Aggressive Corporate Tax Behavior versus Decreasing Probability of Fiscal Control*

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Abstract

We provide a model of strategic interaction between the Internal Revenue Service (IRS) and the firms, that analyzes the impact of the increasing financial sophistication, and respectively, of the book profits reporting and its audit, on tax compliance and fiscal control. In this simple framework we describe basic scenarios in which decreasing IRS audit rates and weaker fiscal discipline appear endogenously, that is, when growing financial sophistication is paralleled by changes in the information on book profits available to the tax authority, or by changes in the distribution of the book profits. In contrast to other views, these scenarios involve simple explicative mechanisms that do not rely on the idea of relative changes in the IRS resources or in the applied penalties.

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1 Introduction

Starting with the early 1990's, both policy makers and the accounting literature began to signal that, in the United States, the government tax receipts have been increasing much less than the size and profitability of firms. This evolution generated growing concern regarding the possibility of an increase in tax sheltering; it also raised questions about its causes.

The growing gap between the book income and the taxable income can be interpreted as indirect empirical evidence of the expanding tax sheltering activity. The increasing discrepancy began in 1992 and evolved independently of the business cycle fluctuations (Slemrod [12]). For instance, the ratio of book income to tax income of the companies with assets greater than \$1 billion grew from 1 to 1.4 between 1991 and 1996 (Desai [3]). As already mentioned, this trend can be simply explained by the increase of corporate tax shelters; nevertheless, there are other appealing alternative explanations. First, it could be due to a change in the size of the items that normally account for this difference (the foreign operations, the different methodology of computing depreciation, the employee stock options compensation, etc.). Second, the increasing divergence could stem from an increased level of book profits manipulation.

However, for the 1990's decade, Desai [3] quantifies the contribution of the items that presumably create the gap between the book and the tax income, and shows that less than half of the aggregate gap is explained by these sources of distinction. The author claims that, although it is not possible to disregard fraudulent book profit reporting as one of the sources of this discrepancy, the micro analysis suggests that the breakdown in the relationship between tax and book income is more consistent with increasing levels of tax sheltering. He concludes that firms became more fiscally aggressive during this decade, and mentions as possible causes either lower probabilities of detection, or lower perceived penalties.

A 1999 report of the U.S. Department of Treasury [14] identifies several directions in which the phenomenon of tax sheltering evolves. An important trend is given by the increasing financial sophistication, i.e. the availability of software and low-cost technologies to carry out complicated transactions, the growing complexity of financial markets, the development of financial innovations, the increasing supply of tax specialists, etc. The expanding financial sophistication provides more opportunities for firms to avoid taxes without breaking the law. More and more firms can reduce tax liabilities to non-natural levels, while they perceive very low probabilities of detection, since they are aware that their tax avoidance activities cannot be proven illegal. Such a trend can be kept under control mainly through improved regulations, and not through changes in the probabilities of control, or in the system of penalties.

Another important trend identified in the report is that there are more and more corporations resorting to fraud (that is, tax behavior that can be proven illegal by an Internal Revenue Service (IRS) audit). In the authors' view, such abusive tax behavior is closely related to the changes in the IRS audit probabilities. Lower rates of fiscal audit lead to a decrease in the perceived risk of the taxpayer to be audited, hence, to more aggressive tax positions of the firms. Indeed, there is significant evidence for the decrease in the rates of the fiscal audit, especially among the companies with large levels of assets. A relevant example is that the IRS audit rates for the group of companies with assets greater than \$100 million decreased from 59 percent in 1990 to only 35 percent in 1997. The overall audit rate decreased as well,

although not that drastically, from 2.9 percent in 1992 to 2.0 percent in 1998. The study further suggests that the decline in the overall audit rates is a consequence of the relative contraction of the IRS resources with respect to the size dynamics of the economy.

Nevertheless, for the 1990's decade, evidence of a decline in the IRS resources can be found only after 1996 (Slemrod [12], Steuerle [13]). Moreover, there is no evidence of changes in the system of penalties applied by the IRS during the decade, that we are aware of. We do not discard the idea that a decline in the IRS budget or in the applied penalties contribute to the increasing fiscal aggressiveness of the corporations; however, we are motivated to consider alternative mechanisms that can replicate the observations that the fiscal audit rates decrease, while the fiscal discipline weakens. The starting point is provided by the same report of the U.S. Department of Treasury [14]. It argues that, due to the growing financial sophistication, more and more firms, which in the absence of any tax avoidance activities would be characterized by high tax liabilities, are assimilated instead to the firms with low liabilities, since they are able to conceal due taxes by means of transactions without any business purpose that the IRS cannot prove illegal. Such a phenomenon may further induce a decrease in the fiscal discipline in the market, in the sense that more corporations, which cannot achieve such low levels of tax liabilities within the margins of the law, will resort to fraud.

In the present paper we try to assess the impact the increasing financial sophistication may have on the corporate fiscal discipline, and on the probability of fiscal control. We provide a stylized framework in which an immediate result is that, in equilibrium, increasing financial sophistication induces, ceteris paribus, a decrease in the level of fiscal discipline in the market. However, contrary to the empirical evidence, it also induces an increase in the total rate of the fiscal control. If besides the effects of the expanding financial sophistication, we also consider the impact that the accounting reporting and its audit may have on tax compliance and control, then we can provide simple scenarios such that decreasing IRS audit rates and weaker fiscal discipline appear both endogenously and simultaneously.

In contrast to the mechanisms suggested by the report [14] and described above, our model is based on the following premises. First, the IRS maximizes the expected net revenue without facing a budget constraint, and therefore, by assumption, the change in the rates of the fiscal audit cannot appear as a consequence of a contraction in the IRS resources. Second, we consider in our analysis an exogenous and constant penalty function. Third, we account for the level of financial sophistication exclusively by the scale at which it allows the companies to avoid taxes without breaking the law. Hence, in our model, changes in the financial sophistication are assimilated to changes in the distribution of firm types. Finally, we assume that the tax administration has access to both tax and book profit reports, as well as to the results of a (possible) audit of the latter.

The theoretical literature on tax evasion is substantial (for a comprehensive survey, see Andreoni et al [2]), but most of it concentrates on the individual taxpayer, and relatively little has been done to analyze the behavior of the firm. Nevertheless, studying the firm is relevant, given that firms face different circumstances than the individuals. An important fact is that, besides taxes, the firm also reports the level of book profits. Book and tax profits are correlated, but they represent different concepts and are computed using different methodologies. Book profits should give outsiders a good idea about the performance of the firm, whereas taxes collect equitably revenues for the budget and can be used by the state as an instrument

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for encouraging or discouraging certain activities (Hanlon and Shevlin [7]). In the case of public corporations, both tax and book profit reports can be subject to manipulation by the firm. In order to deter these types of manipulative behavior, the tax report can be audited by the IRS, while the book profit report can be audited by the Securities and Exchange Commission (SEC).

Mills and Sansing [10] provide a game theoretical model in which the IRS takes advantage of the correlation between the book and tax profit levels in its audit decisions. The authors abstract from the fact that, in reality, the firm also has incentives to manipulate the report on book profits, and assume that the IRS knows with certainty the real value of the book profits. They show that, in equilibrium, the higher is the difference between the book profits and the reported taxes, the higher is the probability of a fiscal audit. Mills [9], and Mills and Sansing [10] find empirical evidence that supports the above theoretical result.

However, there is empirical evidence that firms can also misreport their level of book profits, mainly for other purposes than tax evasion (see Erickson et al [4]). A theoretical paper that takes into account the possibility of manipulating both reports is Goerke [5]. He analyzes the trade-off between overreporting book profits and underreporting taxes, and shows that, in some cases, firms have incentives to pay extra taxes if this allows them to inflate the reported value of their book profit. Nevertheless, tax overreporting is marginal with respect to the growing phenomenon of underreporting. For instance, Rice [11] finds in a US sample, that more than two thirds of the firms underreported their tax liabilities, while only 6 percent actually overreported.

The contribution of the present paper is two-folded. It represents a first attempt to model the strategic interaction between the IRS and the firms, taking into account the influence of the accounting reporting and of the audit activity of the SEC, on the behavior of the players. Additionally, the paper provides a simple framework to analyze the impact of the increasing financial sophistication on tax compliance and control. In this setup we can depict straightforward scenarios in which decreasing IRS audit rates and weaker fiscal discipline appear both endogenously and simultaneously (although we do not rule out the existing explanations of the coexistence of these two phenomena). For example, if growing financial sophistication is paralleled by changes in the information on book profits available to the tax authority (i.e. an increase in the audit rate of the SEC), or changes in the distribution of the book profits, then a decrease in the total fiscal audit rate can appear at the same time with an increase in the tax aggressiveness of firms.

The rest of the paper is organized as follows. In Section 2, we provide three versions of the model. The first assumes away the role of the SEC audit; the second one takes as given the SEC probability of audit. Finally, the third version assigns an objective function to the active SEC. Section 3 studies the comparative statics of the equilibrium in each of the models, and Section 4 discusses the results and concludes. The proofs are relegated to the Appendix.

2 The model

Within an audit class, firms can be of three types, given by their level of book profits and taxes. The book profit B can take two values, $B \in \{B_1, B_2\}$,

$B_2 > B_1$, with probabilities $1 - q$ and q. We understand by B the level of profits computed for financial reporting purposes that comply with the Generally Accepted Accounting Principles (GAAP). The level of taxes is $T \in \{0, 1\}$. For a given firm, we interpret T as the minimum level of tax liabilities that the firm can achieve by transactions that do not break the current tax legislation. More precisely, if the firm is audited by the IRS, these transactions cannot be proved illegal, given the current tax law and the level of book-tax conformity this legislation induces. We assume that when the firms are characterized by low book profits, they can always reduce their taxes to the minimal value within the audit class, which we normalize to 0. In the group of firms characterized by high book profits, we assume that a percentage p can reduce their taxes to $T = 0$ by transactions that could not be proved illegal by an IRS audit. The rest of the firms have a tax liability $T = 1$.

The fact that a fraction p of high book profits firms can reduce their taxes to $T = 0$ can be explained in the following way. In order to save on taxes, firms can exploit discontinuities and loopholes in the tax law, by undertaking a series of transactions that do not have an underlying business purpose, but the avoidance of taxes. The increasing complexity of the financial markets and the financial innovations, the greater supply of tax experts, and the availability of software and low-cost technologies to carry out complicated transactions breed opportunities for firms to lower their taxes without breaking the law. Whether the individual firm can exploit these opportunities or not depends on the particular circumstances it faces. The tax laws and procedures are updated frequently, while tax avoidance requires planning and time-consuming operations, therefore some of the firms (in our model, the fraction $1 - p$ of the high book profit firms) will not be able to take advantage of the law at the same scale as others (the fraction p of the high book profit firms).

We understand by financial sophistication all the previously mentioned factors that create opportunities for reducing taxes in a legal way. As argued in the introduction, the expanding financial sophistication may have as effect a simple rescaling to the left of the range of values for the tax liabilities, but also a change in the distribution of these liabilities, in the sense that more and more firms with large amounts of income, are assimilated to the firms with low liabilities. We capture the latter effect through the parameter p. Throughout the paper, an increase in p will always be associated to the phenomenon of expanding financial sophistication.

We do not model here the process by which firms lower their tax liability, or the costs that these transactions might cause to the firm. We assume that firms perfectly understand the circumstances they face, and know what procedures they can undertake to reduce taxes. Therefore, in our model firms know their type, given by (B, T). Note that the type $(B_2, 1)$ cannot reduce the tax liabilities below $T = 1$ without breaking the law. The only possibility to lower its taxes is by evasion, in which case an IRS audit would identify the illegal tax shelter. We further assume that the tax authority knows the distribution of the accounting profits and tax liabilities, but does not know the true type of each firm. The SEC knows only the distribution of the accounting profits.

The timing and the information structure of the game played by the firms, the IRS and the SEC is as follows. After the firm observes its type, it gives a report (x, y), where $x \in \{B_1, B_2\}$ and $y \in \{0, 1\}$. The model assumes that there is no overlapping between audit classes, in the sense that the firms of an audit class can never report taxes lower than the normalized

zero level, or profits higher than B_2.[1] The SEC can observe and audit only the accounting report x, while the IRS can see the report (x, y) and can audit the tax report y. In ours setup, if the SEC audits the report x, the IRS has access to the result of the audit before it makes its decision. Furthermore, any audit conducted by the SEC or by the IRS leads to full disclosure of the corresponding true value.

An important assumption related to the information structure is that the IRS has access to the information reported to the SEC, but not the other way round. This is motivated by the following facts. On the one hand, in the US the book profit reports are quarterly. They are public, and any of them can be subject to a SEC audit, whose result is public as well. (Note that we simplify the model by combining all the relevant quarterly book profit reports in a single variable x.) Moreover, the firm must also give financial accounting details when it files its tax return to the IRS. On the other hand, the tax reports and audits are yearly. Therefore, we assume that the results of a SEC audit are available to the IRS before the latter institution makes its own audit decisions. An implicit assumption is that, if a SEC audit discloses the real value of any quarterly book profit level, then the IRS can use the information as a perfect indicator of the value of the current year book profits. The tax liability of a firm is not publicly disclosed and it is very difficult to estimate; hence we assume that the SEC does not know the taxable income of the firms.[2]

The tax sheltering decisions are not made in an instant of time, at the end of the year, but throughout the year by means of time-consuming operations. Thus, it seems reasonable to assume that, by the time the SEC audit becomes public, these decisions have been already taken, and therefore the report y on taxes cannot be modified by the firm after a SEC audit. Hence, in our model the firm decides jointly the report (x, y), under uncertainty with respect to whether there will be a SEC audit of x or not.

We consider that the firm is risk neutral and maximizes the following payoff. When there is no audit, the payoff equals reported accounting profits less reported taxes, $\pi = x - y$. If only the IRS audits, then $\pi = x - T - F \cdot 1_{\{T > y\}}$, where F is the fine that the IRS applies for evasion. If only the SEC conducts the audit, then $\pi = B - y$. Finally, when both institutions perform the audit, $\pi = B - T - F \cdot 1_{\{T > y\}}$. This payoff function is the simplest way to convey the idea that the management of the firm has incentives to overreport book profits and to underreport tax liabilities. On the one hand, firms have substantial incentives to inflate their book earnings, because the higher the reported book profits, the higher is the market value of the firm and the bonus to the managers that accrues from good performance (Erickson et al. [4]). On the other hand, saving on taxes decreases one of the most significant costs of a firm. There is increasing evidence that firms put more and more emphasis on the activity of their tax departments, while low tax liabilities are considered a measure of performance [14].

If the fiscal audit reveals that taxes were underreported, the IRS applies a fine F.[3] Because of the strong assumption of non-overlapping audit classes, the firms with low taxes will never have incentives to evade, hence they will

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[1]We consider that the SEC and the IRS identify the audit class the firm belongs to, based on parameters like the size, the type of business etc. If the firm reports profits and taxes that do not characterize its audit class, it will prompt for sure the corresponding audit.

[2]For a detailed discussion, see Hanlon [6], and Hanlon and Shevlin [7]

[3]Generally, the fine is proportional to the amount evaded. Since we have a discrete model with only 2 levels of taxes, this is achieved in a trivial way.

never be fined by the IRS. A SEC audit discloses the real book profits to the financial markets and the shareholders. Therefore, when there is a SEC audit, the firm's payoff will depend on the real book profits. We neglect any possible fine paid to the SEC, because we are interested mainly in how the extra information provided by the SEC audit can influence the interaction between the IRS and the firms. Furthermore, before the Sarbanes-Oxley Act of 2002, that increased the penalties for accounting fraud, it was rather typical for the SEC to avoid applying a fine or to minimize it, given that the firm committed to take remedial measures.[4]

We assume that $B_1 < B_2 - 1$. This assumption matches the empirical evidence that, in general, large profit firms do no have incentives to report lower profits in order to pay less taxes (see Erickson et al [4]). It also makes the equilibrium analysis more tractable. If this condition holds, the book profits report does not play any role in the IRS decision, in the absence of the SEC audit. This stems from the fact that, when this condition is fulfilled, the IRS sees in equilibrium only one accounting report ($x = B_2$).

We consider that the IRS is risk neutral. If it does not audit, the IRS gets y, where y is the reported tax income. When it does audit, it gets the true amount of due taxes, plus a fine F if the firm underreported taxes. We assume that the IRS applies the following audit technology. Based on the information it has (the reports and the SEC audit), the IRS divides into subclasses the firms within an audit class. We take the view that auditing a relevant subclass implies a certain specialization of the IRS personnel. Thus, we assume that, for each informational subclass, the IRS chooses independently the probability of audit, by maximizing the expected net revenue. Greater probability of audit requires greater effort, which means higher costs. Finally, like in Reinganum and Wilde[8], we do not impose a fixed budget for the IRS, hence the tax authority can conduct as many audits as it desires.

Let the cost of audit be $c(\rho)$, where ρ is the probability of audit. Assume that the continuously differentiable cost function $c : [0, \infty) \to [0, \infty)$ satisfies the following properties:

(i) $c'(0) = 0$; $c'(\rho) > 0, \rho > 0$; $c''(\rho) > 0, \rho \geq 0$;
(ii) $c'^{-1}([0, 2]) \subseteq [0, 1]$;
(iii) $c'^{-1}(2) > \frac{1}{2}$.

The first property states the convexity of the cost function. Since there are obvious time constraints on the activity of the specialized teams of IRS, it is reasonable to think that auditing more firms in the same amount of time gives rise to increasing marginal costs. The last two properties are technical.

2.1 A model without the SEC (model A)

We study the effect of introducing the SEC audit by considering the benchmark model where this institution does not exist. The timing is the following. Nature chooses the type that is revealed to the firm, then the firm submits the pair of reports (x, y). The IRS sees the reports and audits the tax liabilities; finally, payoffs are realized.

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[4]We are aware of the fact that, although the SEC does not apply a fine, the payoff can decrease below the value $B - y$ (respectively, $B - T - F \cdot 1_{\{T > y\}}$). This can happen if the financial markets react to the lie of the firm, or if the shareholders penalize the management. We abstract from these considerations for the reasons explained above.

2.2 The passive SEC (model B)

In this version of the model the SEC is not an active agent. The timing of the game is as follows: nature chooses the levels of accounting and taxable profits that give the firm type. The firm submits the pair of reports (x, y). The value of σ is realized, where σ is the exogenous probability of a SEC audit. Given the result of the SEC audit, the IRS makes its own audit decisions. Payoffs are realized.

2.3 The active SEC (model C)

In this case, the SEC probability of audit is endogenous. The SEC is the institution that supervises the well functioning of the security market, and is concerned primarily with promoting the disclosure of important market-related information, maintaining fair dealing, and protecting against fraud. We assign the SEC the truth-telling objective function, $-\nu \cdot (B - x)^2 - \frac{1}{2}\sigma^2$, where $\frac{1}{2} \cdot \sigma^2$ is the quadratic cost of auditing and ν is a rescaling constant. The timing of the game is the same as in model B, with the SEC deciding the probability σ of inspecting after seeing the report x.

3 Results

3.1 Solving model A

The condition $B_1 < B_2 - 1$ together with the fact that there is no audit of the book income report, implies that, for all types, $x = B_2$. Also, whenever $T = 0$, the firm will report truthfully $y = 0$. Therefore, in equilibrium, only two types of reports appear: $(x, y) = (B_2, 0)$ and $(x, y) = (B_2, 1)$.

The IRS will not audit the report $(B_2, 1)$. The report $(B_2, 0)$ can be filed by any type of firm, and the IRS maximizes $\rho \cdot \frac{q \cdot (1-p) \cdot \alpha}{1 - q \cdot (1-p) + \alpha \cdot q \cdot (1-p)} \cdot (1 + F) - c(\rho)$, where α is the probability that the type $(B_2, 1)$ reports $(B_2, 0)$. The solution of the above maximization problem, for any given $(F, q, p) \in (0, 1)^3$ and $\alpha \in [0, 1]$, has the form:

$$\rho_0(F, q, p, \alpha) = c'^{-1}\left((1 + F) \cdot \frac{q \cdot (1 - p) \cdot \alpha}{1 - q \cdot (1 - p) + q \cdot (1 - p) \cdot \alpha}\right) \quad (1)$$

We define the function $\rho_0 : (0, 1)^3 \times [0, \infty) \to R$, given by equation (1), and the function $g_1 : (0, 1)^3 \times [0, \infty) \to R$, $g_1(F, q, p, \alpha) = (1 + F) \cdot \rho_0(F, q, p, \alpha)$. Consider the following relations:

$$g_1(F, q, p, \alpha) = 1 \quad (2)$$

$$g_1(F, q, p, 1) \leq 1 \quad (3)$$

The firm of type $(B_2, 1)$ compares $B_2 - 1$ with $B_2 - \rho_0 \cdot (1 + F)$, hence equation (2) and $0 < \alpha < 1$ are the conditions for a mixed equilibrium strategy. Inequality (3) represents the condition for the existence of a pure equilibrium strategy $\alpha = 1$. It is trivial to prove that there is no equilibrium with $\alpha = 0$.

3.2 Solving model B

By the same argument as in model A, and since the SEC does not apply any fine, in equilibrium there are only two types of reports: $(x, y) = (B_2, 0)$ and $(x, y) = (B_2, 1)$. All types with $T = 0$ declare $x = 0$. Denote α the probability that type $(B_2, 1)$ reports $(B_2, 0)$.

The IRS will never audit a report $(B_2, 1)$, and it will not audit a firm whose book income was disclosed to be B_1 after a SEC audit. The IRS solves two independent maximization problems, for any given $(F, q, p) \in (0, 1)^3$ and $\alpha \in [0, 1]$, with the solutions:

$$\rho_1(F, p, \alpha) = c'^{-1}\left((1 + F) \cdot \frac{(1 - p) \cdot \alpha}{p + (1 - p) \cdot \alpha}\right) \tag{4}$$

$$\rho_2(F, q, p, \alpha) = c'^{-1}\left((1 + F) \cdot \frac{q \cdot (1 - p) \cdot \alpha}{1 - q \cdot (1 - p) + q \cdot (1 - p) \cdot \alpha}\right) \tag{5}$$

The audit rate of the IRS when the report is $(B_2, 0)$ and the SEC audit revealed $B = B_2$ is ρ_1, and the audit rate of the IRS when the report is $(B_2, 0)$ and the SEC did not perform an audit is ρ_2. Note that, within the group of firms that have not been audited by the SEC, the audit rate of the IRS has the same expression as in model A. Define the functions $\rho_1 : (0, 1)^2 \times [0, \infty) \to R$ and $\rho_2 : (0, 1)^3 \times [0, \infty) \to R$, given by equations (4) and (5).

Also define $g_2 : (0, 1)^4 \times [0, \infty) \to R$, $g_2(F, q, p, \sigma, \alpha) = (1 + F) \cdot [\sigma \cdot \rho_1(F, q, p, \alpha) + (1 - \sigma) \cdot \rho_2(F, q, p, \alpha)]$. Consider the following relations:

$$g_2(F, q, p, \sigma, \alpha) = 1 \tag{6}$$

$$g_2(F, q, p, \sigma, 1) \leq 1 \tag{7}$$

As in the previous model, equation (6) and $0 < \alpha < 1$ are the mixed equilibrium conditions. Inequality (7) is the condition for a pure equilibrium $\alpha = 1$, and there is no equilibrium with $\alpha = 0$.

3.3 Solving model C

The discussion is identical to the previous case, with σ given by the solution of the SEC maximization problem. The SEC payoff does not depend on the behavior of the IRS, therefore the IRS decisions are irrelevant to the problem of the SEC. Note that in equilibrium the only type that lies about the book profits is $(B_1, 0)$, in which case the SEC loses $(B_2 - B_1)^2$ if it does not audit. Irrespective of whether it audits or not the other types, the SEC always get zero revenues. Hence, the SEC audits the report (B_2) with probability σ, where σ is the solution to:

$$\max_{\sigma \in [0, 1]} \quad -\nu \cdot (B_2 - B_1)^2 \cdot (1 - \sigma) \cdot (1 - q) - \frac{1}{2} \cdot \sigma^2.$$

Like in model B, $\alpha = 0$ cannot be a solution. The mixing equilibrium solution is obtained when equation (6) is satisfied, and $\alpha = 1$ when inequality (7) is true, with σ given by $\sigma(q) = \nu \cdot (B_2 - B_1)^2 \cdot (1 - q)$. In order $\sigma(q)$ to be a probability, we impose the condition that the constant ν to be such that $\nu \cdot (B_2 - B_1)^2 < 1$.

3.4 Comparative statics

Recall that p is the probability that a high book profit firm is able to shelter its tax liabilities up to the minimal level $T = 0$, such that a fiscal control cannot prove it illegal. The scalar α represents the rate of evasion (and the average evasion) among the firms of type $(B_2, 1)$. In all three models, the equilibrium value of α will be denoted by α^*. The total rate of evasion is given by $\beta = q \cdot (1 - p) \cdot \alpha$, which also represents the total level of evaded taxes, relative to the size of the economy. The equilibrium value of β will be denoted by β^*, in any of the three models. In the analysis that follows, the scalars α^* and β^* are interpreted as measures of the tax aggressiveness of the firms, in the sense that they are indicators of an abusive tax behavior, which the IRS could theoretically control through better enforcement.

In the following paragraphs, we define the total rate of the fiscal audit. Before that, it is important to notice that the total rate of the fiscal control in an audit class is different from the risk of being audited that the firms within the class perceive. The latter is given by a convex combination of the conditional probabilities of being audited inside the informational subclasses to which the IRS assigns a given firm, based on their reports and the existing SEC audit results. The weights in the convex combination are given by the probabilities with which the individual firm expects to be assigned to the respective subclasses. The total rate of the fiscal audit is a convex combination of the same conditional probabilities of audit, but the weights are given by the sizes of the corresponding informational subclasses.

In model A, the type $(B_2, 1)$ chooses the probability α to evade, when facing the risk of being audited ρ_0; however, the total audit rate applied by the IRS is given by the function $\rho_3 : (0, 1)^3 \times [0, 1] \to R$:

$$\rho_3(F, q, p, \alpha) = \rho_0(F, q, p, \alpha) \cdot [1 - q \cdot (1 - p) + q \cdot (1 - p) \cdot \alpha] \qquad (8)$$

In the versions B and C of the model, the perceived risk of fiscal control of a firm $(B_2, 1)$ that evades is given by $\sigma \cdot \rho_1 + (1 - \sigma) \cdot \rho_2$ (with the corresponding analytical form for σ in model C); the total rate of the fiscal audit is given by the function $\rho_4 : (0, 1)^4 \times [0, 1] \to R$:

$$\rho_4(F, q, p, \sigma, \alpha) = \begin{aligned} &(1 - \sigma) \cdot \rho_2(F, q, p, \alpha) \left[1 - q \cdot (1 - p) \cdot (1 - \alpha)\right] + \\ &+ \sigma \cdot \rho_1(F, p, \alpha) \cdot [q \cdot p + q \cdot (1 - p) \cdot \alpha] \end{aligned} \qquad (9)$$

The equilibrium values of ρ_3 and ρ_4 are obtained by plugging in the corresponding equilibrium value α^*. We shall denote by ρ^* the equilibrium value of ρ_3 in model A, respectively the equilibrium value of ρ_4 in model B or model C. It is also useful to recall that the probability of a fiscal control perceived by a firm $(B_2, 1)$ which evades depends in equilibrium only on the penalty value F (as the equations (2) and (6) show). With these observations at hand, the intuition of the first proposition is straightforward: an increase in financial sophistication induces, ceteris paribus, an increase in the tax aggressiveness of firms and in the total rate of the fiscal audit.

Proposition 1 *In any of the models A, B, C, an increase in p induces, ceteris paribus and up to a maximum threshold, an increase in α^* within the interval $(0, 1)$, and respectively an increase in both β^* and ρ^*. Above the maximum threshold, we have that $\alpha^* = 1$, while β^* and ρ^* are decreasing in p.*

The proof of the proposition is an immediate consequence of Lemma 1 and Lemma 2, (i)-(ii) in the Appendix, and it is left to the reader. The result of interest is the first part of the proposition, which describes the effect of increasing p up to the maximum threshold. In the following analysis, we shall focus on the regions where $\alpha^* \in (0, 1)$, since on the regions where $\alpha^* = 1$, the values of β^* and ρ^* will depend in a trivial way on the parameters of the model. Moreover, on those regions where the values of the parameters are such that α^* equals 1, all firms susceptible of evasion actually do it, which means that neither α^* nor β^* reflect changes in the tax aggressiveness of firms.

The message of Proposition 1 is simple. When the financial sophistication expands, the effect is that relatively more firms with high book profits are assimilated to the firms with low liabilities, since they are able to conceal large amounts of due taxes by means of transactions that the IRS cannot prove illegal. Under these circumstances, the firms with large book profits that cannot achieve such low levels of tax liabilities within the margins of the law will find it easier to 'hide' when underreporting, hence they will evade with higher probability. Moreover, these companies may resort to fraud to such an extent that their total evasion can increase, offsetting the effect that they are relatively fewer. Indeed, in our model, because of the constant penalties, the risk of fiscal control perceived by a firm $(B_2, 1)$ that evades remains constant in equilibrium.[5] Suppose by contradiction that $q \cdot (1-p) \cdot \alpha^*$ decreases. Because the percentage $1 - q$ of the firms that inflate their book profits does not change (nor the audit rate of the SEC, in models B and C), then a decrease in $(1 - p) \cdot \alpha^*$ would invariably lead to lower probabilities of fiscal control, at all informational subclasses that an evading firm $(B_2, 1)$ can be assigned to. This would further imply that the risk of fiscal control perceived by an evading firm $(B_2, 1)$ cannot remain constant in equilibrium, which provides the contradiction. Hence, we obtain that an increase in p can only induce, ceteris paribus, an increase in β^*.

There is a stronger version of Proposition 1 for model A: as far as the penalty function does not change, there does not exist any scenario such that the total audit rate of the IRS has different monotonicity than the tax aggressiveness of the firms. This result is stated in the next proposition.

Proposition 2 *Consider the economy in model A, characterized by the vector of parameters $e = (F, q, p) \in (0, 1)^3$. There cannot be found two vectors $e_1 = (F, q_1, p_1)$ and $e_2 = (F, q_2, p_2)$, such that a shift from e_1 to e_2 induces in equilibrium the following effect: either α^* or β^* moves in an opposite direction than ρ^* (as far as α^* remains within the interval $(0, 1)$).*

The proof of the second proposition is given in the Appendix. The main difference between model A and models B or C is the inclusion of the audit activity of the SEC. In the B model, the SEC audit rate is exogenously given. An intuitive result that can be easily proved (see Lemma 2,(iii).2-3) is that an increase in the audit rate of the SEC induces, ceteris paribus, a decrease in the total rate of the IRS audit and strengthens the fiscal discipline in the market (in the sense that α^* and β^* decrease).

We know from Proposition 1 that an increase in p determines, ceteris paribus, an increase in the total rate of the fiscal audit and weakens the

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[5]Note that, in our setup, constant penalties imply that in equilibrium the firms which evade perceive the same risk of being audited. This contradicts the view expressed in the report [14], i.e. that there is necessarily a causal relationship from lower overall fiscal audit rates to lower perceived risk of being audited.

fiscal discipline in the market. These remarks raise the following question: as far as the penalty does not change, is it possible that the increasing audit rates of the SEC have stronger effect on the total rate of the fiscal audit, but weaker on the fiscal discipline, with respect to the effect that the increasing financial sophistication has? The next proposition positively answers to this question, in an open set of values for the parameters of model B.

Proposition 3 *Consider the economy in model B, characterized by the vector of parameters $e = (F, q, p, \sigma) \in (0,1)^4$. There exists a non-empty open set $U^* \subset (0,1)^3$ such that for all $(F, q, \sigma_i) \in U^*, i \in 1, 2$ with $\sigma_2 > \sigma_1$, there can be found $1 > p_2 > p_1 > 0$, such that a shift from $e_1 = (F, q, p_1, \sigma_1)$ to $e_2 = (F, q, p_2, \sigma_2)$ induces in equilibrium the following effect: α^* increases within the interval $(0,1)$, and ρ^* decreases. Depending on the choice of the function $c(\cdot)$, it can also be obtained that β^* increases while ρ^* decreases.*

We provide a proof for Proposition 3 in the Appendix. In the same model B, suppose now that there is an increase in $1 - q$ (or equivalently, a decrease in q). A possible scenario for the decrease in q is that the type of business associated to the audit class experiences new entry, and there is a higher probability that a new entrant is of low profitability within the audit class (hence, q decreases and the distribution of profits becomes more skewed to the right).

The firms with low real book profits will manipulate this information and declare high book profits. Therefore, if q decreases, this determines, ceteris paribus, some pressure on the IRS to decrease the conditional probability of control at the informational subclass represented by the reports $(B_2, 0)$, in the absence of the audit results from SEC. However, if the penalty function does not change, then the risk of fiscal control perceived by the firms remains the same in equilibrium. Under these circumstances, a firm $(B_2, 1)$ will increase its probability of evasion α^*, given that the SEC does not react to the decrease in q and keeps the audit rate constant (see Lemma 2,(iv).2). Analogously with the case of an increase in p, there are two opposite effects on β^* when q decreases, ceteris paribus. There is the direct effect of a decrease in q, doubled by an indirect effect given by the increase in α^*. It can be proved that, contrary to the case of an increase in p, the direct effect of a decrease in q can be dominant on the monotonicity of β^*, no matter what the choice of the cost function $c(\cdot)$ would be (see Lemma 3).

If the SEC audit results are available, the IRS will increase the probability of control at the corresponding informational subclass of reports $(B_2, 0)$. This effect can be offset by the decrease in the conditional probability of control at the informational subclass represented by the reports $(B_2, 0)$, when audit results from SEC are not available. Therefore, with a decrease in q, the total rate of the fiscal audit will generally have opposite monotonicity with respect to α^*, but the same with respect to β^*. However, the next proposition shows that the change in the distribution of book profits given by a decrease in q can have stronger effect on the total rate of the fiscal audit, but weaker on β^*, with respect to the effect that the increasing financial sophistication has. (Again, the latter result is not valid for any function $c(\cdot)$ with the properties (i) to (iii), but it depends on this choice.)

Proposition 4 *Consider the economy in model B, characterized by the vector of parameters $e = (F, q, p, \sigma) \in (0,1)^4$. The following assertions hold:*

(i) *There exists a non-empty open set of vectors e such that a decrease in q induces in equilibrium the following effect: α^* increases within the interval $(0,1)$, and ρ^* decreases.*

(ii) *Depending on the choice of the function $c(\cdot)$, it can be obtained that there exists a non-empty open set $U^* \subset (0,1)^3$ such that for all $(F, q_i, \sigma) \in U^*, i \in 1, 2$ with $q_1 > q_2$, there can be found $1 > p_2 > p_1 > 0$, such that a shift from $e_1 = (F, q_1, p_1, \sigma)$ to $e_2 = (F, q_2, p_2, \sigma)$ induces in equilibrium the following effect: both α^* and β^* increase within the interval $(0,1)$, and ρ^* decreases.*

A proof of this proposition is provided in the Appendix. Consider now that the audit rate of the SEC is sensitive to increasing levels of manipulative behavior from the low book profit firms, as in model C. In this case, a decrease in q induces an increase in the audit rate of the SEC. An analogous of Proposition 3 can be stated for model C.

Proposition 5 *Consider the economy in model C, characterized by the vector of parameters $e = (F, q, p) \in (0,1)^3$. There exists a non-empty open set $U^* \subset (0,1)^2$ such that for all $(F, q) \in U^*, i \in 1, 2$ with $q_2 < q_1$, there can be found $1 > p_2 > p_1 > 0$, such that a shift from $e_1 = (F, q_1, p_1)$ to $e_2 = (F, q_2, p_2)$ induces in equilibrium the following effect: α^* increases within the interval $(0,1)$, and ρ^* decreases. Depending on the choice of the function $c(\cdot)$, it can also be obtained that β^* increases while ρ^* decreases.*

Propositions 3 to 5 show that introducing the activity of SEC in the models B and C generates different results with respect to model A (see also Proposition 2). More important, these propositions depict two basic scenarios in which both the decreasing IRS audit rates and the weaker fiscal discipline appear endogenously, that is, if growing financial sophistication is paralleled by changes in the information on book profits available to the tax authority, or changes in the distribution of the book profits. These scenarios involve simple explicative mechanisms that do not rely on the idea of relative changes in the IRS resources or in the applied penalties.

4   Discussion and concluding remarks

The paper presents a simple model of tax compliance where firms decide jointly on the book income and taxable income reports. We take into account the firms' incentives to manipulate both reports, as well as the influence that the disclosure of the real value of book profits by a SEC audit has on the tax compliance game. In this framework, we study the impact of increasing financial sophistication on fiscal discipline and on the audit rate of the IRS.

We prove that, as far as no changes in the applied penalties are assumed, increasing financial sophistication by itself cannot replicate the empirical evidence that the audit rates of the IRS have decreased while tax aggressiveness has increased. The tax aggressiveness referred to in this paper represents abusive fiscal behavior that can be proven illegal by an IRS audit. If we take into account a possible increase in the audit rates of the SEC, that parallels the increase in financial sophistication, then the above mentioned result appear for a range of parameters. The same can happen when there is a shift in the distribution of the firms, in the sense that there is a decrease in the proportion of large book profit firms.

Contrary to the claims of the US Treasury report [14], the mechanisms presented in this paper do not rely on the idea of relative changes in the IRS resources or in the applied penalties. The channel by which the fiscal aggressiveness increases is different from the one suggested by the report. Both the low probabilities of fiscal control and the weaker fiscal discipline appear endogenously, from the way the agents and the tax authority strategically react to the structural changes induced by the financial sophistication and the accounting audit, within the economic environment.

Although it is very simple, the model provides useful insights in studying the corporate tax evasion. First, we show that if the IRS takes into account the auditing activity of the SEC, then there is some degree of substitution between the audit of the SEC and of the IRS. When the audit rate of the SEC increases, the IRS will decrease its equilibrium audit rate. This effect occurs because the disclosure of the real book profits helps the IRS to better identify the type of firm.[6] For the same reason, the tax aggressiveness of the firms will decrease when the SEC audits more. This raises new questions, related to the possible cooperation between the SEC and the IRS. By coordinating their audit, the two institutions might improve efficiency and curb evasion. However, a comprehensive research should be based on a detailed account of the costs and benefits generated by the activity of each of these two organizations.

The increase in the SEC enforcement cases during the 1990's may suggest that its audit rate has increased[7]. However, we should look at the data cautiously, because this is an absolute change that might not be significant, given that the size of the economy has also increased. Moreover, these numbers reflect only the cases where some enforcement action was taken against the firms, and not the total audit activity. The increase in enforcement cases may simply mean that the audit activity is constant, but there is more accounting fraud or the SEC applies a more severe enforcement policy, in the sense that, ceteris paribus, it is more likely to penalize a given type of misconduct. Therefore, for the time being, we consider that we do not have adequate data to test the substitution effect between the IRS audit activity and the one of the SEC.

Second, the crucial prediction of our model is that the increase in financial sophistication has a magnifying effect on the corporate tax compliance. On the one hand, it offers more opportunities for firms to decrease their taxes in such a way that an IRS audit could not prove that the underlying transactions are illegal. On the other hand, it encourages those firms that, because of the particular circumstances they face cannot resort to this type of methods, to undertake more pure evasion. This happens because, as the distribution of the tax liabilities changes, for those firms doing fraud it is easier to go undetected among those that can legally decrease their taxes. The pure evasion of such firms can increase in total amount, despite the fact that they may be relatively fewer. This conclusion has interesting policy implications. Given the magnifying effect that the financial sophistication has, it may be better to act directly upon it in order to curb evasion. That is, the government should keep up with the technological changes by investing resources in improving regulation and outlawing those transactions than only serve the purpose of concealing taxes. In this way, both legal tax

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[6]A caveat is that, although ex-post it is always better that the IRS uses the information given by the SEC, this is not always ex-ante efficient. If audit costs are quadratic, for some values of the parameters the IRS would be better off if it can commit to ignore any information on the book profit report.

[7]See the annual reports on the SEC web page: www.sec.gov.

avoidance and pure evasion would decrease.

We are aware of certain particular features and limitations of our setup. The small number of types and the strong assumption $B_1 < B_2 - 1$ determine the property that, in equilibrium, smaller firms do only overreporting of the accounting profits, while the larger ones do only underreporting of tax liabilities. The model ignores the system of penalties of the SEC, as well as the firms' cost of undertaking transactions that reduce their taxes. The empirically documented fact that there is a change in the distribution of the tax liabilities as an effect of the expanding financial sophistication, is reflected in our model by the following assumption: a fraction p of the high book profit firms can reduce their due taxes to the same level as the low book profit firms. In this way, an increase in the financial sophistication is translated in the increase in p. However, we do not investigate the possible mechanisms that are beneath such an assumption, like the hypothesis that firms with larger amounts of income find it easier to tax avoid more income.

Despite its limitations, the present paper is a first step in modelling the phenomenon of corporate tax evasion, that allow for the influence of the financial reporting on the behavior of the firms and of the IRS. Moreover, it suggests several directions for future research. An interesting empirical question is whether the scenario presented in Proposition 4 fits the case of certain businesses. As an example, the IT-Web business sector experienced massive entry in the 1990's and in the aftermath of the dot.com bubble crush it was discovered that largely overreporting of the book profits was as widespread as tax avoidance activities (see [1]). We can also think of various relevant questions that can be addressed by expanding the present model. For example, it would be interesting to see how the stringency of the level of the accounting disclosure standards, that varies from country to country, and the conformity between book income and taxable income induced by these, influence the tax compliance and the audit rates of the IRS. More stringent disclosure standards means more information available to the IRS. Another important application is studying the incorporation and listing decisions of companies when they take into account both the taxation system and the financial disclosure standards. Finally, it also raises the question of competition between countries. Although international tax competition has been widely studied, we have no knowledge of a model that constructs the competition on two dimensions: taxes and accounting standards.

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A   Appendix

Lemma 1 *Consider the set $U = \left\{(F,q) \in (0,1)^2 : g_1(F,q,0,1) > 1\right\} \neq \emptyset$. Then, the following assertions hold:*

(i) *For every fixed $(F,q) \in U$, there is a unique scalar $p_{max}(F,q)$ in the interval $(0,1)$ such that:*

  1. *For every $p \in (0, p_{max}(F,q))$, inequality (3) is not fulfilled and equation (2) has a unique positive solution, denoted by $\alpha(F,q,p)$, which belongs to the interval $(0,1)$.*

  2. *For $p = p_{max}(F,q)$, inequality (3) is fulfilled with equality and equation (2) has the scalar $\alpha(F,q,p_{max}(F,q)) = 1$ as unique positive solution.*

  3. *For every $p \in (p_{max}(F,q),1)$, inequality (3) is strictly fulfilled and equation (2) has a unique positive solution, denoted by $\alpha(F,q,p)$, which belongs to the interval $(1,\infty)$.*

  4. *The function of p, $f(F,q,p) = min(\alpha(F,q,p),1) : (0,1) \to R$ is continuous, it is bounded by the interval $[0,1]$, it is strictly increasing on the interval $(0, p_{max}(F,q)]$ and it is constant and equal with 1 on the interval $[p_{max}(F,q),1)$.*

5. *The function of p, $f_3(F, q, p) = \rho_3(F, q, p, f(F, q, p)) : (0, 1) \to R$ is continuous , it is bounded by the interval $[0, 1]$, it is strictly increasing on the interval $(0, p_{max}(F, q)]$ and it is strictly decreasing on the interval $[p_{max}(F, q), 1)$.*

(ii) *For every fixed $(F, q) \in (0, 1)^2 \backslash U$, equation (2) does not have solution in the interval $[0, 1)$ and inequality (3) holds, for every $p \in (0, 1)$.*

Proof. The fact that the set U is not empty can be easily checked using the continuity and the property (iii) of the cost function $c(\cdot)$.

(i) Note that for any $(F, q, p) \in U \times (0, 1)$, the solution to the equation (2) is
$\alpha(F, q, p) = \left(\frac{c'(\frac{1}{1+F})}{1+F-c'(\frac{1}{1+F})} \right) \cdot \left(\frac{1}{q \cdot (1-p)} - 1 \right)$. Also define $p_{max}(F, q) = 1 - \frac{c'(\frac{1}{1+F})}{(1+F) \cdot q}$. The rest trivially follows.

(ii) Take some $(F, q) \in (0, 1)^2 \backslash U$. Suppose $\exists\, p_0 \in (0, 1)$ and $\alpha_0 \in [0, 1)$ such that $g_1(F, q, p_0, \alpha_0) = 1$. From $g_1(F, q, p_0, \alpha)$ strictly increasing in α, we have $g_1(F, q, p_0, 1) > 1$. From the decreasing strict monotonicity of $g_1(F, q, p, 1)$ with respect to p, we obtain that $g_1(F, q, 0, 1) > 1$, which provides the contradiction.

Since $g_1(F, q, 0, 1) \leq 1$ and $g_1(F, q, p, 1)$ is decreasing in p on $[0, 1]$, then inequality (3) is fulfilled for every $p \in (0, 1)$.

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Lemma 2 *Consider the set $U = \{(F, q, \sigma) \in (0, 1)^3 : g_2(F, q, 0, \sigma, 1) > 1\} \neq \emptyset$ ($U' = \{(F, q) \in (0, 1)^2 : g_2(F, q, 0, \sigma(q), 1) > 1\} \neq \emptyset$ with $\sigma = \sigma(q)$ for model C). Then, the following assertions hold (assertions (i) and (ii) also hold for the set U' in model C, substituting (F, q, σ) with (F, q), and σ with $\sigma(q)$):*

(i) *For every fixed $(F, q, \sigma) \in U$, there is a unique scalar $p_{max}(F, q, \sigma) \in (0, 1)$ such that:*

1. *For every $p \in (0, p_{max}(F, q, \sigma))$, inequality (7) is not fulfilled and equation (6) has a unique positive solution, denoted by $\alpha(F, q, p, \sigma)$, which belongs to the interval $(0, 1)$.*

2. *For $p = p_{max}(F, q, \sigma)$, inequality (7) is fulfilled with equality and equation (6) has the scalar $\alpha(F, q, p_{max}(F, q, \sigma), \sigma) = 1$ as unique positive solution.*

3. *For every $p \in (p_{max}(F, q, \sigma), 1)$, inequality (7) is strictly fulfilled and equation (6) has a unique positive solution, denoted by $\alpha(F, q, p, \sigma)$, which belongs to the interval $(1, \infty)$.*

4. *The function of p, $f(F, q, p, \sigma) = min(\alpha(F, q, p, \sigma), 1) : (0, 1) \to R$ is continuous, it is bounded by the interval $[0, 1]$, it is strictly increasing on the interval $(0, p_{max}(F, q, \sigma)]$ and it is constant and equal with 1 on the interval $[p_{max}(F, q, \sigma), 1)$.*

5. *The function of p, $f_4(F, q, p, \sigma) = \rho_4(F, q, p, f(F, q, p, \sigma)) : (0, 1) \rightarrow R$ is continuous , it is bounded by the interval $[0, 1]$, it is strictly increasing on the interval $(0, p_{max}(F, q, \sigma)]$ and it is strictly decreasing on the interval $[p_{max}(F, q, \sigma), 1)$.*

(ii) *For every fixed $(F, q, \sigma) \in (0, 1)^3 \backslash U$, equation (6) does not have solution in the interval $[0, 1)$ and inequality (7) holds, for every $p \in (0, 1)$.*

(iii) *For every $(F, q, \sigma_i) \in U, i \in \{1, 2\}$ such that $\sigma_1 < \sigma_2$, we have:*

    1. $p_{max}(F, q, \sigma_1) < p_{max}(F, q, \sigma_2)$.

    2. $\begin{cases} \forall p \in (0, p_{max}(F, q, \sigma_2)), f(F, q, p, \sigma_2) < f(F, q, p, \sigma_1); \\ \forall p \in [p_{max}(F, q, \sigma_2), 1), f(F, q, p, \sigma_2) = f(F, q, p, \sigma_1). \end{cases}$

    3. $\begin{cases} \forall p \in (0, p_{max}(F, q, \sigma_1)], f_4(F, q, p, \sigma_2) < f_4(F, q, p, \sigma_1); \\ f_4(F, q, p_{max}(F, q, \sigma_2), \sigma_2) < f_4(F, q, p_{max}(F, q, \sigma_1), \sigma_1). \end{cases}$

(iv) *For every $(F, q_i, \sigma) \in U, i \in \{1, 2\}$ such that $q_1 < q_2$, we have:*

    1. $p_{max}(F, q_1, \sigma) < p_{max}(F, q_2, \sigma)$.

    2. $\begin{cases} \forall p \in (0, p_{max}(F, q_2, \sigma)), f(F, q_2, p, \sigma) < f(F, q_1, p, \sigma); \\ \forall p \in [p_{max}(F, q_2, \sigma), 1), f(F, q_2, p, \sigma) = f(F, q_1, p, \sigma). \end{cases}$

Proof. The fact that the set U (respectively U') is different from null can be easily checked using the continuity and the property (iii) of the cost function $c(\cdot)$.

(i) Consider a fixed $(F, q, \sigma) \in U$.

The function $g_2(F, q, p, \sigma, 1)$ is continuous and strictly decreasing in p and $g_2(F, q, 0, \sigma, 1) > 1$, $g_2(F, q, 1, \sigma, 1) = 0$. Then, it exists a unique scalar $p_{max}(F, q, \sigma)$ such that $g_2(F, q, p_{max}(F, q, \sigma), \sigma, 1) = 1$.

Moreover:

$g_2(F, q, p, \sigma, 1) > 1$, if $p \in (0, p_{max}(F, q, \sigma))$

and

$g_2(F, q, p, \sigma, 1) < 1$, if $p \in (p_{max}(F, q, \sigma), 1)$.

    1. Fix some $p \in (0, p_{max}(F, q, \sigma))$. The function $g_2(F, q, p, \sigma, \alpha)$ is strictly increasing and continuous in α. Moreover, we have that $g_2(F, q, p, \sigma, 1) > 1$ and $g_2(F, q, p, \sigma, 0) = 0$. Therefore, it exists a unique solution in $(0, 1)$ to the equation $g_2(F, q, p, \sigma, \alpha) = 1$, denoted by $\alpha(F, q, p, \sigma)$.

    2. For $p = p_{max}(F, q, \sigma)$, we have that $g_2(F, q, p_{max}(F, q, \sigma), \sigma, 1) = 1$, hence $\alpha(F, q, p_{max}(F, q, \sigma), \sigma) = 1$.

    3. Fix some $p \in (p_{max}(F, q, \sigma), 1)$. The function $g_2(F, q, p, \sigma, \alpha)$ is strictly increasing and continuous in α. Moreover, we have that $g_2(F, q, p, \sigma, 1) < 1$ and $\lim\limits_{\alpha \to \infty} g_2(F, q, p, \sigma, \alpha) = (1 + F) \cdot c'^{-1}(1 + F)$, which is greater than 1 when $g_2(F, q, 0, \sigma, 1) > 1$. Therefore, it exists a unique solution to the equation $g_2(F, q, p, \sigma, \alpha) = 1$, denoted by $\alpha(F, q, p, \sigma)$, which belongs to the interval $(1, \infty)$.

4. The only non-trivial part to be proved about the function f is that it is strictly increasing on the interval $(0, p_{max}(F, q, \sigma)]$. Define the function $k(p) = (1 - p) \cdot \alpha(F, q, p, \sigma)$ on the interval $(0, p_{max}(F, q, \sigma)]$ and suppose that it exists $p_1 < p_2$ such that $k(p_1) \geq k(p_2)$. Then, the following inequalities hold:

$$\frac{1}{\frac{p_1}{k(p_1)} + 1} > \frac{1}{\frac{p_2}{k(p_2)} + 1} \quad \text{and} \quad \frac{q}{\frac{1-q}{k(p_1)} + q \cdot \frac{p_1}{k(p_1)} + 1} > \frac{q}{\frac{1-q}{k(p_2)} + q \cdot \frac{p_2}{k(p_2)} + 1}.$$

These inequalities imply that:

$g_2(F, q, p_1, \sigma, \alpha(F, q, p_1, \sigma)) > g_2(F, q, p_2, \sigma, \alpha(F, q, p_2, \sigma))$, which provides the contradiction with the definition for $\alpha(F, q, p, \sigma)$. Since the above defined function $k(p)$ is strictly increasing on the interval $(0, p_{max}(F, q, \sigma)]$, then the function $\alpha(F, q, p, \sigma)$ is strictly incresing on the same interval.

5. The only non-trivial part to be proved about the function f_4 is that it is strictly increasing on the interval $(0, p_{max}(F, q, \sigma)]$. First, we will prove that $\rho_1(F, p, \alpha(F, q, p, \sigma))$ is strictly decreasing and the function $\rho_2(F, q, p, \alpha(F, q, p, \sigma))$ is strictly increasing on the same interval.

Suppose that it exists $p_1 < p_2$ such that $\rho_1(F, p_1, \alpha(F, q, p_1, \sigma)) \leq \rho_1(F, p_2, \alpha(F, q, p_2, \sigma))$. Then $\frac{1}{\frac{p_1}{k(p_1)} + 1} \leq \frac{1}{\frac{p_2}{k(p_2)} + 1}$, thus we have that $\frac{p_1}{k(p_1)} \geq \frac{p_2}{k(p_2)}$. Also note that $\frac{1-q}{q \cdot k(p_1)} > \frac{1-q}{q \cdot k(p_2)}$, because the function $k(p)$ is strictly increasing on the definition domain $(0, p_{max}(F, q, \sigma)]$. The above inequalities imply that:

$$\frac{1}{1 + \frac{q \cdot p_1}{k(p_1)} + \frac{1-q}{q \cdot k(p_1)}} < \frac{1}{1 + \frac{q \cdot p_2}{k(p_2)} + \frac{1-q}{q \cdot k(p_2)}}.$$

Hence, $\rho_2(F, q, p_1, \alpha(F, q, p_1, \sigma)) < \rho_2(F, q, p_2, \alpha(F, q, p_2, \sigma))$.

Use also that $\rho_1(F, p_1, \alpha(F, q, p_1, \sigma)) \leq \rho_1(F, p_2, \alpha(F, q, p_2, \sigma))$, and obtain that:

$g_2(F, q, p_1, \sigma, \alpha(F, q, p_1, \sigma)) < g_2(F, q, p_2, \sigma, \alpha(F, q, p_2, \sigma))$, which provides the contradiction.

We have proved that $\rho_1(F, p, \alpha(F, q, p, \sigma))$ is strictly decreasing on the interval $(0, p_{max}(F, q, \sigma)]$. This implies that the function $\rho_2(F, q, p, \alpha(F, q, p, \sigma))$ is strictly increasing on the same interval.

The function of p, $f_4(F, q, p, \sigma)$ can be written under the following form:

$(q \cdot p + q \cdot k(p)) \cdot \frac{1}{1+F} + (1 - \sigma) \cdot (1 - q) \cdot \rho_2(F, q, p, \alpha(F, q, p, \sigma))$.

It is easy to check that for fixed $(F, q, \sigma) \in U$, this function is strictly increasing on the interval $(0, p_{max}(F, q, \sigma)]$.

(ii) Take some $(F, q, \sigma) \in (0, 1)^3 \backslash U$. Suppose $\exists\, p_0 \in (0, 1)$ and $\alpha_0 \in [0, 1)$ such that $g_2(F, q, p_0, \sigma, \alpha_0) = 1$. From $g_2(F, q, p_0, \sigma, \alpha)$ strictly increasing in α, we have that $g_2(F, q, p_0, \sigma, 1) > 1$. From the decreasing strict monotonicity of $g_2(F, q, p, \sigma, 1)$ with respect to p, we obtain that $g_2(F, q, 0, \sigma, 1) > 1$, which provides the contradiction.

Since $g_1(F, q, 0, \sigma, 1) \leq 1$ and $g_1(F, q, p, \sigma, 1)$ is decreasing in p on $[0, 1]$, then inequality (7) is fulfilled for every $p \in (0, 1)$.

(iii) Consider $(F, q, \sigma_i) \in U$, $i \in \{1, 2\}$ such that $\sigma_1 < \sigma_2$.

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1. For the fixed (F, q) considered above, the implicit function theorem can be applied to $p_{max}(F, q, \sigma)$, for all $\sigma \in (\sigma_1, \sigma_2)$. Therefore, $p_{max}(F, q, \sigma)$ is differentiable with respect to σ on the interval (σ_1, σ_2) and:

$$\frac{\partial p_{max}}{\partial \sigma}(F, q, \sigma) = -\frac{\frac{\partial g_2}{\partial \sigma}}{\frac{\partial g_2}{\partial p}}(F, q, p_{max}(F, q, \sigma), \sigma, 1), \forall \sigma \in (\sigma_1, \sigma_2).$$

The partial derivative $\frac{\partial g_2}{\partial p}(F, q, p_{max}(F, q, \sigma), \sigma, 1)$ exists and it is negative. Note that $\frac{\partial g_2}{\partial \sigma}(F, q, p_{max}(F, q, \sigma), \sigma, 1)$ can be written as $(1 + F) \cdot [\rho_1(F, p_{max}(F, q, \sigma), 1) - \rho_2(F, q, p_{max}(F, q, \sigma), 1)]$, which is greater than zero for every $\sigma \in (\sigma_1, \sigma_2)$. Hence, the function $p_{max}(F, q, \sigma)$ is strictly increasing on the interval (σ_1, σ_2) and by continuity it is on the compact $[\sigma_1, \sigma_2]$. Then, $p_{max}(F, q, \sigma_1) < p_{max}(F, q, \sigma_2)$, for every (F, q, σ_i), $i \in \{1, 2\}$ fixed as above.

2. Fix now some $p \in (0, p_{max}(F, q, \sigma_1))$. We can apply the implicit function theorem and prove that for the fixed $(F, q, \sigma_i) \in U$, $i \in \{1, 2\}$ and $p \in (1, p_{max}(F, q, \sigma_1))$, the function $\alpha(F, q, p, \sigma)$ is differentiable with respect to σ on the interval (σ_1, σ_2) and:

$$\frac{\partial \alpha}{\partial \sigma}(F, q, p, \sigma) = -\frac{\frac{\partial g_2}{\partial \sigma}}{\frac{\partial g_2}{\partial \alpha}}(F, q, p, \sigma, \alpha(F, q, p, \sigma)) < 0, \forall \sigma \in (\sigma_1, \sigma_2).$$

Hence, $f(F, q, p, \sigma_2) < f(F, q, p, \sigma_1)$, and this is fulfilled for every $p \in (0, p_{max}(F, q, \sigma_1))$. The properties of f when p belongs to the interval $[p_{max}(F, q, \sigma_1), p_{max}(F, q, \sigma_2))$, or $p \in [p_{max}(F, q, \sigma_2), 1)$, are trivial to prove.

3. Fix now some $p \in (0, p_{max}(F, q, \sigma_1))$. Write $f_4(F, q, p, \sigma_i)$ as the sum of the following two terms:

$[q \cdot p + q \cdot (1 - p) \cdot \alpha(F, q, p, \sigma_i)] \cdot \frac{1}{1+F}$, and respectively,

$(1 - \sigma_i) \cdot (1 - q) \cdot \rho_2(F, q, p, \alpha(F, q, p, \sigma_i))$.

Since $\sigma_1 < \sigma_2$ and $\alpha(F, q, p, \sigma_2) < \alpha(F, q, p, \sigma_1)$, it is easy to see that $f_4(F, q, p, \sigma_2) < f_4(F, q, p, \sigma_1)$, and this is fulfilled for every $p \in (0, p_{max}(F, q, \sigma_1)]$.

Notice as well that $f_4(F, q, \sigma_i, p_{max}(F, q, \sigma_i))$ is given by:

$q \cdot \frac{1}{1+F} + (1 - \sigma_i) \cdot (1 - q) \cdot \rho_2(F, q, p_{max}(F, q, \sigma_i), 1)$.

Since $\sigma_1 < \sigma_2$ and $p_{max}(F, q, \sigma_1) < p_{max}(F, q, \sigma_2)$, it is easy to see that:

$f_4(F, q, p_{max}(F, q, \sigma_2), \sigma_2) < f_4(F, q, p_{max}(F, q, \sigma_1), \sigma_1)$.

(iv) 1. Consider $(F, q_i, \sigma) \in U$, $i \in 1, 2$ such that $q_1 < q_2$. Notice that $(F, q, \sigma) \in U, \forall q \in [q_1, q_2]$.

Applying the implicit function theorem, we obtain:

$$\frac{\partial p_{max}}{\partial q}(F, q, \sigma) = -\frac{\frac{\partial g_2}{\partial q}}{\frac{\partial g_2}{\partial p}}(F, q, p_{max}(F, q, \sigma), \sigma, 1), \forall q \in (q_1, q_2).$$ This is positive $\forall q \in (q_1, q_2)$.

Hence, the function $p_{max}(F, q, \sigma)$ is increasing on (q_1, q_2) and by continuity with respect to q, $p_{max}(F, q_1, \sigma) < p_{max}(F, q_2, \sigma)$.

2. Fix some $p \in (0, p_{max}(F, q_1, \sigma))$. On the interval (q_1, q_2), by applying implicit function theorem, we obtain:

$$\frac{\partial \alpha}{\partial q}(F,q,p,\sigma) = -\frac{\frac{\partial g_2}{\partial q}}{\frac{\partial g_2}{\partial \alpha}}(F,q,p,\sigma,\alpha(F,q,p,\sigma)) < 0, \forall q \in (q_1,q_2).$$

Hence, $\alpha(F,q,p,\sigma)$ is decreasing on (q_1,q_2) and by continuity with respect to q, $\alpha(F,q_1,p,\sigma) > \alpha(F,q_2,p,\sigma)$. This is fulfilled for every $p \in (0,p_{max}(F,q_1,\sigma))$.

The properties of f when p is in the interval $[p_{max}(F,q_2,\sigma),1)$, or $[p_{max}(F,q_1,\sigma),p_{max}(F,q_2,\sigma))$, are trivial to prove.

∎

Proof of Proposition 2. Note that as far as α^* belongs to $(0,1)$, then:

$$\alpha^* = \alpha(F,q,p) = \frac{c'(\frac{1}{1+F})}{1+F-c'(\frac{1}{1+F})} \cdot \left(\frac{1}{q \cdot (1-p)} - 1\right)$$

$$\beta^* = q \cdot (1-p) \cdot \alpha(F,q,p) = \frac{c'(\frac{1}{1+F})}{1+F-c'(\frac{1}{1+F})} \cdot (1 - q \cdot (1-p))$$

$$\rho^* = \frac{1}{1+F} \cdot \left(1 + \frac{c'(\frac{1}{1+F})}{1+F-c'(\frac{1}{1+F})}\right) \cdot (1 - q \cdot (1-p)).$$

The conclusion immediately follows. ∎

Proof of Proposition 3. Consider the set $U^* = U$, where U is defined as in Lemma 2, and take $(F,q,\sigma_i) \in U, i \in \{1,2\}$ with $\sigma_2 > \sigma_1$. Lemma 2,(i).5 implies that the function of p, $f_4(F,q,p,\sigma_1)$ has the property that it is strictly increasing and bounded on the interval $(0,p_{max}(F,q,\sigma_1)]$. Denote by $l_1(0)$ the right limit in 0 of this function.

Define $p^* = 0$ if $f_4(F,q,p_{max}(F,q,\sigma_2),\sigma_2) \leq l_1(0)$. If the opposite inequality is true, then define $p^* < p_{max}(F,q,\sigma_1)$ as the unique point such that $f_4(F,q,p^*,\sigma_1) = f_4(F,q,p_{max}(F,q,\sigma_2),\sigma_2)$. If $f_4(F,q,p_{max}(F,q,\sigma_2),\sigma_2) > l_1(0)$, then such a point exists and it is unique in $(0,p_{max}(F,q,\sigma_1))$ with this property. This is because, from Lemma 2,(i).5, the function $f_4(F,q,p,\sigma_1)$ is strictly increasing and continuous on $(0,p_{max}(F,q,\sigma_1)]$, and from Lemma 2,(iii).3, $f_4(F,q,p_{max}(F,q,\sigma_2),\sigma_2) < f_4(F,q,p_{max}(F,q,\sigma_1),\sigma_1)$. Consider any two points $p_1 < p_2$ such that $p^* < p_1 < p_{max}(F,q,\sigma_1)$ and $p^* < p_2 < p_{max}(F,q,\sigma_2)$ and $0 < \alpha(F,q,p_1,\sigma_1) < \alpha(F,q,p_2,\sigma_2) < 1$. It is always possible to find such two points, using the properties described in Lemma 2,(i).4 and (iii).1-2. However, from Lemma 2,(i).5 and (iii).3 and the definition of p^*, for any two points (p_1,p_2) with $p^* < p_1 < p_2 < p_{max}(F,q,\sigma_2)$, we have that $f_4(F,q,p_2,\sigma_2) < f_4(F,q,p_1,\sigma_1)$. This proves the first part of the proposition, since $\alpha(F,q,p,\sigma)$ is just α^* and $f_4(F,q,p,\sigma)$ is nothing else but ρ^*.

If $c(\rho) = \rho^2$, then β^* and ρ^* are the same up to an expression that depends on the constant F. Therefore, when $c(\cdot)$ is quadratic, it is impossible to obtain that β^* and ρ^* have different monotonicity with respect to the parameters of the model. We define the set $U_{|c(\rho)=\rho^2} = \{(F,q,\sigma) \in (0,1)^3 : g_{2|c(\rho)=\rho^2}(F,q,0,\sigma,1) > 1\}$ as the corresponding set of Lemma 2 computed for $c(\cdot)$ quadratic.

The set $U = \{(F,q,\sigma) \in (0,1)^3 : g_2(F,q,0,\sigma,1) > 1\}$ is defined as in Lemma 2, for the actual cost function $c(\cdot)$. On the set $U \cap U_{|c(\rho)=\rho^2}$, both β^* and ρ^* have the properties of f_4 described in Lemma 2. Consider p^* previously defined. For $\sigma_1 < \sigma_2$ fixed and close enough, where $(F,q,\sigma_i) \in$

$U \cap U_{|c(\rho)=\rho^2}, i \in \{1, 2\}$ (we assume for the moment that this open set is not empty), remember that p^* is the unique point such that $0 < p^* < p_{max}(F, q, \sigma_1)$ and $f_4(F, q, p^*, \sigma_1) = f_4(F, q, p_{max}(F, q, \sigma_2), \sigma_2)$. Define p^{**} as the unique point such that $0 < p^{**} < p_{max}(F, q, \sigma_1)$ and $q \cdot (1 - p^{**}) \cdot \alpha(F, q, p^{**}, \sigma_1) = q \cdot (1 - p_{max}(F, q, \sigma_2))$.

If $p^* < p^{**}$, then there exist two points $p_1 < p_2$ such that $p^* < p_1 < p_{max}(F, q, \sigma_1)$, $p^{**} < p_2 < p_{max}(F, q, \sigma_2)$ and $q \cdot (1 - p_1) \cdot \alpha(F, q, p_1, \sigma_1) < q \cdot (1 - p_2) \cdot \alpha(F, q, p_2, \sigma_2)$. From the construction of p^*, p^{**}, p_1, p_2 and the properties of f_4 described by Lemma 2, one can easily obtain that $f_4(F, q, p_2, \sigma_2) < f_4(F, q, p_1, \sigma_1)$. Therefore, for the fixed and close enough $\sigma_1 < \sigma_2$, the condition $p^* < p^{**}$ is sufficient to find $p_1 < p_2$ such that β^* increases while ρ^* decreases (notice that β^* is just $q \cdot (1 - p) \cdot \alpha(F, q, p, \sigma)$ and ρ^* is just $f_4(F, q, p, \sigma)$).

It remains to give an example of an actual function $c(\cdot)$ and $(F_0, q_0, \sigma_i), i \in \{1, 2\}$ in an open ball of the corresponding set $U \cap U_{|c(\rho)=\rho^2}$ with the property that $p^* < p^{**}$. The assumption of nicely behaved functions implies that this latter property holds for $\sigma_1' < \sigma_2'$ in the interval (σ_1, σ_2), F in a neighborhood of F_0 and q in a neighborhood of q_0. In this way we can construct the open set $U^* \subset U \cap U_{|c(\rho)=\rho^2}$ of parameters (F, q, σ) such that the statement of the first statement of the proposition is valid for both β^* and α^*.

Consider the function $c(\rho) = A \cdot \rho^\gamma$, where $A = \frac{2}{\gamma} \cdot ((1 - w_1) \cdot 1 + w_1 \cdot 2^{\frac{1}{\gamma - 1}})$, with $\gamma = \frac{3}{2}$ and $w_1 = \frac{1}{10}$. The properties (i) to (iii) of the function $c(\cdot)$ are all fulfilled. Consider the points $(F_0, q_0, \sigma_i) \in U \cap U_{|c(\rho)=\rho^2}, i \in \{1, 2\}$, with $F_0 = 0.8$, $q_0 = 0.3$, $\sigma_1 = 0.7$ and $\sigma_2 = 0.75$. The points (p^*, p^{**}) are such that $p^* < 0.368$ and $0.375 < p^{**}$. Therefore, $p^* < p^{**}$. An example of a pair of points (p_1, p_2) constructed as suggested before is $(0.37, 0.432)$. (For the corresponding shifts in σ and p, β^* increases in equilibrium from approximately 0.1682 to 0.1698, ρ^* decreases from approximately 0.2275 to 0.2264. The indicator α^* increases as well, remaining though within the interval $(0, 1)$ if the shift in σ is the first one, followed only after by the increase in p).

The condition $p^* < p^{**}$ always insures the construction of (p_1, p_2) with the desired properties in the way we presented before. However, it is possible that (p_1, p_2) with the desired properties can also be found in other regions than $(p^*, p_{max}(F, q, \sigma_1)) \times (p^{**}, p_{max}(F, q, \sigma_2))$. ∎

Lemma 3 *Consider the set $U = \{(F, q, \sigma) \in (0, 1)^3 : g_2(F, q, 0, \sigma, 1) > 1\}$ ($U' = \{(F, q) \in (0, 1)^2 : g_2(F, q, 0, \sigma(q), 1) > 1\}$ for model C). The set $U_0 = \{(F, q, \sigma) \in U : \exists I \ interval \ \subseteq (0, p_{max}(F, q, \sigma)] \ with \ \frac{\partial}{\partial q}(q \cdot \alpha(F, q, p, \sigma)) > 0\}$ ($U_0 = \{(F, q) \in U : \exists I \ interval \ \subseteq (0, p_{max}(F, q, \sigma(q))] \ with \ \frac{\partial}{\partial q}(q \cdot \alpha(F, q, p, \sigma(q))) > 0\}$ for model C) is open and it is not empty.*

Proof. Note that $\frac{\partial}{\partial q}(q \cdot \alpha)$ has the analytical form:

$$\alpha \cdot \left(1 - \frac{(1-\sigma) \cdot t_1 \cdot q}{(1-\sigma) \cdot t_1 \cdot q \cdot (1-q+q \cdot p) + \sigma \cdot t_2 \cdot p \cdot \left(\frac{1-q+q \cdot (p+(1-p) \cdot \alpha)}{q \cdot (p+(1-p) \cdot \alpha)} \right)^2} \right)$$
$$+ \frac{t_3}{(1+F) \cdot (1-p) \cdot \left((1-\sigma) \cdot t_1 \cdot q \cdot (1-q+q \cdot p) + \sigma \cdot t_2 \cdot p \cdot \left(\frac{1-q+q \cdot (p+(1-p) \cdot \alpha)}{q \cdot (p+(1-p) \cdot \alpha)} \right)^2 \right)}, \text{ where}$$

$$t_1 = \left(c'^{-1}\right)' \left((1+F) \cdot \frac{q \cdot (1-p) \cdot \alpha}{1-q+q \cdot (p+(1-p) \cdot \alpha)}\right);$$

$$t_2 = \left(c'^{-1}\right)' \left((1+F) \cdot \frac{(1-p) \cdot \alpha}{(p+(1-p) \cdot \alpha)}\right);$$

$$t_3 = \frac{\partial \sigma}{\partial q} \cdot \left[c'^{-1}\left((1+F) \cdot \frac{q \cdot (1-p) \cdot \alpha}{1-q+q \cdot (p+(1-p) \cdot \alpha)}\right) - c'^{-1}\left((1+F) \cdot \frac{(1-p) \cdot \alpha}{(p+(1-p) \cdot \alpha)}\right)\right]$$

for model C, and $t_3 = 0$ for model B. Notice that in both models $t_3 \geq 0$.

It remains to see that for big enough σ, the term

$$\alpha \cdot \left(1 - \frac{(1-\sigma) \cdot t_1 \cdot q}{(1-\sigma) \cdot t_1 \cdot q \cdot (1-q+q \cdot p) + \sigma \cdot t_2 \cdot p \cdot \left(\frac{1-q+q \cdot (p+(1-p) \cdot \alpha)}{q \cdot (p+(1-p) \cdot \alpha)}\right)^2}\right) \text{ is greater than zero}$$

(notice that, if $(F, q, \sigma_0) \in U$ then $(F, q, \sigma) \in U$ for $\sigma > \sigma_0$; make for instance $\sigma \nearrow 1$; for any (F, q) such that it exists σ_0 with $(F, q, \sigma_0) \in U$, and any suitable choice of p, see that α will converge to a strictly positive value, while the second term converges to 1). ∎

Proof of Proposition 4.

(i) Write $f_4(F, q, p, \sigma_i)$ as the sum of the following two terms:

$[1 - q + q \cdot p + q \cdot (1-p) \cdot \alpha(F, q, p, \sigma(q))] \cdot \frac{1}{1+F}$, and respectively,

$-\sigma \cdot (1-q) \cdot \rho_1(F, p, \alpha(F, q, p, \sigma))$. The derivative with respect to q is:

$\frac{1}{1+F} \cdot (1-p) \cdot \frac{\partial}{\partial q}(\alpha \cdot q) + \sigma \cdot \rho_1 + (-\sigma \cdot (1-q) \cdot \frac{\partial}{\partial q}\rho_1 - (1-p) \cdot \frac{1}{1+F})$. It is easy to check that choosing a vector (F, q, σ) within the set U_0 insures that $\frac{\partial}{\partial q}(\alpha \cdot q)$ is greater than zero even when $p \to p_{max}(F, q, \sigma)$. However $p \to p_{max}(F, q, \sigma)$ determines that $(-\sigma \cdot (1-q) \cdot \frac{\partial}{\partial q}\rho_1 - (1-p) \cdot \frac{1}{1+F})$ goes to a positive number. The rest of the details are left to the reader.

(ii) Consider the set U as defined in Lemma 2. Consider as well the open set $U_1 = \{(F, q, \sigma) \in U : \frac{\partial \beta^*}{\partial q} > 0; \frac{\partial \rho^*}{\partial q} > 0, \forall p \in (0, p_{max}(F, q, \sigma)]\}$, and assume that is not empty. For $q_1 > q_2$ fixed and close enough, where $(F, q_i, \sigma) \in U_1$, $i \in \{1, 2\}$, we define p^* as the unique point such that $0 < p^* < p_{max}(F, q_2, \sigma)$, $f_4(F, q_1, p^*, \sigma) = f_4(F, q_2, p_{max}(F, q_2, \sigma), \sigma)$ and p^{**} is the unique point such that $0 < p^{**} < p_{max}(F, q_2, \sigma)$ and $q_1 \cdot (1 - p^{**}) \cdot \alpha(F, q_1, p^{**}, \sigma) = q_2 \cdot (1 - p_{max}(F, q_2, \sigma))$. Analogously to the proof of the second statement of Proposition 3, it can be proved that the condition $p^* < p^{**}$ is sufficient to find $p_1 < p_2$ such that β^* increases while ρ^* decreases. Moreover, α^* can only increase with a decrease in q and an increase in p. It remains to give an example of an actual function $c(\cdot)$ and (F_0, q_i, σ_0), $i \in \{1, 2\}$ in an open ball of the corresponding U_1 with the property that $p^* < p^{**}$. The assumption of nicely behaved functions implies that this latter property holds for $q_1' > q_2'$ in the interval (q_2, q_1), F in a neighborhood of F_0 and σ in a neighborhood of σ_0. In this way we can construct the open set U^* with the desired property. Finally, consider the example $c(\rho) = A \cdot \rho^\gamma$, where $A = \frac{2}{\gamma} \cdot ((1 - w_1) \cdot 1 + w_1 \cdot 2^{\frac{1}{\gamma-1}})$, with $\gamma = 3$ and $w_1 = \frac{1}{10}$ and the points $(F_0, q_i, \sigma_0) \in U_0, i \in \{1, 2\}$, with $F_0 = 0.9$, $\sigma_0 = 0.8$, $q_1 = 0.3$ and $q_2 = 0.27$. Then $(F_0, q_i, \sigma_0) \in U_1$ and $p^* < 0.107(3) < p^{**}$.

∎

Proof of Proposition 5.   A decrease in q induces an increase in σ. Lemma 2,(iii) shows that, if σ increases in model B, then ρ^* decreases; moreover, Proposition 4,(i) shows that a decrease in q alters ρ^* in a similar way. Therefore, if q decreases in model C, ρ^* can only decrease. The increase in σ puts a downward pressure on α^*, while the decrease in q puts an upward pressure on it. We leave to the reader the exercise to specify an open set U^* of vectors (F, q) such that the indirect effect given by the increase in σ is dominant with respect to the direct effect induced by the decrease in q. Moreover, the set U^* should be constructed in such a way that the properties of $\alpha(F, q, p)$ and $f_4(F, q, p)$ for a decrease in q will be exactly the same with the ones of $\alpha(F, q, \sigma, p)$ and $f_4(F, q, \sigma, p)$ in the proof of Proposition 3 for an increase in σ. Then, the proof of the first statement of this proposition will be analogous with the one of the first statement of Proposition 3.

Regarding the second statement of the proposition, consider the function $c(\rho) = A \cdot \rho^\gamma$, where $A = \frac{2}{\gamma} \cdot ((1-w_1) \cdot 1 + w_1 \cdot 2^{\frac{1}{\gamma-1}})$, with $\gamma = \frac{3}{2}$ and $w_1 = \frac{1}{10}$. Consider as well the points $(F_0, q_i), i \in \{1, 2\}$, with $F_0 = 0.8$, $q_1 = 0.3$ and $q_2 = 0.25$. Choose the rescaling scalar ν such that $\sigma_i = 1 - q_i$. In this way σ_i are exactly the ones in Proposition 3. The corresponding points p^* and p^{**} are such that $p^* < 0.299$ and $p^{**} > 0.302$. The rest of the proof is analogous to the one of Proposition 3. ∎