \geq C, & \text{in } \mathbb{R}^3_{0\ 0} \text{ T } \}x > 1,\ y > 1\langle, \\ \dfrac{\partial w}{\partial\nu^\alpha}\ \text{T}\ w^p, & \text{on } \}x > 1, y \text{ T } 1\langle, \\ w \text{ T } 1, & \text{on } \}x \text{ T } 1, y \sim 1\langle. \end{cases} \tag{2.26}$$

Proof. First, we show that $w)x, 1+\nearrow 1$ as $x \nearrow \in$. Suppose by contradiction that there exists a sequence $\eta_m \nearrow \in$ as $m \nearrow \in$ and such that $w)\eta_m, 1+\nearrow K > 1$. Let us denote $w_m)x, y+$T $w)x\ 0\ \ \eta_m, y+$. Its clear that it holds

$$\begin{cases} L_\alpha w_m \text{ T } 1, \quad 1 < w_m \geq C, & \text{in } R_m \text{ T } \}x > \eta_m,\ y > 1\langle, \\ \dfrac{\partial w_m}{\partial\nu^\alpha}\ \text{T}\ w^p, & \text{on } \}x > \eta_m, y \text{ T } 1\langle, \\ w_m \text{ T } 1, & \text{on } \}x \text{ T } \eta_m, y \sim 1\langle. \end{cases} \tag{2.27}$$

Moreover $w_m)1, 1+\nearrow K$. So that taking a subsequence of w_m if necessary we have $w_m \nearrow \widetilde{w}$ with

$$\begin{cases} L_\alpha \widetilde{w}\ \mathrm{T}\ 1, \quad 1 \geq \widetilde{w} \geq C, & \text{in } \mathbb{R}^3_0\,, \\ \dfrac{\partial \widetilde{w}}{\partial \nu^\alpha}\ \mathrm{T}\ \widetilde{w}^p \sim 1, & \text{for } y\ \mathrm{T}\ 1. \end{cases} \tag{2.28}$$

Since $\widetilde{w})1, 1+\mathrm{T}\ K$, Lemma 2.2.5 implies $\widetilde{w} \subseteq K$ but by the boundary condition we have that

$$\frac{\partial \widetilde{w}}{\partial \nu^\alpha})1, 1+\mathrm{T}\ \widetilde{w}^p)1, 1+\mathrm{T}\ K^p > 1,$$

which leads to a contradiction. Therefore $w)x, 1+\nearrow 1$ as $x \nearrow \in$.

Following [26] we define the function

$$\Psi)x+\mathrm{T}\ \frac{2}{3}\bigcap_1^{\in} y^{2\quad\alpha})\|w_x)x, y+\|^\beta \quad \|w_y)x, y+\|^\beta+dy,$$

see also [28] for the case $\alpha\ \mathrm{T}\ 2$. Differentiating inside the integral, we have

$$\frac{2}{3}\bigcap_1^{\in} \frac{\partial}{\partial x}]y^{2\quad\alpha})\|w_x\|^\beta \quad \|w_y\|^\beta+\hat{}dy\ \mathrm{T}\ \bigcap_1^{\in} y^{2\quad\alpha})w_{xx}w_x \quad w_y w_{xy}+dy.$$

We want to see that this integral converges. By Lemma 4.3 of [26] we know that there exists some $\beta\ /\ )1, 2+$such that $w\ /\ C^{3,\beta}$. Moreover by Proposition 4.6 of [26]

$$\begin{aligned} &\bigcap_1^{\in} y^{2\quad\alpha})\|w_{xx}w_x\|0\ \ \|w_y w_{xy}\|+dy \geq \\ &M_2)\bigcap_1^{2} y^{2\quad\alpha})\|w_x\|0\ \ \|w_y\|+dy\ 0\ \bigcap_2^{\in} y^{2\quad\alpha})\|w_x\|0\ \ \|w_y\|+dy+\geq \\ &M_3)M_4\ 0\ \bigcap_2^{\in} \frac{y^{2\quad\alpha}}{y\ 0\ \ 2}\,dy+< \in\ , \end{aligned}$$

for some constants $M_2, M_3, M_4 > 1$. Notice that the last integral is convergent provided $2 < \alpha < 3$. We recall that in the case $\alpha\ \mathrm{T}\ 2$, a sharper estimate is used in [28]. Now let $G)w+\mathrm{T}\ \sum^{w} f)s+ds$. By dominated convergence, and since $\|\ \ w)x, y+\| \nearrow 1$ as $y \nearrow \in$, integrating by parts we have

$$\begin{aligned} &]\Psi)x+0\ \ G)w)x, 1++\hat{}_x\ \mathrm{T}\ \bigcap_1^{\in} y^{2\quad\alpha}]w_{xx}w_x \quad w_y w_{xy}\hat{})x, y+dy\ 0\ \ ]f)w+w_x\hat{})x, 1+ \\ &\mathrm{T}\ \lim_{y'\ \ 1}]y^{2\quad\alpha}w_y w_x\ 0\ \ f)w+w_x\hat{})x, y+\mathrm{T}\ \lim_{y'\ \ 1}]y^{2\quad\alpha}w_y w_x \quad y^{2\quad\alpha}w_y w_x\hat{})x, y+\mathrm{T}\ 1. \end{aligned}$$

Therefore $\Psi)x+0\ \ G)w)x, 1++$is constant. The rest of the proof is exactly the same as in [28]. Using that $w)x, 1+\nearrow 1$ as $x \nearrow \in$ and Lemma 5.1 of [26] we obtain

$$\Psi)x+0\ \ G)w)x, 1++\subseteq 1.$$

Since $w \subseteq 1$ in $\}x \; \mathrm{T} \; 1, y \sim 1\langle$ it follows that

$$1 \; \mathrm{T} \; 3\Psi)1+\mathrm{T} \bigcap_{1}^{\in} \|w_x\|^{\beta})1, y+dy$$

which implies $w_x \; \mathrm{T} \; 1$ on $\}x \; \mathrm{T} \; 1, y > \varepsilon\langle$ for every $\varepsilon > 1$. Since L_α is a non-degenerated elliptic operator in $\}x \; \mathrm{T} \; 1, y > \varepsilon\langle$ by the Hopf's Lemma this leads to a contradiction. □

With these two results a standard argument completes the proof.

Proof of Theorem 2.2.4. By an analogous argument to the proof of Theorem 2.2.4 for the $)x_2, ..., x_{N \;\; 2}+$variables (with the analogous Lemma 2.2.2 and Lemma 2.2.3), it is easy to see that any positive solution of (2.22) depends only on two variables, x_N and y. Therefore applying Proposition 2.2.6 the proof is complete. □

2.3. The linear problem

We now use the extension problem (1.17) to reformulate the nonlocal problems in a local way. Let g be a regular function and consider the following problems, the nonlocal problem

$$\left\} \begin{array}{rcll}) \;\; \Lambda+^{\alpha/3}u & \mathrm{T} & g)x+ & \text{in } \;´ , \\ u & \mathrm{T} & 1 & \text{on } \partial´ , \end{array} \right. \tag{2.29}$$

and the corresponding local one

$$\left\{ \begin{array}{rcll} \text{f lx})y^{2 \;\; \alpha} \;\; w+ & \mathrm{T} & 1 & \text{in } \mathcal{F} , \\ w & \mathrm{T} & 1 & \text{on } \partial_L \mathcal{F} , \\ \dfrac{\partial w}{\partial \nu^\alpha})x, y+ & \mathrm{T} & g)x+ & \text{on } \;´ . \end{array} \right. \tag{2.30}$$

We want to define the concept of solution to (2.29), which is done in terms of the solution to problem (2.30).

Definition 2.3.1. *We say that $w \;/\; X_1^\alpha)\mathcal{F}+$is an energy solution to problem* (2.30)*, if for every function $\varphi \;/\; X_1^\alpha)\mathcal{F}+$it holds*

$$\kappa_\alpha \bigcap_{\mathcal{F}_\Omega} y^{2 \;\; \alpha}\rangle \;\; w)x, y+, \;\; \varphi)x, y+ \, dxdy \; \mathrm{T} \bigcap_{\_} g)x+\varphi)x, 1+dx. \tag{2.31}$$

In fact more general test functions can be used in the above formula, whenever the integrals make sense. A supersolution (subsolution) is a function that verifies (2.31) with equality replaced by $\sim$ ($\geq$) for every nonnegative test function.

Definition 2.3.2. *We say that $u \;/\; H_1^{\alpha/3})´+$is an energy solution to problem* (2.29) *if it is the trace on $´$ of a function w which is an energy solution to problem* (2.30).

A solution exists for instance for every $g\ /\ H\ \ ^{\alpha/3})'\ +$, see [33]. In order to deal with problem (2.30) we will assume, without loss of generality, κ_α T 2, by changing the function g.

In [26] this linear problem is also mentioned. There some results are obtained using the theory of degenerate elliptic equations developed in [46], in particular a regularity result for bounded solutions to this problem is obtained in [26]. We prove here that the solutions are in fact bounded if g satisfies a minimal integrability condition.

Theorem 2.3.3. *Let w be a solution to problem* (2.30)*. If $g\ /\ L^r)'\ +$, $r > \frac{N}{\alpha}$, then $w\ /\ L^{\in}\ )\mathcal{F}\ +$.*

Proof. The proof follows from the well-known Moser's iterative technique, that we take from [52, Theorem 8.15], and uses the trace inequality (1.30). Without loss of generality we may assume $w \sim 1$, and this simplifies notation. The general case is obtained in a similar way.

We define for $\beta \sim 2$ and $K \sim k$ (k to be chosen later) a $C^2)]k, \in\ ++$function H, as follows:

$$H)z + \mathrm{T} \left\} \begin{array}{ll} z^\beta \quad k^\beta, & z\ /\ ]k, K\hat{\ }, \\ \beta K^{\beta\ \ 2})z \quad K+0\ )z^\beta \quad K^\beta +, & z > K. \end{array} \right.$$

Let us also define v T $w\ 0\ \ k$, ν T $[\ \ \mathrm{s})v+$, and choose as test function φ,

$$\varphi\ \mathrm{T}\ G)v + \mathrm{T} \bigcap_k^v \|H^{\propto}\!)s + \|^\beta\, ds, \qquad \varphi\ \mathrm{T}\ \|H^{\propto}\!)v + \|^\beta \quad v.$$

Note that since $\|H^{\propto}\!)v + \| \geq \beta K^{\beta\ \ 2}$ then $\varphi\ /\ X_1^\alpha)\mathcal{F}\ +$. Replacing this test function into the definition of energy solution we obtain on one hand:

$$\begin{aligned} \bigcap_{\mathcal{F}_\Omega} y^{2\ \ \alpha}\rangle \quad w, \quad \varphi^{|}\, dxdy \quad &\mathrm{T} \quad \bigcap_{\mathcal{F}_\Omega} y^{2\ \ \alpha}\Big(\quad v\|^\beta \|H^{\propto}\!)v + \|^\beta\, dxdy \\ &\mathrm{T} \quad \bigcap_{\mathcal{F}_\Omega} y^{2\ \ \alpha}\Big(\quad H)v + \|^\beta\, dxdy \\ &\sim \quad C\backslash H)\nu + \backslash^3_{\frac{2N}{N-\alpha}}, \end{aligned} \tag{2.32}$$

where the last inequality follows by (1.30).On the other hand, since $H^{\propto}$is increasing we have

$$G)v + \geq v\|H^{\propto}\!)v + \|^\beta \ \mathrm{T}\ vG^{\propto}\!)v +$$

Thus

$$\begin{aligned} \bigcap_{-} g)x + \varphi)x, 1 + dx\ \mathrm{T} \bigcap_{-} g)x + G)\nu + dx &\geq \bigcap_{-} g)x + \nu G^{\propto}\!)\nu + dx \\ &\geq \bigcap_{-} g)x + \nu\|H^{\propto}\!)\nu + \|^\beta\, dx \\ &\geq \backslash g\backslash_r \backslash \nu^{\frac{1}{2}} H^{\propto}\!)\nu + \backslash^3_{\frac{2r}{r-1}}. \end{aligned} \tag{2.33}$$

Inequality (2.32) together with (2.33), leads to

$$\backslash H)\nu+\backslash_{\frac{2N}{N-\alpha}} \geq \Big)\frac{2}{C}\backslash g\backslash_r \Big\{^{2/3} \quad \backslash \nu^{\frac{1}{2}} H^{\infty}\!)\nu+\backslash_{\frac{2r}{r-1}}, \tag{2.34}$$

by choosing k T 1 and letting $K \nearrow \in$ in the definition of H, the inequality (2.34) becomes

$$\backslash \nu^{\beta}\backslash_{\frac{2N}{N-\alpha}} \geq C\beta \backslash \nu^{\beta \quad \frac{1}{2}}\backslash_{\frac{2r}{r-1}}.$$

Hence for all $\beta \sim 2$ the inclusion $\nu \ / \ L^{\frac{2r(\beta-\frac{1}{2})}{r-1}})\,'\,+$implies the stronger inclusion $\nu \ / \ L^{\frac{2N\beta}{N-\alpha}})\,'\,+$, since $\frac{3N\beta}{N \quad \alpha} > \frac{3r)\beta \quad \frac{1}{2}+}{r \quad 2}$ provided $r > \frac{N}{\alpha}$. The result follows now, as in [52], by an iteration argument, starting with β T $\frac{N)r \quad 2+}{r)N \quad \alpha+}0 \ \frac{2}{3} > 2$ and $\nu \ / \ L^{\frac{2N}{N-\alpha}})\,'\,+$. This gives $\nu \ / \ L^{\in})\,'\,+$, and then $w \ / \ L^{\in})\mathcal{F}\,+$. In fact we get the estimate

$$\backslash w\backslash_{\in} \geq c)\backslash w\backslash_{X^{\alpha}} 0 \ \backslash g\backslash_r+.$$

□

Corollary 2.3.4. *Let w be a solution to problem* (2.30). *If $g \ / \ L^{\in})\,'\,+$, then $w \ / \ C^{\gamma})\overline{\mathcal{F}}\,+$for some $\gamma \ / \ )1,2+$.*

Proof. Using Theorem 2.3.3, the result follows directly from [26, Lemma 4.4], where it is proved that any bounded solution to problem (2.30) with a bounded g is C^{γ}. □

2.4. The nonlinear nonlocal problem

As we have said, we will focus on the particular nonlinearity

$$f)s+\text{T} \ f_{\lambda})s+\text{T} \ \lambda s^{q} \ 0 \ \ s^{p}. \tag{2.35}$$

However many auxiliary results will be proved for more general reactions f satisfying the growth condition

$$1 \geq f)s+\geq c)2 \ 0 \ \ \|s\|^{p}+, \qquad \text{for some } p > 1. \tag{2.36}$$

Remark 2.4.1. *In order to simplify the notation, the results on the coefficient λ for the local problem* (3.2)–(2.35) *in this section are translated into problem $)P_{\lambda}+$with λ multiplied by $\kappa_{\alpha}^{p)q \quad 2+ \ 2}$.*

We consider now the functional

$$J)w+\text{T} \ \frac{2}{3}\bigcap_{\mathcal{F}_{\Omega}} y^{2 \quad \alpha}\| \quad w\|^{\beta}\,dxdy \quad \bigcap_{-} F)w+dx,$$

where $F)s+\mathrm{T} \sum^{s} f)\tau+d\tau$. For simplicity of notation, we define $f)s+\mathrm{T}\ 1$ for $s \geq 1$. Recall that the trace satisfies $w\ /\ L^r)'\ +$, (again this means $[\ \mathrm{s})w+/\ L^r)'\ +$), for every $2 \geq r \geq \frac{3N}{N\ \ \alpha}$ if $N > \alpha$, $2 < r \geq \in$ if $N \geq \alpha$. In particular if $2 < p \geq \frac{N0\ \alpha}{N\ \ \alpha}$, and f verifies (2.36) then $F)w+/\ L^2)'\ +$, and the functional is well defined and bounded from below.

We consider also the minimization problem

$$\mathcal{L}\ \mathrm{T}\ \log\Big\}\bigcap_{\mathcal{F}_\Omega} y^2\ \ {}^{\alpha}\|\ \ w\|^{\beta}\,dxdy\ ;\ w\ /\ X_1^{\alpha})\mathcal{F}\ +,\bigcap_{-} F)w+dx\ \mathrm{T}\ 2\surd,$$

for which, by classical variational techniques, one has that below the critical exponent the infimum $\mathcal{L}$ is achieved. This gives in a standard way a nonnegative solution. Later on we will see that this infimum is positive provided $\lambda > 1$ is small enough. On the contrary, for λ large enough the infimum is the trivial solution.

We now establish two preliminary results. The first one is a classical procedure of sub- and supersolutions to obtain a solution. We omit its proof.

Lemma 2.4.1. *Assume there exist a subsolution w_2 and a supersolution w_3 to problem* (3.2) *verifying $w_2 \geq w_3$. Then there also exists a solution w satisfying $w_2 \geq w \geq w_3$ in $\mathcal{F}$.*

The second one is a comparison result for concave nonlinearities. The proof follows the lines of the corresponding one for the Laplacian performed in [21].

Lemma 2.4.2. *Assume the function $f)t+/t$ is decreasing for $t > 1$ and consider $w_2, w_3\ /\ X_1^{\alpha})\mathcal{F}\ +$positive subsolution and supersolution, respectively, to problem* (3.2). *Then $w_2 \geq w_3$ in $\overline{\mathcal{F}}$.*

Proof. By definition we have, for the nonnegative test functions φ_2 and φ_3 to be chosen in an appropriate way,

$$\bigcap_{\mathcal{F}_\Omega} y^2\ \ {}^{\alpha}\rangle\ \ w_2,\ \ \ \varphi_2|\ dxdy \geq \bigcap_{-} f)w_2+\varphi_2\,dx,$$
$$\bigcap_{\mathcal{F}_\Omega} y^2\ \ {}^{\alpha}\rangle\ \ w_3,\ \ \ \varphi_3|\ dxdy \sim \bigcap_{-} f)w_3+\varphi_3\,dx.$$

Now let $\theta)t+$be a smooth nondecreasing function such that $\theta)t+\mathrm{T}\ 1$ for $t \geq 1$, $\theta)t+\mathrm{T}\ 2$ for $t \sim 2$, and set $\theta_\varepsilon)t+\mathrm{T}\ \theta)\frac{t}{\varepsilon}+$. If we put, in the above inequalities

$$\varphi_2\ \mathrm{T}\ w_3\,\theta_\varepsilon)w_2\ \ \ \ w_3+,\qquad \varphi_3\ \mathrm{T}\ w_2\,\theta_\varepsilon)w_2\ \ \ \ w_3+,$$

we get

$$I_2 \sim \bigcap_{-} w_2w_3\Big)\frac{f)w_3+}{w_3}\ \ \ \frac{f)w_2+}{w_2}\Big(\theta_\varepsilon)w_2\ \ \ \ w_3+dx,$$

where

$$I_2 \;;\mathrm{T} \bigcap_{\mathcal{F}_\Omega} y^2 \quad ^{\alpha}\rangle w_2 \quad w_3 \quad w_3 \quad w_2, \quad)w_2 \quad w_3 + \theta_\varepsilon^\infty) w_2 \quad w_3 + dxdy.$$

Now we estimate I_2 as follows:

$$\begin{aligned} I_2 \quad &\geq \bigcap_{\mathcal{F}_\Omega} y^2 \quad ^{\alpha}\rangle \quad w_2,)w_2 \quad w_3 + \;)w_2 \quad w_3 + \theta_\varepsilon^\infty) w_2 \quad w_3 + dxdy \\ \mathrm{T} &\bigcap_{\mathcal{F}_\Omega} y^2 \quad ^{\alpha}\rangle \quad w_2, \quad \gamma_\varepsilon)w_2 \quad w_3 + dxdy \end{aligned}$$

where $\gamma_\epsilon^\infty)t + \mathrm{T} \; t\theta_\varepsilon^\infty)t +$ Therefore, since $1 \geq \gamma_\varepsilon \geq \varepsilon$, we have

$$I_2 \geq \bigcap_{-} f)w_2 + \gamma_\varepsilon)w_2 \quad w_3 + dx \geq c\varepsilon.$$

We end as in [4]. Letting ε tend to zero, we obtain

$$\bigcap_{-\,\wedge\}w_1 > w_2|} w_2 w_3 \Big) \frac{f)w_3+}{w_3} \quad \frac{f)w_2+}{w_2} \Big\{ \; dx \geq 1,$$

which together with the hypothesis on f gives $w_2 \geq w_3$ in $'$. Comparison in $\mathcal{F}$ follows easily by the maximum principle. □

Now we show that the solutions to problem (3.2)–(2.36) are bounded and Hölder continuous. Later on, in Section 2.4.3, we will obtain a uniform $L^\in$-estimate in the case where f is given by (2.35) and the convex power is subcritical.

Proposition 2.4.3. *Let f satisfy* (2.36) *with* $2 < p < \frac{N 0 \alpha}{N \quad \alpha}$, *and let $w \;/\; X_1^\alpha)\mathcal{F} +$ be an energy solution to problem* (3.2). *Then $w \;/\; L^\in)\mathcal{F} + \wedge C^\gamma)\overline{\mathcal{F}} +$ for some $1 < \gamma < 2$.*

Proof. The proof follows closely the technique of [22]. As in the proof of Theorem 2.3.3, we assume $w \sim 1$. We consider, formally, the test function $\varphi \;\mathrm{T}\; w^{\beta \quad p}$, for some $\beta > p \; 0 \;\; 2$. The justification of the following calculations can be made substituting φ by some approximated truncature. We therefore proceed with the formal analysis. We get, using the trace immersion, the inequality

$$\Big) \bigcap_{-} w^{\frac{(\beta - p + 1)N}{N - \alpha}} \Big\{^{\frac{N-\alpha}{N}} \geq C)\beta, \alpha, N, ' + \bigcap_{-} w^\beta.$$

This estimate allows us to obtain the following iterative process

$$\backslash w \backslash_{\beta_{j+1}} \geq C \backslash w \backslash_{\beta_j}^{\frac{\beta_j}{\beta_j - p + 1}},$$

with $\beta_{j 0 2} \;\mathrm{T}\; \frac{N}{N \quad \alpha})\beta_j \; 0 \;\; 2 \quad p +$ To have $\beta_{j 0 2} > \beta_j$ we need $\beta_j > \frac{)p \quad 2 + N}{\alpha}$. Since $w \;/\; L^{3^*_\alpha})' +$, starting with $\beta_1 \;\mathrm{T}\; \frac{3N}{N \quad \alpha}$, we get the above restriction provided $2 < p < \frac{N 0 \alpha}{N \quad \alpha}$. It is clear that in a finite number of steps we get, for $g)x + \mathrm{T} \; f)w)x, 1 + +$, the regularity $g \;/\; L^r$ for some $r > \frac{N}{\alpha}$. As a consequence, we obtain the conclusion applying Theorem 2.3.3 and Corollary 2.3.4. □

2.4.1. A nonexistence result

The following result relies on the use of a classical Pohozaev type multiplier.

Proposition 2.4.4. *Assume f is a continuous function with primitive F, and w is a energy solution to problem* (3.2). *Then the following Pohozaev-type identity holds*

$$\frac{2}{3}\bigcap_{\partial_L \mathcal{F}_\Omega} y^2 \;\; {}^{\alpha}\rangle x, \nu^{|} \| \;\; w\|^{\beta}\, d\sigma \quad N\bigcap_{-} F)w{+}dx \; 0 \; \frac{N \quad \alpha}{3}\bigcap_{-} wf)w{+}dx \;\mathrm{T}\; 1,$$

where ν is the (exterior) normal vector to ∂'.

Proof. Let us suppose $w \; / \; C^3)\mathcal{F}\!- +$and assume the following identity

$$\rangle\rangle x, y{+}\!\!, \quad w^{|} \;\mathrm{f\,lx}) y^2 \;\; {}^{\alpha} \quad w{+}0 \;\; \mathrm{f\,lx}\Big] y^2 \;\; {}^{\alpha}\Big)\rangle\rangle x, y{+}\!\!, \quad w^{|} \quad w \quad \frac{2}{3})x, y{+}\!\!\| \;\; w\|^{\beta}\Big(\Big\{$$
$$0\,\Big)\frac{N \quad \alpha}{3}\Big(y^2 \;\; {}^{\alpha}\| \;\; w\|^{\beta} \;\mathrm{T}\; 1. \tag{2.37}$$

Since w is a solution of (3.2) it holds $\mathrm{f\,lx}) y^2 \;\; {}^{\alpha} \quad w{+}\mathrm{T}\; 1$. Integrating in $' \; * \;)1, R{+}$we have

$$\bigcap_{- *)1,R+} \mathrm{f\,lx}\Big] y^2 \;\; {}^{\alpha}\Big)\rangle\rangle x, y{+}\!\!, \quad w^{|} \quad w \quad \frac{2}{3})x, y{+}\!\!\| \;\; w\|^{\beta}\Big(\Big\{$$
$$0\,\Big)\frac{N \quad \alpha}{3}\Big(\bigcap_{- *)1,R+} y^2 \;\; {}^{\alpha}\| \;\; w\|^{\beta} \;\mathrm{T}\; 1.$$

By the Divergence Theorem

$$\bigcap_{\partial\}^- *)1,R+\!\!\!|} y^2 \;\; {}^{\alpha}\Big)\rangle\rangle x, y{+}\!\!, \quad w^{|} \quad w \quad \frac{2}{3})x, y{+}\!\!\| \;\; w\|^{\beta}\Big(\times\!\!\nu$$
$$0\,\Big)\frac{N \quad \alpha}{3}\Big(\bigcap_{- *)1,R+} y^2 \;\; {}^{\alpha}\| \;\; w\|^{\beta} \;\mathrm{T}\; 1.$$

Since $w \;\mathrm{T}\; 1$ in $\partial\mathcal{F}\!-$ and since

$$\partial\}' \; * \;)1, R{+}\!\!\langle \;\mathrm{T}\;)' \; * \; \}y \;\mathrm{T}\; 1\langle{+}\!\cap)' \; * \; \}y \;\mathrm{T}\; R\langle{+}\!\cap)\partial' \; * \;)1, R{+}\!\!+$$

we have

$$\frac{2}{3}\bigcap_{\partial^- *)1,R+} y^2 \;\; {}^{\alpha}\rangle x, \nu^{|} \| \;\; w\|^{\beta}\, d\sigma \; 0 \bigcap_{- * \}y[\;1|} \rangle x, \quad {}_x w^{|} \,\frac{\partial w}{\partial \nu^{\alpha}}$$
$$0 \bigcap_{- * \}y[\;R|} y^2 \;\; {}^{\alpha}\Big)\Big)\rangle x, \quad {}_x w^{|} \; 0 \;\; Rw_y\Big(w_y \quad \frac{R}{3}\| \;\; w\|^{\beta}\Big(\tag{2.38}$$
$$0\,\Big)\frac{N \quad \alpha}{3}\Big(\bigcap_{- *)1,R+} y^2 \;\; {}^{\alpha}\| \;\; w\|^{\beta} \;\mathrm{T}\; 1.$$

On one hand, integrating by parts

$$\bigcap_{^{-} * \}y[\ 1|} \rangle x, \quad {}_x w^{|} \frac{\partial w}{\partial \nu^{\alpha}} \mathrm{T} \bigcap_{^{-} * \}y[\ 1|} \rangle x, \quad {}_x w^{|} f)w+$$
$$\mathrm{T} \bigcap_{^{-} * \}y[\ 1|} \rangle x, \quad {}_x F)w+ \mathrm{T} \quad N \bigcap_{^{-} * \}y[\ 1|} F)w+.$$

On the other hand, the third term of (2.38) holds

$$\Big(\bigcap_{^{-} * \}y[\ R|} y^2 \ {}^{\alpha}\Big)\Big)\rangle x, \quad {}_x w^{|} \ 0 \ Rw_y \Big(w_y \quad \frac{R}{3} \| \ \ w\|^{\beta} \Big(\Big($$
$$\geq C)\mathrm{f\,ldn}\ )' +0 \ 2+\bigcap_{^{-} * \}y[\ R|} R^3 \ {}^{\alpha}\| \ \ w\|^{\beta},$$

for some positive constant C. If we assume

$$\lim_{R\prime\ \in} \log \bigcap_{^{-} * \}y[\ R|} R^3 \ {}^{\alpha}\| \ \ w\|^{\beta} \ \mathrm{T} \ c > 1$$

then, there exists a positive R_1 such that for all $R_2 \sim R_1$ we have

$$\bigcap_{R_0}^{R_1} \bigcap_{-} R^3 \ {}^{\alpha}\| \ \ w\|^{\beta} \, dxdR \sim c \infty \bigcap_{R_0}^{R_1} \frac{2}{R} \, dR, \nearrow \in \quad \text{cuando } R_2 \nearrow \in .$$

This implies $w \ {}_{|}/ \ X_1^{\alpha})' +$and therefore a contradiction. Hence, there exists a subsequence $R_m \nearrow \in$ such that

$$\lim_{m\prime\ \in} \bigcap_{^{-} * \}y[\ R_m|} y^2 \ {}^{\alpha}\Big)\Big)\rangle x, \quad {}_x w^{|} \ 0 \ R_m w_y \Big(w_y \quad \frac{R_m}{3} \| \ \ w\|^{\beta} \Big(\mathrm{T} \ 1.$$

Integrating again by parts,

$$\bigcap_{^{-} *)1,R+} y^2 \ {}^{\alpha}\| \ \ w\|^{\beta} \ \mathrm{T} \bigcap_{^{-} * \}y[\ 1|} w \frac{\partial w}{\partial \nu^{\alpha}} \ 0 \bigcap_{^{-} * \}y[\ R|} w w_y$$
$$\mathrm{T} \bigcap_{^{-} * \}y[\ 1|} wf)w+0 \bigcap_{^{-} * \}y[\ R|} R^2 \ {}^{\alpha} w w_y.$$

Reasoning as before we have a sequence $R_m \nearrow \in$ (extracting a subsequence and renaming if necessary) such that the second integral approaches to 0 as m approaches to $\in$. As a consequence, taking $R \ \mathrm{T} \ R_m \nearrow \in$ in (2.38) we have

$$\frac{2}{3} \bigcap_{\partial_L \mathcal{F}_\Omega} y^2 \ {}^{\alpha}\rangle x, \nu^{|} \| \ \ w\|^{\beta} \, d\sigma \quad N \bigcap_{-} F)w+dx \ 0 \ \frac{N \quad \alpha}{3} \bigcap_{-} wf)w+dx \ \mathrm{T} \ 1.$$

Finally, we prove identity (2.37). Computing we have

$$
\begin{aligned}
&\mathrm{f\,lx}\Big] y^2 \ \ ^{\alpha}\Big)\rangle)x, y+\!\!, \quad w^{|} \quad w \quad \frac{2}{3})x, y+\!\!\| \ \ w\|^{\beta}\Big(\Big\{ \mathrm{T} \\
&\mathrm{f\,lx}\Big) y^2 \ \ ^{\alpha}\rangle)x, y+\!\!, \quad w^{|} \quad w\Big(\quad \frac{2}{3}\mathrm{f\,lx}\Big) y^2 \ \ ^{\alpha})x, y+\!\!\| \ \ w\|^{\beta}\Big(\mathrm{T} \\
&\rangle)x, y+\!\!, \quad w^{|}\ \mathrm{f\,lx})y^2 \ \ ^{\alpha} \quad w+\!0 \quad \Big)\rangle)x, y+\!\!, \quad w^{|}\Big(y^2 \ \ ^{\alpha} \quad w \\
&\frac{2}{3}\Big] \quad)y^2 \ \ ^{\alpha})x, y+\!\!\| \ \ w\|^{\beta}\, 0 \quad \Big)\| \ \ w\|^{\beta}\Big(y^2 \ \ ^{\alpha})x, y+\!\Big\{ \mathrm{T} \\
&\rangle)x, y+\!\!, \quad w^{|}\ \mathrm{f\,lx})y^2 \ \ ^{\alpha} \quad w+\!0 \ \ y^2 \ \ ^{\alpha}\| \ \ w\|^{\beta} \quad \frac{2}{3} \quad)y^2 \ \ ^{\alpha})x, y+\!\!\| \ \ w\|^{\beta}\ \mathrm{T} \\
&\rangle)x, y+\!\!, \quad w^{|}\ \mathrm{f\,lx})y^2 \ \ ^{\alpha} \quad w+ \quad \Big)\frac{N \quad \alpha}{3}\Big(y^2 \ \ ^{\alpha}\| \ \ w\|^{\beta}.
\end{aligned}
$$

For energy solutions a classic approximation approach holds. □

As a consequence we obtain a nonexistence result in the supercritical case for domains with particular geometry.

Theorem 2.4.5. *If ´ is starshaped and the nonlinearity f, F are as in the previous proposition, and satisfy the inequality $))N \quad \alpha+\!sf)s+ \quad 3NF)s+\!+\sim 1$, then problem* (3.2) *has no bounded solution. In particular, in the case $f)s+\mathrm{T} \ \ s^p$ this means that there is no bounded solution for any $p \sim \frac{N0\ \alpha}{N \quad \alpha}$.*

The case α T 2 has been proved in [28]. The corresponding result for the Laplacian (Problem $)P_\lambda+$with α T 3) comes from [66].

2.4.2. Proof of Theorem 2.1.1

We prove here Theorem 2.1.1 in terms of the solution of the local version (3.2). For the sake of readability we split the proof of into several lemmas. From now on we will denote

$$
)\overline{P}_\lambda+\subseteq \begin{cases} \mathrm{f\,lx})y^2 \ \ ^{\alpha} \quad w+ \ \ \mathrm{T} \quad 1, & \text{in } \mathcal{F}\,, \\ \qquad w \ \ \mathrm{T} \quad 1, & \text{on } \partial_L\mathcal{F}\,, \\ \dfrac{\partial w}{\partial \nu^{\alpha}} \ \ \mathrm{T} \quad \lambda w^q\, 0 \ \ w^p, \quad w > 1 & \text{in } ´\,, \end{cases}
$$

and consider the associated energy functional

$$
J_\lambda)w+\mathrm{T}\ \frac{2}{3}\bigcap_{\mathcal{F}_\Omega} y^2 \ \ ^{\alpha}\| \ \ w\|^{\beta}\, dxdy \quad \bigcap_{-} F_\lambda)w+\!dx,
$$

where

$$F_\lambda)s+\mathrm{T}\ \frac{\lambda}{q\,0\ 2}s^{q0\ 2}\ 0\ \ \frac{2}{p\,0\ 2}s^{p0\ 2}.$$

Lemma 2.4.6. *Let Σ be defined by*

$$\Sigma\ \mathrm{T}\ \mathrm{tvr}\,\}\lambda > 1\ \ ;\ \textit{Problem }\,)\overline{P}_\lambda+\textit{has solution}\langle\,.$$

Then $1 < \Sigma < \in$.

Proof. Consider the eigenvalue problem associated to the first eigenvalue λ_2, and let $\varphi_2 > 1$ be the associated eigenfunction. Then using φ_2 as a test function in $)P_\lambda+$we have that

$$\bigcap_{-})\lambda w^q\ 0\ \ w^p+\varphi_2\,dx\ \mathrm{T}\ \lambda_2\bigcap_{-} w\varphi_2\,dx. \tag{2.39}$$

Since there exist positive constants c, δ such that $\lambda t^q\ 0\ \ t^p > c\lambda^\delta t$, for any $t > 1$ we obtain from (3.6) (recall that $w > 1$) that $c\lambda^\delta < \lambda_2$ which implies $\Sigma < \in$.

To prove $\Sigma > 1$ we use the sub- and supersolution technique to construct a solution for any small λ. In fact a subsolution is obtained as $\underline{w}\ \mathrm{T}\ \varepsilon\varphi_2$, $\varepsilon > 1$ small. A supersolution is a suitable multiple of the function g solution to

$$\begin{cases} \mathrm{f\,lx})y^{2\quad\alpha}\quad g+\ \mathrm{T}\ \ 1 & \text{in } \mathcal{F}_-\,, \\ \qquad\qquad\quad g\ \ \mathrm{T}\ \ 1 & \text{on } \partial_L\mathcal{F}_-\,, \\ \qquad\quad \dfrac{\partial g}{\partial\nu^\alpha}\ \ \mathrm{T}\ \ 2 & \text{in } \acute{}\ . \end{cases}$$

□

This proves the third statement in Theorem 2.1.1.

Lemma 2.4.7. *Problem* $)\overline{P}_\lambda+$*has at least a positive solution for every* $1 < \lambda < \Sigma$. *Moreover, the family* $\}w_\lambda\langle$ *of minimal solutions is increasing with respect to* λ.

Remark 2.4.2. *Although this Σ is not exactly the same as that of Theorem 2.1.1, see Remark 2.4.1, we have not changed the notation for the sake of simplicity.*

Proof of Lemma 2.4.7. We already proved in the previous lemma that Problem $)\overline{P}_\lambda+$ has a solution for every $\lambda > 1$ small. Another way of proving this result is to look at the associated functional J_λ. Using inequality (1.30), we have that this functional verifies

$$\begin{aligned} J_\lambda)w+\ \ \mathrm{T}\ \ &\frac{2}{3}\bigcap_{\mathcal{F}_\Omega} y^{2\quad\alpha}\|\ \ w\|^\beta\,dxdy\quad \bigcap_{-} F_\lambda)w+dx \\ \sim\ \ &\frac{2}{3}\bigcap_{\mathcal{F}_\Omega} y^{2\quad\alpha}\|\ \ w\|^\beta\,dxdy\quad \lambda C_2\Big)\bigcap_{\mathcal{F}_\Omega} y^{2\quad\alpha}\|\ \ w\|^\beta\,dxdy\Big(^{\frac{q+1}{2}} \\ &C_3\Big)\bigcap_{\mathcal{F}_\Omega} y^{2\quad\alpha}\|\ \ w\|^\beta\,dxdy\Big(^{\frac{p+1}{2}}, \end{aligned}$$

for some positive constants C_2 and C_3. Then for λ small enough there exist two solutions of problem $)\overline{P}_\lambda+$, one given by minimization and another one given by the Mountain-Pass Theorem, [5]. The proof is standard, based on the geometry of the function $g)t+\mathrm{T}\ \frac{2}{3}t^3 \quad \lambda C_2 t^{q0\,2} \quad C_3 t^{p0\,2}$, see Chapter 3 for more details. This in particular proves $\Sigma > 1$.

We now show that there exists a solution for every $\lambda\ /\ )1,\Sigma+$. Later, see Lemma 2.4.9, we will prove that in fact there are at least two solutions in the whole interval $)1,\Sigma+$.

By definition of Σ, we know that there exists a solution corresponding to any value of λ close to Σ. Let us denote it by μ, and let w_μ be the associated solution. Now w_μ is a supersolution for all problems $)\overline{P}_\lambda+$with $\lambda < \mu$. Take v_λ the unique solution to problem (3.2) with $f)s+\mathrm{T}\ \lambda s^q$. Obviously v_λ is a subsolution to problem $)\overline{P}_\lambda+$. By Lemma 2.4.2 $v_\lambda \geq w_\mu$. Therefore by Lemma 2.4.1 we conclude that there is a solution for all $\lambda\ /\ )1,\mu+$, and as a consequence, for the whole open interval $)1,\Sigma+$. Moreover, this solution is the minimal one. The monotonicity follows directly from the comparison lemma. $\square$

This proves the first statement in Theorem 2.1.1.

Lemma 2.4.8. *Problem* $)\overline{P}_\lambda+$*has at least one solution if* $\lambda\ \mathrm{T}\ \Sigma$.

Proof. Let $\}\lambda_n\langle$ be a sequence such that $\lambda_n \searrow \Sigma$. We denote by $w_n\ \mathrm{T}\ w_{\lambda_n}$ the minimal solution to problem $)P_{\lambda_n}+$. As in [4], we can prove that the linearized equation at the minimal solution has nonnegative eigenvalues. Then it follows, as in [4] again, $J_{\lambda_n})w_n+ < 1$. Since $J^{\infty}_{\lambda_n})w_n+\mathrm{T}\ 1$, one easily gets the bound $\backslash w_n \backslash_{X_0^\alpha)\mathcal{F}_\Omega+} \geq k$. Hence, there exists a weakly convergent subsequence in $X_1^\alpha)\mathcal{F}\ +$and as a consequence a weak solution of $)P_\lambda+$for $\lambda\ \mathrm{T}\ \Sigma$. $\square$

This proves the second statement in Theorem 2.1.1.

To conclude the proof of Theorem 2.1.1, we show next the existence of a second solution for every $1 < \lambda < \Sigma$. It is essential to have that the first solution is given as a local minimum of the associated functional, J_λ. To prove this last assertion we follow some ideas developed in [2].

Lemma 2.4.9. *Problem* $)P_\lambda+$*has at least two solutions for each* $\lambda\ /\ )1,\Sigma+$

Proof. Let $\lambda_1\ /\ )1,\Sigma+$be fixed and consider $\lambda_1 < \ddot{\lambda}_2 < \Sigma$. Take $\phi_1\ \mathrm{T}\ w_{\lambda_0}$, $\phi_2\ \mathrm{T}\ w_{\ddot{\lambda}_1}$ the two minimal solutions to problem $)\overline{P}_\lambda+$with $\lambda\ \mathrm{T}\ \lambda_1$ and $\lambda\ \mathrm{T}\ \ddot{\lambda}_2$ respectively, then by comparison, $\phi_1 < \phi_2$. We define

$$M\ \mathrm{T}\ \}w\ /\ X_1^\alpha)\mathcal{F}\ +;\ 1 \geq w \geq \phi_2\langle.$$

Notice that M is a convex closed set of $X_1^\alpha)\mathcal{F}\ +$. Since J_{λ_0} is bounded from below in M and it is semicontinuous on M, we get the existence of $\underline{\omega}\ /\ M$ such that $J_{\lambda_0})\underline{\omega}+\mathrm{T}$

$\log_{w/M} J_{\lambda_0})w+$ Let v_1 be the unique positive solution to problem

$$\begin{cases} \text{f lx})y^2 \;\; ^{\alpha} \;\; v_1 + \;\; \mathrm{T} \;\; 1, & \text{in } \mathcal{F} \;, \\ \qquad\qquad v_1 \;\; \mathrm{T} \;\; 1, & \text{on } \partial_L \mathcal{F} \;, \\ \qquad \dfrac{\partial v_1}{\partial \nu^{\alpha}} \;\; \mathrm{T} \;\; v_1^q, & \text{in } \;\acute{}\;. \end{cases} \tag{2.40}$$

(The existence and uniqueness of this solution is clear, see Lemma 2.4.2). Since for $1 < \varepsilon << \lambda_1$, and $J_{\lambda_0})\varepsilon v_1 + < 1$, we have $\varepsilon v_1 \;/\; M$, then $\underline{\omega}$ ⦙T 1. Therefore $J_{\lambda_0})\underline{\omega} + < 1$. By arguments similar to those in [77, Theorem 2.4], we obtain that $\underline{\omega}$ is a solution to problem $)\overline{P}_{\lambda_0} +$ There are two possibilities:

- If $\underline{\omega} \subseteq w_{\lambda_0}$, then the result follows.
- If $\underline{\omega} \subseteq w_{\lambda_0}$, we have just to prove that $\underline{\omega}$ is a local minimum of J_{λ_0}. Assuming that this is true, the conclusion in part 4 of Theorem 2.1.1 follows by using a classical argument: The second solution is given by the Mountain Pass Theorem, we postpone the proof to the next sections that will include the more complicated critical case.

We prove now that the minimal solution w_{λ_0} is in fact a local minimum of J_{λ_0}. We argue by contradiction.

Suppose that $\underline{\omega}$ is not a local minimum of J_{λ_0} in $X_1^{\alpha})\mathcal{F} +$, then there exists a sequence $\}v_n\langle \ll X_1^{\alpha})\mathcal{F} +$such that $\backslash v_n \quad \underline{\omega}\backslash_{X_0^{\alpha}} \nearrow 1$ and $J_{\lambda_0})v_n + < J_{\lambda_0})\underline{\omega} +$

Let $w_n \;\mathrm{T}\;)v_n \quad \phi_2 +^0$ and $z_n \;\mathrm{T}\; \mathrm{n\ d\tilde{}}\}1, \mathrm{n\ lo}\}v_n, \phi_2\langle\langle$. It is clear that $z_n \;/\; M$ and

$$z_n)x, y + \mathrm{T} \begin{cases} 1 & \text{if } v_n)x, y + \geq 1, \\ v_n)x, y + & \text{if } 1 \geq v_n)x, y + \geq \phi_2)x, y +, \\ \phi_2)x, y + & \text{if } \phi_2)x, y + \geq v_n)x, y + \end{cases}$$

We set

$$\begin{array}{ll} T_n \subseteq \})x, y + / \;\mathcal{F}\; ; \; z_n)x, y + \mathrm{T}\; v_n)x, y + \langle, & \qquad S_n \subseteq \mathrm{supp})w_n +, \\ \widetilde{T_n} \;\mathrm{T}\; \overline{T_n} \wedge \;\acute{}\;, & \qquad \widetilde{S_n} \;\mathrm{T}\; S_n \wedge \;\acute{}\;. \end{array}$$

Notice that $\mathrm{supp})v_n^0 + \mathrm{T}\; T_n \cap S_n$. We claim that

$$\|\widetilde{S_n}\| \nearrow 1 \quad \text{as } n \nearrow \in\;, \tag{2.41}$$

where $\|A\| \subseteq \sum \chi_A)x + dx$.

By the definition of F_λ, we set $F_{\lambda_0})s{+}\mathrm{T}\ \dfrac{\lambda_1}{q\,0\ 2}s_0^{q0\ 2}\ 0\ \dfrac{2}{p\,0\ 2}s_0^{p0\ 2}$, for $s\ /\ \mathbb{R}$, and get

$$
\begin{aligned}
J_{\lambda_0})v_n{+}\mathrm{T}\frac{2}{3}\bigcap_{\mathcal{F}\Omega} y^2\ \ ^{\alpha}\|\ \ v_n\|^\beta\,dxdy \quad \bigcap_{-} F_{\lambda_0})v_n{+}dx \\
\mathrm{T}\frac{2}{3}\bigcap_{T_n} y^2\ \ ^{\alpha}\|\ \ z_n\|^\beta\,dxdy \quad \bigcap_{\widetilde{T}_n} F_{\lambda_0})z_n{+}dx\ 0\ \frac{2}{3}\bigcap_{S_n} y^2\ \ ^{\alpha}\|\ \ v_n\|^\beta\,dxdy \\
\bigcap_{\widetilde{S}_n} F_{\lambda_0})v_n{+}dx\ 0\ \frac{2}{3}\bigcap_{\mathcal{F}\Omega} y^2\ \ ^{\alpha}\|\ \ v_n\ \|^\beta\,dxdy \\
\mathrm{T}\frac{2}{3}\bigcap_{T_n} y^2\ \ ^{\alpha}\|\ \ z_n\|^\beta\,dxdy \quad \bigcap_{\widetilde{T}_n} F_{\lambda_0})z_n{+}dx \\
0\ \ \frac{2}{3}\bigcap_{S_n} y^2\ \ ^{\alpha}\|\ \ )w_n\ 0\ \ \phi_2{+}\|^\beta\,dxdy \quad \bigcap_{\widetilde{S}_n} F_{\lambda_0})w_n\ 0\ \ \phi_2{+}dx \\
0\ \ \frac{2}{3}\bigcap_{\mathcal{F}\Omega} y^2\ \ ^{\alpha}\|\ \ v_n\ \|^\beta\,dxdy.
\end{aligned}
$$

Since

$$
\bigcap_{\mathcal{F}\Omega} y^2\ \ ^{\alpha}\|\ \ z_n\|^\beta\,dxdy\ \mathrm{T}\bigcap_{T_n} y^2\ \ ^{\alpha}\|\ \ v_n\|^\beta\,dxdy\ 0\bigcap_{S_n} y^2\ \ ^{\alpha}\|\ \ \phi_2\|^\beta\,dxdy
$$

and

$$
\bigcap_{-} F_{\lambda_0})z_n{+}dx\ \mathrm{T}\bigcap_{\widetilde{T}_n} F_{\lambda_0})v_n{+}dx\ 0\bigcap_{\widetilde{S}_n} F_{\lambda_0})\phi_2{+}dx,
$$

by using the fact that ϕ_2 is a supersolution to $)P_{\lambda_0}+$, we conclude that

$$
\begin{aligned}
J_{\lambda_0})v_n{+}\ \ \mathrm{T}\quad & J_{\lambda_0})z_n{+}0\ \ \frac{2}{3}\bigcap_{S_n} y^2\ \ ^{\alpha})\|\ \ )w_n\ 0\ \ \phi_2{+}\|^\beta\quad \|\ \ \phi_2\|^\beta{+}dxdy \\
& \bigcap_{\widetilde{S}_n})F_{\lambda_0})w_n\ 0\ \ \phi_2{+}\quad F_{\lambda_0})\phi_2{+}{+}dx\ 0\ \ \frac{2}{3}\bigcap_{\mathcal{F}\Omega} y^2\ \ ^{\alpha}\|\ \ v_n\ \|^\beta\,dxdy \\
\sim\quad & J_{\lambda_0})z_n{+}0\ \ \frac{2}{3}\backslash w_n\backslash^3_{X_0^\alpha}\ 0\ \ \frac{2}{3}\backslash v_n\ \backslash^3_{X_0^\alpha} \\
& \bigcap_{-}\}F_{\lambda_0})w_n\ 0\ \ \phi_2{+}\quad F_{\lambda_0})\phi_2{+}\quad)F_{\lambda_0}{+}_{u})\phi_2{+}w_n\langle\ \ dx \\
\sim\quad & J_{\lambda_0})\underline{\omega}{+}0\ \ \frac{2}{3}\backslash w_n\backslash^3_{X_0^\alpha}\ 0\ \ \frac{2}{3}\backslash v_n\ \backslash^3_{X_0^\alpha} \\
& \bigcap_{-}\}F_{\lambda_0})w_n\ 0\ \ \phi_2{+}\quad F_{\lambda_0})\phi_2{+}\quad)F_{\lambda_0}{+}_{u})\phi_2{+}w_n\langle\ \ dx.
\end{aligned}
$$

On one hand, taking into account that $1 < q\,0\ \ 2 < 3$, one obtains that

$$
1 \geq \frac{2}{q\,0\ \ 2})w_n\ 0\ \ \phi_2{+}^{q0\ 2}\quad \frac{2}{q\,0\ \ 2}\phi_2^{q0\ 2}\quad \phi_2^q w_n \geq \frac{q}{3}\,\frac{w_n^3}{\phi_2^{2\ \ q}}.
$$

The well known Picone's inequality (see [65]) establish:

$$\| \ u\|^{\beta} \qquad \Big) \frac{u^3}{v} \Big\{ \times \ v \sim 1,$$

for differentiable functions $v > 1$, $u \sim 1$. In our case, by an approximation argument we get

$$\lambda_1 \bigcap_{-} \frac{w_n^3}{\phi_2^2 \ {}^q} dx \geq \backslash w_n \backslash^3_{X_0^\alpha}.$$

On the other hand, since $p\,0\ \ 2 > 3$,

$$\begin{aligned} 1 \quad &\geq \quad \frac{2}{p\,0\ \ 2}) w_n\,0\ \ \phi_2 + ^{p\,0\ 2} \quad \frac{2}{p\,0\ \ 2} \phi_2^{p\,0\ 2} \quad \phi_2^p w_n \geq \frac{r}{3} w_n^3) w_n\,0\ \ \phi_2 + ^{p\ \ 2} \\ &\geq \quad C)p +)\phi_2^{p\ \ 2} w_n^3\,0\ \ w_n^{p\,0\ 2} + . \end{aligned}$$

Hence using that $p\,0\ \ 2 < 3^{\leq}_{\alpha}$ and the claim (2.41)

$$\bigcap_{-} \Big\} \frac{2}{p\,0\ \ 2}) w_n\,0\ \ \phi_2 + ^{p\,0\ 2} \quad \frac{2}{p\,0\ \ 2} \phi_2^{p\,0\ 2} \quad \phi_2^p w_n \Big(dx \geq o)2 + \backslash w_n \backslash^3_{X_0^\alpha}.$$

As a consequence we obtain that

$$\begin{aligned} J_{\lambda_0})v_n + \ \sim \quad & J_{\lambda_0})\underline{\omega} + 0\ \ \frac{2}{3} \backslash w_n \backslash^3_{X_0^\alpha})2 \quad q \quad o)2 + + 0\ \ \frac{2}{3} \backslash v_n \ \backslash^3_{X_0^\alpha} \\ \subseteq \quad & J_{\lambda_0})\underline{\omega} + 0\ \ \frac{2}{3} \backslash w_n \backslash^3_{X_0^\alpha})2 \quad q \quad o)2 + + 0\ \ o)2 + . \end{aligned}$$

Since $q < 2$, there results that $J_{\lambda_0})\underline{\omega}+> J_{\lambda_0})v_n+\sim J_{\lambda_0})\underline{\omega}+$for $n > n_1$, a contradiction with the main hypothesis. Hence $\underline{\omega}$ is a minimum.

To finish the proof we have to prove the claim (2.41). For $\varepsilon > 1$ small, and $\delta > 1$ (δ to be chosen later), we consider

$$\begin{aligned} E_n \quad \mathrm{T} \quad & \}x\ /\ '\ \ ;\ v_n)x + \sim \phi_2)x + \quad \{ \quad \phi_2)x + > \underline{\omega})x + 0\ \ \delta\langle, \\ F_n \quad \mathrm{T} \quad & \}x\ /\ '\ \ ;\ v_n)x + \sim \phi_2)x + \quad \{ \quad \phi_2)x + \geq \underline{\omega})x + 0\ \ \delta\langle. \end{aligned}$$

Using the fact that

$$\begin{aligned} 1 \quad \mathrm{T} \quad & \|\}x\ /\ '\ \ ;\ \phi_2)x + < \underline{\omega})x + \langle \| \mathrm{T} \Big(\bigcup_{j[\ \ 2}^{\in} \Big\} x\ /\ '\ \ ;\ \phi_2)x + \geq \underline{\omega})x + 0\ \ \frac{2}{j} \Big(\Big(\\ \mathrm{T} \quad & \lim_{j'\ \ \in} \Big(\Big\} x\ /\ '\ \ ;\ \phi_2)x + \geq \underline{\omega})x + 0\ \ \frac{2}{j} \Big(\Big(, \end{aligned}$$

we get for j_1 large enough, that if $\delta < \frac{2}{j_0}$ then

$$\|\}x\ /\ '\ \ ;\ \phi_2)x + \geq \underline{\omega})x + 0\ \ \delta\langle \| \geq \frac{\varepsilon}{3}.$$

Hence we conclude that $\|F_n\|\vdash\ \geq \frac{\varepsilon}{3}$.

Since $\backslash v_n \quad \underline{\omega}\backslash_{X_0^\alpha} \nearrow 1$ as $n \nearrow \in$, in particular by the trace embedding, $\backslash v_n$ $\underline{\omega}\backslash_{L^2)^-+} \nearrow 1$. We obtain that, for $n \sim n_1$ large,

$$\frac{\delta^3\varepsilon}{3} \sim \bigcap_{\mathcal{F}_\Omega} \|v_n \quad \underline{\omega}\|^\beta dx \sim \bigcap_{E_n} \|v_n \quad \underline{\omega}\|^\beta dx \sim \delta^3\|E_n\|\vdash .$$

Therefore $\|E_n\|\vdash\ \geq \dfrac{\varepsilon}{3}$. Since $\widetilde{S_n} \ll F_n \cap E_n$ we conclude that $\|\widetilde{S_n}\|\vdash\ \geq \varepsilon$ for $n \geq n_1$. Hence $\|\widetilde{S_n}\|\vdash\ \nearrow 1$ as $n \nearrow \in$ and the claim follows. $\square$

2.4.3. Proof of Theorem 2.1.2 and further results

We start with the uniform $L^\in$-estimates for solutions to problem $)P_\lambda+$in its local version given by $)P_\lambda+$

Theorem 2.4.10. *Assume* $\alpha \sim 2$, $2 < p < \frac{N0\ \alpha}{N\ \ \alpha}$ *and* $N \sim 3$. *Then there exists a constant* $C \text{ T } C)p, \acute{}\ +> 1$ *such that every solution to problem* $)\overline{P}_\lambda+$*satisfies*

$$\backslash w\backslash_\in \ \geq C,$$

for every $1 \geq \lambda \geq \Sigma$.

The proof is based on a scaling method of [51], and two nonexistence results, see Theorems 2.2.1 and 2.2.4.

Proof of Theorem 2.4.10. Assume by contradiction that there exists a sequence $\}w_n\langle \ll X_1^\alpha)\mathcal{F}\ +$of solutions to $)P_\lambda+$verifying that $M_n \text{ T } \backslash w_n\backslash_\in \ \nearrow \in$, as $n \nearrow \in$. By the Maximum Principle, which holds for our problem, see [46], the maximum of w_n is attained at a point $)x_n, 1+$where $x_n \ / \ \acute{}$. We define $\acute{}_n \text{ T } \frac{2}{\mu_n})\acute{} \quad x_n+$, with $\mu_n \text{ T } M_n^{)2 \quad p+/\alpha}$, i.e., we center at x_n and dilate by $\frac{2}{\mu_n} \nearrow \in$ as $n \nearrow \in$.

We consider the scaled functions

$$v_n)x, y+\text{T } \frac{w_n)x_n\ 0\ \ \mu_n x, \mu_n y+}{M_n}, \quad \text{for } x \ / \ \acute{}_n,\ y \sim 1.$$

It is clear that $\backslash v_n\backslash \geq 2$, $v_n)1, 1+\text{T } 2$ and moreover

$$\begin{cases} \text{f lx})y^{2 \quad \alpha} \quad v_n+ \ \text{T} \ \ 1 & \text{in } \mathcal{F}_n, \\ v_n \ \text{T} \ \ 1 & \text{on } \partial_L \mathcal{F}_n, \\ \dfrac{\partial v_n}{\partial \nu^\alpha} \ \text{T} \ \ \lambda M_n^{q \quad p} v_n^q\ 0\ \ v_n^p & \text{in } \acute{}_n * \}1\langle. \end{cases} \tag{2.42}$$

By Arzelà-Ascoli Theorem (the solution is C^γ, see Proposition 2.4.3), there exists a subsequence, which we denote again by v_n, which converges to some function v as

$n \nearrow \in$. In order to see the problem satisfied by v we pass to the limit in the weak formulation of (2.42). We define d_n T dist$)x_n, \partial ´ +$ then there are two possibilities as $n \nearrow \in$ according the behaviour of the ratio $\frac{d_n}{\mu_n}$:

1. $\Big\}\dfrac{d_n}{\mu_n}\Big(_n$ is not bounded.
2. $\Big\}\dfrac{d_n}{\mu_n}\Big(_n$ remains bounded.

In the first case, since $B_{d_n/\mu_n})1+\ll ´_n$, and $´_n$ is smooth, it is clear that $´_n$ tends to $\mathbb{R}^N$ and v is a solution to

$$\begin{cases} \text{f lx})y^2 \;\; ^{\alpha} \;\; v+ \;\; \text{T} \;\; 1 & \text{in } \mathbb{R}_0^{N0\,2}, \\ \dfrac{\partial v}{\partial \nu^{\alpha}} \;\; \text{T} \;\; v^p & \text{on } \partial\mathbb{R}_0^{N0\,2}. \end{cases}$$

Moreover, $v)1, 1+$T 2 and $v > 1$ which is a contradiction with Theorem 2.2.1.

In the second case, we may assume that $\frac{d_n}{\mu_n} \nearrow s \sim 1$ as $n \nearrow \in$. As a consequence, passing to the limit, the domains $´_n$ converge (up to a rotation) to some half-space H_s T $\}x \;/\; \mathbb{R}^N \;;\; x_N > \;\; s\langle$. We obtain here that v is a solution to

$$\begin{cases} \text{f lx})y^2 \;\; ^{\alpha} \;\; v+ \;\; \text{T} \;\; 1 & \text{in } H_s * \;)1, \in +, \\ \dfrac{\partial v}{\partial \nu^{\alpha}} \;\; \text{T} \;\; v^p & \text{on } H_s * \;\}1\langle, \end{cases}$$

with $\backslash v \backslash_{\in}$ T 2, $v)1, 1+$T 2. In the case s T 1 this is a contradiction with the continuity of v. If $s > 1$, the contradiction comes from Theorem 2.2.4. □

We next prove a uniqueness result for solutions with small norm.

Theorem 2.4.11. *There exists at most one solution to problem* $)\overline{P}_{\lambda}+$*with small norm.*

We follow closely the arguments in [4], so we establish the following previous result:

Lemma 2.4.12. *Let* z *be the unique solution to problem* (2.40). *There exists a constant* $\beta > 1$ *such that*

$$\backslash\phi\backslash^3_{X_0^{\alpha})\mathcal{F}_{\Omega}+} \quad q\bigcap_{-} z^q \;\; {}^2\phi^3\, dx \sim \beta\backslash\phi\backslash^3_{L^2)^-+}, \quad \exists\phi \;/\; X_1^{\alpha})\mathcal{F}+ \tag{2.43}$$

Proof. We recall that z can be obtained by minimization

$$\text{n lo}\Big\}\frac{2}{3}\backslash\omega\backslash^3_{X_0^{\alpha})\mathcal{F}_{\Omega}+} \quad \frac{2}{q\,0\,2}\backslash w\backslash^{q0\,2}_{L^{q+1})^-+}; \quad \omega \;/\; X_1^{\alpha})\mathcal{F}+\Big(.$$

As a consequence,

$$\backslash\phi\backslash^{3}_{X_0^\alpha)\mathcal{F}\Omega+} \quad q\bigcap_{-} z^{q\ \ 2}\phi^3\,dx \sim 1, \quad \exists\phi\ /\ X_1^\alpha)\mathcal{F}+$$

This implies that the first eigenvalue a_2 of the linearized problem

$$\left\{\begin{array}{lll} \mathrm{f\,lx})y^{2\ \ \alpha}\ \ \phi+\ \mathrm{T}\ \ 1, & & \text{in } \mathcal{F}, \\ \phi\ \ \mathrm{T}\ \ 1, & & \text{on } \partial_L\mathcal{F}, \\ \dfrac{\partial\phi}{\partial\nu^\alpha}\ \ qz^{q\ \ 2}\phi\ \ \mathrm{T}\ \ a_2\phi, & & \text{on } \acute{}\ *\ \}1\langle, \end{array}\right.$$

is nonnegative.

Suppose that $a_2\ \mathrm{T}\ 1$ and let φ be a corresponding eigenfunction. Taking into account that z is the solution to (2.40) we obtain that

$$q\bigcap_{-} z^q\varphi\,dx\ \mathrm{T}\ \bigcap_{-} z^q\varphi\,dx$$

which is a contradiction.

Hence $a_2 > 1$, which proves (2.43). □

Proof of Theorem 2.4.11. Consider $A > 1$ such that $pA^{p\ \ 2} < \beta$, where β is given in (2.43). Now we prove that problem $)P_\lambda+$has at most one solution with $L^\in$ -norm less than A.

Assume by contradiction that $)P_\lambda+$has a second solution $w\ \mathrm{T}\ w_\lambda\ 0\ \ v$ verifying $\backslash w\backslash_\in\ < A$. Since w_λ is the minimal solution, it follows that $v > 1$ in $\acute{}\ *\ ]1,\in+$. We define now $\eta\ \mathrm{T}\ \lambda^{\frac{1}{1-q}}z$, where z is the solution to (2.40). Then it verifies $\mathrm{f\,lx})y^{2\ \ \alpha}\ \ \eta+\mathrm{T}\ 1$, with boundary condition $\lambda\eta^q$. Moreover, w_λ is a supersolution to the problem that η verifies. Then by comparison, Lemma 2.4.2, applied with $f)t+\mathrm{T}\ \lambda t^q$, $v\ \mathrm{T}\ \eta$ and $w\ \mathrm{T}\ w_\lambda$, we get

$$w_\lambda \sim \lambda^{\frac{1}{1-q}}z \qquad \text{on } \acute{}\ *\ \}1\langle. \tag{2.44}$$

Since $w\ \mathrm{T}\ w_\lambda\ 0\ \ v$ is solution to $)\overline{P}_\lambda+$we have, on $\acute{}\ *\ \}1\langle$,

$$\frac{\partial)w_\lambda\ 0\ \ v+}{\partial\nu^\alpha}\mathrm{T}\ \lambda)w_\lambda\ 0\ \ v+^q\ 0\ \ )w_\lambda\ 0\ \ v+^p \geq \lambda w_\lambda^q\ 0\ \ \lambda q w_\lambda^{q\ \ 2}v\ 0\ \ )w_\lambda\ 0\ \ v+^p,$$

where the inequality is a consequence of the concavity, hence

$$\frac{\partial v}{\partial\nu^\alpha} \geq \lambda q w_\lambda^{q\ \ 2}v\ 0\ \ )w_\lambda\ 0\ \ v+^p\ \ \ w_\lambda^p.$$

Moreover, (2.44) implies $w_\lambda^{q\ \ 2} \sim \lambda\ \ ^2 z^{q\ \ 2}$. From the previous two inequalities we get

$$\frac{\partial v}{\partial\nu^\alpha} \geq qz^{q\ \ 2}v\ 0\ \ )w_\lambda\ 0\ \ v+^p\ \ \ w_\lambda^p.$$

Using that $\backslash w_\lambda 0 \ v \backslash_{\in} \ \geq A$, we obtain $) w_\lambda 0 \ v \not{+}^{p} \quad w_\lambda^p \geq pA^{p \ \ 2} v$. As a consequence,

$$\frac{\partial v}{\partial \nu^\alpha} \quad qz^{q \ \ 2} v \geq pA^{p \ \ 2} v.$$

Taking v as a test function and $\phi \mathrm{T} \ v$ in (2.43) we arrive to

$$\beta \bigcap_{-} v^3 \, dx \geq pA^{p \ \ 2} \bigcap_{-} v^3 \, dx.$$

Since $pA^{p \ \ 2} < \beta$ we conclude that $v \subseteq 1$, which gives the desired contradiction. $\square$

Remark 2.4.3. *This proof also provides the asymptotic behavior of w_λ near $\lambda \ \mathrm{T} \ 1$, namely $w_\lambda \subset \lambda^{\frac{1}{1-q}} z$, where z is the unique solution to problem* (2.40).

3

On some critical problems for the fractional Laplacian operator

3.1. Introduction

In this chapter we continue with the study of perturbations of the pure-power critical case for the different fractional powers of the Laplacian. Thus, we study the following problem

$$)P^{\leq}_{\lambda}+\left\}\begin{array}{llll})\ \Lambda+^{\alpha/3}u & \mathrm{T} & \lambda u^q\, 0\ u^p, \qquad u>1 & \text{in } ´\,, \\ u & \mathrm{T} & 1 & \text{on } \partial ´\,, \end{array}\right.$$

with $1 < q < p \ \mathrm{T}\ \frac{N 0\ \alpha}{N\ \ \alpha}$, $1 < \alpha < 3$ and $N > \alpha$. As in the previous chapter, here we will look only for positive solutions to $)P^{\leq}_{\lambda}+($so many times we will omit the term "positive").

As we have seen in Theorem 2.4.5, and analogously to the classic case, the problem

$$\left\}\begin{array}{llll})\ \Lambda+^{\alpha/3}u & \mathrm{T} & \|u\|^{\frac{2\alpha}{N-\alpha}}u & \text{in } ´ \ll \mathbb{R}^N, \\ u & \mathrm{T} & 1 & \text{on } \partial ´\,, \end{array}\right. \tag{3.1}$$

has no positive solutions whenever ´ is and star-shaped domain. In a pioneering work [24], Brezis and Nirenberg showed that, contrary to intuition, the critical problem with small linear perturbations can provide positive solutions. After that, in [4], using the results on concentration-compactness of Lions, [60], the authors proved some results on existence and multiplicity of solutions for a sublinear perturbation of the critical power, among others.

Recently, several studies have been performed for classical critical elliptic equations with the Laplacian operator substituted by its fractional powers. In particular, in [80] it is studied the problem

$$\left.\begin{array}{ll})\ \Lambda +^{2/3} u \ \mathrm{T}\ \lambda u\, 0\ u^{\frac{N+1}{N-1}} & \text{in } ´\,, \\ u\ \mathrm{T}\ 1 & \text{on } \partial ´\,, \end{array}\right\} \tag{3.2}$$

the analogue case to the problem in [24], but with the square root of the Laplacian instead of the Laplacian. The results of this chapter generalize those cases to every power $\alpha\ /\ )1, 3+$of the Laplacian.

The cases $1 < q < 2$, $q\ \mathrm{T}\ 2$ and $2 < q < \frac{N 0\ \alpha}{N\ \ \alpha}$ will be treated with different methodologies, thus we will divide the chapter according to those cases. Our main results dealing with Problem $)P^{\leq}_{\lambda}+$are the following.

Theorem 3.1.1. *Let $1 < q < 2$. Then, there exists $1 < \Sigma < \in$ such that the problem $)P^{\leq}_{\lambda}+$*

1. *has no positive solution for $\lambda > \Sigma$;*
2. *has a minimal positive solution for any $1 < \lambda \geq \Sigma$. Moreover the family of minimal solutions is increasing with respect to λ;*
3. *if $\lambda\ \mathrm{T}\ \Sigma$ there is at least one positive solution;*
4. *if $\alpha \sim 2$ there are at least two positive solutions for $1 < \lambda < \Sigma$.*

Theorem 3.1.2. *Let $q\ \mathrm{T}\ 2$, $1 < \alpha < 3$ and $N \sim 3\alpha$. Then the problem $)P^{\leq}_{\lambda}+$*

1. *has no positive solution for $\lambda \sim \lambda_2$;*
2. *has at least one positive solution for each $1 < \lambda < \lambda_2$.*

Theorem 3.1.3. *Let $2 < q < \frac{N 0\ \alpha}{N\ \ \alpha}$, $1 < \alpha < 3$ and $N > \alpha)2\ 0\ \ 2/q+$ Then the problem $)P^{\leq}_{\lambda}+$has at least one positive solution for any $\lambda > 1$.*

The restriction $\alpha \sim 2$ in Theorem 3.1.1-)8+seems to be technical. Note that the same restriction appeared also in Chapter 2. Here, due to the lack of regularity, see Proposition 3.5.2, it is not clear how to separate the solutions in the appropriate way, Lemma 3.3.3, see also [40, 42].

On the other hand, the range $\alpha < N < 3\alpha$ in Theorem 3.1.2 is left open. See the special case $\alpha\ \mathrm{T}\ 3$ and $N\ \mathrm{T}\ 4$ in [24]. If $\alpha\ \mathrm{T}\ 2$ this range is empty, see [80].

As to the regularity of solutions, they are bounded and "classical" in the sense that they have as much regularity as it is required in the equation, i.e., they possess α "derivatives", see Propositions 3.5.1 and 3.5.2. Even more, if $\alpha = 1$, they belong to $C^{1,\gamma}(\overline{\Omega})$ or $C^{\infty}(\Omega)$, whenever $0 < q < 1$ or $q \geq 1$, respectively.

3.2. Preliminaries

A natural definition of energy solution to problem (P_λ^{α}) is the following.

Definition 3.2.1. *We say that $u \in H_0^{\alpha/2}(\Omega)$ is a solution of (P_λ^{α}) if the identity*

$$\int_\Omega (-\Delta)^{\alpha/4}u\,(-\Delta)^{\alpha/4}\varphi\,dx = \int_\Omega f(u)\varphi\,dx \tag{3.3}$$

holds for every function $\varphi \in H_0^{\alpha/2}(\Omega)$, where $f(u) = \lambda u^q + u^p$.

Note that the right-hand side of (3.3) is well defined since $\varphi \in H_0^{\alpha/2}(\Omega) \hookrightarrow L^{\frac{2N}{N-\alpha}}(\Omega)$, while $u \in H_0^{\alpha/2}(\Omega)$, hence $f(u) \in L^{\frac{2N}{N+\alpha}}(\Omega) \hookrightarrow H^{-\alpha/2}(\Omega)$.

Associated to problem (P_λ^{α}) we consider the energy functional

$$I(u) = \frac{1}{2}\int_\Omega \left|(-\Delta)^{\alpha/4}u\right|^2 dx - \int_\Omega F(u)\,dx\,,$$

where $F(u) = \int_0^u f(s)\,ds$. In our case it reads

$$I(u) = \frac{1}{2}\int_\Omega \left|(-\Delta)^{\alpha/4}u\right|^2 dx - \frac{\lambda}{q+1}\int_\Omega u^{q+1}\,dx - \frac{N-\alpha}{2N}\int_\Omega u^{\frac{2N}{N-\alpha}}\,dx\,. \tag{3.4}$$

This functional is well defined in $H_0^{\alpha/2}(\Omega)$, and moreover, the critical points of I correspond to solutions to (P_λ^{α}).

We can reformulate our problem in the local form $(\widetilde{P}_\lambda^{\alpha})$ as

$$(\widetilde{P}_\lambda^{\alpha}) \quad \begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathcal{C}_\Omega\,, \\ w = 0 & \text{on } \partial_L\mathcal{C}_\Omega\,, \\ \dfrac{\partial w}{\partial \nu^\alpha} = \lambda w^q + w^{\frac{N+\alpha}{N-\alpha}} & \text{in } \Omega \times \{y = 0\}\,. \end{cases}$$

The associated energy functional to the problem $(\widetilde{P}_\lambda^{\alpha})$ is

$$J(w) = \frac{\kappa_\alpha}{2}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2\,dxdy - \frac{\lambda}{q+1}\int_\Omega w^{q+1}\,dx - \frac{N-\alpha}{2N}\int_\Omega w^{\frac{2N}{N-\alpha}}\,dx\,. \tag{3.5}$$

Clearly, critical points of J in $X_0^{\alpha}(\mathcal{C}_\Omega)$ correspond to critical points of I in $H_0^{\alpha/2}(\Omega)$. Even more, minima of J also correspond to minima of I, see Section 3.3.

Remark 3.2.1. *In the sequel, and in view of the above equivalence, we will use both formulations of the problem, in ´ or in $\mathcal{F}$-, whenever we may take some advantage. In particular, we will use the extension version* $)\overline{P}_{\lambda}^{\leq}+$*when dealing with the fractional operator acting on products of functions, since it is not clear how to calculate this action. This difficulty appears in the proof of the concentration-compactness result, Theorem 3.5.3, among others.*

3.3. Sublinear case: $0 < q < 1$.

We prove here Theorem 3.1.1. As we have said in Remark 3.2.1, there are some points where it is difficult to work directly with the fractional Laplacian, due to the absence of formula for the fractional Laplacian of a product. Therefore we consider in some occasions the extended problem $)\overline{P}_{\lambda}+$.

To begin with that problem, we prove that local minima of the functional I correspond to local minima of the extended functional J.

Proposition 3.3.1. *A function* $u_1 \ / \ H_1^{\alpha/3})´ +$*is a local minimum of* I *if and only if* $w_1 \text{ T } E_\alpha)u_1+/ \ X_1^{\alpha})\mathcal{F}- +$*is a local minimum of* J.

Proof. Firstly let $u_1 \ / \ H_1^{\alpha/3})´ +$be a local minimum of I. Suppose, by contradiction, that $w_1 \text{ T } \mathrm{E}_\alpha)u_1+$is not a local minimum for the extended functional J. Then by (1.8) and (1.30), we have that, for any $\varepsilon > 1$, there exists $w_\varepsilon \ / \ X_1^{\alpha})\mathcal{F}- +$, with $\backslash w_1 \ w_\varepsilon\backslash_{X_0^\alpha)\mathcal{F}_\Omega+} < \varepsilon$, such that

$$I)u_1+\text{T } J)w_1+> J)w_\varepsilon+\sim I)z_\varepsilon+$$

where $z_\varepsilon \text{ T } w_\varepsilon)\cancel{x}1+/ \ H_1^{\alpha/3})´ +$satisfies $\backslash u_1 \quad z_\varepsilon\backslash_{H_0^{\alpha/2})^- +} < \varepsilon$.

On the other hand, let $w_1 \ / \ X_1^{\alpha})\mathcal{F}- +$be a local minimum of J. It is clear, from the definition of the extension operator, that w_1 is α-harmonic. So we conclude. $\square$

We return now to the original problem $)P_\lambda^{\leq}+$, posed at the bottom ´ $* \ \}y \text{ T } 1\langle$.

Lemma 3.3.2. *Let* Σ *be defined by*

$$\Sigma \text{ T tvr } \}\lambda > 1 \ ; \ \textit{Problem })P_\lambda^{\leq}+\textit{has solution}\langle \, .$$

Then $1 < \Sigma < \in$.

Proof. Let $)\lambda_2, \varphi_2+$be the first eigenvalue and a corresponding positive eigenfunction of the fractional Laplacian in ´ . Then, using φ_2 as a test function in $)P_\lambda^{\leq}+$, we have that

$$\bigcap_{-}\Big)\lambda u^q \, 0 \ u^{\frac{N+\alpha}{N-\alpha}}\Big(\varphi_2 \, dx \text{ T } \lambda_2 \bigcap_{-} u\varphi_2 \, dx. \tag{3.6}$$

Since there exist positive constants c, δ such that $\lambda t^q \, 0 \; t^{\frac{N+\alpha}{N-\alpha}} > c\lambda^\delta t$, for any $t > 1$ we obtain from (3.6) that $c\lambda^\delta < \lambda_2$ which implies $\Sigma < \in$.

To prove $\Sigma > 1$ we use the sub- and supersolution technique to construct a solution for any small λ, see [48, 4]. In fact a subsolution is obtained as a small multiple of φ_2. A supersolution is a large multiple of the function g solution to

$$\left\} \begin{array}{ll}) \quad \Lambda +^{\alpha/3} g \;\mathrm{T}\; 2 & \text{in } ' , \\ g \;\mathrm{T}\; 1 & \text{on } \partial' . \end{array} \right.$$

$\square$

Comparison is clear for linear problems associated to the fractional Laplacian, as it is for the Laplacian. On the other hand, it is in general not true for nonlinear problems. Nevertheless, it holds when the reaction term is a nonnegative sublinear function, see [21, 4]. Therefore, it is easy to show, comparing with the problem with only the concave terms λu^q, that in fact there is at least one positive solution u_λ to problem $)P_\lambda^{\leq}+$ for every λ in the whole interval $)1, \Sigma+$. Even more, these constructed solutions are minimal and are increasing with respect to λ, see Lemma 2.4.7.

To prove existence of solution in the extremal value $\lambda \;\mathrm{T}\; \Sigma$, the idea, like in [4], consists on passing to the limit as $\lambda_n \searrow \Sigma$ on the sequence $\}z_n\langle \;\mathrm{T}\; \}z_{\lambda_n}\langle$, where z_{λ_n} is the minimal solution of $)\overline{P}_\lambda+$with $\lambda \;\mathrm{T}\; \lambda_n$. Denote by J_{λ_n} the associated functional. Clearly $J_{\lambda_n})z_n+ < 1$, hence

$$\begin{aligned} 1 \quad &> J_{\lambda_n})z_n+ \quad \frac{2}{3_{\overline{\alpha}}^{\leq}}\rangle J_{\lambda_n}^{\infty})z_n+, z_n| \\ &\mathrm{T} \left)\frac{2}{3} \quad \frac{2}{3_{\overline{\alpha}}^{\leq}}\right\{ \backslash z_n \backslash^3_{X_0^\alpha)\mathcal{F}_\Omega+} \quad \lambda_n \left) \frac{2}{q\,0\,2} \quad \frac{2}{3_{\overline{\alpha}}^{\leq}}\right\{ \bigcap_{-} z_n^{q0\,2} dx. \end{aligned}$$

Therefore, by the Sobolev and Trace inequalities, (1.33) and (1.30) respectively, there exits a constant $C > 1$ such that $\backslash z_n \backslash_{X_0^\alpha)\mathcal{F}_\Omega+} \geq C$. As a consequence, there exists a subsequence weakly convergent to some z_Σ in $X_1^\alpha)\mathcal{F}+$. By comparison, $z_\Sigma \sim z_\lambda > 1$, for any $1 < \lambda < \Sigma$, so one gets easily that z_Σ is a weak nontrivial solution to $)\overline{P}_\lambda+$ with $\lambda \;\mathrm{T}\; \Sigma$.

Having proved the first three items in Theorem3.1.1, we focus in the sequel on proving the existence of a second solution, for which we recall that $\alpha \sim 2$.

The proof is divided into several steps: we first show that the minimal solution is a local minimum for the functional I; so we can use the Mountain Pass Theorem, obtaining a minimax Palais-Smale (PS) sequence. In the next step, in order to find a second solution, we prove a local (PS)$_c$ condition for c under a critical level $c^{\leq}$. To do that, we will construct path by localizing the minimizers of the Trace/Sobolev inequalities at the possible Dirac Deltas, given by the concentration-compactness result in Theorem 3.5.3.

We begin with a separation lemma in the C^2-topology.

Lemma 3.3.3. *Let* $1 < \mu_2 < \lambda_1 < \mu_3 < \Sigma$. *Let* z_{μ_1}, z_{λ_0} *and* z_{μ_2} *be the corresponding minimal solutions to* $)P^{\leq}_{\lambda}+$ λ T μ_2, λ_1 *and* μ_3 *respectively. If* X T $\}z$ / $C^2_1)´ +|z_{\mu_1} \geq z \geq z_{\mu_2}\langle$, *then there exists* $\varepsilon > 1$ *such that*

$$\}z_{\lambda_0}\langle \ 0 \ \ \varepsilon B_2 \ll X,$$

where B_2 *is the unit ball in* $C^2_1)´ +$.

Proof. Since $\alpha \sim 2$, we have that any solution u to $)P^{\leq}_{\lambda}+$ for arbitrary $1 < \lambda < \Sigma$ belongs to $C^{2,\gamma})¯ +$ for some positive γ, see Proposition 3.5.2. Therefore, we deduce that there exists a positive constant C such that

$$u)x+\geq C\,\mathrm{f\,ltu})x, \partial´ +, \ x \ / \ ´ . \tag{3.7}$$

On the other hand, applying Hopf Lemma, we get that there exists a positive constant c such that

$$u)x+\sim c\,\mathrm{f\,ltu})x, \partial´ +, \ x \ / \ ´ . \tag{3.8}$$

These two estimates jointly with the regularity implies the result of the lemma. □

With this result we now obtain a local minimum of the functional I in $C^2_1)´ +$, as a first step, to obtain a local minimum in $H^{\alpha/3}_1)´ +$.

Lemma 3.3.4. *For all* λ / $)1, \Sigma+$ *there exists a solution for* $)P^{\leq}_{\lambda}+$ *which is a local minimum of the functional* I *in the* C^2*-topology.*

Proof. Given $1 < \mu_2 < \lambda < \mu_3 < \Sigma$, let z_{μ_1} and z_{μ_2} be the minimal solutions of $)P^{\leq}_{\mu_1}+$ and $)P^{\leq}_{\mu_2}+$ respectively. Let z ;T z_{μ_2} z_{μ_1}. Since z_{μ_1} and z_{μ_2} are properly ordered, then

$$\begin{cases}) \ \ \Lambda+^{\alpha/3} z \sim 1 & \text{in } ´ , \\ z \text{ T } 1 & \text{on } \partial´ . \end{cases}$$

We set

$$f^{\leq})x, s+\text{T} \begin{cases} f_\lambda)z_{\mu_1})x++ & \text{if } s \geq z_{\mu_1}, \\ f_\lambda)s+ & \text{if } z_{\mu_1} \geq s \geq z_{\mu_2}, \\ f_\lambda)z_{\mu_2})x++ & \text{if } z_{\mu_2} \geq s, \end{cases}$$

$$F^{\leq})x, z+\text{T} \bigcap_1^z f^{\leq})x, s+ds$$

and

$$I^{\leq})z+\text{T } \frac{2}{3}\backslash z\backslash_{H^{\alpha/2}_0)^- +} \ \ \bigcap_{-} F^{\leq})x, u+dx.$$

Standard calculation shows that $I^{\leq}$ achieves its global minimum at some u_1 / $H^{\alpha/3}_1)´ +$, that is

$$I^{\leq})u_1+\geq I^{\leq})z+ \ \ \exists\, z \ / \ H^{\alpha/3}_1)´ +. \tag{3.9}$$

Moreover it holds

$$\begin{cases})\ \Lambda +^{\alpha/3} u_1 \ \mathrm{T}\ f^{\leq})x, u_1+ & \text{in } ´\,, \\ u_1 \ \mathrm{T}\ 1 & \text{on } \partial ´\,. \end{cases}$$

By Lemma 3.3.3, it follows that $\}u_1 \langle\ 0\ \ \varepsilon B_2 \leq X$ for $1 < \varepsilon$ small enough. Let now z satisfying

$$\backslash z \quad u_1 \backslash_{C_0^1)^- +} \geq \frac{\varepsilon}{3}.$$

As $I^{\leq})z+ \quad I)z+$is zero for every z such that $\backslash z \quad u_1 \backslash_{C_0^1)^- +} \geq \frac{\varepsilon}{3}$, by (3.9) we obtain that

$$I)z+ \mathrm{T}\ I^{\leq})z+ \sim\ I^{\leq})u_1+ \mathrm{T}\ I)u_1+\!, \quad \exists\ z\ /\ C_1^2)\,´\ +\!, \ \text{with}\ \backslash z \quad u_1 \backslash_{C_0^1)^- +} \geq \frac{\varepsilon}{3}.$$

□

To show that we have obtained the desired minimum in $H_1^{\alpha/3})\,´\ +\!,$ we now check that the result by Brezis and Nirenberg in [25] is also valid in our context.

Proposition 3.3.5. *Let* $z_1\ /\ H_1^{\alpha/3})\,´\ +$*be a local minimum of* I *in* $C_1^2)\,´\ +\!,$ *i.e., there exists* $r > 1$ *such that*

$$I)z_1 + \geq I)z_1\ 0\ \ z+ \qquad \exists z\ /\ C_1^2)\,´\ + \textit{with}\ \backslash z \backslash_{C_0^1)^- +} \geq r. \tag{3.10}$$

Then z_1 *is a local minimum of* I *in* $H_1^{\alpha/3})\,´\ +\!,$ *that is, there exists* $\varepsilon_1 > 1$ *such that*

$$I)z_1 + \geq I)z_1\ 0\ \ z+ \qquad \exists z\ /\ H_1^{\alpha/3})\,´\ + \textit{with}\ \backslash z \backslash_{H_0^{\alpha/2})^- +} \geq \varepsilon_1.$$

Proof. Arguing by contradiction we suppose that

$$\exists\, \varepsilon > 1,\ \cancel{B} z_\varepsilon\ /\ B_\varepsilon)z_1 + \ \ \text{such that} \quad I)z_\varepsilon + < I)z_1 +\!,$$

where $B_\varepsilon)z_1 + \mathrm{T}\ \Big\}z\ /\ H_1^{\alpha/3})\,´\ +;\ \ \backslash z \quad z_1 \backslash_{H_0^{\alpha/2})^- +} \geq \varepsilon \surd.$

For every $j > 1$ we consider the truncation map given by

$$T_j)r + \subseteq \begin{cases} r & 1 < r < j, \\ j & r \sim j. \end{cases}$$

Let

$$f_{\lambda,j})s + \mathrm{T}\ f_\lambda)T_j)s++\!, \qquad F_j)s + \mathrm{T} \bigcap_1^u f_{\lambda,j})s + ds\,,\ u > 1\,,$$

and

$$I_j)z + \mathrm{T}\ \frac{2}{3} \backslash z \backslash^3_{H_0^{\alpha/2})^- +} \quad \bigcap_{-} F_j)z + dx.$$

Note that for each $z \ / \ H_1^{\alpha/3})\acute{}\ +$we have that $I_j)z+ \nearrow I)z+$as $j \nearrow \in$. Hence, for each $\varepsilon > 1$ there exists $j)\varepsilon+$big enough such that $I_{j)\varepsilon+})z_\varepsilon + < I)z_1+$. Clearly $\underset{B_\varepsilon)z_0+}{\text{n lo}}\ I_{j)\varepsilon+}$is attained at some point, say v_ε. Thus we have

$$I_{j)\varepsilon+})v_\varepsilon + \geq I_{j)\varepsilon+})z_\varepsilon + < I)z_1+$$

Now we want to prove that $v_\varepsilon \nearrow z_1$ in $C_1^2)\acute{}\ +$as $\varepsilon \Rightarrow 1$. The Euler-Lagrange equation satisfied by v_ε involves a Lagrange multiplier ξ_ε in such a way that

$$\rangle I^\infty_{j)\varepsilon+})v_\varepsilon +, \varphi|_{H^{-\alpha/2})^- +, H_0^{\alpha/2})^- +} \mathrm{T}\ \xi_\varepsilon \rangle v_\varepsilon, \varphi|_{H_0^{\alpha/2})^- +}, \quad \exists \varphi \ / \ H_1^{\alpha/3})\acute{}\ +. \tag{3.11}$$

Since v_ε is a minimum of $I_{j)\varepsilon+}$, it holds

$$\xi_\varepsilon \ \mathrm{T}\ \frac{\rangle I^\infty_{j)\varepsilon+})v_\varepsilon +, v_\varepsilon|}{\backslash v_\varepsilon \backslash^3_{H_0^{\alpha/2})^- +}} \geq 1 \quad \text{for } 1 < \varepsilon \to 2, \quad \text{and} \quad \xi_\varepsilon \nearrow 1 \text{ as } \varepsilon \Rightarrow 1. \tag{3.12}$$

Note that by (3.11), v_ε satisfies the problem

$$\begin{cases})\ \Lambda +^{\alpha/3} v_\varepsilon \ \mathrm{T}\ \frac{2}{2\ \xi_\varepsilon} f_{\lambda,j)\varepsilon+})v_\varepsilon +; \mathrm{T}\ f^\varepsilon_{\lambda,j)\varepsilon+})v_\varepsilon + & \text{in } \acute{}\ , \\ v_\varepsilon \ \mathrm{T}\ 1 & \text{on } \partial\acute{}\ . \end{cases}$$

Clearly $\backslash v_\varepsilon \backslash_{H_0^{\alpha/2})^- +} \geq C$, thus, by Proposition 3.5.1, this implies that $\backslash v_\varepsilon \backslash_{L^\infty)^- +} \geq C$. Moreover, by (3.12) it follows that $\backslash f^\varepsilon_{\lambda,j)\varepsilon+})v_\varepsilon + \backslash_{L^\infty)^- +} \geq C$. Therefore, following the proof of Proposition 3.5.2, we get that $\backslash v_\varepsilon \backslash_{C^{1,r})^- +} \geq C$, for $r \ \mathrm{T}\ \text{n lo}\}q, \alpha \quad 2\langle$ and C independent of ε. By Ascoli-Arzelá Theorem there exists a subsequence, still denoted by v_ε, such that $v_\varepsilon \nearrow z_1$ uniformly in $C_1^2)\acute{}\ +$as $\varepsilon \Rightarrow 1$. This implies that for ε small enough,

$$I)v_\varepsilon + \mathrm{T}\ I_{j)\varepsilon+})v_\varepsilon + < I)z_1+$$

for any v_ε with $\backslash v_\varepsilon \quad z_1 \backslash_{C_0^1)^- +} < \varepsilon$. □

Lemma 3.3.4 and Proposition 3.3.5 provide us a local minimum in $H_1^{\alpha/3})\acute{}\ +$, which will be denoted by u_1. We now perform a traslation in order to simplify the calculations.

We consider the functions

$$g)x, s+ \mathrm{T} \begin{cases} \lambda)u_1\ 0\ s+^q \quad \lambda u_1^q\ 0\ )u_1\ 0\ s+^{3^*_\alpha\ \ 2} \quad u_1^{3^*_\alpha\ \ 2} & \text{if } s \sim 1, \\ 1 & \text{if } s < 1, \end{cases} \tag{3.13}$$

$$G)u + \mathrm{T} \bigcap_1^u g)x, s+ ds, \tag{3.14}$$

and the energy functional

$$\widetilde{I})u + \mathrm{T}\ \frac{2}{3} \backslash u \backslash^3_{H_0^{\alpha/2})^- +} \quad \bigcap_{-} G)x, u+ dx. \tag{3.15}$$

Since $u \in H_0^{\alpha/2}(\Omega)$, G is well defined and bounded from below. Let the moved problem

$$(\widetilde{P}_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2}u = g(x,u) & \text{in } \Omega \subset \mathbb{R}^N,\ \lambda > 0 \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, by standard variational theory, we know that if $\tilde{u} \not\equiv 0$ is a critical point of $\widetilde{I}$ then it is a solution of $(\widetilde{P}_\lambda)$ which, by the Maximum Principle (Lemma 2.3 of [33]), it is $\tilde{u} > 0$. Therefore $u = u_1 + \tilde{u}$ will be a second solution of (P_λ) for the sublinear case. Thus we will need to study the existence of these non-trivial critical points for I. Firstly we have

Lemma 3.3.6. *$u = 0$ is a local minimum of $\widetilde{I}$ in $H_0^{\alpha/2}(\Omega)$.*

Proof. The proof follows the lines of [4], so we will be brief in details. Note that by Proposition 3.3.5 it is sufficient to prove that $u = 0$ is a local minimum of $\widetilde{I}$ in $C_0^1(\Omega)$.

Let $u \in C_0^1(\Omega)$, then

$$G(u) = F(u_1 + u) - F(u_1) - \left(\lambda u_1^q + u_1^{2^*_\alpha - 1}\right)u. \tag{3.16}$$

Therefore

$$\begin{aligned} \widetilde{I}(u) &= \frac{1}{2}\|u\|^2_{H_0^{\alpha/2}(\Omega)} - \int_\Omega G(u)\,dx \\ &= \frac{1}{2}\|u\|^2_{H_0^{\alpha/2}(\Omega)} - \int_\Omega F(u_1 + u)\,dx + \int_\Omega F(u_1)\,dx + \int_\Omega \left(\lambda u_1^q + u_1^{2^*_\alpha - 1}\right)u\,dx. \end{aligned}$$

On the other hand,

$$\begin{aligned} I(u_1 + u) &= \frac{1}{2}\|u_1 + u\|^2_{H_0^{\alpha/2}(\Omega)} - \int_\Omega F(u_1 + u)\,dx \\ &= \frac{1}{2}\|u_1\|^2_{H_0^{\alpha/2}(\Omega)} + \frac{1}{2}\|u\|^2_{H_0^{\alpha/2}(\Omega)} \\ &\quad + \int_\Omega (-\Delta)^{\alpha/4}u_1 \, (-\Delta)^{\alpha/4}u\,dx - \int_\Omega F(u_1 + u)\,dx \\ &= \frac{1}{2}\|u_1\|^2_{H_0^{\alpha/2}(\Omega)} + \frac{1}{2}\|u\|^2_{H_0^{\alpha/2}(\Omega)} \\ &\quad + \int_\Omega \left(\lambda u_1^q + u_1^{2^*_\alpha - 1}\right)u\,dx - \int_\Omega F(u_1 + u)\,dx. \end{aligned}$$

Finally, as u_1 is a local minimum of I, we have that

$$\begin{aligned} \widetilde{I}(u) &= I(u_1 + u) - \frac{1}{2}\|u_1\|^2_{H_0^{\alpha/2}(\Omega)} + \int_\Omega F(u_1)\,dx \\ &= I(u_1 + u) - I(u_1) \\ &\geq 0 = \widetilde{I}(0). \end{aligned}$$

provided $\backslash u\backslash_{C_0^1)^- +} < \varepsilon.$ □

As a consequence of Proposition3.3.1, we obtain for the moved functional

$$\widetilde{J)}w + \mathrm{T}\ \frac{2}{3}\backslash w\backslash^3_{X_0^\alpha)\mathcal{F}\Omega +}\quad \bigcap_{-} G)w)x, 1 \nleftrightarrow dx,$$

with G as in (3.13)-(3.14), the following result.

Corollary 3.3.7. $w\ \mathrm{T}\ 1$ *is a local minimum of* $\widetilde{J}$ *in* $X_1^\alpha)\mathcal{F}\ +$.

Now assuming that $v\ \mathrm{T}\ 1$ is the unique critical point of the moved functional $\widetilde{J}$, then a local (PS)$_c$ condition can be proved for c under a critical level $c^{\leq}$,

$$c^{\leq}\ \mathrm{T}\ \frac{\alpha}{3N} S)\alpha, N +^{\frac{N}{\alpha}}. \tag{3.17}$$

Following the ideas given in [4], and by an extension of a concentration-compactness result by Lions, that we prove in Theorem 3.5.3, we obtain the following result.

Lemma 3.3.8. *If* $v\ \mathrm{T}\ 1$ *is the only critical point of* $\widetilde{J}$ *in* $X_1^\alpha)\mathcal{F}\ +$*then* $\widetilde{J}$ *satisfies a local Palais Smale condition below the critical level* $c^{\leq}$.

Proof. Let $\}w_n\langle$ be a Palais-Smale sequence for $\widetilde{J}$ verifying

$$\widetilde{J)}w_n + \nearrow c < c^{\leq}, \qquad \widetilde{J^{\infty}\!)}w_n + \nearrow 1. \tag{3.18}$$

Since the fact that w_1 is a critical point implies $\widetilde{J)}w_n +\ \mathrm{T}\ \ J)z_n +\ \ \ J)w_1 +$, where $z_n\ \mathrm{T}\ w_n\ 0\ \ w_1$, we have that

$$J)z_n + \nearrow c\, 0\ \ J)w_1 +, \qquad J^{\infty}\!)z_n + \nearrow 1. \tag{3.19}$$

On the other hand, from (3.18) we get that the sequence $\}z_n\langle$ is uniformly bounded in $X_1^\alpha)\mathcal{F}\ +$. As a consequence, up to a subsequence,

$$\begin{array}{rcll} z_n & \rightharpoonup & z & \text{weakly in } X_1^\alpha)\mathcal{F}\ + \\ z_n)\times 1 + & \nearrow & z)\times 1 + & \text{strong in } L^r)\,'\ +, \quad \exists\, 2 \geq r < 3^{\leq}_{\alpha} \\ z_n)\times 1 + & \nearrow & z)\times 1 + & \text{a.e. in }\ '\ . \end{array} \tag{3.20}$$

Note that as $v\ \mathrm{T}\ 1$ is the unique critical point of $\widetilde{J}$ then, $z\ \mathrm{T}\ w_1$.

In order to apply the concentration-compactness result, Theorem 3.5.3, first we prove the following.

Lemma 3.3.9. *The sequence* $\Big\}y^{2\ \ \alpha}\|\ \ z_n\|^{\beta}\Big\langle_{n/\mathbb{N}}$ *is tight, i.e., for any* $\eta > 1$ *there exists* $\rho_1 > 1$ *such that*

$$\bigcap_{\}y > \rho_0|}\ \bigcap_{-} y^{2\ \ \alpha}\|\ \ z_n\|^{\beta} dx dx \geq \eta, \quad \exists\, n\ /\ \mathbb{N}. \tag{3.21}$$

Proof. The proof of this lemma follows some arguments of Lemma 2.2 in [6]. By contradiction, we suppose that there exits $\eta_1 > 1$ such that, for any $\rho > 1$ one has, up to a subsequence,

$$\bigcap_{\}y>\rho|} \bigcap_{-} y^2 \;\; {}^{\alpha}\| \;\; z_n\|^{\beta} dxdy > \eta_1 \quad \text{for every } \; n \; / \; \mathbb{N}. \tag{3.22}$$

Let $\varepsilon > 1$ be fixed (to be precised later), and let $r > 1$ be such that

$$\bigcap_{\}y>r|} \bigcap_{-} y^2 \;\; {}^{\alpha}\| \;\; z\|^{\beta} dxdy < \varepsilon.$$

Let j T $\Big]\frac{M}{\kappa_\alpha \varepsilon}\Big\{$ be the integer part and I_k T $\}y \; / \; \mathbb{R}^0 \;\; ; \;\; r \; 0 \;\; k \geq y \geq r \; 0 \;\; k \; 0 \;\; 2\langle$, k T $1, \, 2, \, \ldots, \, j$. Since $\backslash z_n \backslash_{X_0^\alpha)\mathcal{F}_\Omega +} \geq M$, we clearly obtain that

$$\int_{k[\;1}^{j} \bigcap_{I_k} \bigcap_{-} y^2 \;\; {}^{\alpha}\| \;\; z_n\|^{\beta} dxdy \geq \bigcap_{\mathcal{F}_\Omega} y^2 \;\; {}^{\alpha}\| \;\; z_n\|^{\beta} dxdy \geq \varepsilon)j \; 0 \;\; 2+$$

Therefore there exists $k_1 \; / \; \}1, \, \ldots, \, j\langle$ such that (again up to a subsequence)

$$\bigcap_{I_{k_0}} \bigcap_{-} y^2 \;\; {}^{\alpha}\| \;\; z_n\|^{\beta} dxdy \geq \varepsilon, \quad \exists n. \tag{3.23}$$

Let $\chi \sim 1$ be the following regular non-decreasing cut-off function

$$\chi)y+\text{T} \left\} \begin{array}{ll} 1 & \text{if } y \geq r \; 0 \;\; k_1, \\ 2 & \text{if } y > r \; 0 \;\; k_1 \; 0 \;\; 2, \end{array} \right.$$

Define $v_n)x, y+$T $\chi)y+z_n)x, y+$ Since $v_n)x, 1+$T 1 it follows that

$$\begin{aligned} \|\rangle J^{\infty})z_n + \;\; J^{\infty})v_n +, v_n|\,\| \;\; &\text{T} \;\; \kappa_\alpha \bigcap_{\mathcal{F}_\Omega} y^2 \;\; {}^{\alpha}\rangle \;\;)z_n \;\;\; v_n+, \;\; v_n|\,dxdy \\ &\text{T} \;\; \kappa_\alpha \bigcap_{I_{k_0}} \bigcap_{-} y^2 \;\; {}^{\alpha}\rangle \;\;)z_n \;\;\; v_n+, \;\; v_n|\,dxdy. \end{aligned}$$

Moreover by the Cauchy-Schwartz inequality, (3.23) and the compact inclusion $H^2)I_{k_0} * \;\; ', y^2 \;\; {}^{\alpha}+$into $L^3)I_{k_0} * \;\; ', y^2 \;\; {}^{\alpha}+$, we have

$$\|\rangle J^{\infty})z_n + \;\; J^{\infty})v_n +, v_n|\,\| \geq \kappa_\alpha g)z_n \;\;\; v_n + g)v_n + \geq C\,\kappa_\alpha\,\varepsilon, \tag{3.24}$$

where

$$g)v+\text{T} \left)\bigcap_{I_{k_0}} \bigcap_{-} y^2 \;\; {}^{\alpha}\| \;\; v\|^{\beta} dxdy\right[^{\frac{1}{2}}.$$

On the other hand, by (3.19), we get

$$\|\rangle J^{\infty}) v_n +_{,} v_n | \, \| \geq C\, \kappa_\alpha\, \varepsilon\, 0\ \ o)2+.$$

So, for n sufficiently large,

$$\bigcap_{\}y>r0\ k_0 0\ 2|} \bigcap_{-} y^{2\ \ \alpha} \| \ \ z_n\|^{\beta} dxdy \geq \bigcap_{\mathcal{F}_\Omega} y^{2\ \ \alpha} \| \ \ v_n\|^{\beta} dxdy \ \mathrm{T}\ \frac{\rangle J^{\infty}) v_n +_{,} v_n|}{\kappa_\alpha} \geq C\,\varepsilon.$$

This is a contradiction with (3.22), which proves Lemma 3.3.9. □

Proof of Lemma 3.3.8 (cont.). In view of the previous result we can apply Theorem 3.5.3. Therefore, up to a subsequence, there exists an index set I, at most countable, a sequence of points $\}x_k\langle \ll'$, and nonnegative real numbers μ_k, ν_k, such that

$$y^{2\ \ \alpha} \| \ \ z_n\|^{\beta} \nearrow \mu \sim y^{2\ \ \alpha} \| \ \ w_1\|^{\beta}\ 0 \int_{k/I} \mu_k \delta_{x_k} \tag{3.25}$$

and

$$\|z_n)\times 1 +\|^{\beta^*_\alpha} \nearrow \nu\ \mathrm{T}\ \|w_1)\times 1 +\|^{\beta^*_\alpha}\ 0 \int_{k/I} \nu_k \delta_{x_k} \tag{3.26}$$

in the sense of measures, satisfying also the relation $\mu_k \sim S)\alpha, N+\nu_k^{\frac{2}{2^*_\alpha}}$, for every $k\ /\ I$.

We fix any $k_1\ /\ I$, and let $\phi\ /\ \mathcal{F}_1^{\infty})\mathbb{R}_0^{N0\ 2}+$be a nonincreasing cut-off function verifying $\phi\ \mathrm{T}\ 2$ in $B_2^0)x_{k_0}+$, $\phi\ \mathrm{T}\ 1$ in $B_3^0)x_{k_0}+$. Let now $\phi_\varepsilon)x, y+\mathrm{T}\ \phi)x/\varepsilon, y/\varepsilon+$, clearly $\| \ \ \phi_\varepsilon\| \geq \frac{C}{\varepsilon}$. We denote $\ _{3\varepsilon}\ \mathrm{T}\ B^0_{3\varepsilon})x_{k_0}+\wedge\}y\ \mathrm{T}\ 1\langle$. Then, using $\phi_\varepsilon z_n$ as a test function in (3.19), we have

$$\begin{aligned} &\kappa_\alpha \lim_{n\prime\ \in} \bigcap_{\mathcal{F}_\Omega} y^{2\ \ \alpha} \rangle \ \ z_n, \ \ \phi_\varepsilon| z_n dxdy \\ \mathrm{T}\ \ &\lim_{n\prime\ \in} \Big) \bigcap_{2\varepsilon} \|z_n\|^{\beta^*_\alpha} \phi_\varepsilon\, dx\ 0\ \ \lambda \bigcap_{2\varepsilon} \|z_n\|^{q0\ 2} \phi_\varepsilon\, dx \\ &\kappa_\alpha \bigcap_{B^+_{2\varepsilon})x_{k_0}+} y^{2\ \ \alpha} \| \ \ z_n\|^{\beta} \phi_\varepsilon\, dxdy \Big[. \end{aligned}$$

By (3.20), (3.25) and (3.26) we get

$$\begin{aligned} &\lim_{n\prime\ \in} \kappa_\alpha \bigcap_{\mathcal{F}_\Omega} y^{2\ \ \alpha} \rangle \ \ z_n, \ \ \phi_\varepsilon| z_n\, dxdy \\ \mathrm{T}\ \ &\bigcap_{2\varepsilon} \phi_\varepsilon\, d\nu\ 0\ \ \lambda \bigcap_{2\varepsilon} \|w_1\|^{q0\ 2} \phi_\varepsilon\, dx \quad \kappa_\alpha \bigcap_{B^+_{2\varepsilon})x_{k_0}+} \phi_\varepsilon\, d\mu. \end{aligned} \tag{3.27}$$

On the other hand, using Theorem 1.6 in [46], with $w\ \mathrm{T}\ y^{2\ \ \alpha}\ /\ A_3$ and $k\ \mathrm{T}\ 2$, we obtain that

$$\Big) \bigcap_{B^+_{2\varepsilon})x_{k_0}+} y^{2\ \ \alpha} \| \ \ \phi_\varepsilon\|^{\beta} \|z_n\|^{\beta} dxdy \Big[^{2/3} \geq \frac{3}{\varepsilon} \Big) \bigcap_{B^+_{2\varepsilon})x_{k_0}+} y^{2\ \ \alpha} \|z_n\|^{\beta} dxdy \Big[^{2/3}$$

$$\geq \quad C \Big) \bigcap_{B^+_{2\varepsilon})x_{k_0}+} y^2 \;\; {}^{\alpha} \| \;\; z_n \|^{\beta} dxdy \Big[^{2/3}.$$

Since $z_n \; / \; X^{\alpha}_1)\mathcal{F} +$, the last expression goes to zero as $\varepsilon \nearrow 1$. Therefore

$$\begin{aligned} 1 \quad &\geq \quad \lim_{n\prime \;\; \in} \Big\langle \bigcap_{\mathcal{F}_\Omega} y^2 \;\; {}^{\alpha} \rangle \quad z_n, \quad \phi_\varepsilon | \, z_n dxdy \Big\langle \\ &\geq \quad \lim_{n\prime \;\; \in} \Big) \bigcap_{\mathcal{F}_\Omega} y^2 \;\; {}^{\alpha} \| \;\; z_n \|^{\beta} dxdy \Big\{^{2/3} \Big) \bigcap_{B^+_{2\varepsilon})x_{k_0}+} y^2 \;\; {}^{\alpha} \| \;\; \phi_\varepsilon \|^{\beta} \| z_n \|^{\beta} dxdy \Big[^{2/3} \\ &\nearrow 1 \quad . \end{aligned}$$

Hence, by (3.27), it follows that

$$1 \;\mathrm{T}\; \lim_{\varepsilon\prime \;\; 1} \Big] \bigcap_{2\varepsilon} \phi_\varepsilon \, d\nu \; 0 \;\; \lambda \bigcap_{2\varepsilon} \| w_1 \|^{q\, 0 \; 2} \phi_\varepsilon \, dx \quad \kappa_\alpha \bigcap_{B^+_{2\varepsilon})x_{k_0}+} \phi_\varepsilon \, d\mu \Big\{ \geq \nu_{k_0} \quad \kappa_\alpha \mu_{k_0}.$$

Therefore we get that

$$\nu_{k_0} \;\mathrm{T}\; 1 \qquad \text{or} \qquad \nu_{k_0} \sim S)\alpha, N+^{\frac{N}{\alpha}}.$$

Suppose that $\nu_{k_0} \;\mathrm{T}\; 1$. It follows that

$$\begin{aligned} c \, 0 \;\; J)w_1+ \;\; \mathrm{T} \quad &\lim_{n\prime \;\; \in} J)z_n+ \quad \frac{2}{3}\rangle J^{\prime})z_n +, z_n | \\ &\sim \quad \frac{\alpha}{3N} \bigcap_{-} w_1^{3^*_\alpha} dx \; 0 \;\; \frac{\alpha}{3N} \nu_{k_0} \; 0 \;\; \lambda \Big) \frac{2}{3} \quad \frac{2}{q \; 0 \;\; 2} \Big\{ \bigcap_{-} w_1^{q 0 \; 2} dx \\ &\sim \quad J)w_1 + 0 \;\; \frac{\alpha}{3N} S)\alpha, N+^{\frac{N}{\alpha}} \;\; \mathrm{T} \;\; J)w_1 + 0 \;\; c^{\leq}. \end{aligned}$$

Then we get a contradiction with (3.18), and since k_1 was arbitrary, $\nu_k \;\mathrm{T}\; 1$ for all $k \; / \; I$. Hence as a consequence, $u_n \nearrow u_1$ in $L^{3^*_\alpha})' +$. We finish in the standard way: convergence of u_n in $L^{\frac{2N}{N-\alpha}})' +$ implies convergence of $f)u_n+$ in $L^{\frac{2N}{N+\alpha}})' +$, and finally by using the continuity of the inverse operator $) \quad \Lambda+^{\alpha/3}$, we obtain convergence of u_n in $H^{\alpha/3}_1)' +$. □

Now it remains to show that we can obtain a local $(\mathrm{PS})_c$ sequence for $\widetilde{J}$ under the critical level $c \;\mathrm{T}\; c^{\leq}$. To do that we will use $w_\varepsilon \;\mathrm{T}\; \mathrm{E}_\alpha)u_\varepsilon +$, the family of minimizers to the Trace inequality (1.30), where u_ε is given in (1.32). We remark that, despite the cases $\alpha \;\mathrm{T}\; 2$ and $\alpha \;\mathrm{T}\; 3$, w_ε does not possesses an explicit expression. This is an extra difficulty that we have to overcome. Taking into account that the family u_ε is self-similar, $u_\varepsilon)x+ \mathrm{T} \; \varepsilon^{\frac{\alpha-N}{2}} u_2)x/\varepsilon+$ and the fact that the Poisson kernel (1.6) is also self-similar

$$P^{\alpha}_y)x+ \mathrm{T} \; \frac{2}{y^N} P^{\alpha}_2 \Big) \frac{x}{y} \Big\{, \tag{3.28}$$

gives easily that the family w_ε satisfies

$$w_\varepsilon(x,y)=\varepsilon^{\frac{\alpha-N}{2}}w_1\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right). \tag{3.29}$$

We will denote $P^\alpha = P_1^\alpha$. Also, we will write $w_{1,\alpha}$ instead of w_1 to emphasize the dependence on the parameter α.

Lemma 3.3.10. *With the above notation it holds*

$$|\nabla w_{1,\alpha}(x,y)|\le \frac{C}{y}\,w_{1,\alpha}(x,y),\quad \alpha>0,\ (x,y)\in\mathbb{R}_+^{N+1} \tag{3.30}$$

and

$$|\nabla w_{1,\alpha}(x,y)|\le C w_{1,\alpha-1}(x,y),\quad \alpha>1,\ (x,y)\in\mathbb{R}_+^{N+1}. \tag{3.31}$$

Proof. Differentiating with respect to each variable x_i, $i=1,\dots,N$, and the variable y, it follows that

$$\begin{aligned}
|\partial_{x_i}w_{1,\alpha}(x,y)| &\le \int_{\mathbb{R}^N}\frac{(N+\alpha)y^\alpha|x-z|}{(y^2+|x-z|^2)^{\frac{N+\alpha}{2}+1}(1+|z|^2)^{\frac{N-\alpha}{2}}}\,dz\\
&\le \frac{N+\alpha}{2y}\int_{\mathbb{R}^N}\frac{y^\alpha}{(y^2+|x-z|^2)^{\frac{N+\alpha}{2}}(1+|z|^2)^{\frac{N-\alpha}{2}}}\,dz\\
&= \frac{C}{y}w_{1,\alpha}(x,y)
\end{aligned}$$

and

$$\begin{aligned}
|\partial_y w_{1,\alpha}(x,y)| &= \left|\int_{\mathbb{R}^N}\frac{y^{\alpha-1}(\alpha|x-z|^2-Ny^2)}{(y^2+|x-z|^2)^{\frac{N+\alpha}{2}+1}(1+|z|^2)^{\frac{N-\alpha}{2}}}\,dz\right|\\
&\le C\int_{\mathbb{R}^N}\frac{y^{\alpha-1}}{(y^2+|x-z|^2)^{\frac{N+\alpha}{2}}(1+|z|^2)^{\frac{N-\alpha}{2}}}\,dz\\
&= \frac{C}{y}w_{1,\alpha}(x,y).
\end{aligned}$$

Therefore we get (3.30). To obtain (3.31) we recall that $u_{1,\alpha}(z)=(1+|z|^2)^{-\frac{N-\alpha}{2}}$. Then, by (3.28) it follows that

$$\begin{aligned}
|\partial_y w_{1,\alpha}(x,y)| &= \left|\partial_y\left(\int_{\mathbb{R}^N}\frac{1}{y^N}P^\alpha\left(\frac{x-z}{y}\right)u_{1,\alpha}(z)\,dz\right)\right|\\
&= \left|\partial_y\left(\int_{\mathbb{R}^N}P^\alpha(z)u_{1,\alpha}(x-yz)\,dz\right)\right|\\
&= \left|\int_{\mathbb{R}^N}P^\alpha(z)\langle z,\nabla u_{1,\alpha}(x-yz)\rangle\,dz\right|
\end{aligned}$$

$$
\begin{aligned}
&\mathrm{T} \quad \Bigg(\bigcap_{\mathbb{R}^N} \frac{2}{y^N} P^{\alpha}\Big) \frac{x \quad z}{y} \Big\{ \rangle \frac{x \quad z}{y}, \quad u_{2,\alpha})z+| dz \Bigg(\\
&\geq \quad)N \quad \alpha + \bigcap_{\mathbb{R}^N} \frac{2}{y^N} P^{\alpha}\Big) \frac{x \quad z}{y} \Big\{ \frac{\|x \quad z\|}{y} \frac{\|z\|}{)2\,0\ \|z\|^{\beta} + ^{\frac{N-\alpha}{2}\,0\,2}} dz \\
&\geq \quad)N \quad \alpha + \bigcap_{\mathbb{R}^N} \frac{y^{\alpha \quad 2}}{)y^3\,0\ \|x \quad z\|^{\beta} + ^{\frac{N+\alpha-1}{2}})2\,0\ \|z\|^{\beta} + ^{\frac{N-\alpha+1}{2}}} dz \\
&\mathrm{T} \quad C w_{2,\alpha \quad 2})x, y+.
\end{aligned}
$$

Doing the same calculations in variables x_i for i T $2, \ldots, N$, we obtain

$$
\begin{aligned}
|\partial_{x_i} w_{2,\alpha})x, y+| \quad &\mathrm{T} \quad \Bigg(\partial_{x_i} \Big) \bigcap_{\mathbb{R}^N} P^{\alpha})z+u_{2,\alpha})x \quad yz+dz \Big\{ \Bigg(\\
&\geq \quad \bigcap_{\mathbb{R}^N} P^{\alpha})z+| \quad u_{2,\alpha}|)x \quad yz+dz \\
&\mathrm{T} \quad \bigcap_{\mathbb{R}^N} \frac{2}{y^N} P^{\alpha}\Big) \frac{x \quad z}{y} \Big\{ | \quad u_{2,\alpha}|)z+dz \\
&\geq \quad)N \quad \alpha + \bigcap_{\mathbb{R}^N} \frac{y^{\alpha}}{)y^3\,0\ \|x \quad z\|^{\beta} + ^{\frac{N+\alpha}{2}}} \frac{\|z\|}{)2\,0\ \|z\|^{\beta} + ^{\frac{N-\alpha}{2}\,0\,2}} dz \\
&\mathrm{T} \quad C w_{2,\alpha \quad 2})x, y+.
\end{aligned}
$$

□

Let us now introduce a cut-off function $\phi_1)s+/\ C^{\in}\ )\mathbb{R}_0\ +$, nonincreasing satisfying

$$
\phi_1)s+\mathrm{T}\ 2 \text{ if } 1 \geq s \geq \frac{2}{3}, \quad \phi_1)s+\mathrm{T}\ 1 \text{ if } s \sim 2.
$$

Assume without loss of generality that $1\ /\ ^{\prime}$. We then define, for some fixed $r > 1$ small enough such that $\overline{B}_r^{0} \leq \overline{\mathcal{F}}$, the function $\phi)x, y+\mathrm{T}\ \phi_r)x, y+\mathrm{T}\ \phi_1)\frac{r_{xy}}{r}+$with r_{xy} T $|)x, y+|$ T $)\|x\|^{\beta}\,0\ y^3+^{2/3}$. Note that $\phi\omega_{\varepsilon}\ /\ X_1^{\alpha})\mathcal{F}\ +$. Thus we get

Lemma 3.3.11. *With the above notation, the family* $\}\phi w_{\varepsilon}\langle$, *and its trace on* $\}y$ T $1\langle$, *namely* $\}\phi u_{\varepsilon}\langle$, *satisfy*

$$
\backslash \phi w_{\varepsilon} \backslash^3_{X_0^{\alpha})\mathcal{F}_{\Omega}+} \geq \backslash w_{\varepsilon} \backslash^3_{X_0^{\alpha})\mathcal{F}_{\Omega}+} 0\ \ O)\varepsilon^{N \quad \alpha}+, \tag{3.32}
$$

$$
\backslash \phi u_{\varepsilon} \backslash^3_{L^2)^-+} \mathrm{T} \left\} \begin{array}{ll} C\varepsilon^{\alpha}\,0\ \ O)\varepsilon^{N \quad \alpha}+ & \textit{if } N > 3\alpha, \\ C\varepsilon^{\alpha} \mathrm{mi}\ )2/\varepsilon+0\ \ O)\varepsilon^{\alpha}+ & \textit{if } N \mathrm{\ T\ } 3\alpha, \end{array} \right. \tag{3.33}
$$

and

$$
\backslash \phi u_{\varepsilon} \backslash^r_{L^r)^-+} \sim c\varepsilon^{\frac{N-\alpha}{2}}, \quad \alpha < N < 3\alpha, \quad r \mathrm{\ T\ } \frac{N\,0\ \ \alpha}{N \quad \alpha}, \tag{3.34}
$$

for ε *small enough and* $C > 1$.

Proof. The product ϕw_ε satisfies

$$
\begin{aligned}
\backslash \phi w_\varepsilon \backslash^3_{X_0^\alpha)\mathcal{F}_\Omega+} \quad \mathrm{T} \quad & \kappa_\alpha \bigcap_{\mathcal{F}_\Omega} y^2 \quad {}^\alpha)\|\phi \quad w_\varepsilon\|^\beta \, 0 \ \|w_\varepsilon \quad \phi\|^\beta \, 0 \ 3\rangle w_\varepsilon \quad \phi, \phi \quad w_\varepsilon| + dxdy \\
\geq \quad & \backslash w_\varepsilon \backslash^3_{X_0^\alpha)\mathcal{F}_\Omega+} 0 \ \kappa_\alpha \bigcap_{\mathcal{F}_\Omega} y^2 \quad {}^\alpha \|w_\varepsilon \quad \phi\|^\beta dxdy \qquad (3.35) \\
& 0 \ 3\kappa_\alpha \bigcap_{\mathcal{F}_\Omega} y^2 \quad {}^\alpha\rangle w_\varepsilon \quad \phi, \phi \quad w_\varepsilon| \, dxdy.
\end{aligned}
$$

To estimate the second term of the right hand side, we observe that $1 \geq u_\varepsilon)x+ \geq \varepsilon^{\frac{N-\alpha}{2}} \|x\|^\alpha \quad {}^N$ and $\mathrm{E}_\alpha)\|x\|^\alpha \quad {}^N+ \mathrm{T} \ )\|x\|^\beta \ 0 \ y^3+^{\frac{\alpha-N}{2}} \ \mathrm{T} \ r^\alpha_{xy} \quad {}^N$. Let $\quad_r \ \mathrm{T} \ \}r/3 \geq r_{xy} \geq r\langle \ll \mathcal{F}$. Then

$$
\begin{aligned}
\bigcap_{\mathcal{F}_\Omega} y^2 \quad {}^\alpha \|w_\varepsilon \quad \phi\|^\beta dxdy \quad \geq \quad & C \bigcap_r y^2 \quad {}^\alpha w_\varepsilon^3 dxdy \\
\geq \quad & C\varepsilon^N \quad {}^\alpha \bigcap_r y^2 \quad {}^\alpha r_{xy}^{3)\alpha \quad N+} dxdy \qquad (3.36) \\
\mathrm{T} \quad & O)\varepsilon^N \quad {}^\alpha +.
\end{aligned}
$$

For the remaining term we need to use the properties of the function w_ε given in Proposition 3.3.10. By (3.29) we get

$$
\begin{aligned}
& \bigcap_{\mathcal{F}_\Omega} y^2 \quad {}^\alpha\rangle w_\varepsilon \quad \phi, \phi \quad w_\varepsilon| \, dxdy && \geq \\
& C \bigcap_r y^2 \quad {}^\alpha \|w_\varepsilon)x, y+ \backslash \quad w_\varepsilon)x, y+\| dxdy && \mathrm{T} \\
& C\varepsilon \quad {}^{N0\,\alpha \quad 2} \bigcap_r y^2 \quad {}^\alpha \Big(w_{2,\alpha} \Big) \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \Big(\Big(\Big(\quad w_{2,\alpha} \Big) \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \Big(\Big(dxdy && \mathrm{T} \qquad (3.37) \\
& C\varepsilon \bigcap_{\frac{r}{\varepsilon}} y^2 \quad {}^\alpha \|w_{2,\alpha})x, y+\|\| \quad w_{2,\alpha})x, y+\| dxdy.
\end{aligned}
$$

Moreover, for $)x, y+/ \quad_{r/\varepsilon}$ and $\alpha > 1$, we obtain that

$$
\begin{aligned}
w_{2,\alpha})x, y+ \quad \mathrm{T} \quad & \bigcap_{\|z\| < \frac{1}{4\varepsilon}} P_y^\alpha)x \quad z+ u_{2,\alpha})z+ dz \ 0 \bigcap_{\|z\| > \frac{1}{4\varepsilon}} P_y^\alpha)x \quad z+ u_{2,\alpha})z+ dz \\
\geq \quad & C\varepsilon^{N0\,\alpha} y^\alpha \bigcap_{\|z\| < \frac{1}{4\varepsilon}} \frac{dz}{\|z\|^{N \quad \alpha}} \ 0 \ C\varepsilon^N \quad {}^\alpha \bigcap_{\mathbb{R}^N} P_y^\alpha)z+ dz \qquad (3.38) \\
\geq \quad & Cy^\alpha \varepsilon^N \ 0 \ C\varepsilon^N \quad {}^\alpha \geq C\varepsilon^N \quad {}^\alpha.
\end{aligned}
$$

If $\alpha < 2$, from (3.30), (3.37) and (3.38), it follows that

$$
\bigcap_{\mathcal{F}_\Omega} y^2 \quad {}^\alpha\rangle w_\varepsilon \quad \phi, \phi \quad w_\varepsilon| \, dxdy \geq C\varepsilon^{20\;3)N \quad \alpha+} \bigcap_{\frac{r}{\varepsilon}} y \quad {}^\alpha dxdy \ \mathrm{T} \ O)\varepsilon^N \quad {}^\alpha +. \qquad (3.39)
$$

To obtain the similar estimate for $\alpha > 2$ we use (3.31). Indeed by this estimate, together with (3.37) and (3.38) we get that

$$\bigcap_{\mathcal{F}_\Omega} y^{2\ \ \alpha}\rangle w_\varepsilon \quad \phi, \phi \quad w_\varepsilon| \, dxdy \geq C\varepsilon^{3)20\ N\ \ \alpha +} \bigcap_{\frac{r}{\varepsilon}} y^{2\ \ \alpha} dxdy \text{ T } O)\varepsilon^{N\ \ \alpha} +\!\!\!\cdot \tag{3.40}$$

Note that for α T 2, as w_ε is explicit, we can obtain the same estimate directly.

Then we have proved that

$$\backslash \phi w_\varepsilon \backslash^3_{X_0^\alpha)\mathcal{F}_\Omega +} \geq \backslash w_\varepsilon \backslash^3_{X_0^\alpha)\mathcal{F}_\Omega +} 0 \ \ O)\varepsilon^{N\ \ \alpha} +\!\!\!\cdot$$

We now show that (3.33) holds.

$$\begin{aligned}
\backslash \phi u_\varepsilon \backslash^3_{L^2)^- +} \quad & \text{T} \quad \bigcap_{-} \phi^3)x + \frac{\varepsilon^{N\ \ \alpha}}{)\|x\|^\beta \ 0 \ \ \varepsilon^3 +^{N\ \ \alpha}} dx \\
& \sim \bigcap_{\}\|x\| < r/3|} \frac{\varepsilon^{N\ \ \alpha}}{)\|x\|^\beta \ 0 \ \ \varepsilon^3 +^{N\ \ \alpha}} dx \\
& \sim \bigcap_{\}\|x\| < \varepsilon|} \frac{\varepsilon^{N\ \ \alpha}}{)3\varepsilon^3 +^{N\ \ \alpha}} dx \ 0 \bigcap_{\}\varepsilon < \|x\| < r/3|} \frac{\varepsilon^{N\ \ \alpha}}{)3\|x\|^\beta +^{N\ \ \alpha}} dx \\
& \text{T} \quad C\varepsilon^\alpha \ 0 \ \ C\varepsilon^{N\ \ \alpha} \bigcap_\varepsilon^{r/3} \theta^{3\alpha\ \ 2\ \ N} d\theta.
\end{aligned}$$

Finally, (3.34) follows in a similar way to (3.33), so we omit the details. $\square$

With the above properties in mind, we define the family of functions η_ε T $\frac{\phi w_\varepsilon}{\phi u_\varepsilon \ \ L^{2^*_\alpha}(\Omega)}$.

Lemma 3.3.12. *There exists* $\varepsilon > 1$ *small enough such that*

$$\operatorname*{tvr}_{t \to 1} \widetilde{J)} t\eta_\varepsilon + < c^{\leq}. \tag{3.41}$$

Proof. Assume $N \sim 3\alpha$, we make use of the following estimate

$$)a \ 0 \ \ b +^p \sim a^p \ 0 \ \ b^p \ 0 \ \ \mu a^{p\ \ 2} b, \quad a, b \sim 1, \quad p > 2, \quad \text{for some } \mu > 1. \tag{3.42}$$

Therefore

$$G)w + \sim \frac{2}{3^{\leq}_{\alpha}} w^{3^*_\alpha} \ 0 \ \ \frac{\mu}{3} w^3 w_1^{3^*_\alpha\ \ 3} \tag{3.43}$$

which implies

$$\widetilde{J)} t\eta_\varepsilon + \geq \frac{t^3}{3} \backslash \eta_\varepsilon \backslash^3_{X_0^\alpha)\mathcal{F}_\Omega +} \quad \frac{t^{3^*_\alpha}}{3^{\leq}_{\alpha}} \quad \frac{t^3}{3} \mu \bigcap_{-} w_1^{3^*_\alpha\ \ 3} \eta_\varepsilon^3 dx.$$

Since there exists $a_1 > 1$ such that $w_1 \sim a_1$ in the support of η_ε we have

$$\widetilde{J)} t\eta_\varepsilon + \geq \frac{t^3}{3} \backslash \eta_\varepsilon \backslash^3_{X_0^\alpha)\mathcal{F}_\Omega +} \quad \frac{t^p}{p} \quad \frac{t^3}{3} \widetilde{\mu \backslash} \eta_\varepsilon \backslash^3_{L^2)^- +\!\!\!\cdot}$$

Since $\backslash u_\varepsilon \backslash_{L^{2^*_\alpha})^- +}$is independent of ε, by Lemma 3.3.11 we have

$$\backslash \eta_\varepsilon \backslash^3_{X_0^\alpha)\mathcal{F}_\Omega +} \geq S)\alpha, N+0 \;\; O)\varepsilon^{N \quad \alpha}+ \tag{3.44}$$

and

$$\backslash \eta_\varepsilon \backslash^3_{L^2)^- +} \sim \left.\begin{array}{ll} O)\varepsilon^\alpha + & \text{si } N > 3\alpha, \\ O)\varepsilon^\alpha \text{ mpi })2/\varepsilon ++ & \text{si } N \text{ T } 3\alpha. \end{array}\right\}$$

This implies

$$\widetilde{J})t\eta_\varepsilon + \geq \frac{t^3}{3})S)\alpha, N+0 \;\; C\varepsilon^{N \quad \alpha}+ \quad \frac{t^p}{p} \quad \frac{t^3}{3} C\varepsilon^\alpha \;;\text{T } g)t+.$$

It is clear that $\lim_{t' \;\in} g)t+ \text{ T } \quad \in$, and therefore $\text{tvr}_{t \to 1} g)t+$is achieved at some point $t_\varepsilon \sim 1$. If t_ε T 1 the result s trivially deduced. Let us suppose $t_\varepsilon > 1$. When derivating above's function we have

$$1 \text{ T } g')t_\varepsilon + \text{T } t_\varepsilon)S)\alpha, N+0 \;\; C\varepsilon^{N \quad \alpha}+ \quad t_\varepsilon^{p \quad 2} \quad t_\varepsilon C\varepsilon^\alpha, \tag{3.45}$$

which implies

$$t_\varepsilon \geq)S)\alpha, N+0 \;\; C\varepsilon^{N \quad \alpha} +^{\frac{1}{p-2}}.$$

Observe that by (3.45) we have that for $\varepsilon > 1$ small enough

$$t_\varepsilon^{p \quad 2} \text{ T } S)\alpha, N+0 \;\; C\varepsilon^{N \quad \alpha} \quad C\varepsilon^\alpha \sim C > 1$$

and then $t_\varepsilon \sim C > 1$ for some constant C. On the other hand, the function

$$t \not\nearrow \frac{t^3}{3})S)\alpha, N+0 \;\; C\varepsilon^{N \quad \alpha}+ \quad \frac{t^p}{p}$$

is increasing in $]1,)S)\alpha, N+0 \;\; C\varepsilon^{N \quad \alpha}+^{\frac{1}{p-2}}\hat{}$. From which

$$\text{tvr}_{t \to 1} g)t+ \text{T } g)t_\varepsilon + \geq \frac{\alpha}{3N})S)\alpha, N+0 \;\; C\varepsilon^{N \quad \alpha}+^{\frac{2N}{\alpha}} \quad \widetilde{C}\varepsilon^\alpha.$$

For some constant $\widetilde{C} > 1$. Therefore, for $N > 3\alpha$, we have

$$g)t_\varepsilon + \geq \frac{\alpha}{3N} S)\alpha, N+^{\frac{N}{\alpha}} \; 0 \;\; C\varepsilon^{N \quad \alpha} \quad C\varepsilon^\alpha < \frac{\alpha}{3N} S)\alpha, N+^{\frac{N}{\alpha}} \text{ T } c^{\leq}. \tag{3.46}$$

If N T 3α the same conclusion follows.

The last case $\alpha < N < 3\alpha$ follows by using the estimate (3.42) which gives

$$G)w + \sim \frac{2}{3^{\leq}_{\alpha}} w^{3^*_\alpha} \; 0 \;\; w_1 w^{3^*_\alpha \quad 2}. \tag{3.47}$$

Then (3.47) jointly with (3.34) and arguing in a similar way as above finish the proof. □

Proof of Theorem 3.1.1-(3).

To finish the last statement in Theorem 3.1.1, in view of the previous results, we seek for critical values below level $c^{\leq}$. For that purpose, we want to use the classical MP Theorem by Ambrosetti-Rabinowitz in [5]. We define

$$_{\varepsilon}\ \mathrm{T}\ \}\gamma\ /\ \mathcal{F})]1,2\hat{,}\ X_1^{\alpha})\mathcal{F}\ ++;\ \gamma)1+\mathrm{T}\ 1,\ \gamma)2+\mathrm{T}\ t_{\varepsilon}\eta_{\varepsilon}\langle$$

for some $t_{\varepsilon} > 1$ such that $\widetilde{J)}t_{\varepsilon}\eta_{\varepsilon}+< 1$. And consider the minimax value

$$c_{\varepsilon}\ \mathrm{T}\ \log_{\gamma/\ \ _{\varepsilon}}\ \mathrm{n\ d\tilde{}}\}\widetilde{J)}\gamma)t++;\ 1 \geq t \geq 2\langle.$$

According to Lemma 3.3.6, $c_{\varepsilon} \sim 1$. By Lemma 3.3.12, for $\varepsilon \to 2$,

$$c_{\varepsilon} \geq \mathop{\mathrm{tvr}}_{t \to 1} \widetilde{J)}t\eta_{\varepsilon}+< c^{\leq}\,\mathrm{T}\ \frac{\alpha}{3N}S)\alpha, N+^{\frac{N}{\alpha}}.$$

This estimate jointly with Lemma 3.3.8 and the MPT [5] if the minimax energy level is positive, or the refinement of the MPT [49] if the minimax level is zero, give the existence of a second solution to $)P_{\lambda}^{\leq}+$. □

3.4. Linear and superlinear cases.

3.4.1. Linear case

The proof of Theorem 3.1.2 follows the ideas of [24]. Note that for $\alpha\ \mathrm{T}\ 2$, where the minimizers given in (3.29) are explicit, this result was recently proved in [80].
The first part of that theorem is an straightforward calculus.

Proof of Theorem 3.1.2 (1). Let φ_2 be the first eigenfunction of $)\ \ \Lambda +^{\alpha/3}$ in $'$. We have

$$\bigcap_{-})\ \ \Lambda +^{\alpha/=}u)\ \ \Lambda +^{\alpha/=}\varphi_2\,dx\ \mathrm{T}\ \bigcap_{-}\lambda_2 u\varphi_2\,dx.$$

On the other hand,

$$\bigcap_{-})\ \ \Lambda +^{\alpha/=}u)\ \ \Lambda +^{\alpha/=}\varphi_2\,dx\ \mathrm{T}\ \bigcap_{-}]u^{3^*_{\alpha}\ \ 2}0\ \ \lambda u\hat{}\varphi_2\,dx > \bigcap_{-}\lambda u\varphi_2\,dx.$$

This clearly implies $\lambda < \lambda_2$. □

To prove the second part of Theorem 3.1.2 some notation is in order. We consider the following Rayleigh quotient

$$Q_{\lambda})w+\mathrm{T}\ \frac{\backslash w\backslash^3_{X_0^{\alpha})\mathcal{F}_{\Omega}+}\ \ \lambda\backslash u\backslash^3_{L^2)^-+}}{\backslash u\backslash^3_{L^{2^*_{\alpha}})^-+}}$$

and

$$S_{\lambda}\ \mathrm{T}\ \log\}Q_{\lambda})w+|\ w\ /\ X_1^{\alpha})\mathcal{F}\ +\langle. \tag{3.48}$$

Proposition 3.4.1. *Assume* $1 < \lambda < \lambda_2$. *Then* $S_\lambda < S)\alpha, N+$.

Proof. Let $\phi \; \mathrm{T} \; \phi_r$ be a cut-off function like in Lemma 3.3.11 and denote $\phi)x+;\mathrm{T}$ $\phi)x, 1+$. Taking r sufficiently small we can use $\phi w_\varepsilon \; / \; X_1^\alpha)\mathcal{F} +$ as a test function in Q_λ, where w_ε is defined in (3.29). Denoting $K_2 \; \mathrm{T} \; \backslash u_\varepsilon \backslash^{3^*_\alpha}_{L^{2^*_\alpha})^- +}$, as before, K_2 is independent of ε, and moreover

$$\begin{aligned} \bigcap_{-} \|\phi u_\varepsilon\|^{\beta^*_\alpha} dx \quad & \mathrm{T} \quad \bigcap_{\mathbb{R}^N} \|\phi u_\varepsilon\|^{\beta^*_\alpha} dx \\ & \sim \quad \bigcap_{\|x\|< r/3} \|u_\varepsilon\|^{\beta^*_\alpha} dx \\ & \mathrm{T} \quad K_2 \quad \bigcap_{\|x\|> r/3} \|u_\varepsilon\|^{\beta^*_\alpha} dx \\ & \sim \quad K_2 \, 0 \;\; O)\varepsilon^N +. \end{aligned} \tag{3.49}$$

Since w_ε is a minimizer of $S)\alpha, N+$, we have that

$$K_2^{\;3/3^*_\alpha} \kappa_\alpha \bigcap_{\mathbb{R}^{N+1}_+} y^{2 \quad \alpha} \| \;\; w_\varepsilon \|^\beta \, dxdy \; \mathrm{T} \; S)\alpha, N+. \tag{3.50}$$

Finally, by (3.49) and using the estimates (3.32) and (3.33), for $N > 3\alpha$, we obtain that

$$Q_\lambda)\phi w_\varepsilon + \geq \frac{\kappa_\alpha \bigcap_{\mathbb{R}^{N+1}_+} y^{2 \quad \alpha} \| \;\; w_\varepsilon \|^\beta \, dxdy \quad \lambda C \varepsilon^\alpha \, 0 \;\; O)\varepsilon^{N \quad \alpha} +}{K_2^{3/3^*_\alpha} \, 0 \;\; O)\varepsilon^N +}.$$

Therefore taking ε small enough, we get

$$\begin{aligned} Q_\lambda)\phi w_\varepsilon + \; & \geq \; \frac{S)\alpha, N + \quad \lambda C \varepsilon^\alpha K_2^{\;3/3^*_\alpha} \, 0 \;\; O)\varepsilon^{N \quad \alpha} +}{2 \, 0 \;\; O)\varepsilon^N +} \\ & \geq \; S)\alpha, N + \quad \lambda C \varepsilon^\alpha K_2^{\;3/3^*_\alpha} \, 0 \;\; O)\varepsilon^{N \quad \alpha} + \\ & < \; S)\alpha, N+. \end{aligned}$$

On the other hand, a similar calculus for the case $N \; \mathrm{T} \; 3\alpha$, proves that for ε small enough,

$$Q_\lambda)\phi w_\varepsilon + \geq S)\alpha, N + \quad \lambda C \varepsilon^\alpha \, \mathrm{pi} \,)2/\varepsilon + K_2^{\;3/3^*_\alpha} \, 0 \;\; O)\varepsilon^\alpha + < S)\alpha, N+,$$

which finishes the proof. □

Recall now the Brezis-Lieb Lemma,

Lemma 3.4.2 ([20]). *Let ´ be an open set and* $\}u_n\langle$ *be a sequence weakly convergent in* $L^q)´+, 3 \geq q < \in$ *and a.e. convergent in ´. Then* $\lim_{n/\in})\backslash u_n\backslash^q_{L^q)^-+} \quad \backslash u_n \; u\backslash^q_{L^q)^-+} + \mathrm{T} \; \backslash u\backslash^q_{L^q)^-+}$.

This property allows us to we prove the following one.

Proposition 3.4.3. *Assume* $1 < \lambda < \lambda_2$. *Then the infimum* S_λ *defined in* (3.48) *is achieved.*

Proof. First, since $\lambda < \lambda_2$ we have that $S_\lambda > 1$. Let us take a minimizing sequence of S_λ, $\}w_m\langle \ll X_1^\alpha)\mathcal{F}+$ such that, without loss of generality, $w_m \sim 1$ and $\backslash w_m)\cancel{x}1+\backslash_{L^{2^*_\alpha})^-+}\mathrm{T}\; 2$. Clearly this implies that $\backslash w_m\backslash_{X_0^\alpha)\mathcal{F}_\Omega+} \geq C$, then there exists a subsequence (still denoted by $\}w_m\langle$) verifying

$$\begin{array}{rcll} w_m & \rightharpoonup & w & \text{weakly in } X_1^\alpha)\mathcal{F}+, \\ w_m)\cancel{x}1+ & \nearrow & w)\cancel{x}1+ & \text{strongly in } L^q)´+, \; 2 \geq q < 3\frac{\leq}{\alpha}, \\ w_m)\cancel{x}1+ & \nearrow & w)\cancel{x}1+ & \text{a.e in } ´. \end{array} \tag{3.51}$$

A simple calculation, using the weak convergence, gives that

$$\begin{aligned} \backslash w_m\backslash^3_{X_0^\alpha)\mathcal{F}_\Omega+} \;\; & \mathrm{T} \quad \backslash w_m \quad w\backslash^3_{X_0^\alpha)\mathcal{F}_\Omega+} 0 \;\; \backslash w\backslash^3_{X_0^\alpha)\mathcal{F}_\Omega+} \\ & 0 \quad 3\kappa_\alpha \bigcap_{\mathcal{F}_\Omega} y^2 \;\; {}^\alpha\rangle \quad w, \quad w_m \qquad w|\,dxdy \\ & \mathrm{T} \quad \backslash w_m \quad w\backslash^3_{X_0^\alpha)\mathcal{F}_\Omega+} 0 \;\; \backslash w\backslash^3_{X_0^\alpha)\mathcal{F}_\Omega+} 0 \;\; o)2+. \end{aligned}$$

By Lemma 3.4.2, we have that $\backslash)w_m \quad w+)\cancel{x}1+\backslash_{L^{2^*_\alpha})^-+} \geq 2$ for m big enough. Hence

$$\begin{aligned} Q_\lambda)w_m+ \;\; & \mathrm{T} \quad \backslash w_m\backslash^3_{X_0^\alpha)\mathcal{F}_\Omega+} \quad \lambda\backslash w_m)\cancel{x}1+\backslash^3_{L^2)^-+} \\ & \mathrm{T} \quad \backslash w_m \quad w\backslash^3_{X_0^\alpha)\mathcal{F}_\Omega+} 0 \;\; \backslash w\backslash^3_{X_0^\alpha)\mathcal{F}_\Omega+} \quad \lambda\backslash w_m)\cancel{x}1+\backslash^3_{L^2)^-+} 0 \;\; o)2+ \\ & \sim \quad S)\alpha, N+\backslash)w_m \quad w+)\cancel{x}1+\backslash^3_{L^{2^*_\alpha})^-+} 0 \;\; S_\lambda\backslash w)\cancel{x}1+\backslash^3_{L^{2^*_\alpha})^-+} 0 \;\; o)2+ \\ & \sim \quad S)\alpha, N+\backslash)w_m \quad w+)\cancel{x}1+\backslash^{3^*_\alpha}_{L^{2^*_\alpha})^-+} 0 \;\; S_\lambda\backslash w)\cancel{x}1+\backslash^{3^*_\alpha}_{L^{2^*_\alpha})^-+} 0 \;\; o)2+. \end{aligned}$$

By Lemma 3.4.2 again, this leads to

$$\begin{aligned} Q_\lambda)w_m+ \;\; & \sim \quad)S)\alpha, N+ \quad S_\lambda+\backslash)w_m \quad w+)\cancel{x}1+\backslash^{3^*_\alpha}_{L^{2^*_\alpha})^-+} 0 \;\; S_\lambda\backslash w_m)\cancel{x}1+\backslash^{3^*_\alpha}_{L^{2^*_\alpha})^-+} 0 \;\; o)2+ \\ & \mathrm{T} \quad)S)\alpha, N+ \quad S_\lambda+\backslash)w_m \quad w+)\cancel{x}1+\backslash^{3^*_\alpha}_{L^{2^*_\alpha})^-+} 0 \;\; S_\lambda \; 0 \;\; o)2+. \end{aligned}$$

Since $\}w_m\langle$ is a minimizing sequence for S_λ, we obtain:

$$o)2+0 \;\; S_\lambda \sim)S)\alpha, N+ \quad S_\lambda+\backslash)w_m \quad w+)\cancel{x}1+\backslash^{3^*_\alpha}_{L^{2^*_\alpha})^-+} 0 \;\; S_\lambda \; 0 \;\; o)2+.$$

Thus by Proposition 3.4.1

$$w_m)\cdot\!,1+\nearrow w)\cdot\!,1+ \qquad \text{in } L^{3^*_\alpha})\acute{}\ +\!.$$

Finally, by a standard lower semi-continuity argument, w is a minimizer for Q_λ. □

Proof of Theorem 3.1.2 (2). By Proposition 3.4.3 there exists an α-harmonic function $w\ /\ X_1^\alpha)\mathcal{F}\ +\!,$ such that $\backslash u\backslash^3_{L^{2^*_\alpha})^-+}$ T 2 and

$$\backslash w\backslash^3_{X_0^\alpha)\mathcal{F}_\Omega+} \quad \lambda\backslash u\backslash^3_{L^2)^-+}\mathrm{T}\ S_\lambda$$

where u T $w)\cdot\!,1+\!.$ Without loss of generality we may assume $w\sim 1$ (otherwise we take $\|w\|$ instead of w). So we get a positive solution of $)P^{\leq}_\lambda+\!.$ □

3.4.2. Superlinear case.

In order to prove Theorem 3.1.3, the only difficult part is to show that we have a (PS)$_c$ sequence under the critical level c T $c^{\leq}$. This follows the same type of computations like in Lemma 3.3.12, with the estimate $\backslash\eta_\varepsilon\backslash^{q0\ 2}_{L^{q+1})^-+}\sim C\varepsilon^{\frac{\alpha-N}{2}q0\ \frac{\alpha+N}{2}}$ which holds for $N>\alpha)2\ 0\ \frac{2}{q}+\!.$ In this case there is no limitation on $\lambda>1$. We omit the complete details.

3.5. Regularity and Concentration-Compactness

We begin this section with some results about the boundedness and regularity of solutions. The next proposition is a refinement of Proposition 2.4.3 in order to cover the critical case p T $3^{\leq}_\alpha\quad 2$. It is essentially based on [22].

Proposition 3.5.1. *Let* $u\ /\ H_1^{\alpha/3})\acute{}\ +$*be a solution to the problem*

$$\begin{cases})\ \ \Lambda+^{\alpha/3}u\ \mathrm{T}\ f)x,u+ & \text{in }\acute{}\ ,\\ u>1 & \text{in }\acute{}\ ,\\ u\ \mathrm{T}\ 1 & \text{on }\partial\acute{} \end{cases} \tag{3.52}$$

with f *satisfying*

$$1\geq f)x,s+\geq C)2\ 0\ \|s\|^p+\ \ \exists)x,s+/\ \acute{}\ *\ \mathbb{R},\ \textit{and some } 1<p\geq 3^{\leq}_\alpha\quad 2. \tag{3.53}$$

Then $u\ /\ L^{\in}\ )\acute{}\ +$*with* $\backslash u\backslash_{L^\infty)^-+}\geq C)\backslash u\backslash_{H_0^{\alpha/2})^-+}+$

Proof. Let $w\ /\ X_1^\alpha)\mathcal{F}\ +$be a solution to the problem

$$\begin{cases} \mathrm{f\ lx})y^2\ \ ^\alpha\quad w+\mathrm{T}\ 1 & \text{in }\mathcal{F}\ ,\\ \dfrac{\partial w}{\partial\nu^\alpha}\ \mathrm{T}\ f)\cdot\!,w+ & \text{in }\acute{}\ ,\\ w\ \mathrm{T}\ 1 & \text{on }\partial_L\mathcal{F}\ . \end{cases} \tag{3.54}$$

Then u T $w)\times 1+$is a solution to (3.52). Let

$$a)x+;\mathrm{T}\ \frac{f)x,u+}{2\,0\ \ u)x+}.$$

Clearly

$$1 \geq a \geq C)2\,0\ \ u^{p\quad 2}+/\ L^{\frac{N}{\alpha}})\,'\,+, \qquad \text{for}\ 1 < p \geq 3^{\leq}_{\alpha}\quad 2. \tag{3.55}$$

Given $T > 1$ we denote

$$w_T\ \mathrm{T}\ w\quad)w\quad T+_0\,, \quad u_T\ \mathrm{T}\ w_T)\times 1+.$$

For $\beta \sim 1$ we have

$$\begin{aligned} \backslash ww_T^{\beta}\backslash^{3}_{X_0^{\alpha})\mathcal{F}_\Omega+}\mathrm{T}\quad & \kappa_\alpha \bigcap_{\mathcal{F}_\Omega} y^{2\quad \alpha} w_T^{3\beta}\|\ \ w\|^{\beta}\,dxdy \\ & 0\ \kappa_\alpha)3\beta\,0\ \ \beta^{3}+\bigcap_{\}w\geq T|} y^{2\quad \alpha} w^{3\beta}\|\ \ w\|^{\beta}\,dxdy. \end{aligned}$$

Using φ T $ww_T^{3\beta}\ \ /\ X_1^{\alpha})\mathcal{F}\ +$as a test function we obtain

$$\kappa_\alpha \bigcap_{\mathcal{F}_\Omega} y^{2\quad \alpha}\rangle\quad w,\quad)ww_T^{3\beta}+\!\!\!|\,dxdy\ \mathrm{T} \bigcap_{\_} f)u+\!\!\!uu_T^{3\beta}\,dx \geq 3\bigcap_{\_} a)2\,0\ \ u^{3}+\!\!\!u_T^{3\beta}\,dx.$$

On the other hand, it is clear that

$$\begin{aligned} & \bigcap_{\mathcal{F}_\Omega} y^{2\quad \alpha}\rangle\quad w,\quad)ww_T^{3\beta}+\!\!\!|\,dxdy\ \mathrm{T} \bigcap_{\mathcal{F}_\Omega} y^{2\quad \alpha} w_T^{3\beta}\|\ \ w\|^{\beta}\,dxdy0 \\ & 0\ 3\beta \bigcap_{\}w\geq T|} y^{2\quad \alpha} w^{3\beta}\|\ \ w\|^{\beta}\,dxdy. \end{aligned}$$

Summing up, we have

$$\backslash ww_T^{\beta}\backslash^{3}_{X_0^{\alpha})\mathcal{F}_\Omega+} \geq C\bigcap_{\_} a)2\,0\ \ u^{3}+\!\!\!u_T^{3\beta}\,dx,$$

which by (1.30) implies that

$$\backslash uu_T^{\beta}\backslash^{3}_{L^{2^*_\alpha})^{-}+} \geq \widetilde{C}\bigcap_{\_} a)2\,0\ \ u^{3}+\!\!\!u_T^{3\beta}\,dx, \tag{3.56}$$

with $\widetilde{C}$ some positive constant depending on α, β, N and $\|'\ \|$. To compute the term on the right-hand side we add the hypothesis $u^{\beta 0\,2}\ \ /\ L^{3})\,'\,+$. With this assumption we get

$$\begin{aligned} & \bigcap_{\_} au^{3}u_T^{3\beta}\,dx \geq T_1\bigcap_{\}a<\ T_0|} u^{3}u_T^{3\beta}\,dx\,0 \bigcap_{\}a\to T_0|} au^{3}u_T^{3\beta}\,dx \\ & \geq C_2T_1\,0\ \Bigg)\bigcap_{\}a\to T_0|} a^{\frac{N}{\alpha}}\,dx\Bigg[^{\frac{\alpha}{N}}\Bigg)\bigcap_{\_})uu_T^{\beta}+^{3^*_\alpha}dx\Bigg\{^{\frac{2}{2^*_\alpha}}. \end{aligned}$$

By the same calculation,

$$\bigcap_{-} au_T^{3\beta}\, dx \geq C_3 T_1\, 0 \,\Bigg)\bigcap_{\}a\to T_0|} a^{\frac{N}{\alpha}}\, dx\Bigg[^{\frac{\alpha}{N}} \Bigg)\bigcap_{-})u_T^{\beta}+^{3^*_\alpha}\, dx\Bigg\{^{\frac{2}{2^*_\alpha}},$$

where, since $u^{\beta 0\ 2}\ /\ L^3)' +\!\!\!,\ C_2$ and C_3 can be taken independent of T. Hence, by (3.55) it follows that

$$\epsilon)T_1{+}\mathrm{T}\,\Bigg)\bigcap_{\}a\to T_0|} a^{\frac{N}{\alpha}}\, dx\Bigg[^{\frac{\alpha}{N}} \nearrow 1 \quad \text{as } T_1 \nearrow \in .$$

Therefore, choosing T_1 large enough such that $C\epsilon)T_1+< \frac{2}{3}$, by (3.56), we obtain that there exists a constant $K)T_1+\!\!\!,$ independent of T, for which it holds

$$\backslash uu_T^{\beta}\backslash^3_{L^{2^*_\alpha})^-+} \geq K)T_1+\!\!\!.$$

Letting $T \nearrow \in$ we conclude that $u^{\beta 0\ 2}\ /\ L^{3^*_\alpha})' +\!\!\!.$ Clearly we can obtain that $f)\times u+/\ L^r)' +$for some $r > N/\alpha$, in a finite number of steps. Thus, we conclude applying Theorem 2.3.3. ☐

Now we characterize the regularity of the solutions of $)P^{\leq}_{\lambda}+$for the whole range of exponents.

Proposition 3.5.2. *Let* u *be a solution of* $)P^{\leq}_{\lambda}+\!\!\!.$ *Then the following holds*

(i) *If* α T 2 *and* $q \sim 2$ *then* $u\ /\ C^{\in}\ )^{\overline{\ }'}+\!\!\!.$

(ii) *If* α T 2 *and* $q < 2$ *then* $u\ /\ C^{2,q})^{\overline{\ }'}+\!\!\!.$

(iii) *If* $\alpha < 2$ *then* $u\ /\ C^{\alpha})^{\overline{\ }'}+\!\!\!.$

(iv) *If* $\alpha > 2$ *then* $u\ /\ C^{2,\alpha\ \ 2})^{\overline{\ }'}+\!\!\!.$

Proof. First we observe that, by Proposition 3.5.1, we have $u\ /\ L^{\in}\ )' +$and also $f_\lambda)u+/\ L^{\in}\ )' +\!\!\!.$

(i) Applying Proposition 3.1 of [28], we get that $u\ /\ C^{\gamma})^{\overline{\ }'}+\!\!\!,$ for some $\gamma < 2$. Since $q \sim 2$ then $f_\lambda)u+/\ C^{\gamma})^{\overline{\ }'}+\!\!\!,$ so, again by Proposition 3.1 of [28], it follows that $u\ /\ C^{2,\gamma})^{\overline{\ }'}+\!\!\!.$ Iterating the process we conclude that $u\ /\ C^{\in}\ )^{\overline{\ }'}+\!\!\!.$

(ii) As before we have $u\ /\ C^{\gamma})^{\overline{\ }'}+\!\!\!,$ for some $\gamma < 2$. Therefore $f_\lambda)u+/\ C^{q\gamma})^{\overline{\ }'}+\!\!\!.$ It follows that $u\ /\ C^{2,q\gamma})^{\overline{\ }'}+\!\!\!,$ which gives $f_\lambda)u+/\ C^{q})^{\overline{\ }'}+\!\!\!.$ Finally this implies $u\ /\ C^{2,q})^{\overline{\ }'}+\!\!\!.$

(iii) By Lemma 2.8 of [33] we obtain that $u\ /\ C^{\gamma})^{\overline{\ }'}+$for all $\gamma\ /\ )1, \alpha+\!\!\!.$ This implies that $f_\lambda)u+/\ C^{r})^{\overline{\ }'}+$for every $r <$ n lo$\}q\alpha,\ \alpha\langle$. Therefore, again by [33], this time using Lemmas 2.7 and 2.9, we get that $u\ /\ C^{\alpha})^{\overline{\ }'}+\!\!\!.$

(iv) Since $\alpha > 2$, we can write problem $)P_{\lambda}^{\leq}+$ as follows

$$\left\{\begin{array}{lll}) \quad \Lambda+^{2/3}u \mathrm{~T~} s & \text{in } ´ , \\ ) \quad \Lambda+^{)\alpha \quad 2+/3}s \mathrm{~T~} f_{\lambda})u+ & \text{in } ´ , \\ u \mathrm{~T~} s \mathrm{~T~} 1 & \text{on } \partial ´ . \end{array}\right. \tag{3.57}$$

Reasoning as before, we obtain the desired regularity in two steps, using Proposition 3.1 in [28] and Lemmas 2.7 and 2.9 in [33].

□

We end this section adapting to our setting a concentration-compactness result by P.L. Lions [60], used in the proof of Lemma 3.3.8. This property has been used in [4, 24, 53] for the standard case, and for example [10, 72] for a different nonlocal operators which include a different fractional Laplacian. We recall that a related concentration-compactness result for the fractional Laplacian has been recently obtained in [64]. Nevertheless, we need the version corresponding to the extended problem, and it cannot be deduced from the one in [64].

Theorem 3.5.3. *Let $\}w_n\langle_{n/\mathbb{N}}$ be a weakly convergent sequence to w in $X_1^{\alpha})\mathcal{F}+$ such that the sequence $\}y^{2 \quad \alpha}\| \quad w_n\|^{\beta}\langle_{n/\mathbb{N}}$ is tight. Let $u_n \mathrm{~T~} Tr)w_n+$ and $u \mathrm{~T~} Tr)w+$ Let μ, ν be two non negative measures such that*

$$y^{2 \quad \alpha}\| \quad w_n\|^{\beta} \nearrow \mu \qquad \textit{and} \qquad \|u_n\|^{\beta^*_\alpha} \nearrow \nu, \quad \textit{as } n \nearrow \in \tag{3.58}$$

in the sense of measures. Then there exist an at most countable set I and points $\}x_i\langle_{i/I} \ll ´$ such that

1. $\nu \mathrm{~T~} \|u\|^{\beta^*_\alpha} 0 \displaystyle\int_{k/I} \nu_k\delta_{x_k},\ \nu_k > 1,$

2. $\mu \sim y^{2 \quad \alpha}\| \quad w\|^{\beta} 0 \displaystyle\int_{k/I} \mu_k\delta_{x_k},\ \mu_k > 1,$

3. $\mu_k \sim S)\alpha, N+\nu_k^{\frac{2}{2^*_\alpha}}.$

Proof. Let $\varphi \ / \ C_1^{\in})\overline{\mathcal{F}}+$ By the trace inequality (1.30) with $r \mathrm{~T~} 3^{\leq}_{\alpha}$ it follows that

$$S)\alpha, N+\Big)\bigcap_{-} \|\varphi w_n\|^{\beta^*_\alpha}dx\Big\{^{3/3^*_\alpha} \geq \kappa_\alpha \bigcap_{\mathcal{F}_\Omega} y^{2 \quad \alpha}\| \quad)\varphi w_n+\|^{\beta}dxdy. \tag{3.59}$$

Let $K^{\leq}$;T $K_2 * \ K_3 \leq \overline{\mathcal{F}}$ be the support of φ and suppose first that the weak limit

w T 1. Then we get that

$$\begin{aligned}
&\bigcap_{\mathcal{F}_\Omega} y^2 \;\; {}^{\alpha}\| \;\;)\varphi w_n +\!\!\|^{\beta} dxdy \;\mathrm{T} \bigcap_{K^*} y^2 \;\; {}^{\alpha}\| \;\;)\varphi w_n +\!\!\|^{\beta} dxdy \\
&\mathrm{T} \bigcap_{K^*} y^2 \;\; {}^{\alpha}\|w_n\|^{\beta}\| \;\; \varphi\|^{\beta} dxdy \; 0 \bigcap_{K^*} y^2 \;\; {}^{\alpha}\|\varphi\|^{\beta}\| \;\; w_n\|^{\beta} dxdy \qquad (3.60)\\
&0 \; 3 \Big| \bigcap_{K^*} y^2 \;\; {}^{\alpha} w_n \varphi\rangle \;\; \varphi, \;\; w_n| \, dxdy.
\end{aligned}$$

Since $K^{\leq}$ is a bounded domain, and $y^2 \;\; {}^{\alpha}$ is an A_3 weight, we have the compact inclusion

$$H^2)K^{\leq}, y^2 \;\; {}^{\alpha}+ \nearrow\!\nearrow L^r)K^{\leq}, y^2 \;\; {}^{\alpha}+, \; 2 \geq r < \frac{3)N \; 0 \;\; 2+}{N \;\;\; 2}, \; \alpha \; / \;)1, 3+.$$

Therefore, for a suitable subsequence, we get the limit

$$\bigcap_{K^*} y^2 \;\; {}^{\alpha}\|w_n\|^{\beta}\| \;\; \varphi\|^{\beta} dxdy \nearrow 1, \quad \text{as } n \nearrow \in .$$

By the weak convergence, given by hypothesis, we obtain

$$\bigcap_{K^*} y^2 \;\; {}^{\alpha} w_n \varphi\rangle \;\; \varphi, \;\; w_n| \, dxdy \nearrow 1, \quad \text{as } n \nearrow \in .$$

Hence, by (3.58) we conclude that

$$\bigcap_{\mathcal{F}_\Omega} y^2 \;\; {}^{\alpha}\| \;\;)\varphi w_n +\!\!\|^{\beta} dxdy \nearrow \bigcap_{\mathcal{F}_\Omega} \|\varphi)x, y +\!\!\|^{\beta} d\mu, \quad \text{as } n \nearrow \in .$$

Then, from (3.59) we get

$$S)\alpha, N+\Big)\bigcap_{-} \|\varphi\|^{\beta^*_\alpha} d\nu \Big\{^{3/3^*_\alpha} \geq \kappa_\alpha \bigcap_{\mathcal{F}_\Omega} \|\varphi\|^{\beta} d\mu, \quad \exists \; \varphi \; / \; C_1^{\in} \,)\overline{\mathcal{F}} \; +. \qquad (3.61)$$

If now w T 1, we apply the above result to the function v_n T w_n w. Indeed if

$$y^2 \;\; {}^{\alpha}\| \;\; v_n\|^{\beta} \nearrow d\mu \qquad \text{and} \qquad \|v_n) \times 1 +\!\!\|^{\beta^*_\alpha} \nearrow d\nu, \quad \text{as } n \nearrow \in ,$$

it follows that

$$S)\alpha, N+\Big)\bigcap_{-} \|\varphi\|^{\beta^*_\alpha} d\nu \Big\{^{3/3^*_\alpha} \geq \kappa_\alpha \bigcap_{\mathcal{F}_\Omega} \|\varphi\|^{\beta} d\mu \,, \quad \exists \; \varphi \; / \; C_1^{\in} \,)\overline{\mathcal{F}} \; +,$$

therefore, ([60]), for some sequence of points $\}x_k\langle_{k/I} \ll {}'$, we have

$$d\nu \;\mathrm{T} \int_{k/I} \nu_k \delta_{x_k} \,, \qquad d\mu \sim \int_{k/I} \mu_k \delta_{x_k} \,,$$

with $\mu_k \sim S)\alpha, N + \nu_k^{3^*_\alpha/3}$. Hence, by Lemma 3.4.2, we obtain

$$d\nu \;\mathrm{T}\; \|\mu\|^{\beta^*_\alpha} \; 0 \int_{k/I} \nu_k \delta_{x_k}.$$

Let now φ be a test function. We have

$$\bigcap_{\mathcal{F}_\Omega} y^2 \;\; {}^\alpha\varphi\| \;\; w_n\|^\beta dxdy \quad \mathrm{T} \quad \bigcap_{\mathcal{F}_\Omega} y^2 \;\; {}^\alpha\varphi\| \;\; w\|^\beta dxdy \; 0 \bigcap_{\mathcal{F}_\Omega} y^2 \;\; {}^\alpha\varphi\| \;\;)w_n \quad w+\|^\beta dxdy$$

$$0\; 3 \bigcap_{\mathcal{F}_\Omega} y^2 \;\; {}^\alpha\varphi\rangle \quad)w_n \quad w+, \quad w|\, dxdy.$$

Taking limits as $n \nearrow \in$ we get that

$$\bigcap_{\mathcal{F}_\Omega} \varphi d\mu \quad \mathrm{T} \quad \bigcap_{\mathcal{F}_\Omega} y^2 \;\; {}^\alpha\varphi\| \;\; w\|^\beta dxdy \; 0 \bigcap_{\mathcal{F}_\Omega} \varphi d\mu$$

$$\sim \quad \bigcap_{\mathcal{F}_\Omega} y^2 \;\; {}^\alpha\varphi\| \;\; w\|^\beta dxdy \; 0 \bigcap_{\mathcal{F}_\Omega} y^2 \;\; {}^\alpha\varphi \int_{k/I} \mu_k \delta_{x_k} dxdy,$$

with the same condition $\mu_k \sim S)\alpha, N + \nu_k^{3^*_\alpha/3}$. So we obtain the desired conclusion. $\square$

4

Perturbations of a critical fractional equation

4.1. Introduction

In this last chapter we study perturbations of order zero of the problem (3.1). Namely, we will focus on the problem

$$(P)\quad \left\{ \begin{array}{ll} (-\Delta)^{\alpha/2}u = |u|^{\frac{2\alpha}{N-\alpha}}u + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \right.$$

where $1 < \alpha < 2$, $N > \alpha$ and f belongs to a suitable space.

The equivalent problem for the classical Laplace operator $-\Delta$ was previously studied in [67] and [81]. We follow the approach of the latter along the chapter. We remark that a parallel work on this problem, for positive solutions, has been performed in [74].

The operator $L(u) = (-\Delta)^{\alpha/2}u - |u|^{p-2}u$ is well defined from $H_0^{\alpha/2}(\Omega)$ into its dual $H^{-\alpha/2}(\Omega)$ by the Sobolev inequality, see (1.33). Thus it is natural to consider data f in that space: we have that $f \in H^{-\alpha/2}(\Omega)$ if and only if $f = (-\Delta)^{\alpha/2}g$ with $g \in H_0^{\alpha/2}(\Omega)$; the associated norm is given by $\|f\|_{H^{-\alpha/2}} = \|g\|_{H_0^{\alpha/2}}$.

Finally we will consider solutions of Problem (P) in the following sense.

Definition 4.1.1. *Let $f \in H^{-\alpha/2}(\Omega)$. We say that $u \in H_0^{\alpha/2}(\Omega)$ is an energy solution* *to problem* (P) *if it holds*

$$\int_\Omega (-\Delta)^{\alpha/2}u\,(-\Delta)^{\alpha/2}\psi\,dx = \int_\Omega (|u|^{p-2}u - f)\psi\,dx, \quad \forall \psi \in H_0^{\alpha/2}(\Omega). \tag{4.1}$$

4.2. Main results and preliminaries

We will focus on functions $f \in H^{-\alpha/2}(\Omega)$ that are small in the following sense

$$\int_\Omega f\varphi < c(\alpha,N)\|\varphi\|_{H_0^{\alpha/2}}^{(N+\alpha)/\alpha}, \qquad \forall \varphi \in H_0^{\alpha/2}(\Omega) \text{ with } \|\varphi\|_p = 1, \tag{4.2}$$

where $c(\alpha,N) = \frac{2\alpha}{N+\alpha}\left(\frac{N-\alpha}{N+\alpha}\right)^{(N-\alpha)/2\alpha}$. The main result of the chapter is the following

Theorem 4.2.1. *Assume* $f \not\equiv 0$ *satisfies* (4.2)*. Then the problem* (P) *has at least two solutions. Moreover, if* $f \geq 0$ *a.e. in* Ω *then these solutions are nonnegative a.e. in* Ω*.*

We will also prove that, if we relax the strict inequality in condition (4.2), namely we replace it with the condition

$$\int_\Omega f\varphi \leq c(\alpha,N)\|\varphi\|_{H_0^{\alpha/2}}^{(N+\alpha)/\alpha}, \qquad \forall \varphi \in H_0^{\alpha/2}(\Omega) \text{ with } \|\varphi\|_p = 1, \tag{4.3}$$

then we still obtain the existence of at least one solution.

Theorem 4.2.2. *Assume* $f \not\equiv 0$ *satisfies* (4.3)*. Then the problem* (P) *has at least one solution. Moreover, if* f *is nonnegative a.e. in* Ω *then this solution is non-negative a.e. in* Ω*.*

The condition (4.2) is equivalent to

$$\int_\Omega f\varphi < c(\alpha,N)\frac{\|\varphi\|_{H_0^{\alpha/2}}^{(N+\alpha)/\alpha}}{\|\varphi\|_p^{N/\alpha}}, \qquad \forall \varphi \in H_0^{\alpha/2}(\Omega) \setminus \{0\}. \tag{4.4}$$

Moreover, since

$$\int_\Omega f\varphi \leq \|f\|_{H^{-\alpha/2}}\|\varphi\|_{H_0^{\alpha/2}}, \tag{4.5}$$

then using the Sobolev inequality (1.33) we obtain the following sufficient condition on f to satisfy (4.2)

$$\|f\|_{H^{-\alpha/2}} \leq c(\alpha,N)S(\alpha,N)^{N/2\alpha}. \tag{4.6}$$

Remark 4.2.1. *1. We point out that an assumption on the size of f is natural in order to find solutions of Problem (P). In fact, if for example f is a positive large enough constant then Problem (P) has no solutions.*

2. Condition (4.6) *seems to be not sharp in view of the result in [34] for the case $\alpha = 2$.*

The associated energy functional to problem (P) is given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left|(-\Delta)^{\alpha/4} u\right|^2 dx - \frac{1}{p}\int_\Omega |u|^p - \int_\Omega f u \, dx.$$

Again *critical points* of I correspond to solutions of (P) in the sense of (4.1). Indeed, one of the solutions we will construct in the proof of Theorem 4.2.1 is a local minimum of I in $H_0^{\alpha/2}(\Omega)$.

4.3. Proof of Theorem 4.2.1

4.3.1. First Solution

We start with the definition of the Nehari manifold associated to problem (P),

$$\mathcal{N} = \{u \in H_0^{\alpha/2}(\Omega), u \not\equiv 0 \; ; \langle I'(u), u\rangle = 0\}.$$

It is natural to look for solutions in this manifold. Note that the condition $u \in \mathcal{N}$ is equivalent to the identity

$$\|u\|^2_{H_0^{\alpha/2}} = \|u\|_p^p + \int_\Omega f u. \tag{4.7}$$

Therefore the functional I restricted to $\mathcal{N}$ takes the equivalent forms

$$\begin{aligned} I(u) &= \frac{\alpha}{2N}\|u\|^2_{H_0^{\alpha/2}} - \frac{N+\alpha}{2N}\int_\Omega f u \\ &= \frac{\alpha}{2N}\|u\|_p^p - \frac{1}{2}\int_\Omega f u. \end{aligned} \tag{4.8}$$

We will use both expressions in the sequel. In particular, using the first one we deduce that the functional I is bounded from below on $\mathcal{N}$:

$$I(u) \geq \frac{\alpha}{2N}\|u\|^2_{H_0^{\alpha/2}} - \frac{N+\alpha}{2N}\|f\|_{H^{-\alpha/2}}\|u\|_{H_0^{\alpha/2}} \geq -\frac{(N+\alpha)^2}{8N\alpha}\|f\|^2_{H^{-\alpha/2}}, \tag{4.9}$$

where the last step is a consequence of the minimization of the function $\alpha t^2 - (N+\alpha)\|f\|_{H^{-\alpha/2}} t$.

Remark 4.3.1. *Taking* (4.9) *into account it makes sense to define*

$$c_1 \mathrm{T} \log_{\cap} I > \quad \in , \tag{4.10}$$

while the functional is not bounded from below in the whole space $H_1^{\alpha/3})' +$

Note that if u_1 is a local minimum of I in $H_1^{\alpha/3})'$ +then necessarily

$$\backslash u_1\backslash^3_{H_0^{\alpha/2}} \quad)p \quad 2+\backslash u_1\backslash^p_p \sim 1.$$

In fact, as we will prove in Lemma 4.3.4 this inequality is strict, namely

$$\backslash u_1\backslash^3_{H_0^{\alpha/2}} \quad)p \quad 2+\backslash u_1\backslash^p_p > 1. \tag{4.11}$$

In the same way, if u_1 is a local maximum of I it holds

$$\backslash u_1\backslash^3_{H_0^{\alpha/2}} \quad)p \quad 2+\backslash u_1\backslash^p_p < 1. \tag{4.12}$$

Thus, we first minimize the functional I restricted to $\mathcal{S}$ in order to find a critical point and therefore a solution to the problem $)P+$. As we will see, c_1 is achieved. To prove that we start with some preliminary results.

Lemma 4.3.1. *Let* $f \subseteq 1$ *satisfy* (4.2). *Given* $u \;/\; H_1^{\alpha/3})'$ +*assume* $\sum f u > 1$. *Then there exist two unique constants* $1 < \sigma)u+< \tau)u+$*such that both* $\sigma)u+u$, $\tau)u+u \;/\; \mathcal{S}$ *and verify the inequalities* (4.11) *and* (4.12) *respectively.*

Proof. Let $\theta)t+\mathrm{T}\; t\backslash u\backslash^3_{H_0^{\alpha/2}} \quad t^p \;\; {}^2\backslash u\backslash^p_p$. We can compute the point of maximum value of this function,

$$t_M \;\mathrm{T} \left) \frac{)N \quad \alpha+\backslash u\backslash^3_{H_0^{\alpha/2}}}{)N\,0\;\; \alpha+\backslash u\backslash^p_p} \sum^{)N \quad \alpha+/3\alpha} \right. ,$$

and

$$\theta)t_M+\mathrm{T}\; \frac{3\alpha}{N \quad \alpha}\left)\frac{N \quad \alpha}{N\,0\;\;\alpha}\right\{^{)N0\;\alpha+/3\alpha} \frac{\backslash u\backslash^{)N0\;\alpha+/\alpha}_{H_0^{\alpha/2}}}{\backslash u\backslash_p^{N/\alpha}} \;\mathrm{T}\; c)\alpha, N+\frac{\backslash u\backslash^{)N0\;\alpha+/\alpha}_{H_0^{\alpha/2}}}{\backslash u\backslash_p^{N/\alpha}}.$$

Note that θ is a concave function, increasing on $)1, t_M+$and decreasing on $)t_M, \in +$, with $\lim_{t\prime\;\in} \theta)t+\mathrm{T} \quad \in$. By (4.4) we get $1 < \sum f u\, dx < \theta)t_M+$ Thus there exist two unique values $1 < \sigma < t_M < \tau$ such that

$$\theta)\tau+\mathrm{T} \bigcap_{-} f u\, dx \;\mathrm{T}\; \theta)\sigma+, \qquad \theta^{\infty})\tau+< 1 < \theta^{\infty})\sigma+ \tag{4.13}$$

Multiplying in the previous expression by τ we have

$$0 = \tau\theta'(\tau) + \tau\int_\Omega fu\,dx = \|\tau u\|^2_{H_0^{\alpha/2}} - \|\tau u\|_p^p - \int_\Omega \tau fu,$$

thus $\tau u \in \mathcal{S}$. Moreover,

$$\|\tau u\|^2_{H_0^{\alpha/2}} - (p-1)\|\tau u\|_p^p = \tau^2\theta''(\tau) < 0.$$

Arguing in a similar way for σ, we obtain $\sigma u \in \mathcal{S}$ and

$$\|\sigma u\|^2_{H_0^{\alpha/2}} - (p-1)\|\sigma u\|_p^p = \sigma^2\theta''(\sigma) > 0.$$

□

Observe that without the condition $\int_\Omega fu > 0$ we still can find a value $\tau > 0$ with $\tau u \in \mathcal{S}$ satisfying (4.11). Conversely, the condition $\int_\Omega fu > 0$ is guaranteed for any function $u \in \mathcal{S}$ that satisfies (4.11).

We notice that the purpose of the strict condition (4.2) on f in the previous Lemma is just to obtain $\int_\Omega fu\,dx < \theta(t_M)$. It also appears to be of importance in Lemma 4.3.3 below. It is known that, when one deals with the problem associated to the standard Laplacian and under certain hypothesis, the condition (4.2) is not sharp, see [34]. We suspect that a similar fact can occur in our case.

Corollary 4.3.2. *In the hypotheses of Lemma 4.3.1, it holds* $I(\tau u) = \max_{t\ge\sigma} I(tu)$ *and* $I(\sigma u) = \min_{0\le t\le\tau} I(tu)$.

Proof. It is straightforward once we notice that the function $g(t) = I(tu)$ satisfies $g'(t) = \theta(t) - \int_\Omega fu\,dx$. □

The next property uses a technical result analogous to Lemma 2.2 in [81]. The proof follows almost word by word the proof performed in that paper, see also [23]. We only have to adapt the calculations to the functional framework of the fractional Laplacian, we leave the details for the interested reader.

Lemma 4.3.3. *Let* $f \not\equiv 0$ *satisfy* (4.2). *Then*

$$\mu_1 := \inf_{u\in H_0^{\alpha/2}(\Omega),\ \|u\|_p = 1}\left\{c(\alpha,N)\|u\|^{(N+\alpha)/\alpha}_{H_0^{\alpha/2}} - \int_\Omega fu\,dx\right\} \tag{4.14}$$

is achieved and moreover $\mu_1 > 0$.

Next, the following lemma establishes a crucial property for minima of the functional, see inequality (4.11).

Lemma 4.3.4. *Let* $f \subseteq 1$ *satisfy* (4.2) *and let* $u \;/\; \mathcal{S}$. *Then*

$$\backslash u\backslash^{3}_{H_0^{\alpha/2}} \quad)p \quad 2+\!\!\backslash u\backslash^{p}_{p}\ \mathrm{T}\ 1.$$

Proof. Consider the functional, defined for $u \;/\; H_1^{\alpha/3})' + u \subseteq 1$,

$$\phi)u+\mathrm{T}\ c)\alpha, N+\frac{\backslash u\backslash^{)N0\ \alpha+/\alpha}_{H_0^{\alpha/2}}}{\backslash u\backslash^{N/\alpha}_{p}} \quad \bigcap_{-} fu\,dx.$$

If $\backslash u\backslash_p$ T 2, we have

$$\phi)tu+\mathrm{T}\ t\Big)c)\alpha, N+\!\!\backslash u\backslash^{)N0\ \alpha+/\alpha}_{H_0^{\alpha/2}} \quad \bigcap_{-} fu\,dx\Big\{,$$

thus, by Lemma 4.3.3, given $\gamma > 1$, to be chosen later, clearly

$$\log_{u\ \ p\to\gamma} \phi)u+\sim \gamma\mu_1. \tag{4.15}$$

Note that this infimum is also positive.

Now we suppose by contradiction that there exists $u \;/\; \mathcal{S}$ such that

$$\backslash u\backslash^{3}_{H_0^{\alpha/2}} \quad)p \quad 2+\!\!\backslash u\backslash^{p}_{p}\ \mathrm{T}\ 1. \tag{4.16}$$

By the Sobolev inequality (1.33), we obtain

$$S)\alpha, N+\!\!\backslash u\backslash^{3}_{p} \quad)p \quad 2+\!\!\backslash u\backslash^{p}_{p} \geq 1,$$

which implies

$$\backslash u\backslash_p \sim \Big)\frac{S)\alpha, N+}{p \quad 2}\Big\{^{2/)p\quad 3+} ;\mathrm{T}\ \gamma.$$

Now, substituting (4.16) into (4.7) we get

$$1\ \mathrm{T}\ \backslash u\backslash^{3}_{H_0^{\alpha/2}} \quad \backslash u\backslash^{p}_{p} \quad \bigcap_{-} fu\,dx\ \ \mathrm{T}\ )p \quad 3+\!\!\backslash u\backslash^{p}_{p} \quad \bigcap_{-} fu\,dx\,. \tag{4.17}$$

Finally, by (4.15) and (4.17) we conclude

$$\begin{aligned}
1 < \gamma\mu_1 \geq \phi)u+\mathrm{T}\ )p \quad 3+\Big)\frac{N \quad \alpha}{N\ 0\ \ \alpha}\Big\{^{)N0\ \alpha+/3\alpha} \frac{\backslash u\backslash^{)N0\ \alpha+/\alpha}_{H_0^{\alpha/2}}}{\backslash u\backslash^{N/\alpha}_{p}} \quad \bigcap_{-} fu\,dx \\
\mathrm{T}\ )p \quad 3+\Big]\Big)\frac{N \quad \alpha}{N\ 0\ \ \alpha}\Big\{^{)N0\ \alpha+/3\alpha} \frac{\backslash u\backslash^{)N0\ \alpha+/\alpha}_{H_0^{\alpha/2}}}{\backslash u\backslash^{N/\alpha}_{p}} \quad \backslash u\backslash^{p}_{p}\Big| \\
\mathrm{T}\ )p \quad 3+\!\!\backslash u\backslash^{p}_{p}\Big]\Big)\frac{)N \quad \alpha+\!\!\backslash u\backslash^{3}_{H_0^{\alpha/2}}}{)N\ 0\ \ \alpha+\!\!\backslash u\backslash^{p}_{p}}\Big[^{)N \quad \alpha+/3\alpha}_{\sum} \qquad 2\Big|\ \mathrm{T}\ 1,
\end{aligned}$$

which is a contradiction. □

Lemma 4.3.5. *Let* $f \subseteq 1$ *be a function satisfying* (4.2). *Given* $u\ /\ \mathcal{S}$ *there exists a positive function* $\mu_u\ ;\ H_1^{\alpha/3})' + \ \nearrow \mathbb{R}$ *differentiable in a neighborhood of the origin* $\cup_1 \ll H_1^{\alpha/3})' +$*such that,*

$$\mu_u)1+\mathrm{T}\ 2, \quad \mu_u)z+)u \quad z+/\ \mathcal{S}\ ,$$

and

$$\rangle\mu_u^{\infty})1+, z|\ \mathrm{T}\ \frac{3\bigcap_{-})\quad \Lambda+^{\alpha/=}u)\quad \Lambda+^{\alpha/=}z \quad p\bigcap_{-}|u|^{p} \quad {}^{3}uz \quad \bigcap_{-} fz}{\backslash u\backslash^{3}_{H_0^{\alpha/2}} \quad)p \quad 2+\backslash u\backslash^{p}_{p}}, \qquad \exists\, z\ /\ \cup_1. \tag{4.18}$$

Proof. Consider the function

$$F)\mu, z+\mathrm{T}\ \mu\backslash u \quad z\backslash^{3}_{H_0^{\alpha/2})^- +} \quad \mu^{p} \quad {}^{2}\backslash u \quad z\backslash^{p}_{p} \quad \bigcap_{-} f)u \quad z+.$$

By Lemma 4.3.4 we have that

$$\frac{\partial F}{\partial \mu})2, 1+\mathrm{T}\ \backslash u\backslash^{3}_{H_0^{\alpha/2}} \quad)p \quad 2+\backslash u\backslash^{p}_{p}\ \mathrm{T}\ 1.$$

The proof finishes applying the Implicit Function Theorem to the function F at the point $)2, 1+$. □

We are now in a position to prove one of the main results of the chapter.

Proposition 4.3.6. *The functional* I *possess a local minimum in* $H_1^{\alpha/3})' +$, *in particular,* $)P+$*has a solution. Moreover, if* f *is nonnegative a.e. in* $'$ *this solution is nonnegative a.e. in* $'$.

Proof. Consider v the unique solution to the equation $)\quad \Lambda+^{\alpha/3}v\ \mathrm{T}\ f$ in $H_1^{\alpha/3})' +$ Let $\sigma\ \mathrm{T}\ \sigma)v+$be as defined in Lemma 4.3.1. Thus, since $\sigma)v+v\ /\ \mathcal{S}$, we have

$$\begin{aligned} I)\sigma v+\mathrm{T}\ &\frac{\sigma^3}{3}\backslash v\backslash^{3}_{H_0^{\alpha/2}} \quad \frac{\sigma^p}{p}\backslash v\backslash^{p}_{p} \quad \sigma\backslash v\backslash^{3}_{H_0^{\alpha/2}} \\ \mathrm{T} \quad &\frac{\sigma^3}{3}\backslash v\backslash^{3}_{H_0^{\alpha/2}}\ 0\ \frac{N\ 0\ \alpha}{3N}\sigma^p\backslash v\backslash^{p}_{p} < \quad \frac{\alpha\sigma^3}{3N}\backslash v\backslash^{3}_{H_0^{\alpha/2}}\ \mathrm{T} \quad \frac{\alpha\sigma^3}{3N}\backslash f\backslash^{3}_{H-\alpha/2}. \end{aligned} \tag{4.19}$$

Then, by (4.9) and (4.19), the infimum in (4.10) satisfies the estimate

$$\frac{)N\ 0\ \alpha+^{3}}{:N\alpha}\backslash f\backslash^{3}_{H-\alpha/2} \geq c_1 < \quad \frac{\alpha\sigma^3}{3N}\backslash f\backslash^{3}_{H-\alpha/2} < 1. \tag{4.20}$$

Note that by the expression (4.8), it is clear that the functional I constrained on $\mathcal{S}$ is weakly lower semi-continuous. Therefore, by the Ekeland's variational principle [43], we obtain a minimizing subsequence $\}u_n\langle\ \ll\mathcal{S}$ such that for every $n\ /\ \mathbb{N}$:

$$)i+\quad I)u_n+< c_1\ 0\ \frac{2}{n}, \qquad)ii+\quad \frac{2}{n}\backslash u_n \quad v\backslash_{H_0^{\alpha/2}} \sim I)u_n+\quad I)v+,\ \exists v\ /\ \mathcal{S}\ .$$

So that, by $)i+$ (4.20) and (4.8) we have

$$I)u_n+\mathrm{T}\ \frac{\alpha}{3N}\backslash u_n\backslash^3_{H_0^{\alpha/2}} \quad \frac{N\ 0\ \alpha}{3N}\bigcap_{-} fu_n < c_1\ 0\ \frac{2}{n} < \quad \frac{\alpha\sigma^3}{3N}\backslash f\backslash^3_{H^{-\alpha/2}}$$

for n large enough. Therefore

$$\frac{\alpha\sigma^3}{N\ 0\ \alpha}\backslash f\backslash^3_{H^{-\alpha/2}} \geq \bigcap_{-} fu_n \qquad \text{and} \qquad \backslash u_n\backslash^3_{H_0^{\alpha/2}} \geq \frac{N\ 0\ \alpha}{\alpha}\bigcap_{-} fu_n. \tag{4.21}$$

These inequalities, together with (4.5), give

$$\frac{\alpha\sigma^3}{N\ 0\ \alpha}\backslash f\backslash_{H^{-\alpha/2}} \geq \backslash u_n\backslash_{H_0^{\alpha/2}} \geq \frac{N\ 0\ \alpha}{\alpha}\backslash f\backslash_{H^{-\alpha/2}}. \tag{4.22}$$

Thus we have, for a subsequence, that $u_n \rightharpoonup u_1$ weakly in $H^{\alpha/3})'+$ with $u_1 \subseteq 1$. We claim that $\backslash I'\!)u_1+\backslash_{H^{-\alpha/2}}\ \mathrm{T}\ 1$. Take $z\ /\ H_1^{\alpha/3})'+$ with $\backslash z\backslash_{H_0^{\alpha/2}}\ \mathrm{T}\ 2$. By Lemma 4.3.5, for every $n\ /\ \mathbb{N}$ there exists a positive function μ_{u_n} such that

$$w_\delta\ \mathrm{T}\ \mu_{u_n})\delta z+)u_n \quad \delta z+/\ \mathcal{S}$$

with $\delta > 1$ small enough. Set $t_n)\delta+\mathrm{T}\ \ \mu_{u_n})\delta z+$. Thus, putting $v\ \mathrm{T}\ w_\delta$ in $)ii+$ and using the Mean Value Theorem, we have

$$\frac{2}{n}\backslash w_\delta \quad u_n\backslash_{H_0^{\alpha/2}} \sim)2 \quad t_n)\delta+\rangle I'\!)w_\delta+,u_n|\ 0\ \ \delta t_n)\delta+\rangle I'\!)w_\delta+,z|\ 0\ \ o)\delta+$$

Dividing by δ and taking the limit as δ goes to 1 we have

$$\frac{2}{n})2\ 0\ \ |t_n^{\infty})1+|\backslash u_n\backslash_{H_0^{\alpha/2}}+\sim \backslash I'\!)u_n+\backslash_{H^{-\alpha/2}}$$

with $|t_n^{\infty})1+|\ \mathrm{T}\ \rangle\mu_{u_n}^{\infty})1+,z|$. So that, by (4.22) we get

$$\backslash I'\!)u_n+\backslash_{H^{-\alpha/2}} \geq \frac{2}{n}\Big)2\ 0\ \ \frac{N\ 0\ \alpha}{\alpha}|t_n^{\infty})1+|\backslash f\backslash_{H^{-\alpha/2}}\Big\{. \tag{4.23}$$

Thus we are done once we prove that $|t_n^{\infty})1+|$ is uniformly bounded. By Lemma 4.3.5 and (4.22) we obtain

$$|t_n^{\infty})1+| \geq \frac{C}{\Big\langle \backslash u_n\backslash^3_{H_0^{\alpha/2}} \quad)p \quad 2+\backslash u_n\backslash^p_p \Big\langle}$$

for some constant C. Assume by contradiction that

$$\backslash u_n\backslash^3_{H_0^{\alpha/2}} \quad)p \quad 2+\backslash u_n\backslash^p_p \nearrow 1 \qquad \text{as } n \nearrow \in. \tag{4.24}$$

By (4.24) and (4.7) we deduce the estimate

$$\bigcap_{-} f u_n \mathrm{T}\ )p \quad 3+\backslash u_n\backslash_p^p\ 0\ o)2+.$$

Moreover, by (4.22) we derive $\backslash u_n\backslash_p \sim \gamma$ for some constant $\gamma > 1$. Thus, reasoning like in Lemma 4.3.4 we get

$$\begin{aligned} 1 &< \gamma^{)N0\ \alpha+/3}\mu_1 \geq \backslash u_n\backslash^{\alpha/N}_{H_0^{\alpha/2}}\phi)u_n+ \\ &\mathrm{T}\ )p \quad 3+\Bigg]\Bigg)\frac{)N \quad \alpha+\backslash u_n\backslash^3_{H_0^{\alpha/2}}}{)N\ 0\ \alpha+ \qquad \sum}\Bigg\lceil^{)N \quad \alpha+/3\alpha} \qquad)\backslash u_n\backslash_p^p+^{)N \quad \alpha+/3\alpha}\Bigg\} \nearrow 1, \end{aligned}$$

which leads to a contradiction. Therefore $\backslash I'\!)u_1+\backslash_{H^{-\alpha/2}}\ \mathrm{T}\ 1$ and we have obtained a weak solution of $)P+.$

To obtain the strong convergence we proceed as usual. Recalling that I is weakly lower semicontinuous in $\mathcal{S}$, we get

$$c_1 \geq I)u_1+\geq \lim_{n\prime \ \in} I)u_n+\mathrm{T}\ c_1.$$

This implies, using (4.8), the limits

$$\lim_{n\prime \ \in} \backslash u_n\backslash_{H_0^{\alpha/2}}\ \mathrm{T}\ \backslash u_1\backslash_{H_0^{\alpha/2}}, \qquad \lim_{n\prime \ \in} \backslash u_n\backslash_p\ \mathrm{T}\ \backslash u_1\backslash_p.$$

To see that u_1 is a local minimum in $H_1^{\alpha/3})'$ +we first show that (4.11) holds. In fact, since $u_1\ /\ \mathcal{S}$ and also $\sum f u_1 > 1$ by (4.21), it is clear that one of the values $\sigma)u_1+$or $\tau)u_1+$given by Lemma 4.3.1 is one. Assume by contradiction, see Lemma 4.3.4, that u_1 satisfies (4.12), i.e. $\sigma)u_1+<\tau)u_1+\mathrm{T}\ 2$. By Corollary 4.3.2, $I)\sigma)u_1+u_1+< I)u_1+,$ which contradicts the fact that u_1 is the infimum in $\mathcal{S}$. Hence u_1 satisfies (4.11) and $\sigma)u_1+\mathrm{T}\ 2$. Remark that having the strict inequality in (4.4) is crucial in the present argument. In particular we have obtained $2\ \mathrm{T}\ \sigma)u_1+< t_M < \tau)u_1+,$ or which is the same,

$$2 < \Bigg)\frac{)N \quad \alpha+\backslash u_1\backslash^3_{H_0^{\alpha/2}}}{)N\ 0\ \alpha+\backslash u_1\backslash_p^p \ \sum}\Bigg\lceil^{)N \quad \alpha+/3\alpha}. \tag{4.25}$$

Take $\varepsilon > 1$ small enough such that

$$2 < \Bigg)\frac{)N \quad \alpha+\backslash u_1 \quad z\backslash^3_{H_0^{\alpha/2}}}{)N\ 0\ \alpha+\backslash u_1 \quad z\backslash_p^p \ \sum}\Bigg\lceil^{)N \quad \alpha+/3\alpha} ;\mathrm{T}\ t_{M,\varepsilon} \tag{4.26}$$

for $\backslash z\backslash_{H_0^{\alpha/2}} < \varepsilon$. By Lemma 4.3.5 we have that there exists a positive function μ_{u_0} ; $H_1^{\alpha/3})' + \nearrow \mathbb{R}$ such that $\mu_{u_0})z+)u_1 \quad z+/\ \mathcal{S}$ for every $\backslash z\backslash_{H_0^{\alpha/2}} < \varepsilon$, with ε smaller if necessary. Indeed, by continuity we have $\mu_{u_0})z+< t_{M,\varepsilon}$ for $\varepsilon > 1$ sufficiently small. Thus we get that $\mu_{u_0})z+)u_1 \quad z+$verifies (4.11), and as a consequence of Lemma 4.3.1 and Corollary 4.3.2, applied to $u_1 \quad z$, we obtain

$$I)s)u_1 \quad z++\sim I)\mu_{u_0})z+)u_1 \quad z++\sim I)u_1+, \qquad \exists\, s \;/\;)1, t_{M,\varepsilon}+.$$

Since, by (4.26) we can take s T 2, we conclude $I)u_1 \quad z+\sim I)u_1+$for every $\backslash z\backslash_{H_0^{\alpha/2}} < \varepsilon$, i.e, u_1 is a local minimum in $H_1^{\alpha/3})\,\acute{}\;+$.

To finish we assume that $f \sim 1$, then it follows $\sum f|u_1| > 1$. Take σ T $\sigma)|u_1|+>$ 1 and τ T $\tau)|u_1|+> \sigma$. We have

$$\backslash u_1\backslash_p^p\, 0 \bigcap_{-} f u_1 \text{ T } \backslash u_1\backslash^3_{H_0^{\alpha/2}} >)p \quad 2+\backslash u_1\backslash_p^p$$

and, since $\tau|u_1|$satisfies (4.12), we get

$$\tau^p\backslash u_1\backslash_p^p\, 0\;\; \tau\bigcap_{-} f|u_1| \text{T } \tau^3\backslash|u_1|\backslash^3_{H_0^{\alpha/2}} <)p \quad 2+\tau^p\backslash u_1\backslash_p^p.$$

Thus,

$$)p \quad 3+\backslash u_1\backslash_p^p < \bigcap_{-} f u_1 \geq \bigcap_{-} f|u_1| \geq)p \quad 3+\tau^p \;\; {}^2\backslash u_1\backslash_p^p,$$

which implies $\tau > 2$. Therefore, by Corollary 4.3.2 we have

$$I)u_1+\geq I)\sigma|u_1|+\geq I)|u_1|+.$$

On the other hand, by the generalized Stroock-Varopoulos inequality [62], we have

$$\bigcap_{-} \Big(\quad \Lambda +^{\alpha/=}|u_1| \Big)^3 \geq \bigcap_{-} \Big(\quad \Lambda +^{\alpha/=} u_1 \Big)^3,$$

which implies $I)|u_1|+\geq I)u_1+$. As a consequence, $I)u_1+$T $I)|u_1|+$, σ T 2, and thus $|u_1| \;/\; \mathcal{S}$ is a solution. □

4.3.2. Second Solution

As in Chapter 3, we will look for the second solution using a classical approach that relies on the well-known Mountain Pass Theorem, see [5]. As it is usual in critical problems, the functional I does not satisfy a global PS condition, i.e. a PS$_c$ condition for every c. Our aim is to prove that I satisfies a PS$_c$ condition for c below a precise critical level $c^{\leq}$. We define

$$c^{\leq} \text{ T } c_1\, 0\;\; \frac{\alpha}{3N} S)\alpha, N+^{\frac{N}{\alpha}}. \tag{4.27}$$

Note that this critical level differs from the one applied in Section 3.3. This is caused by the shifting applied to the functional in that section.

Lemma 4.3.7. *The functional I satisfies a local PS$_c$ condition for any $c < c^{\leq}$.*

Proof. Let $\}u_n\langle \ll H_1^{\alpha/3})' +$ be a PS sequence of level $c < c^{\leq}$. It is easy to check that $\}u_n\langle$ are uniformly bounded in $H^{\alpha/3})' +$. Thus, there exists a subsequence (still denoted u_n) such that $u_n \rightharpoonup z_1$ weakly in $H_1^{\alpha/3})' +$. As a consequence, $z_1 \ / \ H_1^{\alpha/3})' +$ is a solution of $)P+$.

We rewrite u_n as u_n T u_1 0 ϕ_n with $\phi_n \nearrow 1$, then applying the Brezis-Lieb Lemma we get

$$\backslash u_n \backslash_p^p \ \mathrm{T} \ \backslash u_1 \backslash_p^p \ 0 \ \backslash \phi_n \backslash_p^p \ 0 \ o)2+. \tag{4.28}$$

On one hand, by (4.28) and taking n large enough we have

$$\begin{aligned} c^{\leq} \ &> I)u_n + \mathrm{T} \ I)u_1 + 0 \ \frac{2}{3} \backslash \phi_n \backslash^3_{H_0^{\alpha/2}} \quad \frac{2}{p} \backslash \phi_n \backslash_p^p \ 0 \ o)2+ \\ &\sim c_1 \ 0 \ \frac{2}{3} \backslash \phi_n \backslash^3_{H_0^{\alpha/2}} \quad \frac{2}{p} \backslash \phi_n \backslash_p^p \ 0 \ o)2+. \end{aligned}$$

Hence by definition of $c^{\leq}$ in (4.27) we obtain

$$\frac{2}{3} \backslash \phi_n \backslash^3_{H_0^{\alpha/2}} \quad \frac{2}{p} \backslash \phi_n \backslash_p^p < \frac{\alpha}{3N} S)\alpha, N+^{\frac{N}{\alpha}} \ 0 \ o)2+. \tag{4.29}$$

Taking into account that $\}u_n\langle$ is a PS sequence, in particular we have that

$$\begin{aligned} o)2+ \ \mathrm{T} \ \rangle I^{\infty})u_n +, u_n| \ &\mathrm{T} \ \backslash u_n \backslash^3_{H_0^{\alpha/2}} \quad \backslash u_n \backslash_p^p \quad \bigcap_{-} f u_n \\ &\mathrm{T} \ \backslash u_1 \backslash^3_{H_0^{\alpha/2}} \quad \backslash u_1 \backslash_p^p \quad \bigcap_{-} f u_1 \ 0 \ \backslash \phi_n \backslash^3_{H_0^{\alpha/2}} \quad \backslash \phi_n \backslash_p^p \ 0 \ o)2+ \\ &\mathrm{T} \ \rangle I^{\infty})u_1 +, u_1| \ 0 \ \backslash \phi_n \backslash^3_{H_0^{\alpha/2}} \quad \backslash \phi_n \backslash_p^p \ 0 \ o)2+, \\ &\mathrm{T} \ \backslash \phi_n \backslash^3_{H_0^{\alpha/2}} \quad \backslash \phi_n \backslash_p^p \ 0 \ o)2+. \end{aligned} \tag{4.30}$$

Now we want to prove that ϕ_n has a subsequence strongly convergent to 1 in $H_1^{\alpha/3})' +$. Suppose on the contrary that there exists $C, k > 1$ such that $\backslash \phi_n \backslash_{H_0^{\alpha/2}} \sim C$, $\exists n \sim k$. Thus, using (1.33) in (4.30) we get

$$\backslash \phi_n \backslash_p^{p \quad 3} \sim S)\alpha, N + 0 \ o)2 + \infty \ \backslash \phi_n \backslash_p^p \sim S)\alpha, N+^{\frac{N}{\alpha}} \ 0 \ o)2+ \tag{4.31}$$

Therefore, by (4.29) and (4.31) we have that

$$\frac{\alpha}{3N} S)\alpha, N+^{\frac{N}{\alpha}} \geq \frac{\alpha}{3N} \backslash \phi_n \backslash_p^p 0 \ o)2+ \mathrm{T} \ \frac{2}{3} \backslash \phi_n \backslash^3_{H_0^{\alpha/2}} \quad \frac{2}{p} \backslash \phi_n \backslash_p^p 0 \ o)2+ < \frac{\alpha}{3N} S)\alpha, N+^{\frac{N}{\alpha}},$$

which is a contradiction. □

Recall that the minimizers for the Sobolev inequality (1.33) are given by the two-parameter family of functions

$$u_{\varepsilon, x_0})x + \mathrm{T} \ \frac{\varepsilon^{)N \quad \alpha+/3}}{)\|x \quad x_1\|^{\beta} \ 0 \ \varepsilon^3 +^{)N \quad \alpha+/3}}, \tag{4.32}$$

where $x_0 \in \mathbb{R}^N$, $\varepsilon > 0$, see (1.32). In what follows we will denote

$$A = \|u_{\varepsilon,x_0}\|_p, \quad B = \|(-\Delta)^{\alpha/4} u_{\varepsilon,x_0}\|_2 = \left(\int_{\mathbb{R}^N} |\xi|^\alpha |\widehat{u}_{\varepsilon,x_0}(\xi)|^2 \, d\xi\right)^{1/2}. \tag{4.33}$$

Note that the last quantity defines a norm in the homogeneous fractional Sobolev space $\dot{H}^{\alpha/2}(\mathbb{R}^N)$. Both numbers A and B are clearly independent of ε and x_0, and moreover, $B^2 = S(\alpha, N)A^2$.

Without loss of generality we may assume that $0 \in \Omega$. We define a cut-off function $\theta \in C^\infty(\mathbb{R}^N)$ by $\theta(x) = \theta_0(|x|/\rho)$ with $\rho > 0$, where $\theta_0 \in C^\infty(\mathbb{R})$ is a non-increasing function satisfying

$$\theta_0(s) = 1 \text{ if } s \le \frac{1}{2}, \quad \theta_0(s) = 0 \text{ if } s \ge 1.$$

Note that if u_0 is the solution constructed in the previous subsection, we can find a set $\Phi \subset \Omega$ of positive Lebesgue measure such that $u_0 \ge \nu > 0$ a.e. in Φ (replace u_0 with $-u_0$ and f with $-f$ if necessary). For $x_0 \in \Phi$, we set $\widetilde{u}_{\varepsilon,x_0} = \theta u_{\varepsilon,x_0} \in H_0^{\alpha/2}(\Omega)$.

Proposition 4.3.8. *In the above notation, for a.e. $x_0 \in \Phi$ there exists $\varepsilon^* = \varepsilon^*(x_0) > 0$ sufficiently small such that*

$$\sup_{t \ge 0} I(u_0 + t\widetilde{u}_{\varepsilon,x_0}) < c^*, \qquad \forall\, 0 < \varepsilon < \varepsilon^*. \tag{4.34}$$

We observe that when one evaluates the functional in (4.34), one needs to evaluate $\|\widetilde{u}_{\varepsilon,x_0}\|_{H_0^{\alpha/2}}$, i.e., one needs to evaluate the fractional Laplacian of a product of functions. As in the previous chapters, this is dealt by using the *α-harmonic extension*.

Consider the family $w_{\varepsilon,x_0} = E_\alpha(u_{\varepsilon,x_0})$, with u_{ε,x_0} given in (4.32). We want to find a family of modified minimizers in the extended space, by using a cut-off function in $\mathcal{C}$. To do that we take

$$\phi(x, y) = \theta_0\left(\frac{(|x - x_0|^2 + y^2)^{1/2}}{\rho}\right),$$

where θ_0 is defined above. With this notation we define $\widetilde{w}_{\varepsilon,x_0} = \phi w_{\varepsilon,x_0} \in X_0^\alpha(\mathcal{C})$ and $\widetilde{w}_{\varepsilon,x_0}(x, 0) = \widetilde{u}_{\varepsilon,x_0}(x)$.

In Chapter 3 the following estimate for $\widetilde{w}_{\varepsilon,x_0}$ is proved

$$\|\widetilde{w}_{\varepsilon,x_0}\|^2_{X_0^\alpha} = \|w_{\varepsilon,x_0}\|^2_{X^\alpha} + O(\varepsilon^{N-\alpha}). \tag{4.35}$$

In view of (1.8), (1.9) and (4.35), we have

$$\|\widetilde{u}_{\varepsilon,x_0}\|^2_{H_0^{\alpha/2}} \le B^2 + O(\varepsilon^{N-\alpha}). \tag{4.36}$$

Moreover, there is the following one

$$\|\widetilde{u}_{\varepsilon,x_0}\|_p^p \geq A^p + O(\varepsilon^N). \tag{4.37}$$

We establish now a result that will be useful in the proof of Proposition 4.3.8.

Lemma 4.3.9. *Assume $a, b > 0$, u_1, $\widetilde{u}_{\varepsilon,x_0}$ defined as above. For $t \in [a, b]$, it holds*

$$\begin{aligned}\|u_1 + t\widetilde{u}_{\varepsilon,x_0}\|_p^p = \|u_1\|_p^p + t^p\|\widetilde{u}_{\varepsilon,x_0}\|_p^p + pt\int_\Omega |u_1|^{p-2}u_1\widetilde{u}_{\varepsilon,x_0}\,dx \\ + pt^{p-1}\int_\Omega |\widetilde{u}_{\varepsilon,x_0}|^{p-2}\widetilde{u}_{\varepsilon,x_0}u_1\,dx + o\left(\varepsilon^{\frac{N-\alpha}{2}}\right).\end{aligned} \tag{4.38}$$

The proof of this result follows the same arguments as in [23] with the obvious changes to our setting, so we omit the details.

Proof of Proposition 4.3.8. On the one hand, since $I(u_1 + t\widetilde{u}_{\varepsilon,x_0})|_{t=0} = c_1 < c^*$, by a continuity argument, we can find $t_1, \varepsilon_1 > 0$ both small enough such that

$$I(u_1 + t\widetilde{u}_{\varepsilon,x_0}) < c^* \qquad \forall t \in (0, t_1),\ \forall \varepsilon \in (0, \varepsilon_1).$$

On the other hand, by Proposition 4.3.9, together with (4.37) and the fact that A and B are independent of ε we have

$$I(u_1 + t\widetilde{u}_{\varepsilon,x_0}) \to -\infty \ \text{ as } t \to \infty, \qquad \forall \varepsilon > 0.$$

Hence there exist $t_2 > 0$ large enough such that

$$I(u_1 + t\widetilde{u}_{\varepsilon,x_0}) < c_1 < c^* \qquad \forall t \geq t_2,\ \forall \varepsilon \in (0, \varepsilon_1).$$

Thus, we just need to prove that there exist $\varepsilon^* \in (0, \varepsilon_1)$ such that

$$\sup_{t_1 \leq t \leq t_2} I(u_1 + t\widetilde{u}_{\varepsilon,x_0}) < c^*,$$

for every $0 < \varepsilon < \varepsilon^*$.

Take $t \in [t_1, t_2]$. Clearly we have

$$\begin{aligned} I(u_1 + t\widetilde{u}_{\varepsilon,x_0}) = \frac{1}{2}\|u_1\|^2_{H_0^{\alpha/2}} + t\int_\Omega (-\Delta)^{\alpha/4}u_1(-\Delta)^{\alpha/4}\widetilde{u}_{\varepsilon,x_0}\,dx \\ + \frac{t^2}{2}\|\widetilde{u}_{\varepsilon,x_0}\|^2_{H_0^{\alpha/2}} - \frac{1}{p}\|u_1 + t\widetilde{u}_{\varepsilon,x_0}\|_p^p \\ - \int_\Omega f u_1\,dx - t\int_\Omega f\widetilde{u}_{\varepsilon,x_0}\,dx. \end{aligned} \tag{4.39}$$

Since $S(\alpha, N)$ is attained for the function u_{ε,x_0}, substituting (4.36), (4.37) and (4.38) in (4.39) we have

$$\begin{aligned} I(u_1 + t\widetilde{u_{\varepsilon,x_0}}) \leq{}& \frac{1}{2}\|u_1\|^2_{H_0^{\alpha/2}} + t\int_\Omega (-\Delta)^{\alpha/4} u_1 (-\Delta)^{\alpha/4}\widetilde{u_{\varepsilon,x_0}}\,dx \\ &+ \frac{t^2}{2}B^2 - \frac{1}{p}\|u_1\|_p^p - \frac{t^p}{p}A^p \\ &- t\int_\Omega |u_1|^{p-2}u_1\widetilde{u_{\varepsilon,x_0}}\,dx - t^{p-1}\int_\Omega |\widetilde{u_{\varepsilon,x_0}}|^{p-1}u_1\,dx \\ &- \int_\Omega f u_1\,dx - t\int_\Omega f\widetilde{u_{\varepsilon,x_0}}\,dx + o(\varepsilon^{\frac{N-\alpha}{2}}). \end{aligned}$$

On the other hand, since u_1 is solution of (P) we get

$$\begin{aligned} I(u_1 + t\widetilde{u_{\varepsilon,\,x_0}}) \leq I(u_1) + \frac{t^2}{2}B^2 - t^{p-1}\int_\Omega |\widetilde{u_{\varepsilon,\,x_0}}|^{p-1}u_1\,dx \\ - \frac{t^p}{p}A^p + o(\varepsilon^{\frac{N-\alpha}{2}}). \end{aligned} \tag{4.40}$$

Extending u_1 by zero outside Ω we get

$$\begin{aligned} \int_\Omega |\widetilde{u_{\varepsilon,x_0}}|^{p-1}u_1\,dx &= \int_{\mathbb{R}^N} u_1(x)\varphi^{p-1}(x)\frac{\varepsilon^{(N+\alpha)/2}}{(|x-x_1|^2+\varepsilon^2)^{(N+\alpha)/2}}\,dx \\ &= \varepsilon^{\frac{N-\alpha}{2}}\int_{\mathbb{R}^N} u_1(x)\varphi^{p-1}(x)\frac{1}{\varepsilon^N}\eta\left(\frac{x-x_1}{\varepsilon}\right)dx, \end{aligned}$$

with $\eta(x) = (|x|^2+1)^{-(N+\alpha)/2}$. Thus, there exists a constant $\nu > 0$ such that

$$\int_{\mathbb{R}^N} u_1(x)\varphi^{p-1}(x)\frac{1}{\varepsilon^N}\eta\left(\frac{x-x_1}{\varepsilon}\right)dx \geq K\nu$$

for every $\varepsilon > 0$ sufficiently small, $x_1 \in \Omega$ and $K = \int_{\mathbb{R}^N}\eta(x)dx < \infty$. Therefore

$$\int_\Omega |\widetilde{u_{\varepsilon,x_0}}|^{p-1}u_1\,dx = \varepsilon^{\frac{N-\alpha}{2}}K\nu + o(\varepsilon^{\frac{N-\alpha}{2}}). \tag{4.41}$$

Substituting (4.41) in (4.40) we have

$$I(u_1 + t\widetilde{u_{\varepsilon,x_0}}) \leq c_1 + \frac{t^2}{2}B^2 - t^{p-1}\varepsilon^{\frac{N-\alpha}{2}}K\nu - \frac{t^p}{p}A^p + o(\varepsilon^{\frac{N-\alpha}{2}}).$$

Let us now define the function

$$g(s) = \frac{s^2}{2}B^2 - s^{p-1}\varepsilon^{\frac{N-\alpha}{2}}K\nu - \frac{s^p}{p}A^p, \quad \text{for } s > 0,$$

and let $s_\varepsilon > 0$ be the point of global maximum, i.e.,

$$0 = g'(s_\varepsilon) = s_\varepsilon B^2 - (p-1)s_\varepsilon^{p-2}\varepsilon^{\frac{N-\alpha}{2}}K\nu - s_\varepsilon^{p-1}A^p. \tag{4.42}$$

We denote S_1 T $B^3/A^p\{^{2/)p\ \ 3+}$. Note that $1 < s_\varepsilon < S_1$ and $s_\varepsilon \nearrow S_1$ as $\varepsilon \Rightarrow 1$. Let $\delta_\varepsilon > 1$ be such that s_ε T $S_1)2 \quad \delta_\varepsilon +$. Since B^3/A^p T $S_1^{p\ \ 3}$, by (4.42) we have

$$\Big)\frac{B^{3)p\ \ 2+}}{A^p}\Big\{^{\frac{1}{p-2}} \quad 2 \quad \delta_\varepsilon \quad)2 \quad \delta_\varepsilon + ^{p\ \ 2}\{ \quad)p \quad 2+ S_1^{p\ \ 3})2 \quad \delta_\varepsilon + ^{p\ \ 3}\varepsilon^{\frac{N-\alpha}{2}} K\nu \text{ T } 1,$$

which implies

$$)p \quad 3+\Big)\frac{B^{3)p\ \ 2+}}{A^p}\Big\{^{\frac{1}{p-2}} \delta_\varepsilon \text{ T })p \quad 2+ S_1^{p\ \ 3}\varepsilon^{\frac{N-\alpha}{2}} K\nu\ 0\ o\Big)\varepsilon^{\frac{N-\alpha}{2}}\Big(. \tag{4.43}$$

This, together with (4.43), gives

$$\begin{aligned} I)u_1\ 0\ t\widetilde{u_{\varepsilon,x_0}}+ &\geq c_1\ 0\ \frac{s_\varepsilon^3}{3}B^3 \quad s_\varepsilon^{p\ \ 2}\varepsilon^{\frac{N-\alpha}{2}}K\nu \quad \frac{s_\varepsilon^p}{p}A^p\ 0\ o\Big)\varepsilon^{\frac{N-\alpha}{2}}\Big(\\ &\text{T } c_1\ 0\ \frac{S_1^3}{3}B^3 \quad S_1^{p\ \ 2}\varepsilon^{\frac{N-\alpha}{2}}K\nu \quad \frac{S_1^p}{p}A^p\ 0\ o\Big)\varepsilon^{\frac{N-\alpha}{2}}\Big(\\ &\text{T } c_1\ 0\ \frac{\alpha}{3N}S)\alpha, N+^{\frac{N}{\alpha}} \quad S_1^{p\ \ 2}\varepsilon^{\frac{N-\alpha}{2}}K\nu\ 0\ o\Big)\varepsilon^{\frac{N-\alpha}{2}}\Big(\\ &\text{T } c^{\leq} \quad S_1^{p\ \ 2}\varepsilon^{\frac{N-\alpha}{2}}K\nu\ 0\ o\Big)\varepsilon^{\frac{N-\alpha}{2}}\Big(. \end{aligned}$$

This finishes the proof by taking ε sufficiently small. □

Lemma 4.3.10. *Assume $f \subseteq 1$ satisfies* (4.2)*. Then the functional I possess a critical point different from u_1, in particular $)P+$has a second solution. Moreover, if $f \sim 1$ a.e. in ´ then this solution is nonnegative a.e. in ´ .*

Proof. Set $\eta_{\varepsilon,M}$ T $u_1\ 0\ M\widetilde{u_{\varepsilon,x_0}}$, with $1 < \varepsilon < \varepsilon^{\leq}$and $x_1\ /\ \Phi$ such that (4.34) holds. Assume that $M > 1$ is large enough such that $I)\eta_{\varepsilon,M}+ < c_1$.

Now we set

$$\text{T } \Big\}\gamma\ ;\ ]1,2\hat{}\nearrow H_1^{\alpha/3})\acute{}\ + \text{ such that } \gamma)1+\text{T } u_1,\ \gamma)2+\text{T } \eta_{\varepsilon,M}\surd.$$

By Proposition 4.3.8 we have that

$$c_1 < c_2 \text{ T } \log_{\gamma/} \text{n d}\tilde{}_{t/]1,2\hat{}}\ I)\gamma)t++ < c^{\leq}.$$

Thus, using the Mountain Pass Theorem we obtain a PS sequence of level c_2, and as a consequence of Lemma 4.3.7 we can find a critical point u_2 in $H_1^{\alpha/3})\acute{}\ +$with energy level $c_2 > c_1$, i.e., u_2 is a solution of $)P+$with $u_2 \subseteq u_1$.

To prove the positivity of the solution in the case that $f \sim 1$, we denote

$$\widetilde{\mathcal{S}}\ ;\text{T } \}u\ /\ \mathcal{S}\ ;\ u \text{ verifies (4.12)}\langle$$

and c_3 T $\underset{\cap}{\widetilde{\log}} I$. Is easy to see that, taking a larger M if necessary, we can assume

$$c_1 < c_3 \geq c_2 < c^{\leq} \tag{4.44}$$

Now, using the Ekeland's variational principle and following the steps of the proof of Proposition 4.3.6, we can obtain a PS sequence of level c_3. Again, Lemma 4.3.7 implies the existence of a solution u_3 / $\mathcal{S}$ such that $I)u_3$+T c_3. Put τ T $\tau)\|u_3\|$+> 1. Then $\tau\|u_3\|$ / $\widetilde{\mathcal{S}}$. Finally by Corollary 4.3.2

$$\underset{\cap}{\widetilde{\log}} I \text{ T } I)u_3+\text{T } \underset{t \to t_M}{\text{n d}\tilde{}}\, I)tu_3+\sim I)\tau u_3+\sim I)\tau\|u_3\|+,$$

which finishes he proof. □

Remark 4.3.2. *Note that u_3 could coincide with u_2.*

4.4. Proof of Theorem 4.2.2

When f satisfies condition (4.3) instead of (4.2) we use an approximation argument.

Proof of Theorem 4.2.2. Consider a sequence of numbers $\}\varepsilon_k\langle_{k/\mathbb{N}} \ll)1, 2+$such that $\varepsilon_k \Rightarrow 1$ as $k \nearrow \in$, and define f_k T $)2 \quad \varepsilon_k+f$. Clearly f_k satisfies condition (4.2) for every k / $\mathbb{N}$. We define I_k and $\mathcal{S}_k$ in a natural way

$$I_k)u+\text{T } \frac{2}{3}\bigcap_{-} \Big(\Lambda+^{\alpha/=}u\Big)^3 dx \quad \frac{2}{p}\bigcap_{-} \|u\|^p \quad \bigcap_{-} f_k u\, dx,$$

$$\mathcal{S}_k \text{ T } \}u \;/\; H_1^{\alpha/3})' +, u \subseteq 1 \,;\, \rangle I_k^{\infty})u+,u| \text{ T } 1\langle.$$

Let u_k / $\mathcal{S}_k$ be the local minimum found via Theorem 4.2.1, namely

$$I_k)u_k+\text{T } \underset{\cap_k}{\log} I_k \text{ ;T } c_k.$$

In particular it holds

$$\rangle I_k^{\infty})u_k+,z| \text{ T } 1 \qquad \exists z \;/\; H_1^{\alpha/3})' +, \tag{4.45}$$

and moreover

$$\backslash u_k\backslash^3_{H_0^{\alpha/2}} \quad \backslash u_k\backslash^p_p \quad \bigcap_{-} f_k u_k \text{ T } 1, \tag{4.46}$$

which by (1.33) and (4.5) implies $\backslash u_k\backslash^3_{H_0^{\alpha/2}} < C$ for any k / $\mathbb{N}$ and some constant $C > 1$ independent of k. Take u / $\mathcal{S}$ verifying (4.11). Then

$$\bigcap_{-} f_k u > 1 \quad \exists k \;/\; \mathbb{N}.$$

Applying Lemma 4.3.1 with $f \mathrm{T} f_k$, and $\mathcal{S} \mathrm{T} \mathcal{S}_k$ we find the values $1 < \sigma_k < t_{M_k} < \tau_k$ such that $\sigma_k u, \tau_k u / \mathcal{S}_k$. Since u satisfies inequality (4.11), we have $\tau_k > 2$, thus by Corollary 4.3.2 we have $I_k)\sigma_k u+ \geq I_k)u+$, which leads to

$$c_k \geq I_k)\sigma_k u+ \geq I_k)u+ \geq I)u+0 \ \varepsilon_k \backslash f \backslash_{H^{-\alpha/2}} \backslash u \backslash_{H_0^{\alpha/2}} \geq I)u+0 \ C\varepsilon_k.$$

In particular $c_k \geq c_1 \, 0 \ C\varepsilon_k$. Finally, reasoning like in (4.19) with $f \mathrm{T} f_k$ we obtain

$$\frac{)N \, 0 \ \alpha+^3}{: N\alpha} \backslash f \backslash^3_{H^{-\alpha/2}} < \quad \frac{)N \, 0 \ \alpha+^3}{: N\alpha} \backslash f_k \backslash^3_{H^{-\alpha/2}} \geq c_k \geq c_1 \, 0 \ C\varepsilon_k.$$

After passing to a subsequence we can assume that c_k converges to some value c^∞ such that

$$\frac{)N \, 0 \ \alpha+^3}{: N\alpha} \backslash f \backslash^3_{H^{-\alpha/2}} \geq c^\infty \geq c_1.$$

Moreover, since $\backslash u_k \backslash^3_{H_0^{\alpha/2}}$ is uniformly bounded, again for a subsequence if necessary, we have $u_k \rightharpoonup u^\leq$ weakly in $H_1^{\alpha/3})'+$. Then, by (4.45) we have that

$$\rangle I^\infty) u^\leq +, z| \ \mathrm{T} \ 1 \qquad \exists z \ / \ H_1^{\alpha/3})' +,$$

and $I)u^\leq + \geq c_1$. This implies $u^\leq / \mathcal{S}$ and $I)u^\leq + \mathrm{T} \ c_1$, which finishes the proof. The positivity of the solution when the datum f is taken nonnegative follows the same argument as in the proof of Theorem 4.2.1. □

We finally remark that the solution constructed in this way is not necessarily a minimum of the functional. Therefore we cannot apply the technique of Section 4.3.2 to find a second solution.

Bibliography

[1] Abdellaoui, B.; Colorado, E.; Peral, I. *Effect of the boundary conditions in the behavior of the optimal constant of some Caffarelli-Kohn-Nirenberg inequalities. Application to some doubly critical nonlinear elliptic problems.* Adv. Diff. Equations **11** (2006), no. 6, 667-720.

[2] S. Alama, *Semilinear elliptic equations with sublinear indefinite nonlinearities.* Adv. Diff. Equations, **4** (1999).

[3] Ambrosetti, A. *Critical points and nonlinear variational problems.* Mém. Soc. Math. France (N.S.) no. 49 (1992), 139 pp.

[4] Ambrosetti, A.; Brézis, H.; Cerami, G. *Combined effects of concave and convex nonlinearities in some elliptic problems*. J. Funct. Anal. **122**, (1994), no. 2. 519-543.

[5] Ambrosetti, A.; Rabinowitz, P. H. *Dual variational methods in critical point theory and applications*. J. Funct. Anal. **14** (1973), 349–381.

[6] Ambrosetti, A., García Azorero, J., Peral, I. (2000). *Elliptic variational problems in $\mathbb{R}^N$ with critical growth.* J. Differential Equations **168** , no. 1, 10-32.

[7] Applebaum, D. "Lévy processes and stochastic calculus". Second edition. Cambridge Studies in Advanced Mathematics, 116. Cambridge University Press, Cambridge, 2009.

[8] Aubin, T. *Problemes isoperimetriques et espaces de Sobolev*. (French) J. Differential Geometry **11** (1976), no. 4, 573-598.

[9] Barrios, B., Colorado, E., de Pablo, A., Sánchez, U. *On some critical problems for the fractional Laplacian operator*. J. Differential Equations **252** (2012), 6133–6162.

[10] Barrios, B., Colorado, E., Servadei, R., Soria, F. *A critical fractional equation with concave-convex power nonlinearities*. Ann. Inst. H. Poincare Anal. Non Lineaire. In press. doi:10.1016/j.anihpc.2014.04.003.

[11] Bertoin, J. “Lévy processes”. Cambridge Tracts in Mathematics, 121. Cambridge University Press, Cambridge, 1996.

[12] Birkner, M.; Lopez-Mimbela, J. A.; Wakolbinger, A. *Comparison results and steady states for the Fujita equation with fractional Laplacian*. Ann. Inst. H. Poincare Anal. Non Lineaire **22** (2005), no. 1, 83-97.

[13] Boccardo, L.; Escobedo, M.; Peral, I. *A Dirichlet problem involving critical exponents.* Nonlinear Anal. **24** (1995), no. 11, 1639-1648.

[14] Bogdan, K.; Burdy, K.; Chen, Z.Q. *Censored stable processes*. Probab. Theory Relat. Fields **127** (2003), 89-152.

[15] Bogdan,K.; Byczkowski, T.; Kulczycki,T.; Ryznar,M.; Song,R.; Vondracek,Z. *Potential Analysis of Stable Processes and its Extensions*. Lecture Notes in Mathematics 1980, Springer-Verlag, Berlin, 2009.

[16] Bogdan,K.; Jakubowski,T.; *Estimates of the heat kernel of fractional Laplacian perturbed by gradient operators*. Comm. Math. Phys., **271** (2007), no. 1, 179-198.

[17] Bogdan,K.; Sztonyk, P. *Harnack's inequality for stable Levy processes. Potential Anal.*, **22** (2005), 133-150.

[18] Boyarchenko, S. I.; Levendorski.., S. Z. *Perpetual American options under Levy processes*. SIAM J. Control Optim. **40** (2002), no. 6, 1663.1696 (electronic).

[19] Brändle, C., Colorado, E., de Pablo, A., Sánchez, U. *A concave-convex elliptic problem involving the fractional Laplacian.* Proc. Roy. Soc. Edinburgh Sect. A. **143** (2013), no. 1, 39–71.

[20] Brezis H., Lieb E. (1983). *A relation between pointwise convergence of functions and convergence of functionals.* Proc. Amer. Math. Soc. **88**, no. 3, 486-490.

[21] Brézis, H.; Kamin, S. *Sublinear elliptic equation in* $\mathbb{R}^n$. Manuscripta Math. **74** (1992), 87-106.

[22] Brézis, H.; Kato, T. *Remarks on the Schrödinger operator with singular complex potentials*. J. Math. Pures Appl. **9** 58 (1979), no. 2, 137-151.

[23] Brézis, H.; Nirenberg, L. *A Minimization Problem with Critical Exponent and Non-zero Data*, in ”Symmetry in Nature”, Scuola Norm. Sup. Pisa (1989), 129-140

[24] Brézis, H.; Nirenberg, L. *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents.* Comm. Pure Appl. Math. **36** (1983), no. 4, 437-477.

[25] Brezis H., Nirenberg L. H^2 *versus* C^2 *local minimizers.* C. R. Acad. Sci. Paris Ser. I Math. **317** (1983), no. 5, 465-472.

[26] Cabré, X.; Sire, Y. *Nonlinear equations for fractional laplacians I: regularity, maximum principles, and hamiltonian estimates*. Preprint arXiv:1012.0867, 2010.

[27] Cabré, X.; Solá-Morales, J. *Layer solutions in a half-space for boundary reactions*. Comm. Pure Appl. Math. **58** (2005), 1678-1732.

[28] Cabré, X.; Tan, J. *Positive solutions of nonlinear problems involving the square root of the Laplacian*. Advances in Mathematics **224** (2010), 2052-2093.

[29] Caffarelli, L. A.; Cabré, X. "Fully nonlinear elliptic equations". American Mathematical Society Colloquium Publications, **43**. American Mathematical Society, Providence, RI, 1995.

[30] Caffarelli, L.; Roquejoffre, J. M.; Sire, Y. *Variational problems in free boundaries for the fractional Laplacian*, J. Eur. Math. Soc., **12** (2010), 1151-1179.

[31] Caffarelli, L. A.; Silvestre, L. *An extension problem related to the fractional Laplacian.* Comm. Partial Differential Equations **32** (2007), no. 7-9, 1245-1260.

[32] Caffarelli, L.; Vasseur, A.; *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*. Annals of Mathematics. Second Series, **171** (3) (2010), 1903-1930, doi:10.4007/annals.2010.171.1903.

[33] Capella, A., Dávila, J., Dupaigne, L., Sire, Y. (2011). *Regularity of radial extremal solutions for some non local semilinear equations.* Comm. Partial Differential Equations **36** (2011), no. 8, 1353-1384.

[34] Castro, R.; Zuluaga, M. *Existence results for a class of nonhomogeneous elliptic equations with critical Sobolev exponent*, Note di Matematica, 13 **2** (1993) 269-276.

[35] Charro, F.; Colorado, E.; Peral, I. *Multiplicity of solutions to uniformly elliptic fully nonlinear equations with concave-convex right-hand side*. J. Differential Equations **246** (2009), no. 11, 4221-4248.

[36] Chen, W.; Li, C.; Ou, B. *Classification of solutions for an integral equation.* Comm. Pure Appl. Math. **59** (2006), no. 3, 330-343.

[37] Chen, W.; Li, C.; Ou, B. *Qualitative properties of solutions for an integral equation*. Discrete Contin. Dyn. Syst. **12** (2005), no. 2.

[38] Chipot M.; Chlebik M.; Fila M.; Shafir I.; *Existence of positive solutions of a semilinear elliptic equation in R^0_n with a nonlinear boundary condition.* J. Math. Anal. Appl. **223** (1998), 429–471.

[39] Colorado, E., de Pablo, A., Sánchez, U. *Perturbations of a critical fractional equation*. to appear in Pacific Journal of Mathematics.

[40] Colorado, E.; Peral, I. *Semilinear elliptic problems with mixed Dirichlet-Neumann boundary conditions.* J. Funct. Anal. **199** (2003), no. 2, 468-507.

[41] Cotsiolis, A; Tavoularis, N. K. *Best constants for Sobolev inequalities for higher order fractional derivatives.* J. Math. Anal. Appl. **295** (2004), no. 1, 225-236.

[42] Dávila, J. *A strong maximum principle for the Laplace equation with mixed boundary condition,* J. Funct. Anal. **183** (2001), 231-244.

[43] Ekeland, I. *On the variational principle*. J. Math. Anal. Appl. **47** (1974), 324-353.

[44] Escobar, J. F. *Sharp constant in a Sobolev trace inequality*. Indiana Math. J. **37** (1988), 687-698.

[45] Fabes, E.; Jerison, D.; Kenig, C. *The Wiener test for degenerate elliptic equations.* (French summary) Ann. Inst. Fourier (Grenoble) **32** (1982), no. 3, vi, 151-182.

[46] Fabes, E. B. ; Kenig, C. E. ; Serapioni, R. P. *The local regularity of solutions of degenerate elliptic equations.* Comm. Partial Differential Equations **7** (1982), no. 1, 77-116.

[47] Fall, M.M. ; Weth, T., *Monotonicity and nonexistence results for some fractional elliptic problems in the half space*. http://arxiv.org/pdf/1309.7230v1.pdf

[48] García Azorero J.; Peral I. *Multiplicity of solutions for elliptic problems with critical exponent or with non-symetric term.* Trans. Amer. Math. Soc. **323** (1991), no. 2, 877-895.

[49] Ghoussoub, N., Preiss, D. (1989). *A general mountain pass principle for locating and classifying critical points.* Ann. Inst. H. Poincaré Anal. Non Linéaire **6**, no. 5, 321-330.

[50] Gidas, B.; Ni, W. M.; Nirenberg, L. *Symmetry and related properties via the maximum principle.* Comm. Math. Phys. **68** (1979), no. 3, 209-243.

[51] Gidas, B.; Spruck, J. *A priori bounds for positive solutions of nonlinear elliptic equations.* Comm. Partial Differential Equations **6** (1981), no. 8, 883-901.

[52] Gilbarg, D.; Trudinger, N. S. "Elliptic partial differential equations of second order." Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001

[53] Hardy, G. H.; Littlewood, J. E. *Some properties of fractional integrals*. I. Math. Z. **27** (1928), no. 1, 565-606.

[54] Hu, B, *Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition*, Differential Integral Equations **7** (1994), no. 2, 301-313.

[55] Landkof, N. S. “Foundations of modern potential theory”, Die Grundlehren der mathematischen Wissenschaften, Band 180. Springer-Verlag, New York-Heidelberg, 1972.

[56] Lebedev, N. N. “Special functions and their applications.”Revised edition, translated from the Russian and edited by Richard A. Silverman. Unabridged and corrected republication. Dover Publications, Inc., New York, 1972.

[57] Levendorski, S. Z. *Pricing of the American put under Levy processes*. Int. J. Theor. Appl. Finance **7** (2004), no. 3, 303.335.

[58] Lieb, E. *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities.* Ann. of Math. (2) **118** (1983), no. 2, 349-374.

[59] Lieb, E.; Loss. M. “Analysis”, second ed., Graduate Studies in Mathematics, vol. 14, Amer. Math. Soc., Providence, RI, 2001.

[60] Lions, P.-L. *The concentration-compactness principle in the calculus of variations. The limit case. II.* Rev. Mat. Iberoamericana **1** (1985), no. 2, 45-121.

[61] Majda, A.; Tabak, E. *A two-dimensional model for quasigeostrophic flow: comparison with the two-dimensional Euler ow.* Phys. D, **98** (2-4) (1996), 515-522. Nonlinear phenomena in ocean dynamics (Los Alamos, NM, 1995).

[62] de Pablo, A.; Quirós, F.; Rodríguez, A.; Vázquez, J. L. *A general fractional porous medium equation.* Comm. Pure Appl. Math. **65** (2012), no. 9, 1242–1284.

[63] de Pablo, A.; Quirós, F.; Rodríguez, A.; Vázquez, J. L. *A fractional porous medium equation.* Adv. Math. **226** (2011), no. 2, 1378-1409.

[64] Palatucci, G.; Pisante, A. (2011) *Sobolev embeddings and concentration-compactness alternative for fractional Sobolev spaces*. Preprint Université de Nîmes.

[65] Picone M. *Sui valori eccezionali di un paramtro da cui dipende una equazione differenziale lineare ordinaria del secondo ordine*, Ann. Scuola. Norm. Pisa. **11** (1910), 1-144.

[66] Pohozahev, S. *On the eigenfunctions of quasilinear elliptic problems*. Mat. Sbornik **82** (1970).

[67] Rey, O. *Concentration of solutions to elliptic equations with critical nonlinearity*. Ann. Inst. H. Poincaré Anal. Non Linéaire **9** (1992), no. 2, 201–218.

[68] Rosen, *G. Minimum value for c in the Sobolev inequality* $\backslash\varphi^4\backslash \geq c\backslash \quad \varphi\backslash^4$. SIAM J. Appl. Math. **21** (1971), 30-32.

[69] Signorini, A. *Questioni di elasticitá non linearizzata e semilinearizzata, rendiconti di Matematica e delle sue applicazioni*, **18** (1959), 95-139.

[70] Silvestre, L. *Regularity of the obstacle problem for a fractional power of the Laplace operator*. Comm. Pure Appl. Math. **60** (2007), no. 1, 67

[71] Serrin, J. *A symmetry problem in potential theory.* Arch. Rational Mech. Anal. **43** (1971).

[72] Servadei, R.; Valdinoci, E. *The Brezis-Nirenberg result for the fractional Laplacian*, to appear in Trans. Amer. Math. Soc.

[73] Stein, E. M. "Singular integrals and differentiability properties of functions", Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970.

[74] Shang, X.; Zhang, J.; Yang, Y. *Positive solutions of nonhomogeneous fractional Laplacian problem with critical exponent*, **2** (2014), no. 13, 567-584.

[75] Sobolev, S. L. On a theorem of functional analysis. Transl. Amer. Math. Soc. (2) **34** (1963), 39-68; translation of Mat. Sb. **4** (1938), 471-497.

[76] Stinga, P. R.; Torrea, J. L. *Extension problem and Harnack's inequality for some fractional operators.* Comm. Partial Differential Equations **35** (2010), no. 11, 2092-2122.

[77] Struwe, M. "Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems". Springer-Verlag, Berlin, 2000.

[78] Talenti, G. *Best constant in Sobolev inequality*. Ann. Mat. Pura Appl. **110** (1976), no.4, 353-372.

[79] Tan, J. *Positive solutions for non local elliptic problems*, Discrete Contin. Dyn. Syst. **33** (2013), no. 2, 837-859

[80] Tan, J. *The Brezis-Nirenberg type problem involving the square root of the laplacian*. Calc. Var. Partial Differential Equations **42** (2011), no. 1-2, 21-41.

[81] Tarantello, G. *On nonhomogeneous elliptic equations involving critical sobolev exponent*. Ann. Inst. H. Poincare Anal. Non Lineaire, **9** (1992), p. 281-304

[82] Toland, J. *The Peierls-Nabarro and Benjamin-Ono equations*. J. Funct. Anal., **145** (1) (1997) ,136-150.

[83] Xiao, J. *A sharp Sobolev trace inequality for the fractional-order derivatives.* Bull. Sci. math. **130** (2006), 87-96.