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Alejandro Balbas y Raquel Balbas

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Minimax Strategies and Duality with Applications in Financial Mathematics

Alejandro Balbás

Académico Correspondiente de la Real Academia de Ciencias. University Carlos III of Madrid.

C/ Madrid, 126. 28903 Getafe, Madrid, Spain. alejandro.balbas@uc3m.es

Raquel Balbás

University Complutense of Madrid. Department of Actuarial and Financial Economics. Somosaguas Campus.

28223 Pozuelo de Alarcón, Madrid, Spain. raquel.balbas@ccee.ucm.es

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Abstract

Many topics in Actuarial and Financial Mathematics lead to Minimax or Maximin problems (risk measures optimization, ambiguous setting, robust solutions, Bayesian credibility theory, interest rate risk, etc.). However, minimax problems are usually difficult to address, since they may involve complex vector (Banach) spaces or constraints.

This paper presents an unified approach so as to deal with minimax convex problems. In particular, we will yield a dual problem providing necessary and sufficient optimality conditions that easily apply in practice. Both, duals and optimality conditions are significantly simplified by using a new Mean Value Theorem. Important applications in risk analysis are given.

Key words. Optimization in Banach Spaces, Min-Max Strategies, Duality, Applications in Actuarial and Financial Mathematics.

A.M.S. Classification Subject. 91G80, 90C47.

J.E.L. Classification. C02, G10.

1 Introduction

The notion of general risk measure is becoming more and more important in Actuarial and Financial Mathematics. Many interesting actuarial and financial problems involve the optimization of risk measures. In a recent paper Balbás *et al.* (2010*a*) have provided a new method so as to optimize convex measures of risk. Useful applications of this new methodology have been provided by the same authors. For instance, Balbás *et al.* (2009) dealt with the optimal reinsurance problem of Actuarial Mathematics, Balbás *et al.* (2010*b*) dealt with portfolio choice problems and equilibrium issues in Financial Economics, and Balbás *et al.* (2010*c*) dealt with pricing issues in incomplete markets.

Though former papers dealt with convex analysis and sub-gradient linked optimality conditions in order to minimize risk measures (see, for instance, Ruszczynski and Shapiro, 2006), the novelty in the analysis of Balbás *et al.* (2010*a*) was the incorporation of two elements: Firstly, the representation theorem of convex risk measures permits us to transform a risk minimization problem in a minimax problem. Secondly, the discovery of a new Mean Value Theorem allows us to simplify the dual problem of the minimax one, as well as to provide simple (linear) necessary and sufficient optimality conditions.

There are many classical actuarial and financial problems that are beyond the minimization of risk measures but still lead to minimax or maximin problems. For instance, the minimization of distances, semi-norms or deviations, the Bayesian credibility theory as a experience rating technique (Lemaire, 1995), the incorporation of ambiguity or uncertainty (Klibanoff *et al.*, 2005, Schied, 2007, etc.), the interest in robust solutions (Calafiore, 2007, Zhu and Fukushima, 2009, etc.), some interest rate linked problems (Bierwag and Khang, 1979, Balbás and Romera, 2007, etc.), etc. Therefore, it is worthwhile to study whether the Balbás *et al.* (2010*a*) methodology applies for further analyses.

This paper focuses on a general minimax convex problem and provides both a dual approach and necessary and sufficient optimality conditions, which easily apply in practical applications. In Section 2 we will introduce the general framework and will prove a Mean Value Theorem significantly extending that in Balbás *et al.* (2010*a*). It will characterize some specially important linear and continuous real valued functions on a general Banach space, and the Hahn Banach and the Banach Steinhause Theorems will play a crucial role in the proof. Section 3 will yield a new dual problem and Theorem 3, which states the existence of strong duality between the initial minimax problem and its dual. Some corollaries will focus on particular interesting situations. Section 4 will present four applications: The optimization of risks, problems involving ambiguity and robust optimization, problems involving markets with frictions, and interest rate problems. Finally, Section 5 will conclude the paper.

2 Preliminaries and notations

Let Y and $\mathcal{Y}$ be Banach spaces and Z and $\mathcal{Z}$ their dual spaces. Denote by $Y \times Z \ni (y, z) \rightarrow \langle y, z \rangle \in \mathbb{R}$ and $\mathcal{Y} \times \mathcal{Z} \ni (\gamma, \lambda) \rightarrow \langle \gamma, \lambda \rangle \in \mathbb{R}$ the usual bilinear maps. Suppose that $\mathcal{Y}$ is ordered by the (non necessarily pointed) convex cone $\mathcal{Y}_+$ (and therefore $\mathcal{Z}$ is ordered by the dual cone $\mathcal{Z}_+$) whose interior is non void. For $j = 1, 2, ..., k$ fix $y_j \in Y$, $T_j : Y \rightarrow Y$ linear and continuous, and $\Delta_j \subset Z$ convex and $\sigma(Z, Y)-$compact. Fix finally a convex set $Y_0 \subset Y$ and a convex function $H : Y_0 \rightarrow \mathcal{Y}$. We will deal with the minimax problem

$$\begin{cases} Min \ \rho(y) \\ H(y) \leq 0, \ y \in Y_0 \end{cases} \tag{1}$$

$\rho : Y \rightarrow \mathbb{R}$ given by

$$\rho(y) = Max \ \{\langle T_j(y) + y_j, z_j \rangle ; \ z_j \in \Delta_j, \ j = 1, 2, ..., k\}. \tag{2}$$

Notice that the *weak*$^*-$compactness of every Δ_j guarantees the existence of the maximum in (2). In order to prevent the existence of "duality gaps" (Luenberger, 1969), we will impose the assumption below.

Assumption 1. $\mathcal{Y}_+$ has non void interior and there exists $y \in Y_0$ with $H(y) \in -(\mathcal{Y}_+)^\circ$. In particular, (1) is feasible. □

Expression (2) implies that

$$\rho(\lambda y + (1 - \lambda) y') \leq \lambda \rho(y) + (1 - \lambda) \rho(y')$$

for every $\lambda \in [0,1]$ and every $y, y' \in Y$, and consequently ρ is a convex function and (1) is a convex problem.

As will be seen in section 4, some actuarial and financial problems have the maximin form

$$\begin{cases} Max\ \phi(y) \\ H(y) \leq 0,\ y \in Y_0 \end{cases} \tag{3}$$

where

$$\phi(y) = Min\ \{\langle T_j(y) + y_j, z_j \rangle;\ z_j \in \Delta_j,\ j = 1, 2, ..., k\}. \tag{4}$$

Obviously, (3) and (4) may be trivially reduced to (1) and (2) because one can deal with the equivalent problem

$$\begin{cases} Min\ -\phi(y) \\ H(y) \leq 0,\ y \in Y_0 \end{cases}$$

and

$$-\phi(y) = Max\ \{\langle T_j(y) + y_j, z_j \rangle;\ z_j \in -\Delta_j,\ j = 1, 2, ..., k\}.$$

Denote by $\mathcal{C}(\Delta_j)$ the Banach space composed of the $\mathbb{R}-$valued and $\sigma(Z,Y)-$continuous functions on Δ_j endowed with the usual supremum norm. Denote by $\mathcal{M}(\Delta_j)$ the space of inner regular real valued $\sigma-$additive measures on the Borel $\sigma-$algebra of Δ_j (Δ_j endowed with the *weak** topology), and by $\mathcal{P}(\Delta_j) \subset \mathcal{M}(\Delta_j)$ the set composed of those $\sigma-$additive measures that are probabilities (*i.e.*, if $\mu \in \mathcal{M}(\Delta_j)$ then $\mu \in \mathcal{P}(\Delta_j)$ if $\mu \geq 0$ and $\mu(\Delta_j) = 1$). According to the Riesz Representation Theorem $\mathcal{M}(\Delta_j)$ endowed with the variation norm is the dual space of $\mathcal{C}(\Delta_j)$. In order to simplify some notations, $\mathcal{C}_+(\Delta_j)$ and $\mathcal{M}_+(\Delta_j)$ will represent the usual non-negative cones of $\mathcal{C}(\Delta_j)$ and $\mathcal{M}(\Delta_j)$, respectively.

Next we will prove a Mean Value Theorem which extends particular results of Balbás *et al.* (2009) and (2010*a*). Indeed, Balbás *et al.* (2009) stated this Theorem for expectation bounded risk measures and deviations, and Balbás *et al.* (2010*a*) dealt with more complex convex risk measures.

Lemma 1 *(Mean Value Theorem). For every $\mathbb{P} \in \mathcal{P}(\Delta_j)$ there exists a unique $z_{\mathbb{P}} \in Z$*

such that

$$\int_{\Delta_j} \langle y, z \rangle \, d\mathbb{P}(z) = \langle y, z_{\mathbf{P}} \rangle \tag{5}$$

for every $y \in Y$. *Furthermore,* $z_{\mathbf{P}} \in \Delta_j$.

Proof. Δ_j is $\sigma(Z, Y)-$compact, and therefore the Banach-Steinhause Theorem (Rudin, 1972) shows that it is bounded, *i.e.*, there exists $M \in \mathbb{R}$ such that $\|z\| \leq M$ for every $z \in \Delta_j$. For every $y \in Y$ we have that

$$\left| \int_{\Delta_j} \langle y, z \rangle \, d\mathbb{P}(z) \right| \leq \int_{\Delta_j} |\langle y, z \rangle| \, d\mathbb{P}(z) \leq \left(\int_{\Delta_j} \|z\| \, d\mathbb{P}(z) \right) \|y\| \leq M \|y\|,$$

which implies that

$$Y \ni y \to \int_{\Delta_j} \langle y, z \rangle \, d\mathbb{P}(z) \in \mathbb{R}$$

is a continuous linear function. Thus, there exists a unique $z_{\mathbf{P}} \in Z$ such that (5) holds, and it only remains to see that $z_{\mathbf{P}} \in \Delta_j$. If $z_{\mathbf{P}} \notin \Delta_j$ then $z_{\mathbf{P}}$ and the $\sigma(Z, Y)-$compact set Δ_F can be separated (Hahn-Banach Theorem, Rudin, 1972), and there exists $y^* \in Y$ with

$$\langle y^*, z_{\mathbf{P}} \rangle < Min \; \{\langle y^*, z \rangle ; \; z \in \Delta_j\}.$$

Then, bearing in mind Expression (5) we have that

$$\begin{array}{l} \langle y^*, z_{\mathbf{P}} \rangle < Min \; \{\langle y^*, z \rangle ; \; z \in \Delta_j\} = \\ \int_{\Delta_j} (Min \; \{\langle y^*, z \rangle ; \; z \in \Delta_j\}) \, d\mathbb{P}(z) \leq \\ \int_{\Delta_j} \langle y^*, z \rangle \, d\mathbb{P}(z) = \langle y^*, z_{\mathbf{P}} \rangle \end{array}$$

which is absurd. $\square$

3 Duality for Min-Max problems

Henceforth we will consider the following dual problem of (1).

$$\left\{ \begin{array}{l} Max \; \left(Inf \; \left\{ \left(\sum_{j=1}^{k} \lambda_j \langle T_j(y) + y_j, z_j \rangle \right) + \Lambda \circ H(y); \; y \in Y_0 \right\} \right) \\ z_j \in \Delta_j, \; j = 1, 2, ..., k, \; \Lambda \in \mathcal{Z}_+ \\ \lambda_j \in \mathbb{R}, \; \lambda_j \geq 0, \; j = 1, 2, ..., k, \; \sum_{j=1}^{k} \lambda_j = 1 \end{array} \right. \tag{6}$$

In order to simplify some expressions let us denote by D the dual feasible set, *i.e.*,

$$D=\left\{\left((z_j)_{j=1}^k,\Lambda,(\lambda_j)_{j=1}^k\right)\in\prod_{j=1}^k(\Delta_j)\times\mathcal{Z}_+\times\mathbb{R}^k;\ \sum_{j=1}^k\lambda_j=1 \ and\ \lambda_j\geq 0,\ j=1,2,...,k\right\}$$

We will prove the absence of duality gap and the existence of strong duality between (1) and (6). However, the standard Duality Theory for Convex Programming generates a dual problem much more complex than (6), since some dual variables should involve spaces of inner regular $\sigma-$additive measures (recall that $\mathcal{M}(\Delta_j)$ is the dual of $\mathcal{C}(\Delta_j)$). Thus, let us see that the Mean Value Theorem permits us to simplify the usual dual of (1).

Lemma 2 *Consider Problem*

$$\left\{\begin{array}{l} Max\ \left(Inf_{y\in Y_0}\left\{\left(\sum_{j=1}^k\int_{\Delta_j}\langle T_j(y)+y_j,z\rangle\, dm_j(z)\right)+\Lambda\circ H(y)\right\}\right)\\ \Lambda\in\mathcal{Z}_+,\ m_j\in\mathcal{M}_+(\Delta_j),\ j=1,2,...,k\\ \sum_{j=1}^k\int_{\Delta_j}dm_j=1\end{array}\right.\tag{7}$$

and denote by D_M its feasible set. Consider the correspondence

$$\prod_{j=1}^k(\mathcal{M}_+(\Delta_j))\times\mathcal{Z}_+\ni\left((m_j)_{j=1}^k,\Lambda\right)\to\left((z_j)_{j=1}^k,\Lambda,(\lambda_j)_{j=1}^k\right)\in\prod_{j=1}^k(\Delta_j)\times\mathcal{Z}_+\times\mathbb{R}^k$$

characterized by

$$\begin{array}{ll}\lambda_j=\int_{\Delta_j}dm_j, & j=1,2,...,m\\ \langle y,z_j\rangle=\frac{1}{m_j(\Delta_j)}\int_{\Delta_j}\langle y,z_j\rangle\, dm_j(z)\ \ \forall y\in Y, & if\ m_j(\Delta_j)>0\\ z_j\in\Delta_j\ is\ arbitrary, & if\ m_j(\Delta_j)=0\end{array}\tag{8}$$

This correspondence is surjective from D_M to D, and the (7)$-$objective value in every $\left((m_j)_{j=1}^k,\Lambda\right)$ equals the (6)$-$objective value in its images. Hence,

$$\left((m_j)_{j=1}^k,\Lambda\right)\in\prod_{j=1}^k(\mathcal{M}_+(\Delta_j))\times\mathcal{Z}_+$$

solves (7) if and only if

$$\left((z_j)_{j=1}^k,\Lambda,(\lambda_j)_{j=1}^k\right)\in\prod_{j=1}^k(\Delta_j)\times\mathcal{Z}_+\times\mathbb{R}^k$$

solves (6), and, conversely, every solution of (6) is given by (8) and a solution of (7). Both optimal values coincide.

Proof. The Mean Value Theorem (Lemma 1) shows that the correspondence given by (8) is well defined, and the constraints of (7) show that the element $\left((z_j)_{j=1}^k, \Lambda, (\lambda_j)_{j=1}^k\right)$ in (8) is (6)-feasible. Moreover, the objective function of $\left((z_j)_{j=1}^k, \Lambda, (\lambda_j)_{j=1}^k\right)$ on (6) equals the objective function of $\left((m_j)_{j=1}^k, \Lambda\right)$ on (7). Hence, the result will be proved if we see that every (6)-feasible $\left((z_j)_{j=1}^k, \Lambda, (\lambda_j)_{j=1}^k\right)$ is generated by a (7)-feasible $\left((m_j)_{j=1}^k, \Lambda\right)$. To prove that property, take a (6)-feasible $\left((z_j)_{j=1}^k, \Lambda, (\lambda_j)_{j=1}^k\right)$ and define

$$(m_j)_{j=1}^k = \left(\lambda_j \delta_{z_j}\right)_{j=1}^k \in \prod_{j=1}^k \left(\mathcal{M}_+ \left(\Delta_j\right)\right),$$

every δ_{z_j} denoting the usual Dirac delta concentrating the whole mass on $\{z_j\}$. □

Theorem 3 *Suppose that (1) is bounded.*

a) Problem (6) is solvable (the optimal value is attainable) and both optimal objective values coincide, i.e.,

$$\begin{array}{ll} Inf_{\substack{y \in Y_0 \\ H(y) \leq 0}} & (\rho(y)) = \\ Max_{\left((z_j)_{j=1}^k, \Lambda, (\lambda_j)_{j=1}^k\right) \in D} & \left(Inf \left\{ \sum_{j=1}^k \lambda_j \left(\langle T_j(y) + y_j, z_j \rangle\right) + \Lambda \circ H(y);\ y \in Y_0 \right\} \right). \end{array}$$

b) If $y^* \in Y_0$ and $H(y^*) \leq 0$ then y^* solves Problem (1) if and only if there exists $\left((z_j^*)_{j=1}^k, \Lambda^*, (\lambda_j^*)_{j=1}^k\right) \in D$ such that

$$\begin{array}{l} \sum_{j=1}^k \lambda_j^* \left(\left\langle T_j(y) + y_j, z_j^* \right\rangle\right) + \Lambda^* \circ H(y) \geq \\ \sum_{j=1}^k \lambda_j^* \left(\left\langle T_j(y^*) + y_j, z_j^* \right\rangle\right) + \Lambda^* \circ H(y^*) \end{array} \tag{9}$$

for every $y \in Y_0$, and the complementary slackness condition

$$\begin{cases} \Lambda^* \circ H(y^*) = 0 \\ \lambda_j^* \left(\left\langle T_j(y^*) + y_j, z_j^* \right\rangle - \rho(y^*)\right) = 0, \quad j = 1, 2, ..., k \end{cases} \tag{10}$$

holds. In such a case $\left((z_j^*)_{j=1}^k, \Lambda^*, (\lambda_j^*)_{j=1}^k\right)$ solves Problem (6).

Proof. Expression (2) shows that (1) is equivalent to

$$\left\{\begin{array}{l} Min\ \theta \\ \langle T_1(y)+y_1,z\rangle-\theta\leq 0,\quad \forall z\in\Delta_1 \\ .... \\ \langle T_k(y)+y_k,z\rangle-\theta\leq 0,\quad \forall z\in\Delta_k \\ H(y)\leq 0 \\ \theta\in\mathbb{R},\ y\in Y_0 \end{array}\right. \tag{11}$$

(θ, y) being the decision variable. Indeed, y solves (1) if and only if (θ, y) solves (11) for some $\theta \in \mathbb{R}$, in which case $\theta = \rho(y)$.The first constraints of (11) are $\mathcal{C}(\Delta_j)$ −valued. Since $\mathcal{M}(\Delta_j)$ is the dual space of $\mathcal{C}(\Delta_j)$ the Lagrangian function (Luenberger, 1969)

$$\mathbb{R}\times Y_0\times\prod_{j=1}^{k}(\mathcal{M}_+(\Delta_j))\times\mathcal{Z}_+\ni\left(\theta,y,(m_j)_{j=1}^k,\Lambda\right)\to\mathcal{L}\left(\theta,y,(m_j)_{j=1}^k,\Lambda\right)\in\mathbb{R}$$

becomes

$$\begin{array}{l}\mathcal{L}\left(\theta,y,(m_j)_{j=1}^k,\Lambda\right)= \\ \theta\left(1-\sum_{j=1}^{k}\int_{\Delta_j}dm_j\right)+\sum_{j=1}^{k}\left(\int_{\Delta_j}\langle T_j(y)+y_j,z\rangle\,dm_j(z)\right)+\Lambda\circ H(y).\end{array}$$

According to the Duality Theory in Luenberger (1969), $\left((m_j)_{j=1}^k,\Lambda\right)\in\prod_{j=1}^k(\mathcal{M}_+(\Delta_j))\times\mathcal{Z}_+$ is dual-feasible if and only if

$$Inf_{(\theta,y)\in\mathbb{R}\times Y_0}\mathcal{L}\left(\theta,y,(m_j)_{j=1}^k,\Lambda\right)>-\infty,$$

which implies that $\sum_{j=1}^{k}\int_{\Delta_j}dm_j=1$. In such a case, $\mathcal{L}$ does not depend on θ, and the dual problem of (1) becomes (7). Since the non negative cone of $\mathcal{C}(\Delta_j)$ has non void interior, in order to guarantee that (7) is solvable and there is no duality gap between (1) and (7) it is sufficient to see that (11) satisfies the Slater qualification, *i.e.*, the constraints of this problem are strictly satisfied for at least a feasible element (see Luenberger, 1969). But this is obvious because one can choose a feasible element (θ, y) with $H(y)$ in the interior of $\mathcal{Y}_+$ (Assumption 1) and then take

$$\theta > Max\ \left\{\langle T_j(y)+y_j,z\rangle;\ z\in\Delta_j\right\},$$

$j = 1, 2, ..., k$, so as to ensure that $\langle T_j(y)+y_j,z\rangle-\theta<0$ for every $z\in\Delta_j$ and $j = 1, 2, ..., k$.

$a)$ Statement $a)$ is an obvious consequence of Lemma 2 and the existence of strong duality between (1) and (7).

$b)$ Suppose that $y^* \in Y_0$ and $H(y^*) \leq 0$. y^* solves (1) if and only if there exists $\theta^* \in \mathbb{R}$ such that (θ^*, y^*) solves (11), in which case $\theta^* = \rho(y^*)$. According to Luenberger (1969), (θ^*, y^*) solves (11) if and only if there exists $\left((m_j^*)_{j=1}^k, \Lambda^*\right) \in \prod_{j=1}^k (\mathcal{M}_+(\Delta_j)) \times \mathcal{Z}_+$ such that

$$\begin{array}{l} \sum_{j=1}^{k} \left(\int_{\Delta_j} \langle T_j(y^*) + y_j, z\rangle \, dm_j^*(z)\right) + \Lambda^* \circ H(y^*) \geq \\ \sum_{j=1}^{k} \left(\int_{\Delta_j} \langle T_j(y) + y_j, z\rangle \, dm_j^*(z)\right) + \Lambda^* \circ H(y) \end{array} \tag{12}$$

for every $y \in Y_0$, and the complementary slackness conditions

$$\int_{\Delta_j} \left(\langle T_j(y^*) + y_j, z\rangle - \theta^*\right) dm_j^*(z) = 0, \quad j = 1, 2, ..., k \tag{13}$$

and $\Lambda^* \circ H(y^*) = 0$ hold. Besides, Lemma 2 and its proof prove that (12) holds if and only if (9) holds. Moreover, (13) and $\theta^* = \rho(y^*)$ imply the fulfillment of (10).

Conversely, if (9) and (10) hold then Lemma 2 guarantees the fulfillment of (12). If we show that (13) holds for some θ^* with (θ^*, y^*) (11)$-$feasible then y^* will solve (1). Take $\theta^* = \rho(y^*)$ and (2) guarantees that (θ^*, y^*) is (11)$-$feasible.

Finally, in the affirmative case, $\left((m_j)_{j=1}^k, \Lambda\right)$ solves (7), and then $\left((z_j^*)_{j=1}^k, \Lambda^*, (\lambda_j^*)_{j=1}^k\right)$ solves Problem (6) due to Lemma 2. □

Consider now that Constraint $H(y) \leq 0$ is removed. The new problem become

$$\begin{cases} Min \ \rho(y) \\ \quad y \in Y_0 \end{cases} \tag{14}$$

whose dual will be

$$\begin{cases} Max \left(Inf \left\{\left(\sum_{j=1}^{k} \lambda_j \langle T_j(y) + y_j, z_j\rangle\right); \ y \in Y_0\right\}\right) \\ z_j \in \Delta_j, \ j = 1, 2, ..., k. \\ \lambda_j \in \mathbb{R}, \ \lambda_j \geq 0, \ j = 1, 2, ..., k, \ \sum_{j=1}^{k} \lambda_j = 1 \end{cases} \tag{15}$$

We still have strong duality between (14) and (15).

Corollary 4 *Suppose that (14) is feasible and bounded from below.*

a) Problem (15) is solvable (the optimal value is attainable) and both optimal objective values coincide.

b) If $y^* \in Y_0$ then y^* solves Problem (14) if and only if there exists $\left(\left(z_j^*\right)_{j=1}^k, \left(\lambda_j^*\right)_{j=1}^k\right)$ (15)–feasible such that

$$\sum_{j=1}^{k} \lambda_j^* \left\langle T_j(y) + y_j, z_j^* \right\rangle \geq \sum_{j=1}^{k} \lambda_j^* \left\langle T_j(y^*) + y_j, z_j^* \right\rangle$$

for every $y \in Y_0$, and the complementary slackness condition

$$\lambda_j^* \left(\left\langle T_j(y^*) + y_j, z_j^* \right\rangle - \rho(y^*) \right) = 0, \quad j = 1, 2, ..., k$$

holds. In such a case $\left(\left(z_j^*\right)_{j=1}^k, \left(\lambda_j^*\right)_{j=1}^k\right)$ solves Problem (15).

Proof. It immediately follows from the previous Theorem if one takes $\mathcal{Y} = \mathbb{R}$ and $H = -1$. $\square$

As said in the previous section, some classical problems of Actuarial and Financial Mathematics are Maximin rather than Minimax. In such a case, by using the straightforward modifications indicated in Section 2, we have:

Corollary 5 *the dual problem of (3) is*

$$\begin{cases} Min \left(Sup \left\{ \left(\sum_{j=1}^{k} \lambda_j \left\langle T_j(y) + y_j, z_j \right\rangle \right) - \Lambda \circ H(y);\ y \in Y_0 \right\} \right) \\ z_j \in \Delta_j,\ j = 1, 2, ..., k,\ \Lambda \in \mathcal{Z}_+ \\ \lambda_j \in \mathbb{R},\ \lambda_j \geq 0,\ j = 1, 2, ..., k,\ \sum_{j=1}^{k} \lambda_j = 1 \end{cases} \tag{16}$$

There is no duality gap between both problems and (16) is solvable. Furthermore, if $y^* \in Y_0$ and $H(y^*) \leq 0$, then it solves (3) if and only if there exists $\left(\left(z_j^*\right)_{j=1}^k, \Lambda^*, \left(\lambda_j^*\right)_{j=1}^k\right)$ (16)–feasible such that

$$\sum_{j=1}^{k} \lambda_j^* \left(\left\langle T_j(y) + y_j, z_j^* \right\rangle \right) - \Lambda^* \circ H(y) \geq \sum_{j=1}^{k} \lambda_j^* \left(\left\langle T_j(y^*) + y_j, z_j^* \right\rangle \right) - \Lambda^* \circ H(y^*)$$

for every $y \in Y_0$ and (10) holds. In such a case $\left(\left(z_j^*\right)_{j=1}^k, \Lambda^*, \left(\lambda_j^*\right)_{j=1}^k\right)$ solves (16). $\square$

Remark 1 *Notice that $\rho(y^*)$ arises in (10), which might generate computational problems in some applications. However, in most of the cases the minimum value in (2) will be achieved in a unique (j_0, z_{j_0}). Then*

$$\lambda_j^* \left(\left\langle T_j(y^*) + y_j, z_j^* \right\rangle - \rho(y^*) \right) = 0$$

is equivalent to

$$\begin{array}{l} \lambda_j^* = 0, \; j \neq j_0 \\ \rho(y^*) = \left\langle T_{j_0}(y^*) + y_{j_0}, z_{j_0}^* \right\rangle \end{array}$$

that may be easily applied in practice. $\square$

4 Actuarial and financial applications

As said in the introduction, there are many classical problems in Actuarial and Financial Mathematics fitting in the framework of this paper. Let us devote this section to presenting four examples.

4.1 Risk measures, semi-norms and deviations

General risk measures are becoming very important in finance and insurance. Artzner *et al.* (1999) introduced the axioms and properties of the "Coherent Measures of Risk", and, since then, many authors have extended the discussion. In our setting, particularly important are the expectation bounded risk measures and the deviation measures of Rockafellar *et al.* (2006), because both are particular cases of (2) with $Y = L^p(\Omega, \mathcal{F}, \mathbb{P})$ and $Z = L^q(\Omega, \mathcal{F}, \mathbb{P})$, $p \geq 1$, $q \leq \infty$, $1/p + 1/q = 1$, and $(\Omega, \mathcal{F}, \mathbb{P})$ being a probability space.

The optimization of risk measures and deviations is a very complex problem that has motivated several deep analyses (Nakano (2004), Ruszczynski and Shapiro (2006), Mansini *et al.* (2007), etc.), all of them related to Convex Analysis. As said in the introduction, the approach of this paper is an alternative way that significantly simplifies many applications and allows us to reach further conclusions about the analyzed problems. In Balbás *et al.* (2009), (2010b) and (2010c) one can find actuarial and financial problems involving risk measures, all of then solved with the method proposed in Balbás *et al.* (2010a) that has been extended in Theorem 3 and its corollaries.

4.2 Ambiguity and robust optimization

Ambiguity arises in finance and insurance if we are not sure about the real probability space reflecting the random or stochastic behavior of the variables we are interested in. Recent significant examples are Calafiore (2007), Schied (2007) and Zhu and Fukushima (2009), where portfolio selection problems are studied.

All of these analyses fit in our framework, since, instead of dealing with "variable probability spaces" one can often fix the "true" probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and then introduce the ambiguity by modifying the distribution of some random variables indicating our risk/uncertainty. For instance, let us adapt the optimal reinsurance problem of Balbás *et al.* (2009) to the ambiguous setting. Without ambiguity the problem is as follows: Consider that the insurance company receives the fixed amount S_0 (premium) and will have to pay the random variable

$$C \in L_{+}^{p}(\Omega, \mathcal{F}, \mathbb{P}) = \{y \in L^{p}(\Omega, \mathcal{F}, \mathbb{P}) : \mathbb{P}(y \geq 0) = 1\}$$

within a given period $[0, T]$ (claims). Suppose also that a reinsurance contract is signed in such a way that the company will only pay $y \in L^{p}(\Omega, \mathcal{F}, \mathbb{P})$, whereas the reinsurer will pay $C - y$. If the reinsurer premium principle is given by the continuous linear function,

$$\pi : L^{p}(\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \mathbb{R}$$

and $S_1 > 0$ is the highest amount that the insurer would like to pay at T for the contract, then the insurance company will choose y (optimal retention) so as to solve

$$\begin{cases} Min\ \rho(S_0 - y - \pi(C - y)) \\ \pi(C - y) \leq S_1 \\ 0 \leq y \leq C \end{cases} \tag{17}$$

ρ representing an expectation bounded and coherent risk measure. Problem (17) may incorporate ambiguity if we consider that the total claims C are ambiguous and, therefore, it may be substituted by a convex set

$$\left\{ C + \sum_{j=1}^{k} \alpha_j C_j;\ \sum_{j=1}^{k} \alpha_j = 1,\ \alpha_j \geq 0,\ j = 1, 2, ..., k \right\},$$

where $\{C_j;\ j = 1, 2, ..., k\} \subset L^p(\Omega, \mathcal{F}, \mathbb{P})$ is fixed. In such a case (17) would become

$$\begin{cases} Min\ F(y) \\ \pi(C + C_j - y) \leq S_1,\ j = 1, 2, ..., k \\ 0 \leq y \leq C + C_j,\ j = 1, 2, ..., k \end{cases}$$

where

$$F(y) = Max\ \{\langle (S_0 - y - \pi(C + C_j - y)), z\rangle;\ z \in \Delta,\ j = 1, 2, ..., k\},$$

$\Delta \subset L^p(\Omega, \mathcal{F}, \mathbb{P})$ being the sub-gradient of the risk measure ρ (see Rockafellar *et al.*, 2006).

4.3 Imperfect markets

In practice transaction costs may be significant enough so as to modify the real financial markets behavior. Thus, though linear pricing rules are usual in perfect or friction free markets (Duffie, 1988), convex pricing rules are usual when dealing with imperfections (Jouin and Kallal, 1995, Schachermayer, 2004, etc.). Hence, pricing or portfolio selection problems become convex too under transaction costs. For instance the optimal portfolio problem of Balbás *et al.* (2010*b*) will be

$$\begin{cases} Min\ \Pi(y) \\ y \in C,\ \rho(y) \leq 1, \mathbb{E}\ (y) \geq R \end{cases}$$

where $\rho : L^2(\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \mathbb{R}$ is a expectation risk measure or a deviation measure, $C \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ is the convex cone of reachable pay-offs in a market with frictions, $\Pi : L^2(\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \mathbb{R}$ is a pricing rule that satisfies Expression (2), $R > 0$ is the minimum required expected return, and $\mathbb{E}$ denotes the mathematical expectation of every random variable. The developed theory allows for extensions of those properties found in Balbás *et al.* (2010*b*).

Other interesting minimax problem related to imperfections is the minimization of the price of a given pay-off. Though this problem does not make any sense in frictionless arbitrage free markets, because there is only one price per pay-off, this property does not remain true under frictions, and the corresponding minimax problems becomes

$$\begin{cases} Min\ \Pi(y) \\ y \in C,\ y \geq P \end{cases}$$

P being the desired pay-off.

4.4 Interest rate risk

Interest rate risk hedging is a classical issue in Financial Mathematics. If the Term Structure of Interest Rates ($TSIR$) grows (falls) then bond prices fall (grow) which implies that traders may lose money due to the $TSIR$ evolution. Usually, duration and convexity (Montrucchio and Peccati, 1991, among others) are the portfolio parameters that investors control in order to protect their wealth, but Bierwag and Khang (1979) showed that hedged portfolios are also maximin strategies. Since then, many researchers have extended the discussion, and a very complete analysis may be found, for instance, in Barber and Copper (1998) or Balbás and Romera (2007). In particular, this recent paper has developed a semi-infinite simplex like algorithm (Anderson and Nash, 1987) that leads to the maximin strategy for a wide family of $TSIR$ shifts.

Following Balbás and Romera (2007), consider n arbitrary bonds B_j, $j = 1, 2, ..., n$, and denote by $p = (p_1, p_2, ..., p_n)$, $p_j > 0$, $j = 1, 2, ..., n$, the vector of prices. Suppose that T is a future date such that the bond maturities lie within the interval $[0, T]$. Suppose that $m \in [0, T]$ represents the horizon planning period and K is a set of real valued functions on $[0, T]$ whose elements are admissible shocks on the $TSIR$. The portfolio composed of q_j units of B_j, $j = 1, 2, ..., n$, will be represented by $q = (q_1, q_2, ..., q_n) \in \mathbb{R}^n$ and $\sum_{j=1}^{n} p_j q_j$ will be its price. If $V_j(k)$ is the value of B_j, $j = 1, 2, ..., n$, at m if $k \in K$ takes place, then the real valued function $V : \mathbb{R}^n \times K \longrightarrow \mathbb{R}$ given by

$$V(q, k) = \sum_{j=1}^{n} q_j V_j(k) \tag{18}$$

will provide the value of $q = (q_1, ..., q_n)$ at m if k takes place.

Expression (18) implies that $V(-, k)$ is linear in the q variable. We will assume that K is endowed with an appropriate topology and becomes Hausdorff and compact. For instance, in Balbás and Romera (2007) there are three examples, whose compactness is implied by the Ascoli-Arzelà or the Alaoglu's Theorem (Rudin, 1972). These examples are:

a) $K_1 \subset C[0, T]$ is composed of those continuously differentiable functions k such that $| k(t) | \leq \lambda_1$ and $| k'(t) | \leq \lambda_2$ for every $t \in [0, T]$. K_1 is endowed with the compact-open topology topology.

b) $K_2 \subset L^2[0, T]$ is composed of those functions k such that $| k(t) | \leq \lambda_1$ and $| k(t_2) - k(t_1) | \leq$

λ_2 almost everywhere. K_2 is endowed with the weak topology.

c) If K_3 is a closed ball of $L^2[0,T]$, endowed with the weak topology.

Moreover, $V_j : K \longrightarrow \mathbb{R}$, $j = 1, 2, ..., n$, is assumed to be continuous and, therefore, it follows from (18) that $V(q, -) : K \longrightarrow \mathbb{R}$ is also continuous in the k variable.

Given $q \in \mathbb{R}^n$, we define its guaranteed value at m by

$$\bar{V}(q) = Min\{V(q,k);\ k \in K\},$$

which implies that $\bar{V}$ is the minimum of a family of linear functions. Moreover, the maximin hedging problem is given by

$$\begin{cases} Max\ \bar{V}\,(q) \\ q \in Q \end{cases} \tag{19}$$

where the convex set Q will be defined by real constraints in practical applications. They may be related to budget, short-selling or duration restrictions, liabilities, and other situations. Obviously, (19) is a particular case of (3) and therefore Corollary 5 applies. Hence, Problem (19) may be frequently solved by methods less complex than the semi-infinite simplex like algorithm Balbás and Romera (2007). Since $q \in \mathbb{R}^n$ then for every fixed $k \in K$ the linear function $q \rightarrow V(q,k)$ belongs to $\mathbb{R}^n$ too, *i.e.*, (19) fits in (3) with the finite-dimensional framework $Y = Z = \mathbb{R}^n$. Thus, the conditions of Corollary 5 apply in finite dimensions, which significantly simplifies the previous theoretical and computational analyses dealing with this problem (Barber and Copper, 1998, Balbás and Romera, 2007, etc.).

5 Conclusions

This paper has provided a new Duality Theory for maximin and minimax convex problems. The major finding is Theorem 3, which guarantees strong duality between the minimax problem and its dual, as well as the existence of simple systems of equations characterizing both primal and dual solutions. With respect to former studies, this new approach significantly simplifies the optimality conditions, which become easy to apply in practice. Actuarial and Financial applications have been given. $\square$

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